HOMOLOGY SMALE-BARDEN MANIFOLDS WITH K-CONTACT AND SASAKIAN STRUCTURES

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Kollár has found subtle obstructions to the existence of Sasakian structures on 5-dimensional manifolds. In the present article we develop methods of using these obstructions to distinguish K-contact manifolds from Sasakian ones. In particular, we find the first example of a closed 5-manifold M with $H_1(M) = 0$ which is K-contact but which carries no semi-regular Sasakian structures.

Let (M, η) be a co-oriented contact manifold with a contact form $\eta \in \Omega^1(M)$, that is $\eta \wedge (d\eta)^n > 0$ everywhere, with $\dim M = 2n + 1$. We say that (M, η) is *K-contact* if there is an endomorphism Φ of TM such that:

- $\Phi^2 = -\operatorname{Id} + \xi \otimes \eta$, where ξ is the Reeb vector field of η (that is $i_\xi \eta = 1$, $i_\xi(d\eta) = 0$),
- the contact form η is compatible with Φ in the sense that $d\eta(\Phi X, \Phi Y) = d\eta(X, Y)$, for all vector fields X, Y,
- $d\eta(\Phi X, X) > 0$ for all nonzero $X \in \ker \eta$, and
- the Reeb field ξ is Killing with respect to the Riemannian metric defined by the formula $g(X, Y) = d\eta(\Phi X, Y) + \eta(X)\eta(Y)$.

In other words, the endomorphism Φ defines a complex structure on $\mathcal{D} = \ker \eta$ compatible with $d\eta$, hence Φ is orthogonal with respect to the metric $g|_{\mathcal{D}}$. By definition, the Reeb vector field ξ is orthogonal to $\ker \eta$, and it is a Killing vector field.

Let (M, η, g, Φ) be a K-contact manifold. Consider the contact cone as the Riemannian manifold $C(M) = (M \times \mathbb{R}^{>0}, t^2 g + dt^2)$. One defines the almost complex structure I on $C(M)$ by:

- $I(X) = \Phi(X)$ on $\ker \eta$,
- $I(\xi) = t\frac{\partial}{\partial t}$, $I(t\frac{\partial}{\partial t}) = -\xi$, for the Killing vector field ξ of η.

We say that (M, η, Φ, g, I) is *Sasakian* if I is integrable. Thus, by definition, any Sasakian manifold is K-contact.

There is much interest on constructing K-contact manifolds which do not admit Sasakian structures.

The problem of the existence of simply connected K-contact non-Sasakian compact manifolds (open problem 7.4.1 in [2]) is still open in dimension 5. It was solved for dimensions ≥ 9 in [3] and for dimension 7 in [5] by a combination of various techniques based on the homotopy theory and symplectic geometry. In the least possible dimension the problem appears to be much more difficult. Here one has to use the arguments of [4] which give subtle obstructions associated to the classification of Kähler surfaces. By definition, a simply connected compact oriented 5-manifold is called a *Smale-Barden manifold*. The following problem is still open (open problem 10.2.1 in [2]).

Do there exist Smale-Barden manifolds which carry K-contact but do not carry Sasakian structures?

In the present paper we make the first step towards a positive answer for the above question. A *homology Smale-Barden manifold* is a compact 5-dimensional manifold with $H_1(M,\mathbb{Z}) = 0$. A Sasakian structure is regular if the leaves of the Reeb flow are a folation by circles with the structure of a circle bundle over a smooth manifold. The Sasakian structure is quasi-regular if the foliation is a Seifert circle bundle over a (cyclic) orbifold. It is semi-regular if this foliation has only locus of non-trivial isotropy of codimesion 2, that is, if the base orbifold is a topological manifold. Any manifold admitting a Sasakian structure has also a quasi-regular Sasakian structure. Semi-regularity is only a small extra requirement. With this notions, our main result is:

Theorem 1. *There exists a homology Smale-Barden manifold which admits a semi-regular K-contact structure but which does not carry any semi-regular Sasakian structure.*

To produce K-contact 5-dimensional manifolds we need to produce symplectic 4-dimensional orbifolds with suitable symplectic surfaces spanning the second homology. Such K-contact 5-manifold cannot admit a Sasakian structure if we prove that such configuration of surfaces (genus, disjointness condition and spanning the second homology) cannot be produced for a Kähler orbifold with complex curves.

The proof of the main theorem follows from the fact that manifolds carrying quasi-regular K-contact (Sasakian) structures are obtained as Seifert bundles over symplectic (Kaehler) orbifolds combined with Theorem 2, Corollary 3 and Theorem 4.

Theorem 2. *There exists a simply connected symplectic* 4*-manifold* X *with* $b_2 = 36$ *and with* 36 *disjoint surfaces* $S_1, \dots, S_{36}$ *such that*

(1) $g(S_1) = \dots = g(S_9) = 1$, $g(S_{11}) = \dots = g(S_{19}) = 1$, $g(S_{21}) = \dots = g(S_{29}) = 1$, *and* $S_i \cdot S_i = -1$, *for* $i = 1, \dots, 9, 11, \dots, 19, 21, \dots, 29$*;*
(2) $g(S_{10}) = 3$, $g(S_{20}) = 3$, $g(S_{30}) = 3$, *and* $S_j \cdot S_j = 1$, $j = 10, 20, 30$*;*
(3) $g(S_{31}) = 1$, $g(S_{32}) = 1$, $g(S_{33}) = 2$, *and* $S_{31} \cdot S_{31} = -1$, $S_{32} \cdot S_{32} = -1$, $S_{33} \cdot S_{33} = 1$*;*
(4) $g(S_{34}) = 1$, $g(S_{35}) = 1$, $g(S_{36}) = 2$, *and* $S_{34} \cdot S_{34} = -1$, $S_{35} \cdot S_{35} = -1$, $S_{36} \cdot S_{36} = 1$.

The homology classes $[S_j]$, $j = 1, \dots, 36$, *generate* $H_2(X, \mathbb{Z})$.

Corollary 3. *Take a prime* p, *and* $g_i = g(S_i)$ *as given in Theorem 2. Then there exists a 5-dimensional K-contact manifold* M *with* $H_1(M, \mathbb{Z}) = 0$ *and*

$$H_2(M, \mathbb{Z}) = \mathbb{Z}^{35} \oplus_{i=1}^{36} (\mathbb{Z}/p^i)^{2g_i}.$$

Theorem 4. *Let* S *be a smooth Kähler surface with* $H_1(S, \mathbb{Q}) = 0$ *and containing* $D_1, \dots, D_b$, $b = b_2(S)$, *smooth disjoint complex curves with* $g(D_i) = g_i > 0$, *and spanning* $H_2(S, \mathbb{Q})$*. Assume that:*

- *at least two* g_i *are bigger than* 1,
- $g = \max\{g_i\} \le 3$.

Then $b \le 2g + 3$.

Proposition 5. *Let* M *be a* 5*-dimensional manifold with* $H_1(M, \mathbb{Z}) = 0$ *and*

$$H_2(M, \mathbb{Z}) = \mathbb{Z}^{35} \oplus \bigoplus_{i=1}^{36} (\mathbb{Z}/p^i)^{2g_i}.$$

where $g_i = g(S_i)$ *are the numbers given in Theorem 2, and* p *is a prime number. Then* M *does not admit a semi-regular Sasakian structure.*

References

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