Isaac Newton Institute for Mathematical Sciences

Combinatorics and Statistical Mechanics

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Organisers: P Cameron and B Jackson (Queen Mary, London), A Scott (Oxford), A Sokal (New York University/UCL), D Wagner (Waterloo)

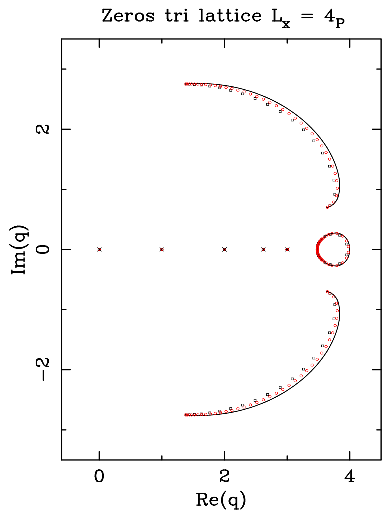
A classical problem from the nineteenth century is the map colouring problem: what is the minimum number of colours required to colour a given map if adjacent countries must receive different colours? A more modern version involves assigning radio frequencies to transmitters so that nearby transmitters use different frequencies. If k colours (or frequencies) are available, then the number of permissible colourings (or frequency allocations) is in fact a polynomial in k, called the *chromatic polynomial*. The minimum number of colours needed for a permissible colouring is thus the smallest positive integer that is not a zero of this polynomial.

The map colouring problem belongs to the branch of mathematics known as *combinatorics*, which is concerned with the study of discrete structures. In this problem the underlying combinatorial structure is a *graph*: its vertices are the countries or transmitters and its edges encode adjacency of countries or transmitters. The chromatic polynomial actually contains much information about the graph beyond counting colourings. For example, evaluated at –1 it counts the orientations of the edges that do not create a cycle. Moreover, the chromatic polynomial is a specialisation of a two-variable polynomial called the *Tutte polynomial* that contains even more information.

A similar situation arises in the branch of physics known as *statistical mechanics*. Suppose, for instance, that each atom in a crystalline lattice can be in one of k states, and that the interaction energy between two adjacent atoms depends only on whether they are in the same or different states. (For k=2 this is the famous Ising model of ferromagnetism, and for arbitrary k it is the Potts model). Each configuration is then assigned a weight, $e^{-\beta E}$ where E is the configuration's total energy and β is the inverse temperature; the sum of these weights over all configurations is termed the *partition function*. It turns out that the Potts-model partition function is simply the Tutte polynomial in disguise.

Physicists are used to the notion of *phase transitions*, whereby some global property changes discontinuously (or non-differentiably, or in general non-analytically) as a parameter is varied: ice melts, an iron bar becomes magnetised, etc.

Such discontinuous changes are not possible in a finite system, but arise in the infinite-volume limit. Combinatorialists are also familiar with phase transitions, for example in random-graph models: as some parameter varies, the typical graph abruptly develops a giant component, or becomes connected.



As Yang and Lee showed in 1952, the properties of phase transitions can be elucidated by studying the zeros of the partition function in the complex plane – even though non-real values of the variable are physically meaningless. This gives a physical motivation for studying the complex zeros of the chromatic polynomial (shown in the figure for a $4\times n$ triangular lattice) and other combinatorial polynomials.

These ideas and connections are beginning to be applied in contexts much wider than graphs. Combinatorial generalisations include matroids, partitions, permutations, partial orders, species, and orbits of infinite permutation groups. Methods and insights from physics are also being applied in contexts such as combinatorial identities and their q-analogues, principles of enumeration (exact, asymptotic or approximate), correlation inequalities, Markov chains and more.

The aims of this programme are to bring together expertise in the areas of combinatorics, statistical mechanics, probability and computer science, to attack unsolved problems and develop new areas of interaction. We also aim to attract young researchers to the field by providing tutorial lectures.