Boundary singularities and non-crystallographic Coxeter groups

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Abstract

Singularities connected with Coxeter groups $I_2(k)$, H_3 and G_2 are studied. At first, such singularities were found by O. Lyashko in the classification of critical points of non-singular functions on a singular hypersurface. We establish a link between these critical points and boundary singularities. We describe a class of deformations of boundary singularities which provides miniversal deformations of critical points of non-singular functions on a singular hypersurface. In particular, Coxeter groups $I_2(k)$ and H_3 turn out to be connected with unimodal boundary singularities B^3_{k-1} and F^4_4, respectively, and group G_2 is connected with simple boundary singularity F_4.

Introduction

Since 1972 it has been known that singularities of holomorphic functions are closely related to the geometry of Coxeter groups. To describe this relation, we recall briefly some data about singularities and about Coxeter groups.

BIFURCATION DIAGRAMS OF SINGULARITIES

Let f be a holomorphic function germ at a critical point. Its *deformation* is a holomorphic family germ $F(\cdot, \lambda)$ such that $F(\cdot, O) = f$, $\lambda \in \mathbb{C}^\mu$. For an equivalence relation on the set of the holomorphic germs, deformation F is called *versal* with respect to the relation, if F contains (for some values of parameter λ close to O)

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representatives of any equivalence class close enough to f. A versal deformation with the minimal dimension of the parameter space $\mathbb{C}^\mu$, is called *miniversal.* The values of λ's such that the corresponding germ in a miniversal deformation has zero as its critical value form a hypersurface in the parameter space $\mathbb{C}^\mu$ called a *bifurcation diagram.* A classical example is an ordinary singularity, which is a class of *stable equivalency.* It is well known, that if it has a finite multiplicity, then miniversal deformations exist, the bifurcation diagram is unique up to a diffeomorphism, and its structure contains a lot of information about the singularity.

MANIFOLDS OF NON-REGULAR ORBITS OF COXETER GROUPS

A finite group generated by reflections in μ-dimensional euclidean space, i.e. a *Coxeter group*, has a basis of invariants, that is, the manifold of the orbits of the complexified action of the group is furnished with a natural structure of a smooth algebraic variety $\mathbb{C}^\mu$. The number of points in the orbit of a typical point is equal to the number of elements in this group. However, some orbits are smaller. These *non-regular* orbits form an algebraic hypersurface in the orbit space called a *manifold of non-regular orbits.* The list of the crystallographic Coxeter groups (or of the Weyl groups of the simple Lie groups) contains groups $A_\mu, D_\mu\ (\mu \geq 4), E_6, E_7, E_8$ having root systems with only roots of equal length, and groups $B_\mu\ (\mu \geq 2), C_\mu\ (\mu \geq 3), F_4$ and G_2 with inhomogeneous root systems. Besides the Weyl groups, the list of Coxeter groups contains non-crystallographic groups $I_2(p)$, $p \geq 5$ (the symmetry groups of regular p-gons), H_3 (the symmetry group of an icosahedron) and H_4 (the symmetry group of a hyper-icosahedron).

In [1] it was established that the bifurcation diagrams of the simple ordinary singularities $A_\mu, D_\mu\ (\mu \geq 4), E_6, E_7, E_8$ are diffeomorphic to the manifolds of the non-regular orbits of the corresponding reflection groups acting on the complex space.

The extension of this connection to include other reflection groups has been a problem stimulating a deep research in the singularity theory.

Deformations which are miniversal with respect to other equivalence relations appear naturally by considering singularities with additional structures as boundaries, obstacles, symmetries etc. Among the corresponding bifurcation diagrams one can recognize the manifolds of the non-regular orbits of other Coxeter groups. After papers [3], [10], this is a "standard" way of finding reflection groups in the singularity theory.

Namely, miniversal deformations of the simple singularities on manifolds with boundary are to be identified with deformations of the simple singularities $A_{2\mu-1}$, $D_{\mu+1}$, E_6 which are miniversal in the class of $\mathbb{Z}_2$-invariant functions ([3], [5]). The corresponding bifurcation diagrams are diffeomorphic to the manifolds of the non-regular orbits of reflection groups B_μ, C_μ, F_4.

A similar example is given by singularities of the distance function in the problem of avoiding an obstacle [10]. The manifolds of the non-regular orbits of Coxeter groups $I_2(5)$, H_3, H_4 are realized as the bifurcation diagrams of deformations of the simple singularities A_4, D_6, E_8, respectively, which are miniversal in the class of the singularities of even multiplicity.

Unlike the previous cases, singularities connected with Coxeter groups $I_2(p)$ ($p \geq 5$), H_3, G_2, have appeared in [7] in a different way. These are singularities of critical points of functions on a singular hypersurface, and the corresponding critical points are not simple but unimodal.

In the present paper we propose a construction that allows us to include groups $I_2(p)$, H_3 and G_2 into the generic framework of the singularity theory described above.

We establish a link between critical points of non-singular functions on a singular hypersurface and boundary singularities. It turns out that there is one-to-one correspondence between simple boundary singularities and simple critical points on a singular surface. We prove that the bifurcation diagram of a critical point on a singular surface is the bifurcation diagram of a certain deformation of the restriction to the surface which is miniversal with respect to the stable equivalence of functions on a singular surface. This deformation is related to the corresponding boundary singularity.

In particular, unimodal singularities $I_2(p)$, $p \geq 5$, and H_3 are stable equivalence classes of critical points of non-singular functions on a hypersurface of types A_{p-1} and A_3, respectively. These critical points correspond to the unimodal boundary singularities B^3_{p-1} and F^4_4, respectively. The bifurcation diagrams of these critical points are the bifurcation diagrams of certain deformations of simple singularities A_3 and A_4 related to the boundary singularities B^3_{p-1} and F^4_4, respectively. The simple critical point G_2 on a hypersurface of type A_2 corresponds to the simple boundary singularity $F_4 \equiv F^2_2$, and the manifold of the non-regular orbits G_2 appears as the bifurcation diagram of a deformation of A_2 related to F_4.

The structure of the paper is as follows. In Sec. 1 we recall the theory of critical points of functions on a singular hypersurface ([7], [4]). Sec. 2 is devoted to the case of non-singular functions on a singular hypersurface. In Sec. 3 we recall some facts from the boundary singularity theory ([5], [11]). In Sec. 4 a connection between non-singular functions on a singular hypersurface and boundary singularities is described. In Sec. 5, the case when the hypersurface has a singularity of type A_k is considered. We prove that the origin is a non-critical point for a germ on a singular hypersurface if and only if the corresponding boundary germ is of type B_k. In Sec. 6, for a wide class of critical points on a singular hypersurface, we prove that the local ring is isomorphic to the local ring of the ordinary singularity given by the restriction of the corresponding boundary germ to the boundary. In Sec. 7 we describe miniversal deformations of a critical point on a singular hypersurface in terms of certain deformations of the restriction of the corresponding boundary germ to the boundary. As a corollary, we get (Sec. 8) that the manifolds of the non-regular orbits of reflection groups $I_2(p)$ and H_3 are diffeomorphic to the generic hyperplane sections of the corresponding bifurcation diagrams. In the last Sec. 9 we consider the critical point of type G_2.

1 Critical points of functions on a singular hypersurface

In this section we recall the theory of critical points of functions on a manifold with singular boundary of [7]. We use slightly different terminology and call these critical points "critical points of functions on a singular hypersurface" in order not to mess them up with critical points on a manifold with boundary participating in our considerations.

A function germ f on a singular hypersurface V is a triple $(f, V, \mathbb{C}^n)$ where

• $f : (\mathbb{C}^n, O) \to (\mathbb{C}, 0)$ is a germ of a holomorphic function;

• $V = \{z \in \mathbb{C}^n \mid h(z) = 0\}$ is a germ of a hypersurface with an isolated singular point at O.

Germs of diffeomorphisms $(\mathbb{C}^n, O) \to (\mathbb{C}^n, O)$ act on the set of the triples, and two triples are *equivalent* if they lie in the same orbit of this action.

For $m > n$, denote by π the natural projection $\pi : \mathbb{C}^m = \mathbb{C}^n \times \mathbb{C}^{m-n} \to \mathbb{C}^n$,

$$\pi(z_1, \ldots, z_n, z_{n+1}, \ldots, z_m) = (z_1, \ldots, z_n).$$

In $\mathbb{C}^m$, we define the hypersurface $\tilde{V} = \{h(z_1, \dots, z_n) + z_{n+1}^2 + \dots + z_m^2 = 0\}$ and the function germ $\tilde{f} = \pi^* f$. The triple $(\tilde{f}, \tilde{V}, \mathbb{C}^m)$ is called *the stabilization* of $(f, V, \mathbb{C}^n)$.

Functions on singular hypersurfaces are *stable equivalent* if they have equivalent stabilizations. Stable equivalent triples have the same singularity of hypersurfaces.

We say that O is a *non-critical* point of $(f, V, \mathbb{C}^n)$ if $V_0 = f^{-1}(O)$ is a germ of a smooth hypersurface (i.e. f is *non-singular* at O) *transversal* to V at O. In the opposite case we say that O is a *critical point of the triple* $(f, V, \mathbb{C}^n)$.

The transversality at a singular point of V means the following. Consider $PT^*\mathbb{C}^n$, the projectivization of the cotangent bundle of $\mathbb{C}^n$, and the canonical projection

$$\rho : PT^*\mathbb{C}^n \to \mathbb{C}^n, \quad \rho(z_1, \dots, z_n, p_1 : \dots : p_n) = (z_1, \dots, z_n).$$

For any subvariety $M = \{\phi(z) = 0\}$ in $\mathbb{C}^n$, denote by PM the image of M under the embedding in $PT^*\mathbb{C}^n$:

$$(z \in M) \mapsto (z, \text{tangent plane to } M \text{ at } z).$$

In other words, PM is the following subvariety of $PT^*\mathbb{C}^n$:

$$PM = \{\phi(z) = 0, \;\; p_i \frac{\partial \phi}{\partial z_j} - p_j \frac{\partial \phi}{\partial z_i} = 0, \;\; 1 \leq i, j \leq n\}.$$

In particular, if V has an isolated critical point at O, then PV is reducible and consists of $PT^*\{O\}$ and $V_1 = \overline{PV \setminus PT^*\{O\}}$, the closure of $PV \setminus PT^*\{O\}$. We say that $V_0 = f^{-1}(O)$ is *transversal* to V at O, if V_0 is smooth and $V_1 \cap PV_0 \cap \rho^{-1}(U) = \emptyset$, where U is a small neighborhood of the origin in $\mathbb{C}^n$. This means that the tangent plane to V_0 at the origin, considered as a point in $PT^*\mathbb{C}^n$, does not belong to V_1.

In the case when hypersurface V has an isolated simple singularity, the classification of critical points on V reduces to the description of the orbits of the group of diffeomorphisms preserving V. The Lie algebra of this group, T_V, consists of the vector fields preserving V. Vector field $\bar{v}$ *preserves hypersurface* V, $\bar{v} \in T_V$, if the directional derivative $L_{\bar{v}} h$ belongs to the principal ideal (h), i.e. $(\bar{v}, \mathrm{grad} h) = gh$ for some smooth germ g. We define the ideal

$$I_{f|V} = \{L_{\bar{v}} f \mid \bar{v} \in T_V\}$$

and the local algebra

$$Q_{f|V} = \mathcal{O}_n / I_{f|V},$$

where $\mathcal{O}_n$ is the ring of the holomorphic germs $(\mathbb{C}^n, O) \to (\mathbb{C}, 0)$. The *multiplicity* $\mu(f, V)$ of the critical point O of $(f, V, \mathbb{C}^n)$ is defined as

$$\mu(f, V) = \dim_{\mathbb{C}} Q_{f|V} - 1.$$

For any stabilization $(\tilde{f}, \tilde{V}, \mathbb{C}^{n+k})$ of $(f, V, \mathbb{C}^n)$, the local algebras $Q_{f|V}$ and $Q_{\tilde{f}|\tilde{V}}$ are isomorphic, and thus $\mu(f, V, \mathbb{C}^n)) = \mu(\tilde{f}, \tilde{V}, \mathbb{C}^{n+k}) = \mu(f, V)$ does not depend on dimension.

By the usual way, the notions of modality, versal deformation, bifurcation diagram can be defined for this situation. In particular, the *modality* of $(f, V, \mathbb{C}^n)$ is the minimal number m such that a small neighborhood of the orbit of this germ (under the action of diffeomorphisms preserving V) is covered by a finite number of m-parameter families of orbits. When m is equal to 0 or 1, a critical point is called *simple* or *unimodal* respectively.

In the case $\mu < \infty$, one can take a versal deformation of $(f, V, \mathbb{C}^n)$ in the form $(F, V, \mathbb{C}^n)$, where F is the family of functions

$$F(z, \lambda) = f(z) + \lambda_0 e_0 + \cdots + \lambda_\mu e_\mu,$$

$e_0, \ldots, e_\mu$ are representatives of a basis of the local algebra $Q_{f|V}$ over $\mathbb{C}$, and $\lambda = (\lambda_0, \ldots, \lambda_\mu) \in \mathbb{C}^{\mu+1}$ is a parameter of versal deformation.

The bifurcation diagram $\Sigma(f, V)$ of the critical point $(f, V, \mathbb{C}^n)$ is a hypersurface in the base of versal deformation, $\mathbb{C}^{\mu+1}$, formed by the parameter values $\lambda \in \mathbb{C}^{\mu+1}$ such that 0 is a critical value of $(F(\cdot, \lambda), V, \mathbb{C}^n)$.

It turns out that simple and unimodal critical points appear only on a hypersurface with a simple singularity of type A_k. In [7], it was proven that for a critical point of modality 1, the bifurcation diagram is analytically trivial along the strata $\mu = \text{const}$, and the classification of the simple and unimodal critical points on a hypersurface of type A_k was obtained.

A part of the classification is connected with reflection groups H_3 and $I_2(p)$. Namely, a critical point of type $I_2(p)$, $p \geq 4$, is given by the germ $x + \epsilon y + z^2$, $\epsilon \neq 0$, on the hypersurface $xy = z^p$ of type A_{p-1}, and a critical point of type H_3 is given by the germ $x + y + \epsilon z^3$ on the hypersurface $xy = z^5$ of type A_4. These critical points are unimodal and ϵ is a parameter along the strata $\mu = \text{const}$. The main result of [7] is the following theorem.

Theorem 1 *The intersection of the bifurcation diagram of a critical point of type $I_2(p)$, $p \geq 4$, or H_3, with a hyperplane in the base of versal deformation which is transversal to the stratum $\mu = \text{const}$, is biholomorphic equivalent to the manifold of the non-regular orbits of the corresponding group generated by reflections, acting on the complexification of the Euclidean space $\mathbb{C}^\mu$.*

2 Non-singular functions on a singular hypersurface

As it was pointed out in [7], if the number of variables is greater then two, then simple and unimodal critical points on a singular hypersurface can appear only for function germs with non-zero 1-jet. Moreover, simple critical points appear only on a hypersurface of type A_k. On a boundary of type D_k or E_k, there are only unimodal non-critical points.

For that reason, in the paper, we study triples $(f, V, \mathbb{C}^n)$ where $f : (\mathbb{C}^n, O) \to (\mathbb{C}, 0)$ is a non-singular germ, i.e. $V_0 = f^{-1}(O)$ is a germ of smooth hypersurface.

Let $z_1, \ldots, z_n$ be coordinates in $\mathbb{C}^n$ such that $\partial f/\partial z_1$ does not vanishes at the origin. The change of variables $x = f$, $y_1 = z_2$, $\ldots$, $y_{n-1} = z_n$, gives an equivalent triple

$$(x, V = \{g(x, y_1, \ldots, y_{n-1}) = 0\}, \mathbb{C}^n).$$

We denote $y = (y_1, \ldots, y_{n-1})$, $g_0(y) = g(0, y_1, \ldots, y_{n-1})$. In this case $V_0 = \{x = 0\}$ and thus

$$PV_0 = \{x = 0,\ p_1 = \cdots = p_{n-1} = 0\}.$$

The tangent plane to V_0 at the origin is point $(0, \ldots, 0; 1 : 0 : \cdots : 0)$ in $PT^*\mathbb{C}^n$ with coordinates $(x, y_1, \ldots, y_{n-1}; p_0 : p_1 : \cdots : p_{n-1})$. Further, PV is given by

$$PV = \{g(x, y) = 0,\ p_0\frac{\partial g}{\partial y_i} = p_i\frac{\partial g}{\partial x},\ p_j\frac{\partial g}{\partial y_i} = p_i\frac{\partial g}{\partial y_j},\ 1 \leq i, j \leq n\}.$$

Therefore the intersection $PV_0 \cap PV$ is

$$PV_0 \cap PV = \{g(x, y_1, \ldots, y_{n-1}) = 0,\ x = 0,\ p_0\frac{\partial g}{\partial y_i} = 0,\ p_i = 0,\ i = 1, \ldots, n-1\}$$

and thus

$$PV_0 \cap V_1 = \{g(x, y_1, \ldots, y_{n-1}) = 0,\ x = 0,\ \frac{\partial g}{\partial y_i} = 0,\ p_i = 0,\ i = 1, \ldots, n-1\}.$$

Example 1 Let $V = \{x^{k+1} = y^2\} \subset \mathbb{C}^2,\ k \geq 2$. Then for the triple $(x,\ V,\ \mathbb{C}^2)$, the origin is a non-critical point, whereas for the triple $(y,\ V,\ \mathbb{C}^2)$, the origin is a critical point. Indeed,

$$PV = \{x^{k+1} = y^2,\ (k+1)x^k p_1 = 2yp_0\} = \{y = x^{\frac{k+1}{2}},\ (k+1)x^k p_1 = 2x^{\frac{k+1}{2}} p_0\}.$$

If $x \neq 0$, we get

$$p_0 = \frac{k+1}{2} p_1 x^{\frac{k-1}{2}}.$$

For $f = x$, we have $PV_x = \{x = p_1 = 0\}$ and the corresponding point in $PT^*\mathbb{C}^2$ is $(0,0;1:0) \notin V_1 = \overline{PV \setminus PT^*\{O\}}$, whereas for $f = y$, $PV_y = \{y = p_0 = 0\}$ and the corresponding point in $PT^*\mathbb{C}^2$ is $(0,0;0:1) \in V_1$.

3 Boundary singularities

Here we recall some basic facts of boundary singularities theory [11].

A *boundary germ* is a triple $(g, Y, \mathbb{C}^n)$, where Y is a germ at O of a smooth hypersurface, called *a boundary*, and $g : (\mathbb{C}^n, O) \to (\mathbb{C}, 0)$ is a holomorphic germ such that both g and $g_0 = g|_Y : (Y, O) \to (\mathbb{C}, 0)$ have isolated critical points. In appropriate local coordinates $(x, y_1, \ldots, y_{n-1})$ of $\mathbb{C}^n$, the boundary is $Y = \{x = 0\}$, and $g_0(y) = g(0, y_1, \ldots, y_{n-1})$.

A *stabilization* of $(g, Y, \mathbb{C}^n)$ is a boundary germ $(\tilde{g}, \tilde{Y}, \mathbb{C}^m)$, $m > n$, where $\tilde{g}(x, y_1, \ldots, y_{m-1}) = g(x, y_1, \ldots, y_{n-1}) + y_n^2 + \cdots + y_{m-1}^2$, $\tilde{Y} = \{x = 0\}$.

A *boundary singularity* is a boundary germ considered up to germs of diffeomorphisms preserving the boundary and up to stabilizations.

The *multiplicity* $\mu(g, Y)$ of the critical point of the boundary germ $(g, Y, \mathbb{C}^n)$ is the dimension over $\mathbb{C}$ of the *local ring* $Q(g, Y) = \mathcal{O}_n / I(g, Y)$, where $I(g, Y)$ is the ideal generated by $x\partial g/\partial x, \partial g/\partial y_1, \ldots, \partial g/\partial y_{n-1}$.

The multiplicity of a boundary germ and of any its stabilization is the same.

In some natural sense, boundary singularity $(g, Y, \mathbb{C}^n)$ is an extension of two ordinary singularities given by germs g and g_0. These two ordinary singularities are called the *decomposition* of $(g, Y, \mathbb{C}^n)$.

Recall that for an ordinary singularity given by a holomorphic germ $f(z_1, \ldots, z_n)$ at the critical point O, the multiplicity is the dimension over $\mathbb{C}$ of the local ring $Q(f) = \mathcal{O}_n/I(f)$, where ideal $I(f)$ is generated by the partial derivatives $\partial f/\partial z_1, \dots, \partial f/\partial z_n$:

$$\mu(f) = \dim_{\mathbb{C}} Q(f).$$

For boundary singularity $(g, Y, \mathbb{C}^n)$ with decomposition (g, g_0), we have $\mu(g, Y) = \mu(g) + \mu(g_0)$.

A versal deformation of boundary singularity $(g, Y, \mathbb{C}^n)$ one can take in the form

$$G(x, y, \lambda) = g(x, y) + \Sigma\lambda_i' e_i' + x\Sigma\lambda_j'' e_j'',$$

where $\{e_i',\ 1 \leq i \leq \mu(g_0)\}$ represent a basis of $Q(g_0)$, and $\{e_j'',\ 1 \leq j \leq \mu(g)\}$ represent a basis of $Q(g)$. The bifurcation diagram of the boundary singularity $(g, Y, \mathbb{C}^n)$ has two irreducible components, which are bifurcation diagrams of g and g_0 respectively multiplying by complex spaces of appropriate dimensions.

For boundary singularity $(g, Y, \mathbb{C}^n)$ with decomposition (g, g_0), we have $\mu(g, Y) = \mu(g) + \mu(g_0)$.

In generic case, the decomposition does not define a boundary singularity (see example 4). But as it was proved in [9], boundary singularities with decomposition of type (A_k, A_l) are well-defined by their decomposition. In particular, if g_0 is a Morse function (i.e. of type A_1) and g is of type A_k, then the boundary singularity is of type B_{k+1}. It can be given by the function germ $x^{k+1} + Q(y)$, where $Q(y) = Q(y_1, \dots, y_{n-1})$ is a Morse function, the boundary is $x = 0$.

4 Link between non-singular functions on singular hypersurfaces and boundary singularities

If $(f(z), V = \{g(z) = 0\}, \mathbb{C}^n)$ is a germ of non-singular function f on a singular hypersurface V, such that $\mu(f, V) < \infty$, then triple $(g(z), V_0 = \{f(z) = 0\}, \mathbb{C}^n)$ is a boundary germ.

Conversely, any boundary germ $(g(x, y), Y = \{x = 0\}, \mathbb{C}^n)$ defines a germ $(x, V = \{g(x, y) = 0\}, \mathbb{C}^n)$ of a non-singular function on a singular hypersurface.

Proposition 1 *Equivalent germs of non-singular functions on a singular hypersurface define the same boundary singularity*

Proof. If $(f_i, V_i, \mathbb{C}^n)$, $i = 1, 2$, are equivalent triples and f_1, f_2 are germs of non-singular functions, then there exist germs of diffeomorphisms $\phi, \phi_1, \phi_2 : (\mathbb{C}^n, O) \to (\mathbb{C}^n, O)$, such that ϕ sends $(f_1, V_1, \mathbb{C}^n)$ to $(f_2, V_2, \mathbb{C}^n)$ and ϕ_i sends $(f_i, V_i, \mathbb{C}^n)$ to $(x, \{g_i(x, y) = 0\}, \mathbb{C}^n)$, $i = 1, 2$.

The corresponding boundary germs are $(g_i(x, y), Y, \mathbb{C}^n)$, with the boundary $Y = \{x = 0\}$, $i = 1, 2$, and one goes to another by the diffeomorphism $\phi_2 \circ \phi \circ \phi_1^{-1}$ which obviously preserves the boundary. □

Comparing the lists of the simple critical points on singular surfaces [7] and of the simple boundary singularities [3], we get the following.

Proposition 2 *The simple critical points on a singular hypersurface are exactly the ones that correspond to the simple boundary singularities C_k, $k \geq 2$, F_4. Non-critical points on a singular hypersurface correspond to the simple boundary singularities of type B_k, $k \geq 2$.*

Example 2

- *$I_2(p)$-singularities.*
  In the Lyashko classification [7], critical points of type $I_2(p), p \geq 4$, appear on a hypersurface of type A_{p-1}. These critical points are unimodal. The normal form of the function germ is $f = x + \epsilon y + z^2$, where $\epsilon \neq 0$ is a parameter along the strata $\mu =$ const. The hypersurface V is given by $V = \{xy = z^p + Q\}$, where Q is a Morse function in additional variables.

  The corresponding boundary singularity has decomposition (A_{p-1}, A_3). Indeed, A_{p-1} is the type of the hypersurface. Consider $f^{-1}(O) = V_0 = \{x + \epsilon y + z^2 = 0\}$. The intersection with the boundary, $V \cap V_0$, is given by
  $$V \cap V_0 = \{z^p + (\epsilon y + z^2)y = 0\}.$$
  We have: $z^p + (\epsilon y + z^2)y = z^p - \alpha z^4 + (\sqrt{\alpha} z^2 + \sqrt{\epsilon} y)^2$, where $\alpha = 1/4\epsilon$, therefore this is a singularity of type A_3.

  Thus the boundary singularity corresponding to the critical point of type $I_2(p)$ is the unimodal boundary singularity of type B^3_{p-1} (in notations of [9]).

- *H_3-singularity.*
  In the Lyashko list, a critical point of type H_3 is given by germ $x + y + \epsilon z^3$ on a

hypersurface $xy = z^5$ of type A_4 (ϵ is a parameter along the strata $\mu = \text{const}$). After the change of variables

$$(x,\ y,\ z)\ \rightarrow\ (X = x + y + \epsilon z^3,\ y,\ z),$$

we get germ $z^5 - (X - y - \epsilon z^3)y$ on the boundary $X = 0$. It is easy to see that this boundary singularity has decomposition (A_4, A_4). This is unimodal boundary singularity F_4^4 (in notations of [9]).

5 Singular hypersurface of type A_k

Here we consider triples $(f, V, \mathbb{C}^n)$ such that f is a non-singular germ, and hypersurface V has a simple singularity of type A_k at the origin. We call such triples *L-germs.* Note that the critical points on a singular hypersurface, which are connected with reflection groups $I_2(p), p \geq 4$, and H_3, are L-germs.

For a L-germ, we can choose coordinates such that the triple is of the form $(f(x, y),\ \{x^{k+1} = Q(y)\},\ \mathbb{C}^n)$, where $Q(y) = Q(y_1, \ldots, y_{n-1})$ is a Morse function.

Theorem 2 *The origin is a non-critical point of the triple* $(f(x,y),\ V = \{x^{n+1} = Q(y)\},\ \mathbb{C}^n)$, *if and only if* $\partial f/\partial x|_O \neq 0$.

Proof As it follows from [2], [8], if $\partial f/\partial x$ does not vanish at the origin, then f can be reduced to x by a diffeomorphism preserving $\{x^{n+1} = Q(y)\}$, and we get a triple which is stable equivalent to the triple $(x, V, \mathbb{C}^2)$ of the example 1.

If $\partial f/\partial x|_O = 0$, then $\partial f/\partial y_i|_O \neq 0$ for some $1 \leq i \leq n-1$ (recall that f is non-singular at the origin). We can assume

$$\frac{\partial f}{\partial y_1}|_O \neq 0,\ \frac{\partial f}{\partial x}|_O = \frac{\partial f}{\partial y_i}|_O = 0,\ i = 2, \ldots, n-1.$$

Indeed, if $f = a_1 y_1 + \cdots + a_{n-1}y_{n-1} + (\text{terms of order } \geq 2)$, and $a_1 \neq 0$, then the required change of variables is

$$(x, y_1, \ldots, y_{n-1}) \mapsto (x, Y_1 = a_1 y_1 + \cdots + a_{n-1}y_{n-1}, y_2, \ldots, y_{n-1}).$$

The tangent plane to $\{f = 0\}$ at the origin corresponds to point $(0, \ldots, 0; 0 : 1 : 0 : \cdots : 0)$. This point is obviously in V_1 (the intersection with plane $\{y_2 = \cdots = y_{n-1} = 0\}$ reduce this case to the case of two variables considered in example 1). $\square$

Theorem 3 *The origin is a non-critical point for a L-germ, if and only if the corresponding boundary germ is of type* B_{k+1}.

Proof. Consider L-germ $(f(x,y),\ \{x^{n+1} = Q(y)\},\ \mathbb{C}^n)$, where Q is a Morse function. The condition that the origin is a non-critical point for the L-germ means that $\partial f/\partial x|_O \neq 0$, i.e. the equation $f = 0$ can be solved with respect to x: $x = \phi(y)$, where $\phi(y)$ is a holomorphic function germ. Then the restriction of the function $x^{k+1} - Q(y)$ to the boundary, $(\phi(y))^{k+1} - Q(y)$, has a non-degenerate quadratic part and therefore defines an ordinary singularity of type A_1. Therefore boundary singularity $(x^{k+1} - Q(y), f = 0, \mathbb{C}^n)$ has decomposition (A_k, A_1), i.e. this is a boundary singularity of type B_{k+1}. □

If the hypersurface, V, has another simple singularity, i.e. a singularity of type D_k $(k \geq 4)$ or E_k $(k = 6,7,8)$, then one can prove that a germ $(x, V = \{g(x,y) = 0\}, \mathbb{C}^{n+1})$ does not have the origin as a critical point if the germ $g_0(y) = g(0,y)$ defines a singularity of type A_2.

Example 3 The following germs on the boundary $x = 0$ define unimodal non-critical points $(x, V = \{g(x,y) = 0\}, \mathbb{C}^2)$ on a hypersurface of type D_k, $k \geq 4$, and E_k, $k = 6,7,8$, respectively:

$$\begin{array}{ll} (D_k, A_2): & g(x,y) = xy^2 + y^3 + ax^{k-1},\ a \neq 0, \\ (E_6, A_2): & g(x,y) = x^4 + y^3 + ax^3y, \\ (E_7, A_2): & g(x,y) = x^3y + y^3 + ax^5, \\ (E_8, A_2): & g(x,y) = x^5 + y^3 + ax^4. \end{array}$$

Consider hypersurface $V = \{x^a + y^b = 0\}$, $a > b$. Direct calculations show that O is a non-critical point for triple $(x, V, \mathbb{C}^2)$ and O is a critical point for triple $(y, V, \mathbb{C}^2)$. More generically, we have the following proposition.

Proposition 3 *Let hypersurface* $V = \{g(x_1, \ldots, x_n) = 0\}$ *be given by a quasi-homogeneous function* g *with* $\deg x_i = \alpha_i$, $1 \leq i \leq n-1$, $\alpha_1 > \alpha_2 \geq \cdots \geq \alpha_n$. *Then the origin is a critical point of* $(x_1, V, \mathbb{C}^n)$.

Proof Note first of all, that if g is a quasi-homogeneous function with $\deg g = d$, then $\partial h/\partial x_i$ is a quasi-homogeneous function as well and $\deg \partial h/\partial x_i = d - \alpha_i, i = 1, \ldots, n$. We have $d - \alpha_1 < d - \alpha_2 \leq \cdots \leq d - \alpha_n$, and therefore for the curve

$$x_i = a_i t^{\alpha_i},\ g(a_1, \ldots, a_n) = 0,$$

lying on V, the tangent planes correspond to the points

$$(x; t^{d-\alpha_1} : t^{d-\alpha_2} : \cdots : t^{d-\alpha_n}) = (x; 1 : t^{\alpha_1-\alpha_2} : \cdots : t^{\alpha_1-\alpha_n}) \to (O; 1 : 0 \cdots : 0)$$

as $t \to 0$. Point $(O; 1 : 0 : \cdots : 0)$ corresponds to the function x_1. □

6 Local rings

It appears that in many cases, the local ring $Q_{f|V} = \mathcal{O}_n/I_{f|V}$ of the triple $(f, V = \{h(z) = 0\}, \mathbb{C}^n)$ is isomorphic to the local ring $Q(h_0)$ of the ordinary singularity given by $h_0 = h|_{f=0}$. First we establish this result in the quasi-homogeneous case.

Theorem 4 *Let $(g(x,y), Y = \{x = 0\}, \mathbb{C}^n)$ be a boundary germ given by a quasi-homogeneous function g, and $V = \{g(x,y) = 0\}$. Then the local rings $Q_{x|V}$ and $Q(g_0)$ are isomorphic.*

Proof. Consider germ $(x, V, \mathbb{C}^n)$ on a singular hypersurface corresponding to the given boundary germ. Let the weights of the variables be $\deg x = \alpha, \deg y_i = \beta_i, i = 1, \ldots, n-1$. Then, as it is proven in [7], the module of tangent vector fields, T_V, is generated by the Euler vector field

$$v_0 = \alpha x \frac{\partial}{\partial x} + \beta_1 y_1 \frac{\partial}{\partial y_1} + \cdots + \beta_{n-1} y_{n-1} \frac{\partial}{\partial y_{n-1}},$$

and by the Hamiltonian vector fields

$$v_k = \frac{\partial g}{\partial x}\frac{\partial}{\partial y_k} - \frac{\partial g}{\partial y_k}\frac{\partial}{\partial x}, \quad 1 \le k \le n-1,$$

$$v_{ij} = \frac{\partial g}{\partial y_i}\frac{\partial}{\partial y_j} - \frac{\partial g}{\partial y_j}\frac{\partial}{\partial y_i}, \quad 1 \le i, j \le n-1.$$

Applying these vector fields to the function x, we get the generators of $I_{x|V}$ which are $x, \partial g/\partial y_k, k = 1, \ldots, n-1$. Therefore

$$Q_{x|V} = \mathcal{O}_n / < x, \frac{\partial g}{\partial y_1}, \ldots, \frac{\partial g}{\partial y_{n-1}} > \cong \mathcal{O}_{n-1}/I(g_0) = Q(g_0).$$ □

Corollary 1 *If $(g(x,y), Y = \{x = 0\}, \mathbb{C}^n)$ is a boundary germ given by a quasi-homogeneous function g, then $\mu(x, V) = \mu(g_0) - 1$*

The similar statement is hold for L-germs of modality 0 or 1.

Theorem 5 *If $(f, V = \{h(z) = 0\}, \mathbb{C}^n)$ is a L-germ of modality 0 or 1, then:*

(i) $Q_{f|V} \cong Q(h|_{f=0})$*;*

(ii) $\mu(f, V) = \mu(h|_{f=0}) - 1$.

The proof of the theorem can be obtained by direct calculations using the list of normal forms of simple and unimodal critical poins and their versal deformations given in [7].

We say that $(f, V = \{h(z) = 0\}, \mathbb{C}^n)$ is *a good triple* if it satisfies condition *(i)* (and, hence, *(ii)*). In particular, triples of types H_3 and $I_2(p)$ are good.

Using the parametrized Morse lemma, one can prove the similar result for triples $(x, V = \{g(x, y) = 0\}, \mathbb{C}^n)$ such that the corresponding boundary germs $(g(x, y), Y = \{y = 0\}, \mathbb{C}^n)$ have decomposition (A_k, A_l).

Unfortunately we do not know a proof working for all these cases. It would be interesting to get such a proof and to understand general conditions for triple to be good.

One can guess that for L-triples, the equivalence class of a critical point $(f, V = \{h(z) = 0\}, \mathbb{C}^n)$ is defined by the equivalence class of the germ $h|_{f=0}$, but this is not the case as the following example shows.

Example 4 Consider two germs on $\mathbb{C}^3$ with the boundary $\{x = 0\}$:

$$h_1 = xz - z^4 - zy^2 - y^4, \quad h_2 = xz - z^y - y^4 - zy^3.$$

As it follows from [9], they define non-equivalent boundary singularities with the same decomposition (A_3, D_5). This means that the critical points $(x, V_i = \{h_i = 0\}, \mathbb{C}^3)$ $(i = 1, 2)$ on a singular hypersurface of type A_3 have the same singularity, namely D_5, of the restriction $h_i|_{x=0}$, but they are non-equivalent L-germ.

For holomorphic functions, an isolated critical point always has a finite multiplicity. Next example provides a non-singular function having a critical point of infinite multiplicity at an isolated singular point of a hypersurface.

Example 5 Consider the following critical point on a singular hypersurface:

$$(x,\ V = \{x^3 + xz^2 + y^2 z = 0\},\ \mathbb{C}^n).$$

It is clear that hypersurface V has an isolated singularity at O, but the multiplicity of the critical point $\mu(x, V) = \infty$. Indeed, V is given by a homogeneous function, therefore theorem 3 gives $Q_{x|V} \cong \mathcal{O}_{n-1}/I(g_0)$, where $g_0 = y^2 z$. Function g_0 has a non-isolated critical point at 0 (in fact, the line $y = 0$ is the line of critical points), that means that $\dim_{\mathbb{C}} Q_{x|V} = \infty$.

7 Versal deformations and bifurcation diagrams of good triples

Let $L = (x, V = \{g(x,y) = 0\}, \mathbb{C}^n)$ be a good triple. A versal deformation of L one can take in the form

$$L_\lambda = (x + \lambda_0 e_0 + \cdots + \lambda_\mu e_\mu, V, \mathbb{C}^n),$$

where $\{e_i = e_i(y),\ 0 \le i \le \mu\}$ represent a basis of the local ring $Q(g_0)$.

We define a deformation $G_L(x, y, \lambda)$ of boundary germ $(g(x,y), \{x = 0\}, \mathbb{C}^n)$ which corresponds to the versal deformation of the germ L as the family

$$G_L(x, y, \lambda) = g(x + \lambda_0 e_0 + \cdots + \lambda_\mu e_\mu, y).$$

The restriction of this family to the boundary,

$$G_L^0(y, \lambda) = G_L(0, y, \lambda) = g(\lambda_0 e_0 + \cdots + \lambda_\mu e_\mu, y),$$

is a deformation of $g_0 = g(0, y)$ related to the boundary germ $(g(x,y)$. We call this deformation *boundary deformation of* g_0 and the corresponding bifurcation diagram *the boundary diagram.* It is easy to see that the boundary diagrams of equivalent boundary germs are diffeomorphic. It turns out that the bifurcation diagram of this deformation is the bifurcation diagram of L.

Theorem 6 *The bifurcation diagram of L is*

$$\Sigma(L) = \{\lambda \in \mathbb{C}^{\mu+1} \mid 0 \text{ is a critical value of } G_L^0(\cdot, \lambda)\}$$

Proof Change of variables

$$(x, y) \mapsto (X = x - \lambda_0 e_0 - \cdots - \lambda_\mu e_\mu, y)$$

gives an equivalent family

$$\tilde{L}_\lambda = (X, V_\lambda = \{g(X + \lambda_0 e_0 + \cdots + \lambda_\mu e_\mu, y) = 0\}, \mathbb{C}^n).$$

The bifurcation diagram of this family is given by

$$\Sigma(L) = \{\lambda \in \mathbb{C}^{\mu+1} \mid 0 \text{ is a critical value of } \tilde{L}_\lambda\}.$$

Consider $PT^*\mathbb{C}^n$ with coordinates $(x, y_1, \dots, y_{n-1}; p_0 : p_1 : \dots : p_{n-1})$. Variety PV_λ of tangent planes to V_λ is given by

$$PV_\lambda = \left\{ G_L(X, y, \lambda) = 0,\ p_0 \frac{\partial G_L}{\partial y_i} = p_i \frac{\partial G_L}{\partial X},\ p_i \frac{\partial G_L}{\partial y_j} = p_j \frac{\partial G_L}{\partial y_i},\ i, j = 1, \dots, n-1 \right\}.$$

The intersection with $PV_0 = \{X = p_1 = \dots = p_{n-1} = 0\}$ gives

$$PV_\lambda \cap PV_0 = \{X = p_1 = \dots = p_{n-1} = 0,\ G_L(X, y, \lambda) = 0,\ p_0 \partial G_L / \partial y_i = 0\}.$$

Thus the intersection $V_1 \cap PV_0$ is given by $V_1 \cap PV_0 = \{G_L(X, y, \lambda) = 0,\ X = 0,\ \partial G_L / \partial y_i = 0,\ p_i = 0,\ i = 1, \dots, n-1\} =$

$$= \{G_L^0(y, \lambda) = 0,\ \partial G_L^0 / \partial y_i = 0,\ p_i = 0,\ i = 1, \dots, n-1\}.$$

It is non-empty if and only if 0 is the critical value of $G_L^0(\cdot, \lambda)$. □

8 Critical points of type H_3 and $I_2(p)$

Consider function $f = x$ having a critical point of type H_3 at the origin on a hypersurface $V = \{x^2 + y^5 = 0\}$. Its versal deformation is $x + \lambda_3 y^3 + \lambda_2 y^2 + \lambda y + \lambda_0$. Thus the corresponding deformation of the unimodal boundary singularity F_4^4 is

$$G_H(x, y, \lambda) = y^5 + (x + \lambda_3 y^3 + \lambda_2 y^2 + \lambda y + \lambda_0)^2.$$

As it is proven in[7], the bifurcation digram of this critical point is analytically trivial along the stratum $\mu =$ const and λ_3 is a parameter along this stratum. The bifurcation diagram of this critical point is the bifurcation diagram of the deformation

$$G_H^0(y, \lambda) = y^5 + (\lambda_3 y^3 + \lambda_2 y^2 + \lambda y + \lambda_0)^2$$

of simple singularity A_4.

A critical point of type $I_2(p)$, $p \geq 5$, can be given by germ $f = x$ on a hypersurface $V = \{(x + y^2)^2 + y^p = 0\}$. Its versal deformation is $x + y^2 + \lambda_2 y^2 + \lambda_1 y + \lambda_0$. Again, as it follows from [7], the bifurcation digram is analytically trivial along the stratum

$\mu =$ const and λ_2 is a parameter along this stratum. The corresponding deformation of the unimodal boundary singularity B^3_{p-1} is

$$G_I(x, y, \lambda) = y^p + (x + y^2 + \lambda_2 y^2 + \lambda_1 y + \lambda_0)^2.$$

The bifurcation diagram of this critical point is the bifurcation diagram of the deformation

$$G_I^0(y, \lambda) = y^p + (y^2 + \lambda_2 y^2 + \lambda_1 y + \lambda_0)^2$$

of simple singularity A_3.

Triples H_3 and $I_2(p)$ are good, therefore λ_3 (resp. λ_2) is a parameter along the stratum $\mu =$ const for the deformation G_H (resp. G_I) of the unimodal boundary singularity F_4^4 (resp. B^3_{p-1}) as well. The bifurcation diagram of critical point H_3 (resp. $I_2(p)$) is the component of the bifurcation diagram of G_H (resp. G_I) which corresponds to the restriction to the boundary. Thus get the following result.

Theorem 7 *The bifurcation diagram of the deformation G_I^0 (G_H^0 resp.) of a simple singularity A_3 (A_4 resp.) is diffeomorphic to the manifold of the non-regular orbits of the group $I_2(p)$ (H_3 resp.) multiplying by a complex line.*

9 Critical point of type G_2

The simple critical point of type G_2 appears in the Lyashko classification for non-singular germ $x + y$ on a singular surface $xy = z^3$. The corresponding simple boundary singularity F_4 is given by germ $x^2 + y^3$ and boundary $\{x = 0\}$. The boundary deformation is given by $G_G^0(y, \lambda) = (\lambda_1 + \lambda_2 y)^2 + y^3$, and it is easy to check that the boundary diagram is exactly the manifold of the non-regular orbits of group G_2.

References

[1] Arnold, V.I., Normal forms of functions near degenerate critical points, the Weyl groups A_k, D_k, E_k and Lagrangian singularities, *Functional Anal. Appl.* **6** (1972), 254–272.

[2] Arnold, V.I., Wave front evolution and equivariant Morse lemma, *Comm. Pure Appl. Math.*, **29** (1976), 557–582.

[3] Arnold, V.I., Critical points of function on a manifold with a boundary, the simple Lie groups B_k, C_k, F_4, and singularities of evolutes, *Russian Math. Surveys*, **33** (1978), 91–105.

[4] Arnold, V.I., Goryunov, V.V., Lyashko, O.V., Vassiliev, V.A, *Singularities II*, Encycl. Math. Sci., vol.39, Springer Verlag, Berlin and New York, 1993.

[5] Arnold, V.I., Gusein-Zade, S.M., Varchenko, A.N., *Singularities of Differentiable Maps*, Vol. I, Birkhäuser, Basel, 1985.

[6] Bourbaki, N., *Groupes et algèbres de Lie*, Chapters IV, V and VI, Hermann, Paris, 1968.

[7] Lyashko O., Classification of critical points of functions on a manifold with singular boundary, *Functional Anal. Appl.* **17:3** (1983), 187–193.

[8] Lyashko O., Geometry of bifurcation diagrams, *Journal of Soviet Math.*, **27** (1984), 2536–2759.

[9] Scherbak I., Boundary singularities with simple decomposition, *Journal of Soviet Math.*, **60** (1992), 1681–1693.

[10] Scherbak O., Wave fronts and reflection groups, *Russian Math. Surveys*, **43:3** (1988), 149–194.

[11] Scherbak I., Szpirglas A., Boundary singularities: topology and duality, *Advances in Soviet Math.*, **21** (1994), 213–223.

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