On the Initial-Boundary Value Problems for Soliton Equations

A. Degasperis[1,2,§], S. V. Manakov[3,§] and P. M. Santini[1,2,§]

[1] *Dipartimento di Fisica, Università di Roma "La Sapienza" Piazz.le Aldo Moro 2, I-00185 Roma, Italy*

[2] *Istituto Nazionale di Fisica Nucleare, Sezione di Roma P.le Aldo Moro 2, I-00185 Roma, Italy*

[3] *Landau Institute for Theoretical Physics, Moscow, Russia*

[§] e-mail: `antonio.degasperis@roma1.infn.it` `manakov@itp.ac.ru` `paolo.santini@roma1.infn.it`

November 13, 2001

We present a novel approach to solve initial-boundary value problems on the segment and on the half line for soliton equations. Our method is illustrated by solving a prototype, and widely applicable, dispersive soliton equation: the celebrated nonlinear Schroedinger equation. It is well-known that the basic difficulty associated with boundaries is that some coefficients of the evolution equation of the (x-) scattering matrix $S(k,t)$ depend on unknown boundary data. In this paper we overcome this difficulty by expressing the unknown boundary data in terms of elements of the scattering matrix itself, so obtaining a nonlinear integro - differential evolution equation for $S(k,t)$. We also sketch an alternative approach, in the semiline case, based on a nonlinear equation for $S(k,t)$ which does not contain unknown boundary data; in this way, the "linearizable" boundary value problems correspond to the cases in which $S(k,t)$ can be found by solving a linear Riemann - Hilbert problem.

PACS: 02.60.Lj, 03.40.Kf

1. Initial Boundary Value (IBV) problems for partial differential equations play an important role in applications to Physics and, in general, to the Natural Sciences.

Since the discovery of the inverse scattering (spectral) transform method to solve the IBV problem on the infinite line with vanishing boundary conditions for a class of distinguished nonlinear evolution equations, like the Korteweg de Vries (KdV), nonlinear Schrödinger (NLS) and sine Gordon (SG) equations (see, f.i., [1]), several attempts have been made to extend this method to the case of more complicated IBV problems, in which Dirichelet and/or Neumann boundary conditions are prescribed on the semi - infinite line or on the segment.

It is well-known that the basic difficulty associated with these problems is that the evolution equation of the traditional x - scattering matrix $S(k,t)$, as given by the Lax equations, cannot be integrated in most of the cases because its coefficients depend on unknown boundary data.

Different approaches to the study of IBV problems for soliton equations have been developed during the last few years. In [2], an "elbow scattering" in the (x,t)- plane has been introduced to deal with the semiline problem for KdV, leading to a Gel'fand - Levitan - Marchenko formulation. In [3, 4] a different approach, based on a simultaneous x-t spectral transform, has been introduced and rigorously developed [5, 6] to solve IBV problems for soliton equations on the semiline. It allows one for a rigorous asymptotics [7] and captures in a natural way the known cases of linearizable boundaries. In the case in which the Dirichelet condition $q(0,t)$ is given for the semiline problem for NLS, the unknown boundary $q_x(0,t)$ is obtained through a nonlinear Volterra equation, whose solution exists without a small norm assumption [5]. This approach has also proven to be useful to treat linear equations on arbitrary convex polygonal domains [8, 9, 10].

In some distinguished cases of soliton equations corresponding to singular dispersion relations, like the stimulated Raman scattering (SRS) equations and the SG equation in light cone coordinates, the evolution equation of the scattering matrix does not contain unknown boundary data. The SG equation on the semiline has been treated using the x-t spectral transform [3, 6]; the SRS and the SG equations on the semiline have also been treated using a more traditional x- transform method respectively in [11] and in [12]; the x- spectral data used in this last approach satisfy a nonlinear evolution equation of Riccati type.

We finally remark that IBV problems for C - integrable equations have also been considered [13]. In particular, for the Burgers equation, the problem reduces to that of solving a nonlinear Volterra equation.

In this paper we present a novel approach to the solution of IBV problems on the segment and on the half line for soliton equations. In our approach we overcome the difficulty associated with unknown boundaries using the analyticity properties of the spectral matrix, which allow one to express the unknown boundary data in terms of elements of the scattering matrix itself, so obtaining a closed nonlinear integro-differential evolution equation of novel type for the spectral matrix $S(k,t)$. The approach is illustrated on the prototype example of the NLS equation

$$iq_t + q_{xx} + c|q|^2 q = 0, \quad q = q(x,t), \tag{1}$$

where c is an arbitrary real parameter, which describes the amplitude modulation of a wave packet in a strongly dispersive and weakly nonlinear medium, but applies as well to most of the known examples of dispersive soliton equations in $1+1$ dimensions, like the KdV and the modified KdV equations. When applied to the SG equation in light cone coordinates, the spectral matrix used in our approach satisfies a linear evolution equation.

For the NLS equation (1) we consider the following basic IBV problems.
The NLS equation on the segment. We look for the solution $q(x,t)$ of the NLS equation (1) in the closed domain $0 \le x \le L,\ 0 \le t \le T$, satisfying the initial condition $q(x,0) = u(x)$ and one of the following three boundary conditions:

$$q(0,t) = v_0(t), \qquad q(L,t) = v_L(t). \tag{2}$$

$$q_x(0,t) = w_0(t), \qquad q_x(L,t) = w_L(t). \tag{3}$$

$$q(0,t) + a_0 q_x(0,t) = f_0(t), \quad q(L,t) + a_L q_x(L,t) = f_L(t). \tag{4}$$

where a_0, a_L are arbitrary real constants.

The problems on the half line: $x \in [0,\infty)$ and on the line: $x \in (-\infty,\infty)$ can obviously be viewed as limiting cases of that on the segment.

2. To solve the above problems we make essential use of the fact that the NLS equation (1) is the integrability condition of the following system of linear 2×2 matrix equations (the well-known Lax pair) [14]:

$$\Psi_x = (ik\sigma_3 + Q)\Psi, \qquad \Psi_t = (2ik^2\sigma_3 + \tilde{Q})\Psi + \Psi C \tag{5}$$

where $\sigma_3 = \mathrm{diag}(1,-1)$, C is an arbitrary x-independent matrix and

$$Q(x,t) = \begin{pmatrix} 0 & -c\bar{q}(x,t) \\ q(x,t) & 0 \end{pmatrix}, \qquad \tilde{Q}(x,t) = 2kQ - i\sigma_3 Q_x + iQ^2\sigma_3 \tag{6}$$

As in the tradition of the spectral transform method, the Jost solutions $\Psi_+(x,t,k)$ and $\Psi_-(x,t,k)$ of (5) are defined by the conditions:

$$\Psi_-(0,t,k) = I, \quad \Psi_+(L,t,k) = e^{ikL\sigma_3} \tag{7}$$

and the scattering matrix $S(k,t)$ is introduced by the relation:

$$\Psi_+(x,t,k) = \Psi_-(x,t,k)S(k,t). \tag{8}$$

It is well-known that the Jost solutions and the scattering matrix have unit determinant; it is also well-known that, with Q given by Eq.(6), they have the following structures:

$$\Psi_\pm(x,k) = \begin{pmatrix} \psi_{\pm 11}(k) & -c\overline{\psi_{\pm 21}(\bar{k})} \\ \psi_{\pm 21}(k) & \overline{\psi_{\pm 11}(\bar{k})} \end{pmatrix}, \qquad S(k) = \begin{pmatrix} \alpha(k) & -c\overline{\beta(\bar{k})} \\ \beta(k) & \overline{\alpha(\bar{k})} \end{pmatrix} \tag{9}$$

It is standard to show that $M := \Psi_+(x,t,k)e^{-ikx\sigma_3}$, $N := \Psi_-(x,t,k)e^{-ikx\sigma_3}$ and $S(k,t)$ are entire analytic functions of k. The first column of N is analytic in the lower half k plane (LHP) with asymptotics:

$$\begin{pmatrix} N_{11}(k) \\ N_{21}(k) \end{pmatrix} = \begin{pmatrix} 1 + O(k^{-1}) \\ \frac{q}{2ik} + O(k^{-2}) \end{pmatrix} \tag{10}$$

and grows exponentially like: $e^{-2ikx}O(k^{-1})$ in the upper half plane (UHP), combining both behaviors (power decay and exponential oscillation) on the real axis. The first column of M is analytic in the UHP with the asymptotics (10) and grows exponentially in the LHP like: $e^{2ik(L-x)}O(k^{-1})$, combining both behaviors (power decay and exponential oscillation) on the real axis. The analyticity properties and the asymptotics of the second columns follow from equations (9). The scattering matrix $S(k,t) = M(0,t,k)$ shares the analyticity properties of M and its asymptotics are written down below in some detail for future use.

$$\alpha(k) \doteq \begin{cases} 1 + \frac{c}{2ik}\int\limits_0^L dx|q|^2 + O(k^{-2}) & \mathrm{Im}k > 0, \\ e^{2ikL}(-\frac{c}{(2ik)^2}\bar{v}_0 v_L + O(k^{-3})) & \mathrm{Im}k < 0. \end{cases} \tag{11}$$

$$\beta(k) = \begin{cases} \frac{1}{2ik}v_0 - \frac{1}{(2ik)^2}\beta_0 + O(k^{-3}) & \mathrm{Im}k > 0, \\ e^{2ikL}(-\frac{1}{2ik}v_L + \frac{1}{(2ik)^2}\beta_L + O(k^{-3})) & \mathrm{Im}k < 0. \end{cases} \tag{12}$$

where $\beta_0 = w_0 - cv_0 \int\limits_0^L dx|q|^2$ and $\beta_L = w_L + cv_L \int\limits_0^L dx|q|^2$.

The Direct Problem is the mapping from the initial condition $q(x,0) = u(x)$ to the elements $\alpha(k,0), \beta(k,0)$ of the scattering matrix at $t = 0$.

The Inverse Problem is the mapping from the evoluted elements $\alpha(k,t), \beta(k,t)$ of the scattering matrix (or, more precisely, from the evoluted spectral data) to the NLS field $q(x,t)$. This problem does not differ from the case of NLS on the line and we refer to the classical literature [14, 1] for details.

3. The intermediate step, the t-evolution of the scattering data, is where our approach introduces some important novelties. Using equation (5b), one can show that the time evolution of the scattering matrix $S(k,t) = \Psi_+(0,t,k)$ is gouverned by the following matrix equation

$$S_t = 2ik^2[\sigma_3, S] + \tilde{Q}(0,t,k)S - Se^{-ikL\sigma_3}\tilde{Q}(L,t,k)e^{ikL\sigma_3} \tag{13}$$

which takes the following form, for the relevant components:

$$\begin{gathered} \alpha_t(k) = c[i(|v_L|^2 - |v_0|^2)\alpha(k) + (2kv_L + iw_L)e^{2ikL}\overline{\beta(\bar{k})} - (2k\bar{v}_0 - i\bar{w}_0)\beta(k)], \\ \beta_t(k) = -4ik^2\beta(k) + (2kv_0 + iw_0)\alpha(k) - (2kv_L + iw_L)e^{2ikL}\overline{\alpha(\bar{k})} + \\ ic(|v_L|^2 + |v_0|^2)\beta(k). \end{gathered} \tag{14}$$

This system of equations depends on known as well as unknown boundary data and cannot be used, as it is, to obtain $\alpha(k,t)$ and $\beta(k,t)$. This basic difficulty can be simply overcome using the analytic properties of α, β and their asymptotic expansions (11, 12) which allow one to express the unknown boundary data in terms of known ones and α, β, thus obtaining the desired closed evolution equation. It is easy to show that the following formula take place:

$$w_0(t) = -\frac{2}{\pi}v_0(t)\int\limits_{-\infty}^{\infty} dk[\alpha_+(k,t)-1] + \frac{4i}{\pi}\int\limits_{-\infty}^{\infty} dk[k\beta_-(k,t) + \frac{iv_0(t)}{2} - \frac{iv_L(t)}{2}\cos 2kL],$$

$$w_L(t) = \frac{2}{\pi}v_L(t)\int\limits_{-\infty}^{\infty} dk[\alpha_+(k,t)) - 1] + \qquad (15)$$

$$\frac{4i}{\pi}\int\limits_{-\infty}^{\infty} dk[k\cos 2kL\beta_-(k,t) - ik\sin 2kL\beta_+(k,t) + \frac{iv_0(t)}{2}\cos 2kL - \frac{iv_L(t)}{2}],$$

$$v_0(t) = -\frac{2}{\pi}\int\limits_{-\infty}^{\infty} dk\beta_+(k,t),$$

$$v_L(t) = -\frac{2}{\pi}\int\limits_{-\infty}^{\infty} dk[\cos 2kL\beta_+(k,t) - i\sin 2kL\beta_-(k,t)], \qquad (16)$$

where $\alpha_\pm(k)$ ($\beta_\pm(k)$) are the even and odd parts of $\alpha(k)$ ($\beta(k)$): $\alpha_\pm(k) = (\alpha(k) \pm \alpha(-k))/2$, $\beta_\pm(k) = (\beta(k) \pm \beta(-k))/2$.

Therefore the t - evolution of the scattering data corresponding to the initial-boundary value problems (2), (3) or (4) is given by the system of equations (14) (with the initial conditions $\alpha(k,0)$ and $\beta(k,0)$ obtained from the direct problem) in which the unknown boundary data are replaced by expressions (15) or (16).

For example, the time evolution of $S(k,t)$ for the boundary condition (2) is given by equation (14) with $w_0(t)$, $w_L(t)$ replaced by expressions (15) etc.

Of course, our method of solution applies also to the *linear* Schroedinger equation. In analogy with this case, it is possible to prove that the solutions $\alpha(k,t)$, $\beta(k,t)$ of the above nonlinear integro-differential evolution equations exist unique. This is in full agreement with PDE theory, in which the boundary values (2), (3) and (4) are necessary and sufficient to obtain one and only one solution $q(x,t)$ with given initial condition. It is interesting to remark that, if we replaced not only the unknown boundary conditions, but also the assigned ones by their spectral representations, then the nonlinear evolutions would loose uniqueness and the solutions would depend on arbitrary functions of time (which could therefore be interpreted as the "illegitimately suppressed" boundary data).

The initial-boundary value problems for NLS on the semiline $x \in [0,\infty)$ are easily obtained from the above treatment in the limit $L \to \infty$, assuming a sufficiently fast vanishing at $x \to \infty$ of the relevant fields and hence setting $v_L = w_L = f_L = 0$.

In the semiline case the first columns of $M(x,t,k)$ and $S(k,t)$ are analytic in the UHP. The asymptotics of the scattering matrix read:

$$\alpha(k) = 1 + \frac{c}{2ik}\int\limits_0^\infty dx|q|^2 + O(k^{-2}), \quad \mathrm{Im}k \geq 0,$$

$$\beta(k) = \frac{1}{2ik}v_0 - \frac{1}{(2ik)^2}(w_0 - cv_0\int\limits_0^\infty dx|q|^2) + O(k^{-3}), \quad \mathrm{Im}k \geq 0, \qquad (17)$$

The inverse problem is unchanged and the t-evolutions of the scattering data, corresponding to our initial-boundary value problems are given now by equation

$$S_t = 2ik^2[\sigma_3, S] + \tilde{Q}(0,t,k)S, \tag{18}$$

which takes the following form for its relevant components:

$$\begin{aligned} \alpha_t(k) &= -c[i|v_0|^2\alpha(k) + (2k\bar{v}_0 - i\bar{w}_0)\beta(k)], \\ \beta_t(k) &= -4ik^2\beta(k) + (2kv_0 + iw_0)\alpha(k) + ic|v_0|^2\beta(k) \end{aligned} \tag{19}$$

supplemented respectively by the following spectral representations of the unknown boundary data:

$$w_0(t) = -\frac{2}{\pi}v_0(t)\int_{-\infty}^{\infty} dk[\alpha_+(k,t) - 1] + \frac{4i}{\pi}\int_{-\infty}^{\infty} dk[k\beta_-(k,t) + \frac{iv_0(t)}{2}]; \tag{20}$$

$$v_0(t) = -\frac{2}{\pi}\int_{-\infty}^{\infty} dk\beta_+(k,t); \tag{21}$$

$$v_0(t) = -\frac{2}{\pi}\int_{-\infty}^{\infty} dk\beta_+(k,t), \qquad w_0(t) = \frac{1}{a_0}f_0(t) + \frac{2}{\pi a_0}\int_{-\infty}^{\infty} dk\beta_+(k,t). \tag{22}$$

4. Here we describe, for the semiline case, an alternative approach. It is based on the equation:

$$\begin{aligned} &\left(e^{-2ik^2t\sigma_3}S^{-1}(-k,t)[a_1 I + 2ia_2k\sigma_3]S(k,t)e^{2ik^2t\sigma_3}\right)_t = \\ &4ke^{-2ik^2t\sigma_3}S^{-1}(-k,t)[a_1Q(0,t) + a_2Q_x(0,t)]S(k,t)e^{2ik^2t\sigma_3}, \end{aligned} \tag{23}$$

where a_1, a_2 are arbitrary real parameters, which follows directly from (18). If we assign the boundary condition

$$a_1q(0,t) + a_2q_x(0,t) = f(t), \tag{24}$$

(which describes, due to the arbitrariness of the real parameters a_1, a_2, all the three IBV problems), equation (23), together with the analyticity properties of the scattering matrix, allows one to construct $S(k,t)$ through a nonlinear integral equation in both variables k and t. The main advantage of this approach is that equation (23) does not contain unknown boundary data; a discussion of this approach is out of the scope of the present paper and will be reported in a subsequent one.

Another advantage of this approach is that it captures in a natural way the known cases of linearizable boundary conditions [15, 16, 17], summarized by the equation: $a_1q(0,t) + a_2q_x(0,t) = 0$. Indeed, in this case, the right hand side of equation (23) is zero; therefore:

$$S^{-1}(-k,t)[a_1 I + 2ia_2 k\sigma_3]S(k,t) =$$
$$e^{2ik^2t\sigma_3}\left(S^{-1}(-k,0)[a_1 I + 2ia_2 k\sigma_3]S(k,0)\right)e^{-2ik^2t\sigma_3} \qquad (25)$$

and, using the analyticity properties of $S(k,t)$, equation (25) can be interpreted as a linear RH problem for the columns of $S(k,t)$. Assuming that, for simplicity, the upper function $J(k) = (a_1 + 2ia_2k)\alpha(k,0)\overline{\alpha(-\bar{k},0)} + c(a_1 - 2ia_2k)\beta(k,0)\overline{\beta(-\bar{k},0)}$ has no zeros for $\mathrm{Im}k \geq 0$, then the RH problem (25) can be expressed in terms of the following system of integral equations:

$$\begin{pmatrix} \alpha(k,t) \\ \beta(k,t) \end{pmatrix} + \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \frac{dk'}{k' - (k+i0)}\gamma(k')e^{-4ik'^2t}\begin{pmatrix} -c\overline{\beta(k',t)} \\ \overline{\alpha(k',t)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad (26)$$

where

$$\gamma(k) = \frac{(a_1 + 2ia_2k)\alpha(k,0)\beta(-k,0) - (a_1 - 2ia_2k)\alpha(-k,0)\beta(k,0)}{J(-k)}, \qquad (27)$$

which allows one to construct $S(k,t)$ from the initial condition $S(k,0)$ through a linear system of equations.

The present work has been carried out during several visits and meetings. We gratefully acknowledge the financial contributions provided by the RFBR Grant 01-01-00929, the INTAS Grant 99-1782 and by the following Institutions: the University of Rome "La Sapienza" (Italy), the Istituto Nazionale di Fisica Nucleare (Sezione di Roma), the Landau Institute for Theoretical Physics, Moscow (Russia) and the Isaac Newton Institute, Cambridge (UK), within the programme "Integrable Systems".

References

[1] M.J.Ablowitz and H.Segur, *Solitons and the inverse scattering transform* SIAM, Philadelphia, 1981; V.E.Zakharov, S.V.Manakov,S.P.Novikov and L.P.Pitaevsky, *Theory of solitons*, Nauka, 1980 [Consultants Bureau (Plenum Publ.), New York, 1984].

[2] P. C. Sabatier, J. Math. Phys. **41**, 414 (2000).

[3] A. S. Fokas, Proc. Roy. Soc. Lond. A, **53**, 1411 (1997).

[4] A. S. Fokas, J. Math. Phys. **41**, 4188 (2000).

[5] A. S. Fokas, A. R. Its and L. Y. Sung, preprint (in preparation). Private communication.

[6] A. S. Fokas, Integrable Nonlinear Evolution Equations on the Half-Line, preprint (October 2001).

[7] A. S. Fokas and A. R. Its, Phys. Rev. Lett. **68**, 3117 (1992).

[8] A. S. Fokas, Proc. Roy. Soc. Lond. A, **457**, 371 (2001).

[9] A. S. Fokas and B. Pelloni, Math. Proc. Camb. Phil. Soc. (in press).

[10] A. S. Fokas and B. Pelloni, Proc. Roy. Soc. Lond. A, **454**, 654 (1998); A. S. Fokas and B. Pelloni, Phys. Rev. Lett. **84**, 4785 (2000).

[11] J. Leon and A. V. Mikhailov, Phys. Lett. A **253**, 33 (1999).

[12] J. Leon and A. Spire, Preprint nlin.PS/0105066.

[13] F. Calogero and S. Delillo, J. Math. Phys. **32**, 99 (1991); F. Calogero and S. Delillo, Inverse Problems **4**, L33-37 (1988).

[14] V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP **37**, 62 (1972).

[15] M. J. Ablowitz and H. Segur, J. Math. Phys. **16**, 1054 (1975).

[16] E. K Sklyanin, Functional Anal.Appl. **21**, 86 (1987).

[17] A. S. Fokas, Physica D, **35**, 167 (1989).