An extension of Prandtl-Batchelor closed streamline theory and consequences for chaotic advection

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Abstract

We extend the Prandtl-Batchelor theory of steady laminar motion at large Reynolds number and combine it with ergodic theory to show that flows with strong Beltrami property (e.g. ABC flows) can not be a paradigm for chaotic advection in inertia-dominated three-dimensional flows. Our results indicate that viscous forces are responsible for chaotic advection in steady, three-dimensional Navier-Stokes flows at large Reynolds numbers.

I. INTRODUCTION

The flows discussed in chaotic advection studies are mostly kinematic models (e.g. ABC maps of [1]), solutions of Stokes equations (e.g. [2,3]), or weak solutions based on singular vortex distributions (e.g. [4–6]). However, the first results in the field of chaotic advection were based on the analysis of velocity fields that were smooth solutions of Euler equations - the so-called ABC flows [7]. Recently, a few studies ([8–12]) appeared that took account of the restrictions imposed by the fact that Newtonian fluid flows satisfy Navier-Stokes equations.

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One way of interpreting suggestions on importance of ABC flows is that viscous perturbations to Euler flows can be taken to be small away from the boundaries and, due to integrability of Euler flows that do not have velocity and vorticity proportional, chaotic motion can be only caused by an ABC-type flow. The idea is that in the region where inertial forces are dominant, a Navier-Stokes flow can be decomposed into an Euler and a small viscous part. If the Euler part is chaotic, the Navier Stokes flow will be chaotic by structural stability. Here we show that at large Reynolds numbers, the strong Beltrami condition

$$\omega = c\mathbf{v},$$

between velocity $\mathbf{v}$ and vorticity ω of the part of the flow satisfying Euler's equation, with constant c, is *inconsistent* with requirements on $\mathbf{v}$ that arise from a theory of steady laminar ergodic motion in 3-D at large Reynolds number which we obtain as an extension of Prandtl-Batchelor theory for flows with closed streamlines [13].

II. AN EXTENSION OF BATCHELOR'S THEORY

Consider the steady Navier-Stokes equation in three dimensional, bounded space:

$$-\mathbf{v} \times \omega + \nabla H = \nu \nabla \times \omega, \tag{1}$$

where ν is the kinematic viscosity and $H = p/\rho + \mathbf{v}^2/2$. We assume that a solution of (1) exists and has bounded velocity and pressure. Let us multiply (1) by $\mathbf{v}$ from the right and average over time along an arbitrary trajectory starting at $\mathbf{x}$:

$$\frac{1}{T}\int_0^T \nabla H \cdot \mathbf{v} dt = \nu \frac{1}{T}\int_0^T \nabla \times \omega \cdot \mathbf{v} dt$$

We take the limit as $T \to \infty$ to obtain

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \nabla H \cdot \mathbf{v} dt = \nu \lim_{T\to\infty} \frac{1}{T}\int_0^T \nabla \times \omega \cdot \mathbf{v} dt, \tag{2}$$

Now we show that the left-hand side of (2) is 0. We have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \nabla H\cdot \mathbf{v}dt = \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{dH}{dt}dt \equiv (\frac{dH}{dt})^*(\mathbf{x}),$$

where $f^*(\mathbf{x})$ denotes the time average of a function f on a trajectory starting at $\mathbf{x}$. If the time average of dH/dt is not zero, than the function H grows without bound on that trajectory which contradicts our assumption that velocity and pressure are bounded. Thus, we obtain

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \nabla\times\omega\cdot\mathbf{v}dt = 0. \tag{3}$$

Note that (3) reduces to

$$\int_\gamma \nabla\times\omega\mathbf{dl} = 0,$$

when evaluated over a closed orbit γ, where $\mathbf{dl}$ is a line element of γ, which is exactly the condition obtained by Batchelor [13]. The condition (3) must be satisfied for every trajectory of a steady three-dimensional Navier-Stokes flow.

III. CHAOTIC STRONG BELTRAMI FLOWS

Now we assume that the Navier-Stokes flow can be split into an Euler part and a small viscous part in the region where inertial forces are dominant. Proceeding as in [13], we evaluate (3) using the Euler part of the flow. Assume that the Euler flow is strong Beltrami, i.e. $\omega = c\mathbf{v}$. Then (3) becomes

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \nabla\times\omega\cdot\mathbf{v}dt = c^2\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbf{v}^2dt.$$

Assume now that the Beltrami flow is chaotic in the sense that is possesses a region A of non-zero volume such that it is ergodic in that region. By ergodicity,

$$0 = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbf{v}^2dt = \int_A \mathbf{v}^2dV.$$

But, the velocity is not zero almost everywhere in A, because that would contradict the assumption of ergodicity. Thus, (3) can not be satisfied by a Navier-Stokes flow that is ergodic in a subvolume of the domain and whose inertial part is Beltrami.

IV. CONCLUSIONS

The above calculation leads to the conclusion that at high Reynolds number, the splitting of a steady Navier-Stokes solution reads $\mathbf{v} = \mathbf{v}_E + f(\nu)\mathbf{w}$, where $\mathbf{v}_E$ is integrable, $f(\nu) \to 0$ as $\nu \to 0$. The implications of this for chaotic advection are extensively discussed in [12]. The conclusion there is that chaotic motion is, in this limit, predominantly "caused" by viscous forces. The flow $\mathbf{v}_E$ is, by the theorem of Arnold [14] split into domains in which particle motion is restricted to two-dimensional tori or cylinders. The condition (3) might be of some independent interest. For example, a similar calculation in two dimensions lead Prandtl and Batchelor to flows with patches of constant vorticity as Euler flows satisfying a condition analogous to (3). Just like in the case of axisymmetric swirling flows [13] it is not easy to write down explicitly Euler flows satisfying (3). However, our results indicate clearly that the topological nature of such flows is simple.

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