A New Integrable Equation with Peakon Solutions

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Abstract

We consider a new partial differential equation, of a similar form to the Camassa-Holm shallow water wave equation, which was recently obtained by Degasperis and Procesi using the method of asymptotic integrability. We prove the exact integrability of the new equation by constructing its Lax pair, and we explain its connection with a negative flow in the Kaup-Kupershmidt hierarchy via a reciprocal transformation. The infinite sequence of conserved quantities is derived together with a proposed bi-Hamiltonian structure. The equation admits exact solutions in the form of a superposition of multi-peakons, and we describe the integrable finite-dimensional peakon dynamics and compare it with the analogous results for Camassa-Holm peakons.

1 Introduction

Since its discovery [1] there has been a considerable amount of interest in the Camassa-Holm shallow-water equation,

$$u_t + 2\kappa u_x + \gamma u_{xxx} - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{1.1}$$

It was originally derived as an approximation to the incompressible Euler equations, and found to be completely integrable with a Lax pair and associated bi-Hamiltonian structure [2]. It was subsequently shown to have a hodograph link to the KdV hierarchy and an interpretation within the framework of hereditary recursion operators [9]. In [7], the CH equation with linear dispersion was shown to be one full order more accurate in asymptotic approximation beyond Korteweg-de Vries (KdV) for shallow water waves. Yet, it still preserves KdV's soliton properties such as complete integrability

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via the inverse scattering transform (IST) method. 2+1-dimensional analogues have also been studied [14, 16]. A particularly unusual feature of (1.1) is that in the dispersionless limit $\kappa \to 0$ it admits peaked solitons or peakons. The single peakon takes the form

$$u(x,t) = ce^{-|x-ct|}, \tag{1.2}$$

while the N-peakon solution is just a simple superposition,

$$u(x,t) = \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|}, \tag{1.3}$$

where the canonical positions q_j and momenta p_j satisfy a completely integrable finite-dimensional Hamiltonian system.

The current work was motivated by the question of which equations of similar form to (1.1) are integrable. In the recent study [4] the method of asymptotic integrability was applied to the family of third order dispersive PDE conservation laws,

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{xxt} = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x, \tag{1.4}$$

whose right-hand side is the derivative of a quadratic differential polynomial. Within this family, only three equations that satisfy asymptotic integrability conditions up to third order are singled out, namely KdV ($\alpha = c_2 = c_3 = 0$), Camassa-Holm ($c_1 = -\frac{3}{2}c_3/\alpha^2, c_2 = c_3/2$) and one new equation ($c_1 = -2c_3/\alpha^2, c_2 = c_3$) which can be scaled to the following form:

$$u_t + u_x + 6uu_x + u_{xxx} - \alpha^2 \left(u_{xxt} + \frac{9}{2} u_x u_{xx} + \frac{3}{2} u u_{xxx} \right) = 0\,. \tag{1.5}$$

(Unfortunately, the equation in [4] appears with the typographical error $3/2 \to 2/3$.) However, by its nature the method of [4] only isolates necessary conditions for integrability and is not sufficient to prove it. In the following we demonstrate that the new equation (1.5) is integrable by explicitly constructing its Lax pair.

A simple observation that proved very useful in our investigations was that by rescaling, shifting the dependent variable and applying a Galilean boost, the equation (1.5) may be transformed to the dispersionless form

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{1.6}$$

but integrability is unaffected by such transformations. (We have set the parameter $\alpha = 1$, but it will sometimes be convenient to reintroduce it when considering various limits.) The next key observation was that in this form the equation (1.6) also has the single peakon (1.2) as an exact travelling wave solution. This led us to consider the family of equations

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}. \tag{1.7}$$

for real parameter b, which includes both Camassa-Holm ($b = 2$) and the new equation (1.6) ($b = 3$) as special cases. It turns out that all the equations in this family possess not just the peakon solution (1.2) but also the multi-peakon solutions (1.3). In the case of arbitrary b the p_j, q_j are not canonical variables, but satisfy the dynamical system

$$\dot{p}_j = -(b-1)\,\frac{\partial G_N}{\partial q_j}, \qquad \dot{q}_j = \frac{\partial G_N}{\partial p_j}, \tag{1.8}$$

where the generating function G_N is

$$G_N = \frac{1}{2}\sum_{j,k=1}^{N} p_j p_k\, e^{-|q_j - q_k|}\,; \tag{1.9}$$

this takes the canonical Hamiltonian form only in the special case $b = 2$ (Camassa-Holm).

The asymptotic analysis in [4] implies that $b = 2, 3$ should be the only possible integrable cases within the family (1.7). In [10] a class of PDEs including all of the equations (1.7) was tested for integrability using Painlevé analysis. However, because both Camassa-Holm equation (1.1) and the new equation (1.6) are examples of integrable systems with algebraic branching in their solutions (the weak Painlevé property of [18]), they were explicitly excluded by the various Painlevé tests applied in [10], and in fact all of the equations in that class failed the combination of tests. The authors of [10] noted that the (strong) Painlevé property is destroyed by changes of variables, and thus a transformation may be required before applying the test. For the case of Camassa-Holm a reciprocal transformation provides a link with a negative flow of KdV (see [9, 12]), which restores the Painlevé property. In a forthcoming article [3] we shall generalize this transformation for the whole family (1.7). Applying Painlevé analysis to the transformed family of equations then isolates the cases $b = 2, 3$ as the only candidates for integrability.

In this note we restrict our attention to the new integrable equation (1.6). We first present the associated linear system with a third order Lax operator. We explain the connection with a negative flow of the Kaup-Kupershmidt hierarchy [11], and derive infinitely many conservation laws and a proposed bi-Hamiltonian structure. The dynamics of multi-peakon solutions, described by the finite-dimensional system (1.8) in the particular case $b = 3$, is also analysed.

2 Reciprocal transformation

We consider the equation (1.6) which we write in the form

$$m_t + m_x u + 3 m u_x = 0, \qquad m = \mathcal{L}u := (1 - \partial_x^2)u. \tag{2.1}$$

In this form it is clear that we have the conservation law

$$(m^{1/3})_t = -(m^{1/3}u)_x. \tag{2.2}$$

The construction of the other conservation laws (infinitely many) will be presented in Section 4.

Our strategy is to make use of a reciprocal transformation which maps between systems of conservation laws [15, 19, 20], since it is known that such transformations can be used to connect systems with algebraic branching in their solutions to equations which have only pole type singularities. The relevant example here is the Camassa-Holm equation, which has a reciprocal transformation to the first negative flow in the KdV hierarchy, namely the equation

$$R \cdot V_T = 0, \tag{2.3}$$

where $R = \partial_X^2 + 4V + 2V_X\partial_X^{-1}$ is the KdV recursion operator (for more details see [12] and references). Camassa-Holm has weak Painlevé expansions with cube root branching [10], while KdV solutions have only double pole singularities.

To define a reciprocal transformation for the equation (2.1) we introduce the dependent variable

$$p = -m^{1/3} \tag{2.4}$$

which is related to u by

$$p^3 = (\partial_x^2 - 1)u\,. \tag{2.5}$$

We then use the conservation law (2.2) in the form

$$p_t = -(pu)_x \tag{2.6}$$

to define the new independent variables X, T via

$$dX = p\,dx - pu\,dt, \qquad dT = dt. \tag{2.7}$$

By transforming the derivatives we find the new conservation law

$$(p^{-1})_T = u_X. \tag{2.8}$$

Replacing ∂_x by $p\partial_X$ in (2.5) and using (2.8) to eliminate derivatives of u leads to the identity $u = -p(\log p)_{XT} - p^3$. Hence, (2.8) can be written in terms of p alone, viz:

$$(p^{-1})_T + (p(\log p)_{XT} + p^3)_X = 0. \tag{2.9}$$

It then turns out that this equation for p can be written in conservation form in another way, namely

$$\left(-\frac{p_{XX}}{p} + \frac{p_X^2}{2p^2} - \frac{1}{2p^2}\right)_T = \frac{3}{2}(p^2)_X. \tag{2.10}$$

The equation (2.10) is most conveniently written as the system

$$\begin{aligned} V_T &= \tfrac{3}{4}(p^2)_X, \\ pp_{XX} - \tfrac{1}{2}p_X^2 + 2Vp^2 + \tfrac{1}{2} &= 0, \end{aligned} \tag{2.11}$$

where the second equation above (Ermakov-Pinney) defines V. A direct consequence of Ermakov-Pinney is

$$B \cdot p = 0, \qquad B \equiv R \cdot \partial_X = \partial_X^3 + 4V\partial_X + 2V_X. \tag{2.12}$$

For a summary of results and references on the Ermakov-Pinney equation see [13]. Having obtained the equation (2.9) from (2.1) by using the reciprocal transformation (2.7), one may check that the standard WTC Painlevé test [22] is satisfied. In fact the original equation (2.1) admits expansions with square root branching, but such generalized (weak [18]) Painlevé expansions were specifically excluded from the classification of [10]. Further details of the Painlevé analysis will be given elsewhere [3].

3 The Lax pair

Equation (2.9), which is equivalent to the system (2.11), is integrable since it admits the Lax pair

$$\begin{aligned} \psi_{XXX} + 4V\psi_X + (2V_X - \lambda)\psi &= 0, \\ \psi_T + \lambda^{-1}(p^2\psi_{XX} - pp_X\psi_X - (pp_{XX} - p_X^2 + \tfrac{2}{3})\psi) &= 0. \end{aligned} \tag{3.1}$$

The third order operator above is the Lax operator of the Kaup-Kupershmidt (KK) hierarchy [11], and so the system (2.11) may be considered as the first negative flow in this hierarchy. As far as we are aware this system has not been studied before, although it turns out to be an integrable generalization of the Tzitzeica equation. (The constant 2/3 in the T part of (3.1) can be gauged away, but allows for easy comparison with reference [11]).

However, it is not the most general negative KK flow derived from such a Lax pair with an inverse power of λ in the T part. Indeed, starting from a more general time evolution of the form

$$\psi_T + \lambda^{-1}\left(W\psi_{XX} - \frac{1}{2}W_X\psi_X + \frac{1}{6}(W_{XX} + 16VW)\psi\right) = 0, \tag{3.2}$$

which is a reduction of the time part of a 2+1-dimensional non-isospectral KK hierarchy introduced in [11], the compatibility conditions of the third order Lax equation with (3.2) yields the two conditions

$$V_T = \frac{3}{4}W_X, \qquad \tilde{B} \cdot W = 0, \tag{3.3}$$

where $\tilde{B} = \partial_X^5 + 6(\partial_X V \partial_X^2 + \partial_X^2 V \partial_X) + 4(\partial_X^3 V + V \partial_X^3) + 32(\partial_X V^2 + V^2 \partial_X)$. The recursion operator of the KK hierarchy is

$$R_{KK} = B \partial_X^{-1} \tilde{B} \partial_X^{-1}, \tag{3.4}$$

with B given by (2.12); B is a Hamiltonian operator for the KdV, KK and Sawada-Kotera hierarchies. (see [11] and references).

Thus we see that the more general inverse KK flow (3.3) is a system of higher order than (2.11) (while the most general first negative flow is $R_{KK} \cdot V_T = 0$, of even higher order). However, the striking thing is that system (2.11) (or equivalently (2.9)) is a consistent reduction of (3.3). This is a consequence of the following remarkable operator identity:

$$\tilde{B} \cdot p^2 = 2(p\partial_X^2 + 5p_X \partial_X + 10 p_{XX} + 16Vp) \cdot B \cdot p. \tag{3.5}$$

Thus for an arbitrary function $h(T)$ we may substitute

$$W = p^2, \qquad V = -\frac{p_{XX}}{2p} + \frac{p_X^2}{4p^2} - \frac{h(T)}{4p^2} \tag{3.6}$$

into the T part (3.2) of the Lax pair, and then upon scaling $h(T) = 1$ (always possible for $h \not\equiv 0$) the compatibility condition of the Lax pair becomes (2.11). This is consistent with the second condition of (3.3) due to the identity (3.5) together with $B \cdot p = 0$ (2.12). In the special case $h(T) = 0$, which is not relevant to the reciprocal transformation of the equation (2.1), we can integrate the resulting system to obtain

$$p(\log p)_{XT} + p^3 + \tilde{h}(T) = 0 \tag{3.7}$$

with $\tilde{h}$ arbitrary. For $\tilde{h} \not\equiv 0$ we rescale and redefine T so that $\tilde{h} = 1$, in which case (3.7) it is known as the Tzitzeica equation [21], or with dependent variable $\phi = \log p$ the Bullough-Dodd equation [5]

$$\phi_{XT} + e^{2\phi} + e^{-\phi} = 0. \tag{3.8}$$

If $\tilde{h} = 0$ we have Liouville's equation.

Finally we present the Lax pair for the original equation (2.1) (or equivalently (1.6)). Setting

$$\Psi = p^{-1}\psi, \tag{3.9}$$

recalling that $m = -p^3$, and transforming all derivatives in (3.1), we find that Ψ satisfies

$$\begin{aligned} \Psi_x - \Psi_{xxx} - m\lambda\Psi &= 0, \\ \Psi_t + \tfrac{1}{\lambda}\Psi_{xx} + u\Psi_x - \left(u_x + \tfrac{2}{3\lambda}\right)\Psi &= 0. \end{aligned} \tag{3.10}$$

(As before the final constant $2/(3\lambda)$ can be removed by a gauge transformation.)

It is worth considering the effect of a Galilean boost and rescaling on the equation (1.6), which restores some of the free parameters in (1.4), yielding

$$m_t + c_0 u_x + \gamma u_{xxx} + m_x u + 3 m u_x = 0, \quad m = u - \alpha^2 u_{xx}. \tag{3.11}$$

In that case, with a rescaled spectral parameter $\mu = \alpha^2\lambda$, the Lax pair is

$$\begin{aligned} (1-\alpha^2\partial_x^2)\Psi_x &= \mu\left(m + \tfrac{c_0}{3} + \tfrac{\gamma}{3\alpha^2}\right)\Psi, \\ \Psi_t + \tfrac{1}{\mu}\Psi_{xx} &+ \left(u - \tfrac{\gamma}{\alpha^2}\right)\Psi_x - u_x\Psi = 0. \end{aligned} \tag{3.12}$$

In the dispersionless case, if $c_0 = 0 = \gamma$ then up to scaling the equation (3.11) is equivalent to the Riemann shock wave equation

$$u_t + u u_x = 0 \tag{3.13}$$

in both the limits $\alpha^2 \to 0$ and $\alpha^2 \to \infty$, i.e. in the limit of both large and small wavenumbers.

The generalized Tzitzeica equation (2.9) has a variety of interesting solutions, including solitons on a constant background and a reduction to a Painlevé III transcendent. We will treat these solutions in more detail in [3]. Here we mention that the reciprocal transformation takes the one-soliton solution on constant background to a soliton solution of the equation (3.11) with dispersion and vanishing at infinity, while the peakon solution (1.2) can be obtained from the soliton in the limit of zero dispersion (although the transformation (2.7) breaks down for the peakon, when $m = -p^3$ is a Dirac delta function).

4 Conservation laws

In deriving infinitely many conservation laws for the equation (1.6), we first introduce the quantity

$$\rho = (\log \psi)_x \,. \tag{4.1}$$

Then from the spatial part of the Lax pair (3.10) we have

$$\mathcal{L}\rho := (1-\partial_x^2)\rho = 3\rho\rho_x + \rho^3 + \lambda m, \tag{4.2}$$

while (removing the $2/(3\lambda)$ term) the time part yields a conservation law for ρ, namely

$$\rho_t = j_x, \qquad j = u_x - u\rho - \lambda^{-1}(\rho_x + \rho^2). \tag{4.3}$$

The density ρ, given by (4.1), may be written as a formal series in powers of the spectral parameter λ, with coefficients determined recursively from (4.2). Substituting a corresponding expansion for the current j into (4.3) and comparing powers of λ yields an infinite sequence of conservation laws.

However, it turns out that two different expansions are possible, leading to two infinite sequences of conserved quantities for (1.6).

With $m = -p^3$ as before, the first expansion takes the form

$$\rho = p\zeta^{-1} + \sum_{n=0}^{\infty} \rho^{(n)}\zeta^n, \qquad \lambda = \zeta^{-3}, \tag{4.4}$$

with the corresponding expansion of the current being

$$j = -pu\zeta^{-1} + \sum_{n=0}^{\infty} j^{(n)}\zeta^n. \tag{4.5}$$

Thus the leading order (ζ^{-1} term) in (4.3) is just the conservation law (2.6). The first two densities and currents found recursively from (4.2, 4.3) are

$$\begin{array}{ll} \rho^{(0)} = -p_x p^{-1}, & \rho^{(1)} = \frac{2}{3}p_{xx}p^{-2} - p_x^2 p^{-3} + \frac{1}{3p}; \\ j^{(0)} = u_x - u\rho^{(0)}, & j^{(1)} = -p^2 - u\rho^{(1)}. \end{array} \tag{4.6}$$

For brevity we omit the recursion relations for $\rho^{(n)}, j^{(n)}$, but note that the densities take the form $\rho^{(n)} = P^{(n)}/p^{2n+1}$ with $P^{(n)}$ a polyonmial of degree $n+1$ in $p, p_x, \ldots, p_{(n+1)x}$; the even terms $\rho^{(2n)}$ are exact x derivatives. The densities p and $\rho^{(1)}$ correspond to the conserved quantities H_5 and H_7 in (5.1) below.

The second expansion is in positive powers of λ, viz

$$\rho = \sum_{n=0}^{\infty} r^{(n)}\lambda^{n+1}, \qquad j = \sum_{n=0}^{\infty} g^{(n)}\lambda^{n+1}. \tag{4.7}$$

Again details of the recursive formulae for $r^{(n)}, g^{(n)}$ are omitted, but we note that the odd densities $r^{(2n+1)}$ are all exact derivatives, and apart from

$$r^{(0)} = u, \qquad r^{(2)} = u^3 \tag{4.8}$$

(which yield the conserved quantities H_0 and H_{-1} below) all other non-trivial densities in this sequence are non-local since they arise by repeated application of the inverse of the Helmholtz operator $\mathcal{L} = (1 - \partial_x^2)$ appearing on the left hand side of (4.2).

There is another conservation law which does not appear in the above sequences, namely

$$\left(v_{xx}^2 + 5v_x^2 + 4v^2\right)_t = \left(4u^2v - \frac{2}{3}u^3 - 4v\mathcal{L}^{-1}(u^2) - v_x\mathcal{L}^{-1}(u^2)_x\right)_x, \tag{4.9}$$

where v is defined by

$$v := (4 - \partial_x^2)^{-1}u. \tag{4.10}$$

This conservation law yields the conserved quantity H_1 in (5.1) below, since the density on the left hand side of (4.9) differs from the compact expression mv by a total derivative.

5 Hamiltonian structures

As derived above, the equation (1.6) has an infinite sequence of conservation laws. Here we list some of the simplest associated conserved quantities:

$$\begin{array}{rclrcl} H_{-1} & = & -\frac{1}{6}\int u^3\,dx, & H_0 & = & -\frac{9}{2}\int m\,dx, \\ H_1 & = & \frac{1}{2}\int mv\,dx, & H_5 & = & \int m^{1/3}\,dx, \\ H_7 & = & -\frac{1}{2}\int (m_x^2 m^{-7/3} + 9m^{-1/3})\,dx. & & & \end{array} \tag{5.1}$$

In the above we take $u = (1-\partial_x^2)^{-1}m$ as before, and from (4.10) we have $v = (4-\partial_x^2)^{-1}(1-\partial_x^2)^{-1}m$. The labelling is such that H_k generates a flow of weight k with Hamiltonian operator B_0 as in (5.2) below.

If we introduce the skew-symmetric differential operators B_0, B_1 according to

$$B_0 = \partial_x(1-\partial_x^2)(4-\partial_x^2), \quad B_1 = m^{2/3}\partial_x m^{1/3}(\partial_x - \partial_x^3)^{-1} m^{1/3}\partial_x m^{2/3}, \tag{5.2}$$

then we can immediately write the equation (1.6) in terms of the gradient of a conserved quantity in two different ways, namely:

$$m_t = B_0\,\frac{\delta H_{-1}}{\delta m} = B_1\,\frac{\delta H_0}{\delta m}. \tag{5.3}$$

The identity (5.3) is our proposed bi-Hamiltonian form of the equation. However, in order to assert that the pair B_0, B_1 define a genuine bi-Hamiltonian structure it is necessary to check that both are Hamiltonian (Poisson) operators, and that they are compatible [17, 6]. The first operator B_0 is constant coefficient and so the Jacobi identity is trivial and it is clearly Poisson, but for the non-local operator B_1 we have not succeeded in verifying the Jacobi identity.

We have noted other properties of the operator pair (5.2) which strongly suggest that they do define the correct bi-Hamiltonian structure for (1.6). In the list (5.1) we see that H_0 is a Casimir for B_0 and H_5 is a Casimir for B_1. The translational flow in the hierarchy is

$$m_x = B_0\,\frac{\delta H_1}{\delta m} = B_1\,\frac{\delta H_7}{\delta m}, \tag{5.4}$$

where the second equality in (5.4) requires suitable definition of the integral operator ∂_x^{-1} in B_1. Also, the fifth-order vector field $B_0\,\frac{\delta H_5}{\delta m}$ is just the reciprocal transformation of the fifth-order Kaup-Kupershmidt equation. This is strong evidence that the operator $B_0\,B_1^{-1}$ should be the reciprocal transform of the Kaup-Kupershmidt recursion operator (3.4).

6 Peakon dynamics

Here we consider the dynamics of multi-peakon solutions of the equation (1.6), which take the form

$$u(x,t) = \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|}, \quad m = 2\sum_{j=1}^{N} p_j(t)\delta(x - q_j(t)), \tag{6.1}$$

with N, the number of peakons, being arbitrary.

Substituting the expression (6.1) into the equation (1.6) yields the coupled ODEs for $q_j(t), p_j(t)$, namely

$$\begin{array}{rcl} \dot{p}_j &=& 2\sum_{k=1}^{N} p_j p_k \, sgn(q_j - q_k) e^{-|q_j - q_k|}, \\ \dot{q}_j &=& \sum_{k=1}^{N} p_k e^{-|q_j - q_k|}, \end{array} \tag{6.2}$$

which is just the system (1.8) in the particular case $b = 3$. Although the variables q_j, p_j are not canonical, and we have not found a (non-canonical) Hamiltonian structure except in the special case $N = 2$, the dynamical system (6.2) is integrable in the sense that it arises from a Lax equation

$$\dot{\mathsf{L}} = [\mathsf{M}, \mathsf{L}] \tag{6.3}$$

where L and M take the form

$$\mathsf{L} = \frac{1}{2}(\mathsf{A} - \mathsf{A}^T - 1 - 2\mathsf{C})\mathsf{P}, \qquad \mathsf{M} = \mathsf{D} - (\mathsf{A} - \mathsf{A}^T)\mathsf{P}. \tag{6.4}$$

In order to specify the matrices in the Lax pair, which are built out of the strictly lower triangular matrices A, C and the diagonal matrices P, D, we first note that an immediate consequence of the system (6.2) is that $\frac{d}{dt} sgn(q_j - q_k) = 0$, so that the peakons preserve their relative ordering (as for Camassa-Holm peakons). Without loss of generality we take the ordering $q_1 < q_2 < \ldots < q_N$ and then for the matrices appearing in (6.4), the non-zero components of A and C may be given (without modulus signs) as

$$A_{jk} = e^{q_k - q_j}, \quad C_{jk} = 1, \quad j > k, \tag{6.5}$$

while the diagonal elements of P, D are the components of the vectors

$$\mathbf{p} = (p_1, p_2, \ldots, p_N)^T, \qquad \mathbf{d} = (\mathsf{A} - \mathsf{A}^T)\mathbf{p} \tag{6.6}$$

respectively.

In the case of 2-peakon scattering, i.e. the equations (6.2) for N=2, we consider two peakons that are initially well separated with $q_1 < q_2$ and have asymptotic speeds c_1 and c_2 as $t \to -\infty$ with $c_1 > c_2$ and $c_1 > 0$ so that they eventually collide. The situation is shown in Figure 1.

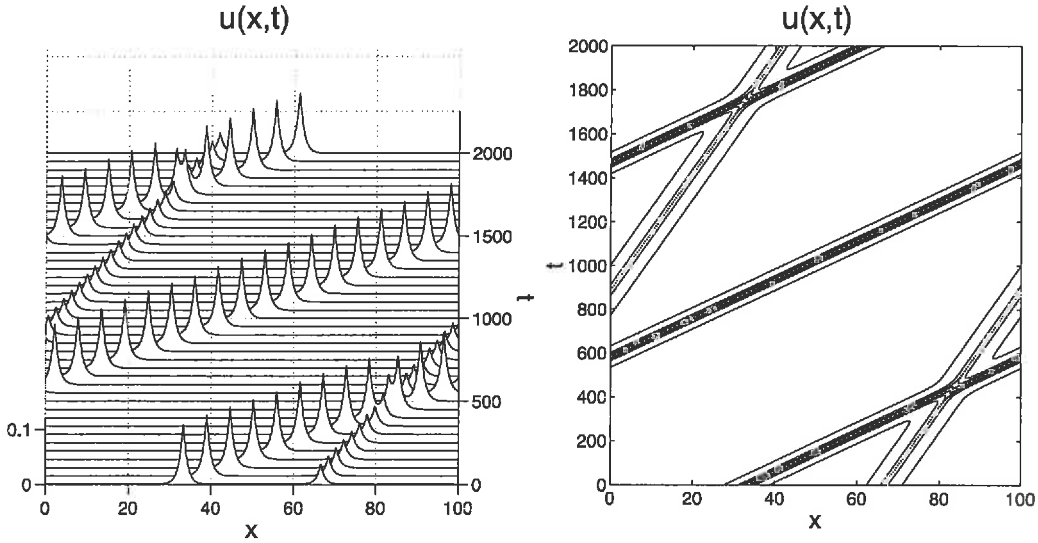

Figure 1: Two rear-end collisionss of $b = 3$ peakons. The initial positions are $x = 33$ and $x = 66$. The faster peakon moves at three times the speed of the slower one. For this ratio of speeds, both collisions result in a phase shift to the right for the faster space-time trajectory, but no phase shift for the slower one.

In this case, we have two conserved quantities

$$P := p_1+p_2 = c_1+c_2, H := p_1^2+p_2^2+2p_1p_2\, e^{q_1-q_2}(2-e^{q_1-q_2}) = c_1^2+c_2^2. \quad (6.7)$$

The equations of motion are then most easily solved in terms of the sum and difference variables P, $Q = q_1 + q_2$, $p = p_1 - p_2$ and $q = q_1 - q_2$, and the quadratures involved are almost the same as in the Camassa-Holm case (see [2]). For the positions of the peaks we find the explicit formulae

$$\begin{aligned} q_1(t) &= c_1\,t + \frac{1}{2}\log(4\Gamma(c_1-c_2)^2) - \log\left(\Gamma e^{(c_1-c_2)\,t} + 4c_2P\right), \quad (6.8)\\ q_2(t) &= c_2\,t - \frac{1}{2}\log(4\Gamma(c_1-c_2)^2) + \log\left(\Gamma e^{(c_1-c_2)\,t} + 4c_1P\right), \quad (6.9)\end{aligned}$$

where Γ is an integration constant.

In the limit $t \to \infty$, these formulas show that the solitons exchange their asymptotic speeds, or equivalently their momenta and amplitudes, as

$$\lim_{t\to\infty} q_1(t) = c_2 t \quad \text{and} \quad \lim_{t\to\infty} q_2(t) = c_1 t\,. \quad (6.10)$$

Thus, the main effect in the peakon scattering is an exchange of momentum, or amplitude, between the two peakons, resulting only in a phase shift at asymptotic times. The phase shift for the faster soliton (the one with speed c_1 in the limit $t \to -\infty$) is defined and evaluated using (6.8) and (6.9) as

$$\Delta q_f = q_2(+\infty) - q_1(-\infty) = \log\left[\frac{c_1(c_1+c_2)}{(c_1-c_2)^2}\right]. \tag{6.11}$$

Likewise, the phase shift for the slower soliton (the one with speed c_2 in the limit $t \to -\infty$) is defined and evaluated using (6.8) and (6.9) as

$$\Delta q_s = q_1(+\infty) - q_2(-\infty) = \log\left[\frac{(c_1-c_2)^2}{c_2(c_1+c_2)}\right]. \tag{6.12}$$

So for $c_1/c_2 > 3$ both peakons experience a forward shift, and for $1 < c_1/c_2 < 3$ the faster peakon shifts forward while the slower one shifts backward; the case $c_1/c_2 = 3$ is the turning point at which the slower peakon has no phase shift. Remarkably, phase shift scattering rules corresponding precisely to these hold for Camassa-Holm peakons, except with $3 \to 2$ [2].

An interesting special case is the peakon-antipeakon collision when c_1 and c_2 have opposite signs; only in this case can the solitons overlap when the variable $p = p_1 - p_2$ diverges. For the perfectly antisymmetric collision $c_1 = -c_2 = c$ the resulting solution of the partial differential equation (2.1) is

$$\begin{aligned} u(x,t) &= \tfrac{c}{1-e^{-2c|t|}}\left[e^{-|x+c|t||} - e^{-|x-c|t||}\right], \\ m(x,t) &= \tfrac{2c}{1-e^{-2c|t|}}\left[\delta(x+c|t|) - \delta(x-c|t|)\right]. \end{aligned} \tag{6.13}$$

In numerical simulations, we also investigated the emergence of peakons from a Gaussian initial condition. The integrable behavior is evidenced in Figure 2 as the peakons collide elastically as they recross the periodic domain.

7 Conclusions

In a forthcoming article [3] we will consider an integro-differential generalization of (1.7), extending the results of [8] to include an extra parameter b, such that there are multi-pulson solutions described by the finite-dimensional system (1.8) with the generating function

$$G_N = \frac{1}{2}\sum_{i,j=1}^{N} p_i p_j \, g(q_i - q_j)\,.$$

for an arbitrary even function $g(x)$ (so that $g(x) = e^{-|x|}$ in the peakons case). It turns out that for arbitrary b values the two-pulson ($N = 2$) dynamics is Hamiltonian and integrable.

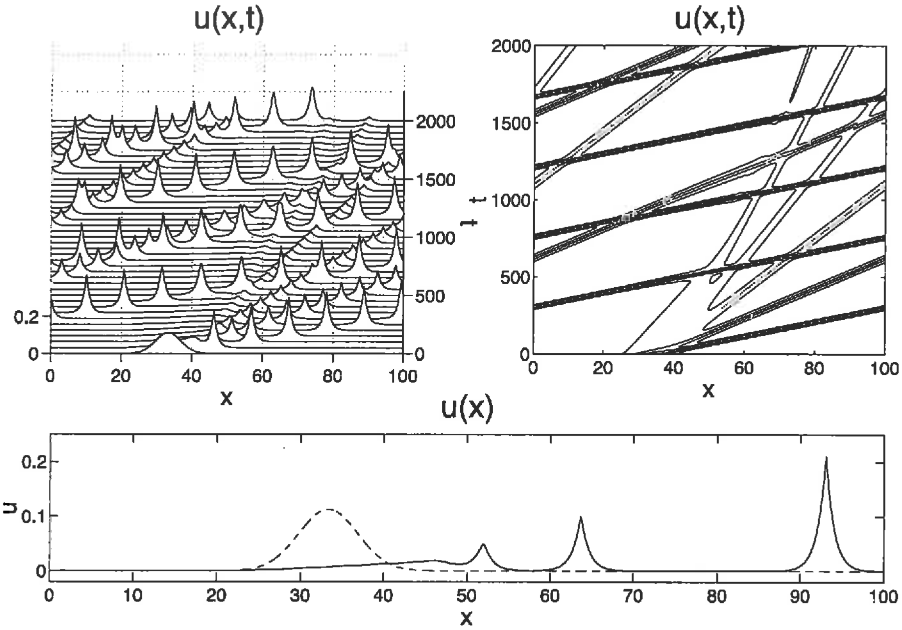

Figure 2: $b = 3$ peakons emerging from a Gaussian of unit area and $\sigma = 5$ centered about $x = 33$ on a periodic domain of length $L = 100$. The fastest peakon recrosses the domain five times and has many elastic interactions with the slower ones.

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