Isaac Newton Institute for Mathematical Sciences

Discrete Integrable Systems

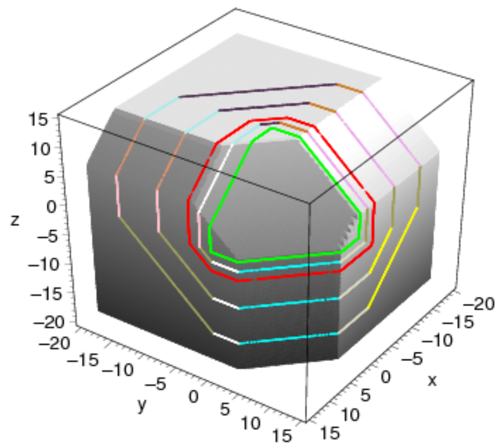
19 January to 3 July 2009

Organisers: RG Halburd (University College London), FW Nijhoff (Leeds), M Noumi (Kobe), GRW Quispel (La Trobe), O Ragnisco (Rome)

Many processes in physics are described mathematically by differential equations that reflect the smoothness of natural processes we experience in our daily lives. However, in modern physics the assumption of a space-time continuum is no longer always adequate and with the advent of quantum mechanics, and in more recent years quantum gravity, we have learned to appreciate the inherent discreteness of physical phenomena at the atomic and subatomic level. To describe these discrete phenomena we need a mathematical machinery comprising *difference equations* rather than differential equations and an inherently discrete theory of functions as the basic mathematical tools. It is a surprising fact that in many respects such a theory still needs to be built.

Difference equations are mathematical equations involving functions whose arguments are shifted by integer or finite steps. If one would take the limit that the step size becomes infinitesimally small one often recovers a corresponding differential equation, but without the limit the equation is essentially nonlocal. This is one reason why the theory of difference equations, in spite of the numerous applications e.g. in numerical analysis, control theory, mathematical biology and economics, has lagged behind the analogous theory of differential equations, as this nonlocality renders such systems both richer as well as more difficult to treat. Furthermore, the fact that several distinct difference equations may reduce to one and the same differential equation after a continuum limit adds to the difficulty of singling out a preferred difference equation as a model for a specific physical phenomenon. It is here that the notion of *integrability* can play a crucial role, as the latter property is very special and may help to select the (in principle) "solvable" equations from the non-solvable ones.

Although there is no accepted general definition of an integrable system, the innumerable examples of such systems (comprising partial and ordinary differential and difference equations, Hamiltonian many-body systems, nonlinear dynamical systems and model systems in quantum mechanics) share a core of distinguishing features such as the existence of complete sets of conservation laws, exact and rigorous solution methods (such as the inverse scattering transform method) and an abundance of explicit, albeit nontrivial solutions. In the past two decades the theory of integrable discrete systems has undergone a true revolution and many novel mathematical phenomena have been discovered through the study of the latter models. It is clear that this area of research is on the verge of some important breakthroughs which will have an impact well beyond the specific field of integrable systems.



An integral manifold with seven plane segments

The programme at the Institute will bring together experts from different fields, such as complex analysis, algebraic geometry, representation theory, Galois theory, spectral analysis, special functions, graph theory, difference geometry and naturally integrable systems, in order for an important cross-fertilisation of ideas and approaches to take place. Building on the developments that have already taken place in recent years (such as the discovery of "exact" difference analogues of the famous Painlevé and Garnier differential equations, and their classification, as well as the development of a theory of difference geometry and corresponding integrable lattice systems) it is expected that substantial new advances in the theory of difference equations will result from the programme.

The programme will focus on a number of aspects which are likely to become of major importance for subsequent developments, such as: the connection between integrable dynamical maps and the algebraic geometry of rational surfaces, the issue of irreducibility of nonlinear special functions defined through discrete equations and the underlying Galois theory of difference equations, the underlying spectral theory and isomonodromic deformations of linear difference equations, the connection with modern developments in representation theory such as cluster algebras and affine Weyl groups, the emergence of Diophantine problems of number theory and p-adic analysis in connection with the integrability of analytic difference equations, the problem of finding symmetries and conservation laws for discrete systems, and the primary role discrete integrable systems play in quantum mechanics, in particular quantum groups and quantum field theory on the space-time lattice.