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Class forcing and Boolean completions

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Class forcing in a generalized context

We consider class forcing in a generalized context which allows more second-order objects than just the definable ones.

Definition 1. A pair $\mathbb{M} = \langle M, \mathcal{C} \rangle$ is a *ground model* if the following statements hold.

1. $\mathcal{C}$ is a countable subset of $\mathcal{P}(M)$.
2. M is a countable transitive model of ZF^- in the language $\mathcal{L}_\in$ enriched with additional predicates for every $A \in \mathcal{C}$, i.e. M satisfies Separation and Replacement for $\mathcal{L}_\in$-formulae in this extended language.
3. If $A_0, \dots, A_{n-1} \in \mathcal{C}$, then $\mathcal{C}$ contains all subsets of M that are definable over $\langle M, \in, A_0, \dots, A_{n-1} \rangle$.

Example.
1. Let M be a countable transitive model of ZF^- and let $\mathrm{Def}(M)$ be the set of all subsets of M that are definable over $\langle M, \in \rangle$. Then $\langle M, \mathrm{Def}(M) \rangle$ is a ground model.
2. If M is countable and transitive, $\mathcal{C}$ is countable and $\mathbb{M} = \langle M, \mathcal{C} \rangle$ is a model of *Kelly-Morse class theory* KM, then $\mathbb{M}$ is a ground model.

Fix a ground model $\mathbb{M} = \langle M, \mathcal{C} \rangle$. By a *class forcing* (for $\mathbb{M}$) we mean a preorder $\mathbb{P} = \langle P, \leq_\mathbb{P}, \mathbb{1}_\mathbb{P} \rangle$ such that $P, \leq_\mathbb{P} \in \mathcal{C}$. Define $M^\mathbb{P}$ to be the set of all $\mathbb{P}$-names contained in M and $\mathcal{C}^\mathbb{P}$ to be the set of all $\mathbb{P}$-names contained in $\mathcal{C}$. We say that a filter G on $\mathbb{P}$ is $\mathbb{P}$-*generic over* $\mathbb{M}$, if G meets every dense subset of $\mathbb{P}$ contained in $\mathcal{C}$. If G is $\mathbb{P}$-generic over $\mathbb{M}$, then we set $M[G] = \{\sigma^G \mid \sigma \in M^\mathbb{P}\}$ and $\mathcal{C}[G] = \{\Gamma^G \mid \Gamma \in \mathcal{C}^\mathbb{P}\}$, and call $\mathbb{M}[G] = \langle M[G], \mathcal{C}[G] \rangle$ a $\mathbb{P}$-*generic extension* of $\mathbb{M}$.
For all $n < \omega$, we let $\mathcal{L}^n$ denote the first-order language that extends the language of set theory $\mathcal{L}_\in$ by unary predicate symbols $A_0, \dots, A_{n-1}$. Furthermore, by $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M)$ we denote the language of infinitary quantifier-free formulae in the forcing language of $\mathbb{P}$ over M, that allows reference to the generic G.

Definition 2.

1. If $\varphi(v_0, \dots, v_{m-1})$ is an $\mathcal{L}^n$-formula, $\vec{\Gamma} = \langle \Gamma_0, \dots, \Gamma_{n-1} \rangle \in (\mathcal{C}^\mathbb{P})^n$, $p \in \mathbb{P}$ and $\sigma_0, \dots, \sigma_{m-1} \in M^\mathbb{P}$, then
$$p \Vdash^{\mathbb{M}, \vec{\Gamma}}_\mathbb{P} \varphi(\sigma_0, \dots, \sigma_{m-1})$$
denotes that $\langle M[G], \in, \Gamma_0^G, \dots, \Gamma_{n-1}^G \rangle \models \varphi(\sigma_0^G, \dots, \sigma_{m-1}^G)$, whenever G is $\mathbb{P}$-generic over $\mathbb{M}$ with $p \in G$.
2. If φ is an $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}$-formula and $p \in \mathbb{P}$, we write $p \Vdash^\mathbb{M}_\mathbb{P} \varphi$ to denote that $\langle M[G], \in, G \rangle \models \varphi^G$ whenever G is $\mathbb{P}$-generic over $\mathbb{M}$ with $p \in G$, where φ^G is the formula obtained by evaluating all $\mathbb{P}$-names in φ.

The forcing theorem

Definition 3. Let $\varphi \equiv \varphi(v_0, \dots, v_{m-1})$ be an $\mathcal{L}^n$-formula.

1. We say that $\mathbb{P}$ *satisfies the definability lemma for* φ *over* $\mathbb{M}$, if for all $\vec{\Gamma} \in (\mathcal{C}^\mathbb{P})^n$,
$$\{\langle p, \sigma_0, \dots, \sigma_{m-1} \rangle \in P \times M^\mathbb{P} \times \dots \times M^\mathbb{P} \mid p \Vdash^{\mathbb{M}, \vec{\Gamma}}_\mathbb{P} \varphi(\sigma_0, \dots, \sigma_{m-1})\} \in \mathcal{C}.$$
2. We say that $\mathbb{P}$ *satisfies the truth lemma for* φ *over* $\mathbb{M}$, if for all $\sigma_0, \dots, \sigma_{m-1} \in M^\mathbb{P}$, $\vec{\Gamma} \in (\mathcal{C}^\mathbb{P})^n$ and every filter G which is $\mathbb{P}$-generic over $\mathbb{M}$ with $\langle M[G], \in \Gamma_0^G, \dots, \Gamma_{n-1}^G \rangle \models \varphi(\sigma_0^G, \dots, \sigma_{m-1}^G)$, there is a $p \in G$ with $p \Vdash^{\mathbb{M}, \vec{\Gamma}}_\mathbb{P} \varphi(\sigma_0, \dots, \sigma_{m-1})$.
3. We say that $\mathbb{P}$ *satisfies the forcing theorem for* φ *over* $\mathbb{M}$, if $\mathbb{P}$ satisfies both the definability lemma and the truth lemma for φ over $\mathbb{M}$.

The definition of the definability lemma, the truth lemma and the forcing theorem for infinitary formulae is analogous.

Definition 4. Let $\mathrm{Fml}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M)$ denote the class of Gödel codes of $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}$-formulae. We say that $\mathbb{P}$ *satisfies the uniform forcing theorem for* $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}$*-formulae*, if $\{\langle p, \ulcorner\varphi\urcorner \rangle \in P \times \mathrm{Fml}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M) \mid p \Vdash^\mathbb{M}_\mathbb{P} \varphi\} \in \mathcal{C}$.

Theorem 5. *If $\mathbb{P}$ satisfies the definability lemma for "$v_0 \subseteq v_1$", then $\mathbb{P}$ satisfies the forcing theorem for every $\mathcal{L}^n$-formula over $\mathbb{M}$.*

Example. Let $\mathbb{P} = \mathrm{Col}(\omega, \mathrm{Ord})$ denote the partial order whose conditions are finite partial functions $p : \omega \xrightarrow{par} \mathrm{Ord}$ ordered by reverse inclusion; Then $\mathbb{P}$ singularizes Ord and hence destroys Replacement. However, $\mathbb{P}$ satisfies the forcing theorem: If $\sigma, \tau \in M^\mathbb{P}$ are names such that all conditions are in $\mathrm{Col}(\omega, \alpha)$ for some $\alpha \in \mathrm{Ord}$, then
$$p \Vdash_\mathbb{P} \sigma \subseteq \tau \iff \pi(p) \Vdash_{\mathrm{Col}(\omega, \alpha+1)} \sigma \subseteq \tau,$$
where $\pi(p) \in \mathrm{Col}(\omega, \alpha+1)$ is the condition obtained by replacing $p(n)$ by α whenever $p(n) > \alpha$. This is definable, since $\mathrm{Col}(\omega, \alpha+1)$ is a set forcing.

Boolean completions

In set forcing, every partial order $\mathbb{P}$ has a Boolean completion whose elements are the regular open sets. Moreover, this completion is unique: If $\mathbb{B}_0$ and $\mathbb{B}_1$ are both Boolean completions of $\mathbb{P}$ with dense embeddings $e_0 : \mathbb{P} \to \mathbb{B}_0$ and $e_1 : \mathbb{P} \to \mathbb{B}_1$, one can define an isomorphism by $f(b) = \sup\{e_1(p) \mid p \in \mathbb{P} \wedge e_0(p) \leq b\}$ which fixes $\mathbb{P}$. This construction clearly fails in class forcing, since the conditions would already be proper classes.

Question.
1. Does every class forcing have a Boolean completion?
2. Are Boolean completions of class forcings unique?

Firstly, we need to clarify what we mean by (unique) Boolean completions in our context.

Definition 6.

1. If $\mathbb{B}$ is a Boolean algebra, then we say that $\mathbb{B}$ is $\mathbb{M}$-*complete*, if the supremum $\sup_\mathbb{B} A$ of all elements in A exists for every $A \in M$ with $A \subseteq \mathbb{B}$.
2. A class forcing $\mathbb{P}$ *has a Boolean completion in* $\mathbb{M}$, if there is an $\mathbb{M}$-complete Boolean algebra such that $\mathcal{C}$ contains $\mathbb{B}$, all Boolean operations of $\mathbb{B}$ and an injective dense embedding from $\mathbb{P}$ into $\mathbb{B} \setminus \{0_\mathbb{B}\}$.
3. A class forcing $\mathbb{P}$ *has a unique Boolean completion in* $\mathbb{M}$, if $\mathbb{P}$ has a Boolean completion $\mathbb{B}_0$ and for any other Boolean completion $\mathbb{B}_1$ of $\mathbb{P}$ there is an isomorphism in V between $\mathbb{B}_0$ and $\mathbb{B}_1$ which fixes $\mathbb{P}$.

Lemma 7. *There is an assignment $\mathrm{Fml}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M) \to M^\mathbb{P} \times M^\mathbb{P}, \ulcorner\varphi\urcorner \mapsto \langle \nu_{\ulcorner\varphi\urcorner}, \mu_{\ulcorner\varphi\urcorner} \rangle$ such that $\{\langle \ulcorner\varphi\urcorner, \nu_{\ulcorner\varphi\urcorner}, \mu_{\ulcorner\varphi\urcorner} \rangle \mid \ulcorner\varphi\urcorner \in \mathrm{Fml}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M)\} \in \mathcal{C}$ and for every $\varphi \in \mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M)$,*
$$\mathbb{1}_\mathbb{P} \Vdash_\mathbb{P} (\varphi \leftrightarrow \nu_{\ulcorner\varphi\urcorner} = \mu_{\ulcorner\varphi\urcorner}).$$

Sketch of the proof. Suppose that every formula is in negation normal form. Some examples are:

- $\nu_{\ulcorner\check{p} \in \dot{G}\urcorner} = \{\langle \check{0}, p \rangle\}$, $\mu_{\ulcorner\check{p} \in \dot{G}\urcorner} = \check{1}$; $\quad \nu_{\ulcorner\sigma \in \tau\urcorner} = \tau, \mu_{\ulcorner\sigma \in \tau\urcorner} = \tau \cup \{\langle \sigma, \mathbb{1}_\mathbb{P} \rangle\}$.
- For φ of the form $\bigwedge_{i \in I} \varphi_i$, let
$$\nu_{\ulcorner\varphi\urcorner} = \{\langle \mathsf{op}(\nu_{\ulcorner\varphi_i\urcorner}, \check{i}), \mathbb{1}_\mathbb{P} \rangle \mid i \in I\} \text{ and}$$
$$\mu_{\ulcorner\varphi\urcorner} = \{\langle \mathsf{op}(\mu_{\ulcorner\varphi_i\urcorner}, \check{i}), \mathbb{1}_\mathbb{P} \rangle \mid i \in I\},$$
where $\mathsf{op}(\sigma, \tau)$ denotes the canonical $\mathbb{P}$-name for the ordered pair $\langle \sigma^G, \tau^G \rangle$.
- Negations can be handled e.g. as $\sigma \notin \tau \equiv \bigwedge_{\langle \pi, p \rangle \in \tau} (\sigma \neq \pi \vee \check{p} \notin \dot{G})$.

□

Boolean completions and the forcing theorem

Theorem 8. *Assume that $\mathbb{M}$ satisfies either global choice or power set and let $\mathbb{P} = \langle P, \leq_\mathbb{P} \rangle$ be a separative class forcing. Then the following statements are equivalent:*

1. *$\mathbb{P}$ satisfies the definability lemma for "$v_0 \subseteq v_1$".*
2. *$\mathbb{P}$ satisfies the forcing theorem for all $\mathcal{L}_\in$-formulae.*
3. *$\mathbb{P}$ satisfies the uniform forcing theorem for all $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M)$-formulae.*
4. *$\mathbb{P}$ has a Boolean completion.*

Sketch of the proof. (1) ⇒ (2) follows from Theorem 5. For (2) ⇒ (3), define the forcing relation for $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M)$-formulae by
$$p \Vdash_\mathbb{P} \varphi \iff p \Vdash_\mathbb{P} \nu_{\ulcorner\varphi\urcorner} = \mu_{\ulcorner\varphi\urcorner}.$$
To show that (3) implies (4), consider the class $\mathrm{Fml}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M)$ of all Gödel codes of infinitary formulae in the forcing language of $\mathbb{P}$ modulo the equivalence relation
$$\ulcorner\varphi\urcorner \approx \ulcorner\psi\urcorner \iff \mathbb{1}_\mathbb{P} \Vdash_\mathbb{P} \varphi \leftrightarrow \psi.$$
For (4) ⇒ (1), let $\mathbb{B}(\mathbb{P})$ be a Boolean completion of $\mathbb{P}$. Then one can define Boolean values for atomic formulae in the usual way by
$$[\![\sigma \in \tau]\!] = \sup_\mathbb{P}\{[\![\sigma = \pi]\!] \wedge p \mid \langle \pi, p \rangle \in \tau\}$$
$$[\![\sigma = \tau]\!] = [\![\sigma \subseteq \tau]\!] \wedge [\![\tau \subseteq \sigma]\!]$$
$$[\![\sigma \subseteq \tau]\!] = \inf_\mathbb{P}\{\neg[\![\pi \in \sigma]\!] \vee [\![\pi \in \tau]\!] \mid \pi \in \mathrm{dom}(\sigma)\}.$$
and thus $p \Vdash_\mathbb{P} \sigma \subseteq \tau$ if and only if $p \leq_{\mathbb{B}(\mathbb{P})} [\![\sigma \subseteq \tau]\!]$. □

Corollary 9. *Every separative pretame class forcing has a Boolean completion.*

Example. Let $\mathbb{F}$ denote the forcing whose conditions are triples $p = \langle d_p, e_p, f_p \rangle$, where d_p is a finite subset of ω, e_p is a binary acyclic relation on d_p and f_p is an injective function with $\mathrm{dom}(f_p) \in \{\emptyset, d_p\}$ and $\mathrm{ran}(f_p) \subseteq M$ such that if $\mathrm{dom}(f_p) = d_p$, then $\langle i, j \rangle \in e_p$ iff $f_p(i) \in f_p(j)$. $\mathbb{F}$ adds a binary relation E on ω such that $\langle \omega, E \rangle$ is isomorphic to $\langle M, \in \rangle$.

Theorem 10. *If M is a countable transitive model of ZF^- and $\mathcal{C} = \mathrm{Def}(M)$, then $\mathbb{F}$ does not satisy the forcing theorem over $\mathbb{M}$. In particular, $\mathbb{F}$ does not have a Boolean completion.*

Non-unique Boolean completions

Let $\mathbb{P} = \mathrm{Col}(\omega, \mathrm{Ord})$ and let $\mathbb{Q} = \mathrm{Col}_\pm(\omega, \mathrm{Ord})$ denote the variant of $\mathrm{Col}(\omega, \mathrm{Ord})$ consisting of finite partial functions $p : \omega \xrightarrow{par} \alpha \cup \{\mathcal{E}, \mathcal{O}\}$ with the ordering defined by $p \leq q$ if and only if $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$ and for all $n \in \mathrm{dom}(q)$, either $p(n) = q(n)$, or $q(n) = \mathcal{E}$ and $p(n)$ is an even ordinal, or $q(n) = \mathcal{O}$ and $p(n)$ is an odd ordinal. By Theorem 8 both $\mathbb{P}$ and $\mathbb{Q}$ have Boolean completions $\mathbb{B}(\mathbb{P})$ and $\mathbb{B}(\mathbb{Q})$ defined as (representatives of) equivalence classes of Gödel codes of infinitary formulae in the forcing language of $\mathbb{P}$ resp. $\mathbb{Q}$.

Notation. Let for the Gödel code $\ulcorner\varphi\urcorner$ of an infinitary formula in the forcing language of $\mathbb{P}$ over M, $\mathrm{supp}(\ulcorner\varphi\urcorner)$ denote the supremum of all ordinals which appear in in the range of some condition in $\ulcorner\varphi\urcorner$. For $b \in \mathbb{B}(\mathbb{P})$, let $\mathrm{supp}(b)$ denote the minimal $\alpha \in \mathrm{Ord}$ such that $\mathrm{supp}(\ulcorner\varphi\urcorner) = \alpha$ for some $\ulcorner\varphi\urcorner$ with $b = [\ulcorner\varphi\urcorner]$.

Observation 11. *For all $\alpha, \beta \in \mathrm{Ord}$ there is an automorphism $\pi_{\alpha\beta} : \mathbb{B}(\mathbb{P}) \to \mathbb{B}(\mathbb{P})$ such that for all $n \in \omega$, $\pi_{\alpha\beta}(\{\langle n, \alpha \rangle\}) = \{\langle n, \beta \rangle\}$ and $\pi_{\alpha\beta}(b) = b$, whenever $\mathrm{supp}(b) < \min\{\alpha, \beta\}$. $\pi_{\alpha\beta}$ is obtained by extending the automorphims on $\mathbb{P}$ which swaps α and β in the range of every condition.*

Theorem 12. *$\mathbb{P}$ has two non-isomorphic Boolean completions, witnessed by $\mathbb{B}(\mathbb{P})$ and $\mathbb{B}(\mathbb{Q})$.*

Sketch of the proof. $\mathbb{B}(\mathbb{P})$ can be considered a subset of $\mathbb{B}(\mathbb{Q})$, since $\mathbb{1}_\mathbb{P} \Vdash_\mathbb{P} \varphi \leftrightarrow \psi$ iff $\mathbb{1}_\mathbb{Q} \Vdash_\mathbb{Q} \varphi \leftrightarrow \psi$ for all $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M)$-formulae φ, ψ. Consider $q = \{\langle 0, \mathcal{E} \rangle\} \in \mathbb{Q} \subset \mathbb{B}(\mathbb{Q})$. Then $q \notin \mathbb{B}(\mathbb{P})$: Otherwise take $\alpha, \beta > \mathrm{supp}(q)$ such that α is even and β is odd. Then $\{\langle 0, \alpha \rangle\} \leq_{\mathbb{B}(\mathbb{P})} q$ and so $\{\langle 0, \beta \rangle\} = \pi_{\alpha\beta}(\{\langle 0, \alpha \rangle\})$ is stronger than $\pi_{\alpha\beta}(q) = q$ contradicting that β is odd.
There is no isomorphism between $\mathbb{B}(\mathbb{P})$ and $\mathbb{B}(\mathbb{Q})$ which fixes $\mathbb{P}$, since such an isomorphism would have to fix all Boolean values and hence such an all of $\mathbb{B}(\mathbb{P})$. □

Class forcing in KM

Recall that KM class theory consists of the set axioms of extensionality, pairing, infinity, union and power set and additionally foundation, extensionality, replacement and separation for classes as well as global choice. In particular, class recursion holds in models of KM.

Fact 13. *If $\mathbb{M}$ satisfies global choice or power set, then every class forcing $\mathbb{P}$ has a separative quotient $\mathbb{S}(\mathbb{P})$. Moreover, $\mathbb{P}$ and $\mathbb{S}(\mathbb{P})$ generate the same generic extensions.*

Lemma 14. *If $\mathbb{M}$ is a model of KM, then every separative class forcing for $\mathbb{M}$ has a Boolean completion.*

Sketch of the proof. Suppose that $\mathbb{P}$ is a separative class forcing. By class recursion, define $\subseteq$-increasing continuous sequences $\langle \mathbb{P}_\alpha \mid \alpha \in \mathrm{Ord} \rangle$ and $\langle \mathbb{Q}_\alpha \mid \alpha \in \mathrm{Ord} \rangle$ of separative class forcings containing $\mathbb{P}$ such that $\{\langle p, \alpha \rangle \mid p \in \mathbb{P}_\alpha\}, \{\langle q, \alpha \rangle \mid q \in \mathbb{Q}_\alpha\} \in \mathcal{C}$. Let $\mathbb{P}_0 = \mathbb{P}$. Given $\mathbb{P}_\alpha$, construct $\mathbb{Q}_\alpha$ by formally adding suprema for all subsets of $\mathbb{P}_\alpha$ which are in M; and given $\mathbb{Q}_\alpha$ define $\mathbb{P}_{\alpha+1}$ by formally adding negations for every $q \in \mathbb{Q}_\alpha$. Then $\bigcup_{\alpha \in \mathrm{Ord}} \mathbb{P}_\alpha = \bigcup_{\alpha \in \mathrm{Ord}} \mathbb{Q}_\alpha$ is in $\mathcal{C}$. To obtain an ordering, take the quotient modulo the equivalence relation $p \approx q$ if and only if there is $\alpha \in \mathrm{Ord}$ such that $p, q \in \mathbb{P}_\alpha$ and $p \leq_{\mathbb{P}_\alpha} q$ and $q \leq_{\mathbb{P}_\alpha} p$. □

Combining Theorems 5 and 8 with Fact 13 we obtain

Theorem 15. *If $\mathbb{M}$ is a model of KM, then every class forcing has a Boolean completion and satisfies the forcing theorem for all $\mathcal{L}^n$-formulae.*

Peter Holy, Regula Krapf, Philipp Lücke, Ana Njegomir and Philipp Schlicht. *Class forcing, Boolean completions and the forcing theorem.* Submitted.

2015 Mathematical Institute
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