RANDOM MATRIX APPROACHES IN NUMBER THEORY

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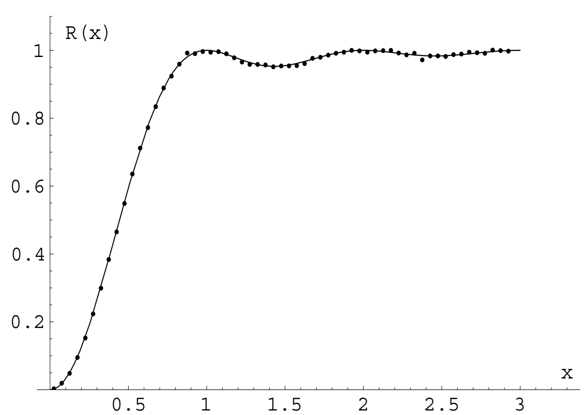
Organizers: Professor JB Conrey (AIM), Professor P Diaconis (Stanford), Dr F Mezzadri (Bristol), Professor P Sarnak (Princeton), Dr NC Snaith (Bristol)

This topic dates back to a chance conversation between a number theorist, Hugh Montgomery, and Freeman Dyson, a physicist, in the 1970s. The non-trivial zeros of the Riemann zeta function are hypothesized to lie on a vertical line in the complex plane, called the critical line. Montgomery was studying the distribution of the zeros high on this line – whether they tend to clump together or to repel each other – and over a cup of tea one day discovered from Dyson that it appears to be identical to the distribution of eigenvalues of matrices pulled at random from an ensemble of unitary matrices called the Circular Unitary Ensemble (CUE).

Extensive numerical investigations performed by Andrew Odlyzko added support to this connection, as illustrated by the figure below. Computing at the formidable height of the 10^{20}-th zero on the critical line, Odlyzko demonstrated the exceptional correspondence between the two-point correlation function of the eigenvalues of random matrices from the CUE and the same statistic computed numerically for the zeros of the Riemann zeta function.



Two-point correlation function of a) the CUE ensemble (solid line) and b) the Riemann zeros, computed numerically (dots). Courtesy of Andrew Odlyzko.

In the following years the connection between number theory and random matrix theory was extended to encompass not only the Riemann zeta function, but other *L*-functions which are also conjectured to possess zeros with the same CUE distribution high on the critical line. Further, the study of statistics of not just one individual *L*-function, but zero statistics of groups, or families, of these functions involves distributions found not just in the Circular Unitary Ensemble, but also in various other ensembles of random matrices.

Random matrix statistics also appear in the distribution of energy eigenvalues of complex nuclear systems, of disordered systems and in the field of quantum chaos. While this programme will consider many of these aspects of random matrix theory, the main aim is to investigate how much random matrix theory can help in answering long-standing and important questions in number theory.

One hint of how powerful random matrix theory can be in this context is provided by the moments of the Riemann zeta function. Mathematicians have studied mean values of powers of the zeta function on the critical line for over a century and have only succeeded in calculating the first two moments. A further two moments have been conjectured, but beyond this number theoretical techniques have proved fruitless. However, there recently arose via random matrix theory a conjecture for all moments that matches exactly all those known and previously conjectured, as well as agreeing with numerical computations.

This is just one indication of the contribution that random matrix theory can make to number theory, and the aim of this programme is to extend this role further.