# Taming the pseudoholomorphic beasts in $\mathbb{R} \times (S^1 \times S^2)$

## *"Gromov invariant" interpretation for the Seiberg-Witten invariants*

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*"It is observed that there are manifolds with nonzero Seiberg-Witten invariants which do not admit symplectic forms. With this understood, one is led to ask whether there is any sort of "Gromov invariant" interpretation for the Seiberg-Witten invariants in the nonsymplectic world."*

— C.H. Taubes, The Seiberg-Witten and Gromov invariants, 1995

**Abstract**

Seiberg-Witten invariants are well-defined for suitable closed 4-manifolds. If such a manifold admits a symplectic structure, these invariants are equal to well-defined counts of pseudoholomorphic curves (Taubes' Gromov invariants). In the absence of a symplectic form there are still nontrivial closed self-dual 2-forms which vanish along a disjoint union of circles and are symplectic elsewhere. This work describes well-defined counts of pseudoholomorphic curves in the complement of the zero-set of such "near-symplectic" forms, which recover the Seiberg-Witten invariants.

## Introduction

Let $X$ be a closed oriented smooth Riemannian 4-manifold such that $b^2_+(X) > 0$. Hodge theory provides nontrivial closed self-dual 2-forms $\omega \in \Gamma(\bigwedge^2 T^*X)$. For suitably generic metrics there exists such a 2-form that vanishes transversally, called a *near-symplectic* form. Its zero-locus $\mathcal{Z} := \omega^{-1}(0)$ is then a disjoint union of embedded circles, and $\omega$ is symplectic on $X - \mathcal{Z}$.

***Poster simplification:*** *Assume $X$ is not a blow-up, and $\mathcal{Z} \approx S^1 \sqcup S^1$. Fix a spin-c structure $\mathfrak{s}$ on $X$ such that the moduli space $\mathfrak{M}(\mathfrak{s})$ of Seiberg-Witten solutions is 0-dimensional.*

*Fine print for the experts: There are two "types" of circles in $\mathcal{Z}$ (orientability of a certain line bundle). Choose $\omega$ so that all circles have "orientable" type (the parity of $b^1(X) + b^2_+(X)$ dictates the parity of the number of such circles).*

When $SW_X(\mathfrak{s}) := \#\mathfrak{M}(\mathfrak{s})$ is nonzero, Taubes proved the existence of a pseudoholomorphic submanifold in $X$ that bounds $\mathcal{Z}$, and when $\mathcal{Z} = \varnothing$ there is a clever count of such submanifolds (Taubes' Gromov invariant) which equals $SW_X(\mathfrak{s})$. But the almost complex structure $J$ (built from $\omega$ and the metric) becomes singular along $\mathcal{Z}$, so it is hard to find an appropriate count of such submanifolds in general. Taubes' issue is better portrayed using the Hofer-Wysocki-Zehnder formalism below, and will be resolved.

## Methodology

1. Decompose $X = \mathcal{N} \cup (X - \mathcal{N})$ for a neighborhood $\mathcal{N}$ of $\mathcal{Z}$, "stretch the neck" along $\partial\mathcal{N}$, and relate Seiberg-Witten solutions on $X$ to those on $X - \mathcal{N}$ using Kronheimer & Mrowka's bible.

2. Cleverly count $J$-holomorphic curves in $X - \mathcal{N}$ using Hutchings' Embedded Contact Homology and Taubes' Gromov invariants.

3. Relate Seiberg-Witten solutions on $X - \mathcal{N}$ to $J$-holomorphic curves in $X - \mathcal{N}$ using (generalizations of) Taubes' isomorphisms.

## Near-symplectic Gromov count

1. Carve out a tubular neighborhood $\mathcal{N}$ of $\mathcal{Z}$ to get a symplectic cobordism from the empty set $\varnothing$ to a disconnected contact 3-manifold

$$(X - \mathcal{N}, \omega|_{X-\mathcal{N}}) : \varnothing \to (S^1 \times S^2, \lambda_{\text{Taubes}}) \sqcup (S^1 \times S^2, \lambda_{\text{Taubes}})$$

2. Use $\omega$ and $\mathfrak{s}$ on $X$ to uniquely determine a relative homology class

$$A_{\mathfrak{s}} \in H_2(X - \mathcal{N}, \partial(X - \mathcal{N}); \mathbb{Z})$$

with $\partial A_{\mathfrak{s}} = (1,1) \in \mathbb{Z}^2 \cong H_1(\partial(X - \mathcal{N}); \mathbb{Z}) = -H_1(S^1 \times S^2; \mathbb{Z})^2$.

3. Let $(\overline{X - \mathcal{N}}, \omega, J)$ be the noncompact symplectic manifold from attaching "symplectization" cylindrical ends to $X - \mathcal{N}$. Let $\Theta = \{(\Theta_i, m_i)\}$ be a set of pairs of embedded Reeb orbits with multiplicities, such that $\sum_i m_i[\Theta_i] = -\partial A_{\mathfrak{s}}$.

4. Study $J$-holomorphic curves in $\overline{X - \mathcal{N}}$ that are asymptotic to $\Theta$, represent $A_{\mathfrak{s}}$, and have "ECH index" 0 (a quantity similar to the Fredholm index of the curve). Such curves may be disconnected and multiply covered, but only care about them as currents (i.e. ignore the specifics about any covering maps except their multiplicities). Denote the moduli of such currents by $\mathcal{M}^J(\Theta; A_{\mathfrak{s}})$.

5. Desire $\mathcal{M}^J(\Theta; A_{\mathfrak{s}})$ to be a finite set of points for generic $J$. *This desire is generally hopeless:* Gromov compactness allows a sequence of currents to converge to a "broken" current in which one piece has negative ECH index, and the moduli of such broken currents is a huge mess. Curves with negative ECH index need to be ruled out.

*But the ECH index depends on the Conley-Zehnder indices of the orbits, and there exist "exceptional" orbits (for $\lambda_{Taubes}$) that have relatively large indices, so negative ECH index curves appear! That was Taubes' issue, expressed in modern language.*

6. Modify $\lambda_{\text{Taubes}}$ (by modifying $\mathcal{N}$) in such a way that the "exceptional" orbits have relatively small Conley-Zehnder indices. Then negative ECH index curves do not arise.

7. Now $\mathcal{M}^J(\Theta; A_{\mathfrak{s}})$ is a finite set (for generic $J$) for certain "admissible" sets $\Theta$, and there are finitely many such $\Theta$ where $\mathcal{M}^J(\Theta; A_{\mathfrak{s}}) \neq \varnothing$. The curves are then either embedded, or they are unbranched covers of tori (which already appear in Taubes' Gromov invariant), or they are unbranched covers of planes (which are asymptotic to an elliptic orbit).

8. Pick orientations and cleverly weight the elements of the moduli spaces, so that we can use Embedded Contact Homology:

$$\sum_{\Theta} \#\mathcal{M}^J(\Theta; A_{\mathfrak{s}}) \cdot \Theta$$

represents an element in

$$ECH_*(S^1 \times S^2, \lambda_{\text{Taubes}}, 1) \otimes ECH_*(S^1 \times S^2, \lambda_{\text{Taubes}}, 1)$$

Then project onto a certain grading (for which the homology is $\mathbb{Z}$) to obtain the "near-symplectic Gromov count" $Gr_X(A_{\mathfrak{s}}) \in \mathbb{Z}$.

## Serving its purpose

1. Using ideas in Kronheimer & Mrowka's bible, recover $SW_X(\mathfrak{s})$ from a certain Seiberg-Witten Floer cobordism map associated to $X - \mathcal{N}$. Morally, there is a gluing theorem, and the space of Seiberg-Witten solutions on $\mathcal{N}$ is rather trivial.

2. This certain Seiberg-Witten Floer cobordism map (which counts Seiberg-Witten solutions on $\overline{X - \mathcal{N}}$) yields an element in

$$\widehat{HM}^{-*}(S^1 \times S^2, \mathfrak{s}_{\lambda_{\text{Taubes}}} + 1) \otimes \widehat{HM}^{-*}(S^1 \times S^2, \mathfrak{s}_{\lambda_{\text{Taubes}}} + 1)$$

3. *Claim:* This element is precisely the one that defines $Gr_X(A_{\mathfrak{s}})$. There is already a grading-preserving isomorphism

$$ECH_*(S^1 \times S^2, \lambda_{\text{Taubes}}, 1) \cong \widehat{HM}^{-*}(S^1 \times S^2, \mathfrak{s}_{\lambda_{\text{Taubes}}} + 1)$$

4. To prove the claim, make use of Taubes' work on $SW = Gr$ and $ECH = SWF$ to establish the desired correspondence between Seiberg-Witten solutions and $J$-holomorphic curves:

Consider the normal bundle to a $J$-holomorphic curve inside $\overline{X - \mathcal{N}}$, and to a Reeb orbit inside $S^1 \times S^2$. Roughly speaking, "almost" Seiberg-Witten solutions are constructed for each given orbit and curve by "gluing in" solutions of the 2-dimensional vortex equations on the 2-dimensional fibers of their normal bundles. There are complications when searching for nearby honest Seiberg-Witten solutions, thanks to:

(1) Curves with multiple punctures asymptotic to the same orbit

(2) Multiply covered tori and planes

While (1) is handled in Taubes' $ECH = SWF$ work, (2) is handled using ideas in Taubes' $SW = Gr$ work (such as Kuranishi structures).

## In general

Allow $\mathfrak{s}$ with $\dim_{\mathbb{R}} \mathfrak{M}(\mathfrak{s}) \neq 0$. Then study $J$-holomorphic curves that have ECH index $\dim_{\mathbb{R}} \mathfrak{M}(\mathfrak{s})$ and are constrained to pass through given loops and points in $X$. This recovers the complete Seiberg-Witten invariant

$$Gr_X = SW_X : \text{Spin}^c(X) \to \Lambda^* H^1(X; \mathbb{Z})$$

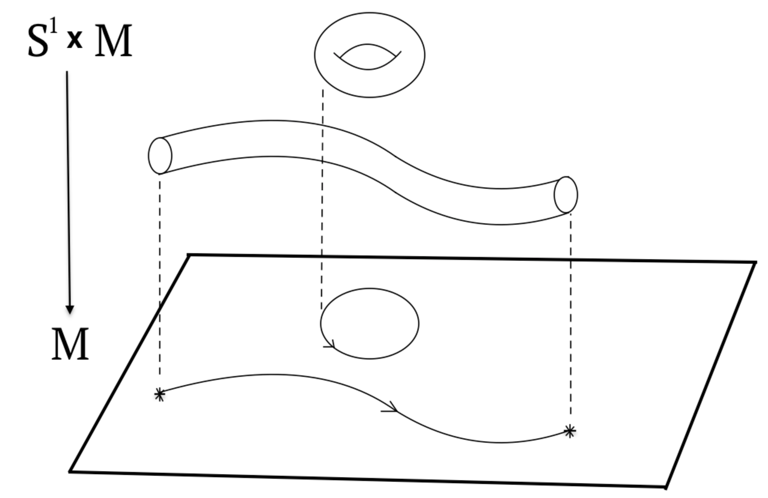
## Example: $S^1$-valued Morse theory

Take a closed oriented 3-manifold $M$, hence $X = S^1 \times M$. Assume $b^1(M) \geq 1$, hence $b^2_+(X) \geq 1$. Fix an $S^1$-valued Morse function $f : M \to S^1$ such that $d^*df = 0$, hence near-symplectic form $\omega = dt \wedge df + *df$. Then $\mathcal{Z} = S^1 \times \text{crit}(f)$, and $S^1$-invariant pseudoholomorphic submanifolds of $X$ have the form $S^1 \times \gamma$ where $\gamma$ is a gradient flowline of $f$ ($k$-periodic closed orbits correspond to $k$-fold covers of holomorphic tori).

Hutchings & Lee built a 3-dimensional invariant $I_3(M)$ which counts such flowlines, and showed that it equals a "torsion" invariant. Turaev showed that this torsion invariant equals the 3-dimensional Seiberg-Witten invariant $SW_3(M)$. Now, the combined result can alternatively be obtained as a dimensional reduction of near-symplectic SW = Gr:

$$I_3(M) = (Gr_{S^1 \times M})^{S^1} = (SW_{S^1 \times M})^{S^1} = SW_3(M)$$

as functions of the spin-c structure on $M$ and the corresponding relative homology class in $H_1(M, \text{crit}(f); \mathbb{Z})$.




## Potential future research

- Bifurcation analysis to recover Taubes' Gromov invariants when a near-symplectic form deforms into a symplectic form.
- Vanishing of near-symplectic Gromov invariants for connected sums.

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