ON A FREE BOUNDARY PROBLEM ARISING FROM POPULATION BIOLOGY

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ABSTRACT. In this paper, using variational inequality techniques, we completely solve the existence and uniqueness question for a free boundary problem arising from the studies of certain population models. We also confirm a conjecture of ours in an earlier paper that this free boundary problem is the correct limiting problem of a degenerate predator-prey model.

1. INTRODUCTION

In this paper, we study the following free boundary problem

$$-\Delta u = \lambda \chi_{\{u<1\}} u,\ u > 0 \text{ in } D;\ u|_{\partial D} = 0,\ \max_{\bar{D}} u = 1, \tag{1.1}$$

and its relations to certain population models, where λ is a constant and D a bounded smooth domain in R^N ($N \geq 2$).

Problem (1.1) arises as the limiting problem of several population models. It is shown in [DDM, Theorem 1.3] that for any fixed $\lambda > \lambda_1^D$, where λ_1^D stands for the first eigenvalue of the problem

$$-\Delta u = \lambda u,\ u|_{\partial D} = 0,$$

the unique positive solution u_p of the steady-state logistic equation

$$-\Delta u = \lambda u - u^p,\ u|_{\partial D} = 0,$$

converges in $C^1(\bar{D})$ to a solution of (1.1) as $p \to \infty$.

In [DD], we conjectured that (1.1) is also the limiting problem of a degenerate predator-prey model. To be more precise, to understand the influence of spatial heterogeneity on population models, we considered in [DD] the degenerate steady-state predator-prey model

$$\begin{cases} -\Delta u = \lambda u - b(x)u^2 - cuv, \\ -\Delta v = \mu v - v^2 + duv, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

where Ω is a bounded smooth domain in R^N, λ, μ, c, d are constants with c, d positive, and $b \in C^\xi(\bar{\Omega})$ $(0 < \xi < 1)$ is a nonnegative function which vanishes on the closure of a subdomain D of Ω and $b(x) > 0$ on $\bar{\Omega} \setminus \bar{D}$. We assume that $\bar{D} \subset \Omega$ and D has smooth boundary.

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We have shown in [DD] that (1.2) behaves like the classical model (i.e., (1.2) with $b(x) \equiv 1$ on Ω) when $\lambda < \lambda_1^D$ but essential changes occur once $\lambda \geq \lambda_1^D$. More precisely, for fixed positive c and d, when $\lambda \leq \lambda_1^\Omega$, (1.2) has no positive solution for any value of μ; when $\lambda_1^\Omega < \lambda < \lambda_1^D$, (1.2) has a positive solution if and only if μ lies in a certain bounded open interval; and when $\lambda \geq \lambda_1^D$, then (1.2) has a positive solution if and only if μ belongs to an unbounded interval of the form $(-\infty, \mu_0)$ with μ_0 a certain positive number. To better understand the change of behavior of (1.2) from the classical case, we fix $\lambda > \lambda_1^D$ and consider the asymptotic behavior of the positive solutions of (1.2) as $\mu \to -\infty$. This is a situation not occurring in the classical model and hence of considerable importance in the understanding of the new behavior of the degenerate model.

Let $\{\mu_n\}$ be a sequence of decreasing negative numbers converging to $-\infty$, and let (u_n, v_n) be a positive solution of (1.2) with $\mu = \mu_n$. We want to know the asymptotic behavior of (u_n, v_n) for large n. It has been proved in [DD, Theorem 2.7] that, among other things, both $\|u_n\|_\infty := \|u_n\|_{L^\infty(\Omega)}$ and $\|u_n\|_{L^1(\Omega)}$ blow up at the rate of $|\mu_n|$, but $\|v_n\|_{L^1(\Omega)}$ stays bounded away from both 0 and ∞. Moreover, if $\{\|v_n\|_{L^q(\Omega)}\}$ is bounded for some $q > 1$, then subject to a subsequence, $u_n/\|u_n\|_\infty$ converges in $L^p(\Omega)$, for any $p \geq 1$, to a solution of (1.1), say u, and v_n converges weakly in $L^q(\Omega)$ to $(\lambda/c)\chi_{\{u=1\}}$. However, we were unable to determine whether we can always find some $q > 1$ such that $\{\|v_n\|_{L^q(\Omega)}\}$ is bounded. Thus, it is unclear whether (1.1) is always a limiting problem for (1.2).

In this paper, we first show that (1.1) has a unique solution whenever $\lambda \geq \lambda_1^D$, and then we show that (1.1) is always the limiting problem for (1.2) as $\mu \to -\infty$. Combined with the uniqueness result for (1.1), this gives a rather complete description of the asymptotic behavior of (1.2). (See Theorem 3.4 in section 3 for details.)

Our approach uses a combination of variational inequalities, fine properties of functions in Sobolev spaces and certain elliptic estimates. The free boundary problem (1.1) is discussed in section 2, and the asymptotic behavior of (1.2) is studied in section 3.

The use of variational inequalities in our approach appears rather indirect but direct arguments do not seem to work. It seems that variational inequality techniques are useful for other limit problems. We intend to return to this.

2. The free boundary problem

This section is devoted to proving the following result.

Theorem 2.1. *Problem (1.1) has a unique solution whenever $\lambda \geq \lambda_1^D$, and it has no solution when $\lambda < \lambda_1^D$.*

Proof. We first prove the nonexistence result. Suppose that (1.1) has a solution u. As $\lambda\chi_{\{u<1\}} \in L^\infty(D)$, it follows from the Harnack inequality that u is strictly positive in D, and a standard regularity consideration shows $u \in H^{2,s}(D) \cap C^1(\bar{D})$, $\forall s \geq 1$. Let $\phi > 0$ be an eigenfunction corresponding to λ_1^D: $-\Delta\phi = \lambda_1^D\phi$, $\phi|_{\partial D} = 0$. Then one easily sees that

$$0<\lambda_1^D\int_D \phi u dx=\int_D \nabla\phi\cdot\nabla u dx=\lambda\int_D \chi_{\{u<1\}}\phi u dx\le\lambda\int_D \phi u dx.$$

It follows that $\lambda\ge\lambda_1^D$. This proves the nonexistence part of the theorem.

When $\lambda=\lambda_1^D$, we see from the above inequality that necessarily $\chi_{\{u<1\}}=1$ almost everywhere in D. Thus any solution must satisfy $-\Delta u=\lambda u$ in D. It follows that u must be the positive normalized eigenfunction corresponding to λ_1^D whenever it is a solution of (1.1) in this case. As such a function is well known to be unique, the uniqueness for this case is proved. Conversely, suppose that ϕ is the unique positive normalized eigenfunction corresponding to λ_1^D, then it is well known that $\nabla\phi$ never vanishes on a set of positive measure in D. (For example, one can prove that if $\nabla\phi$ is zero on a set of positive measure, then $\phi=-\Delta\phi/\lambda_1^D=0$ almost everywhere on this set.) Thus $\{\phi=1\}$ has measure zero and hence $\chi_{\{\phi<1\}}=1$ almost everywhere in D. It follows that ϕ is a solution to (1.1).

We now consider the much more difficult case that $\lambda>\lambda_1^D$. The existence of a solution of (1.1) for this case follows from Theorem 1.3 in [DDM]. (We will also provide a self-contained proof for this later in this section.) It remains to prove uniqueness for this case. Let us note that if u is a solution to (1.1) with $\lambda>\lambda_1^D$, then $\{u=1\}$ must have positive measure, for otherwise, u solves $-\Delta u=\lambda u$, $u|_{\partial D}=0$ and hence $\lambda=\lambda_1^D$, a contradiction. This implies that u satisfies a semilinear equation with a discontinuous nonlinearity.

To cope with the discontinuity of the right hand side of (1.1), we resort to a variational inequality approach. Let

$$K:=\{u\in H_0^1(D): u\le 1 \text{ almost everywhere in } D\}.$$

Then for each $f\in L^2(D)$, the variational inequality

$$\int_D \nabla u\cdot\nabla(v-u)dx\ge\int_D f(v-u)dx,\ \forall v\in K, \tag{2.1}$$

has a unique solution $u\in K$. This follows directly from Theorem 1.4 in chapter III of [KS] on strictly monotone operators. Define the solution operator by $L: u=L(f)$. Then one can easily show that L has the following properties:

(i) $L: L^2(D)\to L^2(D)$ is completely continuous;
(ii) $L(f)\in H^{2,s}(D)\cap C^{1,\mu}(\bar D)$, $\mu=1-(N/s)$, if $f\in L^s(D)$ with $s>N$;
(iii) $L(f)\ge 0$ a.e. in D when $f\ge 0$ a.e. in D;
(iv) $L(f_1)\le L(f_2)$ a.e. in D when $f_1\le f_2$ a.e. in D.
(v) $L(f)=(-\Delta)^{-1}f$ if $f\ge 0$ and $(-\Delta)^{-1}f\le 1$ on D, where $(-\Delta)^{-1}$ denotes the inverse of $-\Delta$ over D under Dirichlet boundary conditions.

These are mostly well-known, but we prove them for completeness.

To see (i), it is enough to show

$$\|L(f_1)-L(f_2)\|_{H_0^1}\le c\|f_1-f_2\|_{L^2}.$$

Denote $u_1 = L(f_1)$ and $u_2 = L(f_2)$. Then, by (2.1),

$$\int_D \nabla u_1 \cdot \nabla(u_2 - u_1)dx \geq \int_D f_1(u_2 - u_1)dx,$$

$$\int_D \nabla u_2 \cdot \nabla(u_1 - u_2)dx \geq \int_D f_2(u_1 - u_2)dx.$$

Adding the above inequalities we obtain

$$\|u_1 - u_2\|_{H_0^1}^2 = \int_D |\nabla(u_1 - u_2)|^2 dx \leq \int_D (f_1 - f_2)(u_1 - u_2)dx$$

$$\leq \|f_1 - f_2\|_{L^2}\|u_1 - u_2\|_{L^2}.$$

By Poincare's inequality, $\|u_1 - u_2\|_{L^2} \leq c\|u_1 - u_2\|_{H_0^1}$ for some $c > 0$. Thus

$$\|u_1 - u_2\|_{H_0^1} \leq c\|f_1 - f_2\|_{L^2},$$

as required.

To see (ii), we note that u solves (2.1) if and only if $w = -u$ solves

(2.2)

$$\int_D \nabla w \cdot \nabla(v - w)dx \geq \int_D (-f)(v - w)dx, \ \forall v \in K' := \{v \in H_0^1(D) : v \geq -1\}.$$

By Theorem 2.3 in chapter IV of [KS], the unique solution w of (2.2) satisfies $w \in H^{2,s}(D) \cap C^{1,\mu}(\bar{D})$ provided that $f \in L^s(D)$, $s > N$. Thus, so is $u = -w$.

Since clearly $L(0) = 0$, we find that (iii) follows from (iv). To prove (iv), we let $w_i = -L(f_i)$, $i = 1, 2$. It is easily seen that w_1 and w_2 are solutions to (2.2) with $f = f_1$ and f_2, respectively. For $0 \leq \xi \in H_0^1(D)$, clearly $w_1 + \xi \in K'$. Therefore, by (2.2) with $f = f_1, w = w_1$ and $v = w_1 + \xi$,

$$\int_D \nabla w_1 \cdot \nabla \xi dx \geq \int_D (-f_1)\xi dx \geq \int_D (-f_2)\xi dx.$$

This is to say that w_1 is a supersolution of $-\Delta + f_2$ and hence, by Theorem 6.4 in chapter II of [KS], $w_1 \geq w_2$, i.e., $L(f_1) \leq L(f_2)$, as required.

Property (v) is self-evident from the definition of $L(f)$.

With these preparations, we are ready to prove the uniqueness of solutions to (1.1). For fixed $\lambda > \lambda_1^D$, we define $L_\lambda : L^2(D) \to L^2(D)$ by

$$L_\lambda f = L(\lambda f).$$

Claim 1: *u is a solution to (1.1) if and only if u is a positive fixed point of L_λ in K.*

To prove the necessity, we assume that u is a solution to (1.1). Clearly $u \in K$. It suffices to show that

$$\int_D \nabla u \cdot \nabla(v - u)dx \geq \int_D \lambda u(v - u)dx, \ \forall v \in K.$$

As u solves (1.1) weakly, we have

$$\int_D \nabla u \cdot \nabla \phi dx = \int_D \lambda \chi_{\{u<1\}} u \phi dx, \ \forall \phi \in H_0^1(D).$$

Taking $\phi = v - u$ with $v \in K$, we obtain

$$\int_D \nabla u \cdot \nabla (v-u) dx = \int_D \lambda \chi_{\{u<1\}} u (v-u) dx$$

$$= \int_D \lambda u(v-u)dx + \int_D \lambda(\chi_{\{u<1\}} - 1)u(v-u)dx$$

$$= \int_D \lambda u(v-u)dx + \int_{\{u=1\}} \lambda(1-v)dx$$

$$\geq \int_D \lambda u(v-u)dx,$$

as we wanted.

We consider next the sufficiency. Suppose that u is a positive fixed point of L_λ in K. Then, $0 \leq u \leq 1$ and hence $\lambda u \in L^\infty(D)$, $\lambda u \geq 0$. From properties (ii) and (iii) of the operator L proved earlier, we find that $0 \leq u \in H^{2,s}(D) \cap C^{1,\mu}(\bar{D})$, $\forall s \geq 1$, $\mu \in (0,1)$.

Using Theorem 6.9 in chapter II of [KS] to $w = -u$, we find that in the distributional sense,

$$-\Delta w = -\lambda u + m \text{ in } D,$$

where m is a nonnegative Radon measure with $\text{supp}(m) \subset \{w = -1\} = \{u = 1\}$. Thus,

$$-\Delta u = \lambda u - m \text{ in } D. \tag{2.3}$$

But from the regularity of u we have $\lambda u + \Delta u \in L^s(D)$ for any $s \geq 1$. Thus m is in fact a function belonging to $L^s(D)$.

If $\|u\|_\infty < 1$, then m has empty support and hence (2.3) reduces to $-\Delta u = \lambda u$, which implies that $\lambda = \lambda_1^D$, a contradiction. Thus $\{u = 1\}$ has positive measure. On this set, by repeatedly using Theorem 6.19 of [LL], we deduce $\Delta u = 0$ almost everywhere. Thus $m = \lambda u - \Delta u = \lambda$ almost everywhere in $\{u = 1\}$. As we already know that $m = 0$ outside this set, we must have $m = \lambda \chi_{\{u=1\}}$ and hence

$$-\Delta u = \lambda u - \lambda \chi_{\{u=1\}} = \lambda \chi_{\{u<1\}} u \text{ for almost every } x \in D.$$

It follows that u is a weak solution to (1.1).

Claim 2: *L_λ has a minimal positive fixed point in K.*

As mentioned before, (1.1) always has a solution. (Recall that we have assumed $\lambda > \lambda_1^D$). Thus, by Claim 1 above, L_λ always has a positive fixed point in K. Let u be an arbitrary positive fixed point of L_λ in K. From equation (1.1) and standard elliptic regularity we know u is C^1 up to the boundary. The Harnack inequality implies $u > 0$ in D. As $u = 0$ on ∂D, we find that $u < 1$ near ∂D, and hence u

solves $-\Delta u = \lambda u$ near ∂D. It follows that u is smooth here and the Hopf boundary lemma gives $\partial u/\partial \nu < 0$ on ∂D, where μ denotes the unit outward normal of ∂D. Thus, $u \geq \epsilon\phi$ on D for all small positive number ϵ, where ϕ denotes the positive eigenfunction corresponding to λ_1^D with $\|\phi\|_\infty = 1$.

If ϵ is small enough, say $\epsilon < \epsilon_0$ for definiteness, we have $v_0 := (\lambda/\lambda_1^D)\epsilon\phi < 1$ on D and v_0 solves, by an easy calculation, $-\Delta v_0 = \lambda\epsilon\phi$. Thus, $L_\lambda(\epsilon\phi) = v_0$ and

$$u = L_\lambda(u) \geq L_\lambda(\epsilon\phi) \geq \epsilon\phi.$$

It follows that $\{(L_\lambda)^n(\epsilon\phi)\}$ is an increasing sequence of functions bounded from above by u. A standard argument shows that

$$w_\epsilon := \lim_{n\to\infty}(L_\lambda)^n(\epsilon\phi)$$

is a fixed point of L_λ satisfying $\epsilon\phi \leq w_\epsilon \leq u$. Thus it is a positive fixed point of L_λ lying in K.

Let us remark that if we replace u by 1 in the above discussion, then since clearly $L_\lambda(1) \leq 1$, we have $(L_\lambda)^n(\epsilon\phi) \leq 1$ for all $n \geq 1$ and therefore the existence of the fixed point w_ϵ of L_λ also follows. Combined with Claim 1, this gives a self-contained proof for the existence of a solution to (1.1) for the case $\lambda > \lambda_1^D$.

We show next that w_ϵ is independent of $\epsilon \in (0, \epsilon_0)$. Let $0 < \epsilon_1 < \epsilon_2 < \epsilon_0$. By property (iv) of the operator L, we easily see that $(L_\lambda)^n(\epsilon_1\phi) \leq (L_\lambda)^n(\epsilon_2\phi)$. It follows that $w_{\epsilon_1} \leq w_{\epsilon_2}$. On the other hand, if ϵ_0 is small enough, then it is easily checked that for any $\epsilon_1 \in (0, \epsilon_0)$, one can find a positive integer n_0 such that

$$\epsilon_0 \leq \epsilon_1(\lambda/\lambda_1^D)^{n_0} < 1.$$

Using

$$(\lambda/\lambda_1^D)\epsilon_1\phi < (\lambda/\lambda_1^D)^2\epsilon_1\phi < \cdots < (\lambda/\lambda_1^D)^{n_0}\epsilon_1\phi < 1,$$

we deduce

$$L_\lambda(\epsilon_1\phi) = (\lambda/\lambda_1^D)\epsilon_1\phi,\ (L_\lambda)^2(\epsilon_1\phi) = (\lambda/\lambda_1^D)^2\epsilon_1\phi, \cdots, (L_\lambda)^{n_0}(\epsilon_1\phi) = (\lambda/\lambda_1^D)^{n_0}\epsilon_1\phi.$$

Therefore,

$$(L_\lambda)^{n_0}(\epsilon_1\phi) \geq \epsilon_0\phi > \epsilon_2\phi.$$

It follows that

$$(L_\lambda)^{n+n_0}(\epsilon_1\phi) \geq (L_\lambda)^n(\epsilon_2\phi),\ \forall n \geq 1,$$

which implies $w_{\epsilon_1} \geq w_{\epsilon_2}$. Thus we must have $w_{\epsilon_1} = w_{\epsilon_2}$, and w_ϵ is independent of ϵ.

Denote by w_0 the common function w_ϵ. Then our previous discussion shows that $w_0 \leq u$ for an arbitrary positive fixed point of L_λ in K. Thus, w_0 is the minimal positive fixed point of L_λ in K.

Claim 3: (1.1) *has a unique solution.*

Otherwise, L_λ has a fixed point in K, say w_1, satisfying $w_1 \geq w_0$ but $w_1 \not\equiv w_0$. Denote $D_i = \{w_i < 1\}$ for $i = 0, 1$. We have $D_1 \subseteq D_0$ and

$$-\Delta w_i = \lambda \chi_{D_i} w_i,\ w_i|_{\partial D} = 0,\ i = 0, 1.$$

It follows that

$$\lambda \int_D \chi_{D_1} w_0 w_1 dx = \int_D \nabla w_1 \cdot \nabla w_0 dx = \lambda \int_D \chi_{D_0} w_0 w_1 dx.$$

Hence

$$\int_D (\chi_{D_1} - \chi_{D_0}) w_0 w_1 dx = 0.$$

As w_0, w_1 are positive on D and $\chi_{D_1} - \chi_{D_0}$ is nonpositive, we necessarily have $\chi_{D_1} = \chi_{D_0}$ almost everywhere in D, i.e., $D_0 \setminus D_1$ has measure zero. Recall that w_0 and w_1 belongs to $H^{2,s}(D)$ for any $s \geq 1$. Hence we have $-\Delta w_i = \lambda w_i$ a.e. in D_0 and $-\Delta w_i = 0$ a.e. in $D \setminus D_0$. If we write $w = w_1 - w_0$, then $-\Delta w = \lambda w$ a.e. in D_0 and $-\Delta w = 0$ a.e. in $D \setminus D_0$. As $w_1 = w_0 = 1$ a.e. in $D \setminus D_0$, we have $w = 0$ a.e. in $D \setminus D_0$, and hence $-\Delta w = \lambda w$ a.e. in D. That is to say that w is a nonnegative nontrivial solution of $-\Delta w = \lambda w, w|_{\partial D} = 0$. Hence we must have $\lambda = \lambda_1^D$, contradicting our assumption. This proves Claim 3 and hence finishes the proof of the theorem. □

Remark 2.2. *For $\lambda \geq \lambda_1^D$, let u_λ denote the unique solution of (1.1). Then the following hold.*

(i) *$\lambda_1^D \leq \lambda \leq \mu$ implies $u_\lambda(x) \leq u_\mu(x)$, $\forall x \in D$.*

(ii) *$\lambda \to u_\lambda$ is continuous as a mapping from $[\lambda_1^D, \infty)$ to $C^1(\bar{D})$.*

(iii) *u_λ, and hence the free boundary $\partial\{u_\lambda = 1\}$, inherits the symmetry of D.*

(iv) *$u_\lambda(x)/\psi_\lambda(d(x, \partial D)) \to 1$ as $\lambda \to \infty$ uniformly in D, where*

$$\psi_\lambda(t) = \begin{cases} \sin\sqrt{\lambda} t, & 0 \leq t < (\pi/2)\lambda^{-1/2}, \\ 1, & t \geq (\pi/2)\lambda^{-1/2}. \end{cases}$$

(v) *For any given $\epsilon > 0$, there exists λ_ϵ such that for $\lambda > \lambda_\epsilon$, the free boundary of u_λ lies between the smooth surfaces*

$$\Gamma_{\lambda,\epsilon}^+ := \{x \in D : d(x, \partial D) = (\frac{\pi}{2} + \epsilon)\lambda^{-1/2}\}$$

and

$$\Gamma_{\lambda,\epsilon}^- := \{x \in D : d(x, \partial D) = (\frac{\pi}{2} - \epsilon)\lambda^{-1/2}\}.$$

The first three conclusions in Remark 2.2 are easy consequences of the uniqueness of u_λ. The last two follow from a careful analysis involving upper and lower solutions.

Remark 2.3. *With minor modifications, our proof of Theorem 2.1 covers the following more general problem*

$$-\Delta u = \lambda \chi_{\{u<1\}} g(u),\ u > 0 \text{ in } D;\ u|_{\partial D} = 0,\ \max_{\bar{D}} u = 1, \tag{2.4}$$

where $g(u)$ is C^1, $g(0) = 0$, $g'(0) = 1$, and $g(u)/u$ is nonincreasing on $(0, 1)$. We have the following conclusions:

Problem (2.4) has a unique solution whenever $\lambda > \lambda_1^D$, and it has no solution when $\lambda < \lambda_1^D$. Moreover, unless $g(u) = u$ on $(0,1)$, (2.4) has no solution when $\lambda = \lambda_1^D$.

Note that if $g(a) = 0$ for some $a \in (0,1]$, then the unique solution of (2.4) is smooth and does not have a free boundary.

3. Limiting behavior of the predator-prey model

To understand the limiting behavior of (1.2) as $\mu \to -\infty$, we need some preparations. The first result we need is the following lemma proved in [DD, Lemma 2.2].

Lemma 3.1. *Suppose $\{u_n\} \subset C^2(\overline{\Omega})$ satisfies*

$$-\Delta u_n \leq \lambda u_n,\ u_n|_{\partial\Omega} = 0,\ u_n \geq 0,\ \|u_n\|_\infty = 1,$$

where λ is a positive constant. Then, there exists $u_\infty \in L^\infty(\Omega) \cap H_0^1(\Omega)$ such that, subject to a subsequence, $u_n \to u_\infty$ weakly in $H_0^1(\Omega)$, strongly in $L^p(\Omega)$, $\forall p \geq 1$, and $\|u_\infty\|_\infty = 1$.

It turns out that we will need some finer properties of the function u_∞ than that given in Lemma 3.1. To this end, let us first recall from [LL] some basic facts on subharmonic functions.

Let $f \in L^1_{loc}(\Omega)$ and we understand that f is a definite, Borel measurable function, not an equivalent class. For each open ball $B_r(x) := \{y \in R^N : |y - x| < r\} \subset \Omega$, with volume denoted by $|B_r(x)|$, let

$$[f]_{x,r} := |B_r(x)|^{-1} \int_{B_r(x)} f(y)dy$$

denote the average of f in $B_r(x)$. If for almost every $x \in \Omega$,

$$f(x) \leq [f]_{x,r}$$

for every r such that $B_r(x) \subset \Omega$, we say that f is subharmonic on Ω. We have the following basic facts. (See Theorem 9.3 of [LL].)

(i) f is subharmonic on Ω if and only if $\Delta f \geq 0$ in the sense of distribution.

(ii) When f is subharmonic on Ω, there exists a function $\tilde{f} : \Omega \to R \cup \{-\infty\}$ satisfying $\tilde{f}(x) = f(x)$ for almost every $x \in \Omega$, and $\tilde{f}(x)$ is upper semicontinuous (u.s.c for short).

(iii) The u.s.c function $\tilde{f}(x)$ can be obtained in the following way:

$$\tilde{f}(x) = \lim_{r \to 0}[f]_{x,r},\ \forall x \in \Omega. \tag{3.1}$$

We next recall several facts on approximation of functions in Sobolev spaces from [H]. Let $C_{1,2}(A)$ denote the $(1,2)$–capacity of the set $A \subset R^N$ given by the following:

(1) If A is compact,

$$C_{1,2}(A) = \inf\{\|\phi\|^p_{H^1(R^N)} : \phi \in C_0^\infty(R^N),\ \phi \geq 1 \text{ on } A\}.$$

(2) If A is open,

$$C_{1,2}(A) = \sup\{C_{1,2}(K) : \ K \subset A, \ K \text{ compact}\}$$

(3) A arbitrary,

$$C_{1,2}(A) = \inf\{C_{1,2}(G) : G \supset A, \ G \text{ open}\}.$$

We have the following facts (see [H] pages 79-80 and 82-83) :

(a) Let $f \in H^1(R^N)$. Then we can choose a representative $\tilde{f}$ in the equivalent class of f so that $\tilde{f}(x) = \lim_{r\to 0}[f]_{x,r}$, $\forall x \in R^N$, except for a set of (1,2)–capacity zero. Note that this choice is local and agrees with (3.1).

(b) Suppose $f \in H^1(R^N)$ and its representative $\tilde{f}$ given in (a) satisfies $\tilde{f} = 0$ on a closed set $A \subset R^N$ except possibly for a subset of A of $(1,2)-$capacity zero. Then there exists a sequence of functions $\phi_n \in H^1(R^N)$ such that each ϕ_n vanishes on a neighbourhood of A and $\phi_n \to \tilde{f}$ in $H^1(R^N)$.

We are now ready to state and prove the needed fine properties of u_∞ in Lemma 3.1.

Lemma 3.2. *The function u_∞ in Lemma 3.1 can be chosen to be u.s.c. Moreover, for any $x_0 \in \Omega$ and any given $\epsilon > 0$, we can find a small ball $B_r(x_0) \subset \Omega$ such that for all large n,*

$$u_n(x) \le u_\infty(x_0) + \epsilon, \ \forall x \in B_r(x_0). \tag{3.2}$$

Proof. Let v_n be given by $-\Delta v_n = \lambda u_n$, $v_n|_{\partial\Omega} = 0$. Clearly $v_n \ge 0$. Since, subject to a subsequence, $u_n \to u_\infty$ in $L^p(\Omega)$ for any $p \ge 1$,

$$v_n = (-\Delta)^{-1}(\lambda u_n) \to v_\infty := (-\Delta)^{-1}(\lambda u_\infty)$$

in $H^{2,p}(\Omega) \cap C^1(\bar{\Omega})$.

Denote $w_n = u_n - v_n$. Then $\Delta w_n \ge 0$ and hence w_n is subharmonic. As $w_n = u_n - v_n \to w_\infty := u_\infty - v_\infty$ in $L^p(\Omega)$, we find easily from the definition of subharmonic functions that w_∞ is subharmonic. Moreover, by defining $\tilde{w}_\infty(x)$ as in (3.1) with $f = w_\infty$, we find that $\tilde{w}_\infty(x) = w_\infty(x)$ almost everywhere in Ω and $\tilde{w}_\infty$ is u.s.c. Therefore $\tilde{u}_\infty := \tilde{w}_\infty + v_\infty$ is u.s.c. and equals u_∞ almost everywhere in Ω.

Assuming now u_∞ is u.s.c, then (3.2) follows from a direct adaptation of the arguments in the proof of lemma 2.2 in [DD]. □

Let us now return to the discussion of the asymptotic behavior of (1.2) as $\mu \to -\infty$. As in the introduction, we assume that $\lambda > \lambda_1^D$, $\{\mu_n\}$ is a sequence of decreasing negative numbers converging to $-\infty$, and (u_n, v_n) is a positive solution of (1.2) with $\mu = \mu_n$. We first recall several facts proved in Theorem 2.7 of [DD] and collect them in the following lemma.

Lemma 3.3. *The following conclusions hold:*

(i) *As $n \to \infty$, $\|u_n\|_\infty/|\mu_n| \to 1/d$, $\|v_n\|_\infty/|\mu_n| \to 0$.*

(ii) *$u_n/\|u_n\|_\infty \to 0$, $v_n \to 0$ uniformly on any compact subset of $\bar{\Omega} \setminus \bar{D}$.*

(iii) *There exist positive numbers $c_1 < c_2$ such that $c_1 \le \|v_n\|_{L^1(\Omega)} \le c_2, \forall n \ge 1$.*

Theorem 3.4. *Let (μ_n, u_n, v_n) be as above. Then*

(a) *On the closed set $\bar{\Omega} \setminus D$, $u_n/\|u_n\|_\infty \to 0$, $v_n \to 0$ uniformly.*
(b) *On the set D, let w be the unique solution of (1.1). Then $u_n/\|u_n\|_\infty \to w$ in $L^p(D)$ for any $p \ge 1$ and $v_n \to (\lambda/c)\chi_{\{w=1\}}$ in the weak* topology in $C(\bar{D})^*$. Moreover, $v_n \to 0$ uniformly on any compact subset of $\bar{D} \setminus \{w = 1\}$.*

Proof. ¿From conclusion (ii) in Lemma 3.3, we find that to prove (a), we only need to concentrate on the part of the region $\bar{\Omega}\setminus D$ that is close to ∂D. Denote $\hat{u}_n = u_n/\|u_n\|_\infty$. Then it is easily seen that $\hat{u}_n$ satisfies the conditions in Lemma 3.1. Hence, subject to a subsequence, $\hat{u}_n \to \hat{u}$ weakly in $H_0^1(\Omega)$ and strongly in $L^p(\Omega)$ for any $p \ge 1$. Moreover, $\|\hat{u}\|_\infty = 1$. It follows from conclusion (ii) in Lemma 3.3 that $\hat{u} = 0$ almost everywhere on $\Omega \setminus D$. It follows that $\hat{u}|_D \in H_0^1(D)$.

Define $\hat{v}$ by

$$-\Delta\hat{v} = \lambda,\ x \in D;\ \hat{v}|_{\partial D} = 0.$$

Clearly $\hat{v} \in C^2(\bar{D})$. On the other hand, $\hat{u}|_D$ satisfies in the weak sense

$$-\Delta\hat{u} = \lambda\hat{u} \le \lambda,\ x \in D;\ \hat{u}|_{\partial D} = 0.$$

Hence we must have $\hat{u}|_D \le \hat{v}$ a.e. on D. Now we extend $\hat{v}$ to R^n by zero outside D and still denote it by $\hat{v}$. We find that $\hat{v}$ is continuous and $\hat{u} \le \hat{v}$ a.e. on Ω. Since $\hat{v}$ is continuous, from (3.1), we find that the u.s.c representative of $\hat{u}$, still denoted by $\hat{u}$, satisfies

$$\hat{u}(x) \le \hat{v}(x),\ \forall x \in \Omega. \tag{3.3}$$

We will from now on assume that $\hat{u}$ has been chosen to be the representative given by (3.1).

By (3.3), we see that $\hat{u}(x) = 0$ for all $x \in \bar{\Omega} \setminus D$. Hence it follows from (3.2) that $\hat{u}_n \to 0$ uniformly on the compact set $\bar{\Omega} \setminus D$. This proves the first part of conclusion (a) in Theorem 3.4. We would like to remark that though our arguments above are carried out only for a subsequence of $\{\hat{u}_n\}$, but as the limit is a definite number 0, it follows that the full original sequence converges to this definite limit. This trick will be repeatedly used in the sequel without being explicitly pointed out.

We show next that subject to a subsequence, $\{v_n\}$ converges to 0 uniformly on any compact subset of $\{\hat{u} < 1\} := \{x \in \bar{\Omega} : \hat{u}(x) < 1\}$. Note that since $\hat{u}$ is u.s.c, $\{\hat{u} < 1\}$ is relatively open in $\bar{\Omega}$. As $\hat{u}(x) = 0$ on $\bar{\Omega} \setminus D$, the second part of conclusion (a) will follow from this.

Let K be an arbitrary compact subset of $\{\hat{u} < 1\}$. Since $\hat{u}$ is u.s.c, we can find $\epsilon > 0$ such that $\hat{u}(x) < 1 - \epsilon$ for all $x \in K$. Let $N(\partial\Omega)$ denote a small neighborhood of $\partial\Omega$ in $\bar{\Omega}$ such that $N(\partial\Omega) \cap \bar{D} = \emptyset$. By (ii) in Lemma 3.3, we know that $v_n \to 0$ uniformly on $N(\partial\Omega)$. Hence it suffices to show that $v_n \to 0$ uniformly on $K_0 := K \setminus N(\partial\Omega)$. To this end, let x_0 be an arbitrary point in K_0. As $\hat{u}(x) < 1 - \epsilon$ on K_0, by Lemma

3.2, we can find a small ball $B_r(x_0)$ such that, for all large n, $\hat{u}_n(x) \le 1 - (\epsilon/2)$ for all $x \in B_r(x_0)$. Therefore, by (i) in Lemma 3.3, we can find $\alpha > 0$ such that, for all large n,

$$\mu_n - v_n + du_n = |\mu_n|(-1 - v_n/|\mu_n| + d\hat{u}_n\|u_n\|_\infty/|\mu_n|) \le -\alpha|\mu_n|, \ \forall x \in B_r(x_0).$$

Therefore, using the equation for v_n in (1.2), we deduce

$$-\Delta v_n \le -\alpha|\mu_n|v_n, \ \forall x \in B_r(x_0).$$

Using (i) in Lemma 3.3, we have $v_n \le |\mu_n|$ on Ω for all large n. In particular, $v_n|_{\partial B_r(x_0)} \le |\mu_n|$ for all large n.

Let

$$w_n(x) = \Pi_{i=1}^N (e^{\sqrt{\alpha|\mu_n|}(x_i - x_i^0)} + e^{-\sqrt{\alpha|\mu_n|}(x_i - x_i^0)}),$$

where $(x_1^0, \cdots, x_N^0) = x_0$. It is easily checked that

$$-\Delta w_n + \alpha|\mu_n|w_n = 0, \ w_n|_{\partial B_r(x_0)} \ge e^{\sqrt{\alpha|\mu_n|}r/\sqrt{N}},$$

where we have used $\max_{1\le i\le N}|x_i - x_i^0| \ge r/\sqrt{N}$ when $|x - x_0| = r$. It follows that

$$w_n^* := |\mu_n|e^{-\sqrt{\alpha|\mu_n|}r/\sqrt{N}} w_n$$

satisfies

$$-\Delta w_n^* + \alpha|\mu_n|w_n^* = 0, \ w_n^*|_{\partial B_r(x_0)} \ge |\mu_n|.$$

Applying the maximum principle to $v_n - w_n^*$ over $B_r(x_0)$, we find that $v_n \le w_n^*$ on $B_r(x_0)$.

Let $\delta > 0$ be such that $\delta < N^{-3/2}$, we find, for $x \in B_{\delta r}(x_0)$,

$$w_n^*(x) \le |\mu_n|e^{-\sqrt{\alpha|\mu_n|}r/\sqrt{N}}(2e^{\sqrt{\alpha|\mu_n|}\delta r})^N$$

$$= |\mu_n|2^N e^{-\sqrt{\alpha|\mu_n|}r(N^{-1/2} - \delta N)} \to 0$$

as $n \to \infty$. Thus, for such δ,

$$0 \le v_n \le w_n^* \to 0, \ \forall x \in B_{\delta r}(x_0).$$

As the compact set K_0 can be covered by finitely many such small balls $B_{\delta r}(x_0)$, we have proved that $v_n \to 0$ uniformly on K_0, as we wanted.

We now set to prove (b). Multiplying the equation for u_n in (1.2) by $\phi/\|u_n\|_\infty$ with $\phi \in C_0^\infty(D)$, and observing that $b(x) = 0$ over D, we obtain

$$\int_D \nabla\hat{u}_n \cdot \nabla\phi dx = \lambda\int_D \hat{u}_n\phi - c\int_D \hat{u}_n v_n \phi dx.$$

It follows that

$$\lim_{n\to\infty}\int_D c\hat{u}_n v_n\phi dx = -\int_D \nabla\hat{u}\cdot\nabla\phi dx + \lambda\int_D \hat{u}\phi dx, \ \forall\phi \in C_0^\infty(D). \tag{3.4}$$

Clearly the right hand side of (3.4) defines a continuous linear functional on $H^1(D)$:

$$T(\phi) = -\int_D \nabla\hat{u}\cdot\nabla\phi dx + \lambda\int_D \hat{u}\phi dx.$$

Using the left hand side of (3.4), we see that $T(\phi)\geq 0$ whenever $\phi\in C_0^\infty(D)$ satisfies $\phi\geq 0$ on D. Moreover, since $v_n\to 0$ uniformly on any compact subset of $\{\hat{u}<1\}$, and $0\leq\hat{u}_n\leq 1$, it follows from the left hand side of (3.4) that $T(\phi)=0$ whenever $\phi\in C_0^\infty(D)$ satisfies $\text{supp}(\phi)\subset\{\hat{u}<1\}$. Using the continuity of T on $H^1(D)$ and the fact that functions in $H_0^1(D)$ can be approximated in the $H^1(D)$ norm by functions in $C_0^\infty(D)$, we find that

$$T(\phi)\geq 0,\ \forall\phi\in H_0^1(D) \text{ satisfying } \phi\geq 0, \tag{3.5}$$

$$T(\phi)=0,\ \forall\phi\in H_0^1(D) \text{ satisfying } \text{supp}(\phi)\subset\{\hat{u}<1\}. \tag{3.6}$$

Note that $\{\hat{u}<1\}$ is open.

Define $K:=\{v\in H_0^1(D) : v\leq 1\}$ and let $\phi\in C_0^\infty(D)$ be chosen such that $0\leq\phi\leq 1$ on D, $\phi=1$ on a $\delta-$neighborhood N_δ of $\{\hat{u}=1\}$. From (3.3) and the fact that $\hat{u}$ has been chosen u.s.c, we know that $\{\hat{u}=1\}$ is a closed set contained in the open set D. Thus the above choice of ϕ is possible. Let $v\in K$ be arbitrary and denote $\hat{v}=\max\{v,\phi\}$. Clearly $0\leq\hat{v}-v\in H_0^1(D)$. Thus, by (3.5),

$$\int_D \nabla\hat{u}\cdot\nabla(v-\hat{u})dx - \lambda\int_D \hat{u}(v-\hat{u})dx = -T(v-\hat{u})$$

$$= T(\hat{v}-v)+T(\hat{u}-\hat{v})\geq T(\hat{u}-\hat{v}).$$

Denote $u^*=\hat{u}-\hat{v}$. Clearly $u^*\in H_0^1(D)$. Now we choose $\psi\in C_0^\infty(D)$ satisfying $0\leq\psi\leq 1$ on D, $\psi=0$ on $D\setminus N_{(2/3)\delta}$, $\psi=1$ on $N_{(1/3)\delta}$. Then clearly

$$\text{supp}((1-\psi)u^*)\subset\bar{D}\setminus N_{(1/3)\delta}\subset\{\hat{u}<1\}.$$

Hence, by (3.6),

$$T(u^*)=T((1-\psi)u^*)+T(\psi u^*)=T(\psi u^*).$$

As $\psi=0$ on $D\setminus N_{(2/3)\delta}$, and $\hat{v}=\max\{v,\phi\}=1$ on N_δ, we find that $\psi u^*=\psi(\hat{u}-1)$ on D. Note that due to the choice of $\hat{u}$, ψu^* is itself the representative obtained by (3.1). Hence by property (b) for $H^1(R^N)$ functions recalled at the beginning of this section we find that ψu^* can be approximated in $H^1(R^N)$ by functions $\phi_n\in H^1(R^N)$ which vanishes in a neighborhood of $\{\hat{u}=1\}$. As ψu^* vanishes outside $N_{(2/3)\delta}$, we may also assume that ϕ_n vanishes outside D including ∂D. Thus, $\phi_n\in H_0^1(D)$ and $\text{supp}(\phi_n)\subset\{\hat{u}<1\}$. By (3.6), $T(\phi_n)=0$ and hence $T(\psi u^*)=\lim_{n\to\infty}T(\phi_n)=0$. It follows finally that

$$\int_D \nabla\hat{u}\cdot\nabla(v-\hat{u})dx - \lambda\int_D \hat{u}(v-\hat{u})dx\geq T(u^*)=T(\psi u^*)=0,\ \forall v\in K.$$

By Claim 1 in the proof of Theorem 2.1, this implies that $\hat{u}$ is the unique solution of (1.1). Hence the full original sequence $\{\hat{u}_n|_D\}$ converges to $\hat{u}$ weakly in $H_0^1(D)$ and strongly in $L^p(D)$ for any $p \geq 1$.

We consider now $\{v_n|_D\}$. Since $\{\|v_n\|_{L^1(\Omega)}\}$ is bounded by part (iii) in Lemma 3.3, v_n defines a sequence of continuous linear functionals on $C(\bar{D})$:

$$l_n(\phi) = \int_D v_n \phi dx,$$

and the norm of l_n has a bound independent of n. As $C(\bar{D})$ is separable, by a well-known result in functional analysis (see, e.g., [HP, Theorem 2.10.1]), subject to a subsequence, $\{l_n\}$ converges in the weak* topology of $C(\bar{D})^*$ to some $l_0 \in C(\bar{D})^*$.

To find an expression for l_0, we multiply the equation for v_n in (1.2) by $\phi/|\mu_n|$ with $\phi \in C_0^\infty(D)$ and deduce

$$\int_D \frac{v_n}{|\mu_n|}(-\Delta\phi)dx = -\int_D v_n\phi dx + \frac{d\|u_n\|_\infty}{|\mu_n|}\int_D \hat{u}_n v_n \phi dx - \int_D \frac{v_n}{|\mu_n|} v_n \phi dx. \tag{3.7}$$

By (i) in Lemma 3.3, we find that the left hand side of the above equation converges to 0 as $n \to \infty$. Using (i) and (iii) of Lemma 3.3, we obtain

$$0 \leq \left|\int_D \frac{v_n}{|\mu_n|} v_n \phi dx\right| \leq \frac{\|v_n\|_\infty}{|\mu_n|}\|\phi\|_\infty \int_D v_n dx \leq c_2\|\phi\|_\infty \frac{\|v_n\|_\infty}{|\mu_n|} \to 0.$$

Thus we obtain from (3.7), (i) in Lemma 3.3, equation (3.4) and the fact that $\hat{u}|_D$ satisfies (1.1),

$$\begin{aligned}
\lim_{n\to\infty}\int_D v_n\phi dx &= \lim_{n\to\infty}\int_D \hat{u}_n v_n \phi dx \\
&= c^{-1}\Big[-\int_D \nabla\hat{u}\cdot\nabla\phi dx + \lambda\int_D \hat{u}\phi dx\Big] \\
&= c^{-1}\Big[-\int_D \lambda\chi_{\{\hat{u}<1\}}\hat{u}\phi dx + \lambda\int_D \hat{u}\phi dx\Big] \\
&= \int_D (\lambda/c)(1-\chi_{\{\hat{u}<1\}})\hat{u}\phi dx \\
&= \int_D (\lambda/c)\chi_{\{\hat{u}=1\}}\phi dx.
\end{aligned}$$

On the other hand,

$$\int_D v_n\phi dx = l_n(\phi) \to l_0(\phi).$$

Thus we have

$$l_0(\phi) = \int_D (\lambda/c)\chi_{\{\hat{u}=1\}}\phi dx, \ \forall \phi \in C_0^\infty(D). \tag{3.8}$$

Consider now an arbitrary $\phi \in C(\bar{D})$. Let $\xi \in C(\bar{D})$ satisfy $\xi = 1$ on a small neighborhood of $\{\hat{u} = 1\}$ and $\xi = 0$ near ∂D. We find that $A :=\text{supp}((1-\xi)\phi) \subset \{\hat{u} < 1\}$, and hence, since $v_n \to 0$ uniformly on A,

$$l_0((1-\xi)\phi) = \lim_{n\to\infty} \int_D v_n(1-\xi)\phi dx = 0.$$

It follows that

$$l_0(\phi) = l_0((1-\xi)\phi) + l_0(\xi\phi) = l_0(\xi\phi).$$

Clearly,

$$\int_D (\lambda/c)\chi_{\{\hat{u}=1\}}\xi\phi dx = \int_D (\lambda/c)\chi_{\{\hat{u}=1\}}\phi dx.$$

As $\xi\phi$ can be approximated by $C_0^\infty(D)$ functions, it follows from (3.8) that

$$l_0(\xi\phi) = \int_D (\lambda/c)\chi_{\{\hat{u}=1\}}\xi\phi dx.$$

Thus,

$$l_0(\phi) = \int_D (\lambda/c)\chi_{\{\hat{u}=1\}}\phi dx, \ \forall \phi \in C(\bar{D}).$$

That is to say that v_n converges to $(\lambda/c)\chi_{\{\hat{u}=1\}}$ in the weak* topology in $C(\bar{D})^*$.

As the uniform convergence to 0 of v_n on compact subset of $\{\hat{u} < 1\}$ has already been proved, the proof of Theorem 3.4 is complete. □

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