Ideal invariant injections (joint work with M. Balcerzak and Sz. Głąb)

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Introduction

We will work with injections from ω to ω. The set of all such injections will be denoted by **Inj**. Fix an ideal $\mathcal{I}$ on ω and let $f \in \mathbf{Inj}$. We say that f is *$\mathcal{I}$-invariant* if $f[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. We say that f^{-1} is *$\mathcal{I}$-invariant* if $f^{-1}[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. If f and f^{-1} are $\mathcal{I}$-invariant, then f is called *bi-$\mathcal{I}$-invariant.* Note that every $f \in \mathbf{Inj}$ is bi-Fin-invariant.

Basic observations

Let $\mathcal{I}$ be an ideal on ω and let $f \in \mathbf{Inj}$.

(i) f^{-1} is $\mathcal{I}$-invariant if and only if $f[A] \notin \mathcal{I}$ for every $A \notin \mathcal{I}$.
(ii) If $f[\omega] \in \mathcal{I}$, then f is $\mathcal{I}$-invariant and it is not bi-$\mathcal{I}$-invariant.
(iii) If $\mathrm{Fix}(f) \in \mathcal{I}^\star$, then f is bi-$\mathcal{I}$-invariant.
(iv) **Inj** is a G_δ subset of ω^ω, hence it is a Polish space.
(v) The set $\{f \in \mathbf{Inj} : \omega \setminus \mathrm{Fix}(f) \in \mathrm{Fin}\}$ is dense in **Inj**. In particular, the set $\{f \in \mathbf{Inj} : f \text{ is bi-}\mathcal{I}\text{-invariant}\}$ is dense in **Inj** for every ideal $\mathcal{I}$ containing all singletons. Moreover, if $\mathcal{I}$ contains infinite sets and all singletons, the set $\{f \in \mathbf{Inj} : f \text{ is not } \mathcal{I}\text{-invariant}\}$ is dense in **Inj** as well.

Easy examples

(i) Let $\mathcal{I}_d$ stand for classical density zero ideal. Note that every increasing injection is $\mathcal{I}_d$-invariant. In particular, $f(n) := n^2$ is $\mathcal{I}_d$-invariant. Moreover, in this case $f[\omega] \in \mathcal{I}_d$, hence f is not bi-$\mathcal{I}_d$-invariant.
(ii) Let $f\colon \omega \to \omega$ be given by the formulas: $f(2n) := 4n$, $f(4n+1) = 4n+2$, $f(4n+3) := 2n+1$ for $n \in \omega$. Then f is a bijection. Consider the ideal $\mathcal{I}$ defined as follows
$$\mathcal{I} := \{A \cup B \colon A \in \mathrm{Fin},\ B \subseteq 2\omega\}.$$
Clearly, f is $\mathcal{I}$-invariant bijection which is not bi-$\mathcal{I}$-invariant.

Basic fact

There are three types of countably generated ideals: Fin, $\mathrm{Fin} \oplus \mathcal{P}(\omega)$ and $\mathrm{Fin} \times \emptyset$.

Invariance with respect to countably generated ideals

(i) If $\mathcal{I} = \mathrm{Fin}$, then each injection is bi-$\mathcal{I}$-invariant.
(ii) If $\mathcal{I} = \mathrm{Fin} \oplus \mathcal{P}(\omega)$, then the sets $\mathcal{I}$-**Inv**, of all $\mathcal{I}$-invariant injections, and **bi-$\mathcal{I}$-Inv**, of all bi-$\mathcal{I}$-invariant injections, are F_σ but not G_δ subsets of **Inj**.
(iii) If $\mathcal{I} = \mathrm{Fin} \times \emptyset$, then $\mathcal{I}$-**Inv** and **bi-$\mathcal{I}$-Inv** are meager of type $F_{\sigma\delta}$ in $\mathbf{Inj} \subseteq (\omega \times \omega)^{\omega\times\omega}$. Moreover, **bi-$\mathcal{I}$-Inv** is $F_{\sigma\delta}$-complete.

Invariance with respect to maximal ideals - main Theorem

Let $\mathcal{I}$ be a maximal ideal on ω and let $f \in \mathbf{Inj}$ be such that $\mathrm{Fix}(f) \notin \mathcal{I}^\star$. Then f is $\mathcal{I}$-invariant if and only if $f[\omega] \in \mathcal{I}$.

Invariance with respect to maximal ideals - Corollary

Let $\mathcal{I}$ be a maximal ideal on ω and $f \in \mathbf{Inj}$. Then:

(i) f is $\mathcal{I}$-invariant if and only if either $\mathrm{Fix}(f) \in \mathcal{I}^\star$ or $f[\omega] \in \mathcal{I}$.
(ii) Condition $\mathrm{Fix}(f) \in \mathcal{I}^\star$ is equivalent to the bi-$\mathcal{I}$-invariance of f.

Invariance with respect to maximal ideals - Example

Let $\mathcal{I}$ and $\mathcal{J}$ be non-isomorphic maximal ideals on ω. Let us define $\mathcal{A} := \mathcal{I} \oplus \mathcal{J}$. Take any $f \in \mathbf{Inj}$. Then f is bi-$\mathcal{A}$-invariant iff $\mathrm{Fix}(f) \in \mathcal{A}^\star$.

Invariance with respect to maximal ideals - Question

What are (reasonable) characterizations of two classes that consist of:

- ideals $\mathcal{I}$ such that every bi-$\mathcal{I}$-invariant injection f satisfies condition $\mathrm{Fix}(f) \in \mathcal{I}^\star$,
- ideals $\mathcal{I}$ such that every $\mathcal{I}$-invariant injection f satisfies either $f[\omega] \in \mathcal{I}$ or $\mathrm{Fix}(f) \in \mathcal{I}^\star$?

Bi-invariance with respect to the ideals $\mathcal{I}_d$ and $\mathcal{I}_{(1/n)}$

Let $\mathcal{I}_{(1/n)}$ stand for classical summable ideal. Let $f\colon \omega \to \omega$ be an increasing injection. The following conditions are equivalent:

(i) f is bi-$\mathcal{I}_d$-invariant;
(ii) $\underline{d}(f[\omega]) > 0$;
(iii) there is $C \in \omega$ such that $f(n) \leq Cn$ for every $n \geq 1$;
(iv) f is bi-$\mathcal{I}_{(1/n)}$-invariant.

Generalized density ideals [2]

Denote by G the set of all functions $g\colon \omega \to [0,\infty)$ satisfying conditions $g(n) \to \infty$ and $n/g(n) \nrightarrow 0$. Then we define the *upper density of weight* $g \in G$ by the formula
$$\overline{d}_g(A) = \limsup_{n\to\infty} \frac{|A \cap \{0,\ldots,n-1\}|}{g(n)} \quad \text{for } A \subseteq \omega.$$
Then consider the following ideal:
$$\mathcal{Z}_g := \{A \subseteq \omega \colon \overline{d}_g(A) = 0\}.$$
In particular, $\mathcal{I}_d = \mathcal{Z}_g$ for $g(n) := n$. Note also that $\mathcal{Z}_g$ is of the form $\mathrm{Exh}(\varphi)$ where $\varphi(A) = \sup_{n\in\omega}(|A \cap \{0,\ldots,n-1\}|/g(n))$ for $A \subseteq \omega$. Every increasing injection is $\mathcal{Z}_g$-invariant.

Question

Take any $g \in G$. Let f be an increasing injection. Then f is $\mathcal{Z}_g$-invariant. Is it true that f is bi-$\mathcal{Z}_g$-invariant iff it is bi-$\mathcal{I}_d$-invariant? Is it true that f is bi-$\mathcal{Z}_g$-invariant iff it is bi-$\mathcal{I}_{(1/g(n))}$-invariant (where $\mathcal{I}_{(1/g(n))}$ is the respective summable ideal)?

Observation

By $\mathbf{Inj}^\uparrow$ we denote the space of all increasing injections in **Inj**. $\mathbf{Inj}^\uparrow$ is a G_δ subset of **Inj** and consequently, $\mathbf{Inj}^\uparrow$ is a Polish space.

Proposition

Let $\mathcal{I} \in \{\mathcal{I}_d, \mathcal{I}_{(1/n)}\}$. The set $B_{\mathcal{I}}^\uparrow$ of all increasing bi-$\mathcal{I}$-invariant injections is a true F_σ meager subset of $\mathbf{Inj}^\uparrow$.

Definition

Given an ideal $\mathcal{I}$ on ω, we say that a sequence $(x_n)_{n\in\omega}$ in a metric space (X,ρ) is *$\mathcal{I}$-convergent* to $x \in X$ (see e.g. [7]) if $\{n \in \omega \colon \rho(x_n, x) \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$. We then write $\mathcal{I}\text{-}\lim_{n\in\omega} x_n = x$ or simply $\mathcal{I}\text{-}\lim_n x_n = x$. Note that $\mathrm{Fin}\text{-}\lim_n x_n = x$ means the usual convergence $\lim_n x_n = x$.

Applications to ideal convergence

Consider the following question. Let $\mathcal{I}$ be an ideal on ω and let $\mathcal{I}\text{-}\lim_n x_n = x$. Does there exist a bi-$\mathcal{I}$-invariant injection f such that $\lim_n x_{f(n)} = x$?

Applications to ideal convergence - Theorem

Let $\mathcal{I}$ be a P-ideal on ω which is not isomorphic to $\mathrm{Fin} \oplus \mathcal{P}(\omega)$. Then for any sequence $(x_n)_{n\in\omega}$ of real numbers which is $\mathcal{I}$-convergent to some x, there exists a bi-$\mathcal{I}$-invariant injection f such that $(x_{f(n)})_{n\in\omega}$ is convergent to x.

Definition

We say that an ideal $\mathcal{I}$ on ω is a *weak P-ideal* if for any sequence (A_n) of sets in $\mathcal{I}$ there exists a set $A \notin \mathcal{I}^\star$ such that for any $n \in \omega$ we have $A_n \subseteq^\star A$ (cf. [8] where weak P-filters were considered). Clearly, every P-ideal is a weak P-ideal. The ideal $\mathrm{Fin} \times \emptyset$ shows that the converse is false.

Applications to ideal convergence - Theorem

Assume that $\mathcal{I}$ is not a weak P-ideal. Then there exists an $\mathcal{I}$-convergent sequence (x_n) such that, for any bi-$\mathcal{I}$-invariant injection f, the sequence $(x_{f(n)})_{n\in\omega}$ is not convergent.

Applications to ideal convergence - Question

What is an exact characterization of ideals $\mathcal{I}$ such that, for any sequence (x_n) of reals, the convergence $\mathcal{I}\text{-}\lim_n x_n = x$ implies $\lim_n x_{f(n)} = x$, for some bi-$\mathcal{I}$-invariant injection f?

Applications to ideal convergence - another problem

How to characterize the $\mathcal{I}$-convergence of a sequence $(x_n)_{n\in\omega}$ in terms of the $\mathcal{I}$-convergence of $(x_{f(n)})_{n\in\omega}$ for the respectively chosen injections f?

Definition

A family $\{f_i \colon i \in K\} \subseteq \mathbf{Inj}$ $(m \in \omega)$ will be called *$\mathcal{I}$-good* if:

(i) every f_i is bi-$\mathcal{I}$-invariant;
(ii) $\bigcup_{i\in K} f_i[\omega] \in \mathcal{I}^\star$.

Clearly, $\{\mathrm{id}\}$ is an $\mathcal{I}$-good family for any ideal $\mathcal{I}$ on ω.

Observation

Let $\mathcal{I}$ be an ideal on ω, and let (x_n) be a sequence of reals. Let $f \in \mathbf{Inj}$ be a bi-$\mathcal{I}$-invariant injection such that $(x_{f(n)})$ is $\mathcal{I}$-convergent to some x. Assume that $card(\omega \setminus f[\omega]) \geq 2$. Take any distinct $n, k \in \omega \setminus f[\omega]$. There exists bi-$\mathcal{I}$-invariant injection $f' \in \mathbf{Inj}$ such that $(x_{f'(n)})$ is $\mathcal{I}$-convergent to x and $f'[\omega] = f[\omega] \cup \{n,k\}$.

Applications to ideal convergence - Theorem

Let $\mathcal{I}$ be an ideal on ω and consider real numbers x and x_n for $n \in \omega$. The following conditions are equivalent:

(i) $\mathcal{I}\text{-}\lim_n x_n = x$;
(ii) $\mathcal{I}\text{-}\lim_n x_{f(n)} = x$ for every $f \in \mathbf{Inj}$ with $\mathcal{I}$-invariant f^{-1};
(iii) $\mathcal{I}\text{-}\lim_n x_{f(n)} = x$ for every bi-$\mathcal{I}$-invariant $f \in \mathbf{Inj}$;
(iv) $\mathcal{I}\text{-}\lim_n x_{f_i(n)} = x$ for every finite $\mathcal{I}$-good family $\{f_i \colon i \in K\} \subseteq \mathbf{Inj}$;
(v) $\mathcal{I}\text{-}\lim_n x_{f_i(n)} = x$ for some finite $\mathcal{I}$-good family $\{f_i \colon i \in K\} \subseteq \mathbf{Inj}$.

Applications to ideal convergence - Theorem

Let (x_n) be a sequence of real numbers and $x \in \mathbb{R}$. Let φ be an lsc submeasure on ω and $\mathcal{I} := \mathrm{Exh}(\varphi)$. Assume that $\{f_i \colon i \in \omega\}$ is an $\mathcal{I}$-good family such that $f_0[\omega], f_1[\omega], \ldots$ are pairwise disjoint and $\sum_{i\in\omega} \varphi(f_i[\omega]) < \infty$. If $\mathcal{I}\text{-}\lim_n x_{f_i(n)} = x$ for every $i \in \omega$, then $\mathcal{I}\text{-}\lim_n x_n = x$.

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