

VANISHING THEOREMS, BOUNDEDNESS AND HYPERBOLICITY OVER HIGHER DIMENSIONAL BASES

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Let C be a smooth projective curve of genus g over an algebraically closed field of characteristic 0, $\Delta \subset C$ a finite subset and $q > 1$ a positive integer. Let $\delta = \#\Delta$.

Shafarevich conjectured that the set of non-isotrivial families of smooth projective curves of genus q over $C \setminus \Delta$ is finite, furthermore if $2g - 2 + \delta \leq 0$, then there are no such families.

This was confirmed by [Parshin68] for the case $\Delta = \emptyset$ and by [Arakelov71] in general cf. [Beauville81]. For a more detailed discussion see [Bedulev-Viehweg00].

Recently there has been a flurry of new results regarding higher dimensional generalizations of this problem; [Migliorini95], [Kovács96], [Kovács97b], [Bedulev-Viehweg00], [Kovács00a], [Viehweg-Zuo01a], and [Kovács00b]. All of these articles study generalizations when the base of the family is a curve.

The main goal of the present article is to extend (some of) these results to higher dimensional bases, i.e., replacing the fixed curve, C , with a fixed smooth projective variety, S , of arbitrary dimension.

An extension like that may be useful for several reasons. A consequence of the results for curves can be phrased the following way: An isotrivial family of curves of genus at least 2 over \mathbb{P}^1 must have at least 3 singular fibers. In other words the corresponding moduli space has a certain hyperbolic property. This of course implies similar statements for families over a base that is covered by rational curves, e.g., \mathbb{P}^k , but not as strong statements as one expects. Consider a family that is smooth over $\mathbb{P}^k \setminus \Delta$ where Δ is a suitable effective divisor. Then the above statement implies that $\deg \Delta > 2$. However, if the hyperbolic behavior holds over higher dimensional bases as well, then one expects that $\deg \Delta > k + 1$, or equivalently that $\omega_{\mathbb{P}^k}(\Delta)$ be ample. In particular this would also say that certain families defined over a plane curve cannot be extended to a family over \mathbb{P}^2 .

One can go even further and conjecture, that if the variation of the family $f : X \rightarrow S$ in moduli is maximal, then $\omega_S(\Delta)$ is big. For a more precise formulation of these conjectures see (5.2.1). These are special cases of Viehweg's more general conjectures [Viehweg00, 6.3]. The last part of Shafarevich's conjecture is a special case of this, and so are the other known generalizations of that.

The proofs of the known cases center around various Kodaira type vanishing theorems. One expects that the first step toward a proof of this conjecture would be a new vanishing theorem.

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The main result of this article is a vanishing theorem. It can be considered a relative version of the logarithmic Kodaira-Akizuki-Nakano vanishing theorem. For the precise statement please see (2.5). The proof of this theorem occupies §1 and §2. §3 is devoted to some positivity results for the push-forward of powers of the relative dualizing sheaf. The proofs here closely follow ideas of Kollár and Viehweg. As an application of the results in §2 and §3, a boundedness result is proved for families over higher dimensional bases in §4. The interesting point here is more the technique than the result itself.

Another application is presented in §5. A couple of theorems are proved for the degeneracy locus of certain families of varieties of general type over a base that's one of the following: an abelian variety, \mathbb{P}^k for $k \leq 3$, or a quadric hypersurface of dimension at most 3.

(5.1.1) is new, it treats a more general case than what was previously known [Kovács97a, 0.1]. (5.2.2) and (5.2.4) are unfortunately not as strong as expected. In particular they also follow from the more general [Viehweg00, 2.6]. Furthermore, in (5.2.2) and (5.2.4), the restrictions on the dimension of the base and fibers respectively should not be necessary. In fact they are not present in the vanishing theorem (2.5). The problem is in the application part. The assumptions of the vanishing theorem hold trivially in the cases presented here, and there is no obvious reason why they would not apply in a more general situation. To resolve this issue will require more work, but one may hope that this method will actually lead to a more general statement along the lines of (5.2.1). Note that since the writing of this article [Viehweg-Zuo01b] achieved stronger results in this direction with different methods.

DEFINITIONS AND NOTATION. Throughout the article the groundfield will always be \mathbb{C} , the field of complex numbers.

Let $f : X \rightarrow S$ be a morphism of schemes. Then X_s denotes the fibre of f over the point $s \in S$ and f_s denotes the restriction of f to X_s . Similarly X_{gen} denotes the generic fibre of f .

Let $f : X \rightarrow S$ be a surjective morphism between smooth projective varieties. $\text{Var}(f)$ denotes the number of effective parameters of the birational equivalence classes of the fibers. For a rigorous definition of $\text{Var}(f)$ the reader is referred to [Kollár87] or [Viehweg83a]. f will be called *isotrivial* if $\text{Var}(f) = 0$.

A line bundle \mathcal{L} on X is called *big* if X is proper and the global sections of \mathcal{L}^m define a birational map for some $m > 0$ and it is called *nef* if $\deg(\mathcal{L}|_C) \geq 0$ for every proper curve $C \subset X$. In particular ample implies nef and big. \mathcal{L} will be called *l-big* if X is proper and the global sections of \mathcal{L}^m define a rational map, $\phi : X \dashrightarrow Y$, for some $m > 0$ such that $\dim X - \dim \phi(X) \leq l$. Obviously 0-big is the same as big. Let U be an open subset of X . \mathcal{L} is called *semi-ample with respect to U* if some positive power of \mathcal{L} is generated by global sections over U , i.e., the natural map $H^0(X, \mathcal{L}^b) \otimes \mathcal{O}_U \rightarrow \mathcal{L}^b|_U$ is surjective for some $b > 0$. It is called *semi-ample* if it is semi-ample with respect to X . Similarly \mathcal{L} is called *l-ample with respect to U* if the global sections of some positive power of \mathcal{L} define a rational map, $\phi : X \dashrightarrow Y$, such that $\phi|_U$ is a morphism (i.e., defined everywhere on U), $\phi^{-1}(\phi(U)) = U$, and $\dim \phi^{-1}(y) \leq l$ for all $y \in \phi(U)$. Let X be a smooth projective variety and $U \subseteq X$ an open subset. Let \mathcal{E} be a locally free sheaf on X , $\pi : P = \mathbb{P}(\mathcal{E}) \rightarrow X$ the projective bundle associated with \mathcal{E} and $\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ the corresponding tautological line bundle. \mathcal{E} is called *semi-positive* or *nef* (resp. *semi-ample with respect to U* , resp. *l-big*, resp. *l-ample with respect to U*) if \mathcal{L} on P is nef (resp. semi-ample with respect to $\pi^{-1}(U)$, resp. *l-big*, resp. *l-ample with respect to $\pi^{-1}(U)$*).

For a coherent sheaf \mathcal{F} , \mathcal{F}^* denotes its dual, $\text{Hom}(\mathcal{F}, \mathcal{O}_X)$.

The dimension of the empty set is $-\infty$

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§1. A VANISHING THEOREM FOR VECTOR BUNDLES

An important ingredient of the main vanishing theorem in §2 is in turn a generalization of the Kodaira-Akizuki-Nakano vanishing theorem. This generalization is essentially a combination of the logarithmic version of Esnault and Viehweg and the vector bundle version of Le Potier. In fact the proof of this combined version does not require any new ideas.

1.1 Theorem. [Esnault-Viehweg92, 6.7], [LePotier75, Théorème 1] *Let X be a smooth projective variety of dimension n , D an effective normal crossing divisor and \mathcal{E} a locally free sheaf of rank r on X . Assume that \mathcal{E} is semi-ample (with respect to X), and l -big and $l + 1$ -ample with respect to $X \setminus D$ for some $l \in \mathbb{N}$. Then for $p + q \geq n + r + l$,*

$$H^q(X, \Omega_X^p(\log D)(-D_{\text{red}}) \otimes \mathcal{E}) = 0.$$

Proof. The following lemma reduces the question to the line bundle case.

1.2 Le Potier's Lemma. (cf. [Shiffman-Sommese85, 5.16]) *Let \mathcal{F} be a coherent sheaf on X , $\pi : P = \mathbb{P}(\mathcal{E}) \rightarrow X$ the projective bundle associated to \mathcal{E} and $\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ the corresponding tautological line bundle on P . Then for all $p, q \in \mathbb{N}$,*

$$H^q(X, \mathcal{F} \otimes \Omega_X^p(\log D) \otimes \mathcal{E}) \simeq H^q(P, \pi^* \mathcal{F} \otimes \Omega_P^p(\log \pi^* D) \otimes \mathcal{L})$$

Proof. The proof of [Shiffman-Sommese85, 5.16] works word by word if one replaces Ω_X and Ω_P with $\Omega_X(\log D)$ and $\Omega_P(\log \pi^* D)$. \square

Now take $\mathcal{F} = \mathcal{O}_X$ and use [Esnault-Viehweg92, 4.11, 6.7] for \mathcal{L} on P . \square

1.2.1 REMARK. An alternative way is to use [Esnault-Viehweg92, 2.3] and induction.

§2. VANISHING THEOREMS FOR FAMILIES

2.1 Let $f : X \rightarrow S$ be a surjective morphism between smooth projective varieties with connected fibers, $\dim X = n$ and $\dim S = k$. Let $\Delta \subset S$ be an effective normal crossing divisor and assume that $D = f^* \Delta \subset X$ is also a normal crossing divisor. Assume that $(X, D) \rightarrow (S, \Delta)$ has the following properties: $X \setminus D \rightarrow S \setminus \Delta$ is smooth, and the sheaf $\Omega_{X/S}(\log D)$ defined by the short exact sequence,

$$0 \rightarrow f^* \Omega_S(\log \Delta) \rightarrow \Omega_X(\log D) \rightarrow \Omega_{X/S}(\log D) \rightarrow 0,$$

is locally free. $\Omega_{X/S}^q(\log D)$ will denote $\bigwedge^q \Omega_{X/S}(\log D)$.

2.2 Let \mathcal{E} be a locally free sheaf of finite rank on S . Then there exists a filtration

$$\mathcal{E} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \dots \supset \mathcal{F}^r = 0$$

of \mathcal{E} such that $\mathcal{F}^{i-1}/\mathcal{F}^i$ is a locally free sheaf of rank at most $k = \dim S$ for all $i = 1, \dots, r$. Choose such a filtration for each \mathcal{E} on S and let

$$\mathfrak{F}(\mathcal{E}) = \{\mathcal{F}^{i-1}/\mathcal{F}^i \mid \mathcal{F}^i \text{ is the chosen filtration of } \mathcal{E}\}.$$

These choices together will be called a *choice of filtrations on S* .

The following is a trivial, but important observation:

2.3 Lemma. *Let $f : X \rightarrow S$ be as in (2.1). If for some $i \in \mathbb{N}$ and \mathcal{G} coherent sheaf on X , $H^i(X, \mathcal{G} \otimes f^*\mathcal{E}) \neq 0$, then there exists an $\mathcal{E}' \in \mathfrak{F}(\mathcal{E})$ such that $H^i(X, \mathcal{G} \otimes f^*\mathcal{E}') \neq 0$. \square*

2.4 Definition. Let $f : X \rightarrow S$ be as in (2.1) and let \mathfrak{A} be a set of locally free sheaves of finite rank on S . Let $\mathfrak{T}^0(\mathfrak{A}) = \mathfrak{A}$ and

$$\mathfrak{T}(\mathfrak{A}) = \mathfrak{T}^1(\mathfrak{A}) = \bigcup \mathfrak{F}(\mathcal{E} \otimes \Omega_S^q(\log \Delta) \otimes \omega_S(\Delta)^{-1}),$$

where the union is taken for all $\mathcal{E} \in \mathfrak{A}$ and for $q = 0, \dots, k$. Furthermore, for $r \geq 1$, let $\mathfrak{T}^r(\mathfrak{A}) = \mathfrak{T}(\mathfrak{T}^{r-1}(\mathfrak{A}))$.

2.5 Theorem. *Let $f : X \rightarrow S$ be as in (2.1). Let \mathfrak{A} be a set of locally free sheaves of rank at most k on S , \mathcal{L} a line bundle on X such that for a suitable choice of filtrations on S , $\mathcal{L}(D_{\text{red}}) \otimes f^*\mathcal{E}'$ is semi-ample (with respect to X), and big and 1-ample with respect to $X \setminus D$ for all $\mathcal{E}' \in \mathfrak{T}^i(\mathfrak{A})$, $i = 0, \dots, n - k$. Then for all $\mathcal{E} \in \mathfrak{A}$,*

$$H^p(X, \Omega_{X/S}^q(\log D) \otimes f^*\omega_S(\Delta) \otimes \mathcal{L} \otimes f^*\mathcal{E}) = 0 \quad \text{for } p + q \geq n.$$

Proof. The statement will be proved by induction using the following Lemma.

2.6 Lemma. *Let $p, q \in \mathbb{N}$ such that $p + q \geq n$ and assume that for every natural number $l > q$ and for every $\mathcal{E}' \in \mathfrak{T}(\mathfrak{A})$, $H^{p-1}(X, \Omega_{X/S}^l(\log D) \otimes f^*\omega_S(\Delta) \otimes \mathcal{L} \otimes f^*\mathcal{E}') = 0$. Then for every $\mathcal{E} \in \mathfrak{A}$, $H^p(X, \Omega_{X/S}^q(\log D) \otimes f^*\omega_S(\Delta) \otimes \mathcal{L} \otimes f^*\mathcal{E}) = 0$.*

Proof. Let $\mathcal{E} \in \mathfrak{A}$. The standard exact sequence of f

$$0 \longrightarrow f^*\Omega_S^1(\log \Delta) \longrightarrow \Omega_X^1(\log D) \longrightarrow \Omega_{X/S}^1(\log D) \longrightarrow 0$$

yields a filtration on $\Omega_X^{q+k}(\log D) \otimes \mathcal{L} \otimes f^*\mathcal{E}$ which in turn induces a spectral sequence,

$$E_1^{r,s} = H^{r+s}(X, \Omega_{X/S}^{q+k-r}(\log D) \otimes f^*\Omega_S^r(\log \Delta) \otimes \mathcal{L} \otimes f^*\mathcal{E}) \Rightarrow H^{r+s}(X, \Omega_X^{q+k}(\log D) \otimes \mathcal{L} \otimes f^*\mathcal{E}).$$

Since $p + q \geq n$, $H^p(X, \Omega_X^{q+k}(\log D) \otimes \mathcal{L} \otimes f^*\mathcal{E}) = 0$ by (1.1). Hence $E_\infty^{r,p-r} = 0$ for all r . In particular $E_\infty^{k,p-k} = 0$. Suppose now that

$$E_1^{k,p-k} = H^p(X, \Omega_{X/S}^q(\log D) \otimes f^*\omega_S(\Delta) \otimes \mathcal{L} \otimes f^*\mathcal{E}) \neq 0.$$

Observe that $E_w^{u,v} = 0$ for every $u > k$ and arbitrary v, w , so $E_\infty^{k,p-k}$ can be zero only if there exists a $t > 0$ such that $E_t^{k-t,p-1-k+t} \neq 0$. Then $E_1^{k-t,p-1-k+t} \neq 0$ for the same t , so $l = q + t > q$ is such that

$$\begin{aligned} E_1^{k-t,p-1-k+t} &= H^{p-1}(X, \Omega_{X/S}^l(\log D) \otimes f^* \Omega_S^{k-t}(\log \Delta) \otimes \mathcal{L} \otimes f^* \mathcal{E}) \neq 0, \text{ i.e.,} \\ H^{p-1}(X, \Omega_{X/S}^l(\log D) \otimes f^* \omega_S(\Delta) \otimes \mathcal{L} \otimes f^*(\mathcal{E} \otimes \Omega_S^{k-t}(\log \Delta) \otimes \omega_S(\Delta)^{-1})) &\neq 0. \end{aligned}$$

Therefore $H^{p-1}(X, \Omega_{X/S}^l(\log D) \otimes f^* \omega_S(\Delta) \otimes \mathcal{L} \otimes f^* \mathcal{E}') \neq 0$, for some $\mathcal{E}' \in \mathfrak{T}(\mathfrak{A})$ by (2.3). This contradicts the hypotheses, hence the statement follows. \square

Proof of (2.5) continued: Let $\mathcal{E} \in \mathfrak{A}$ and suppose that

$$H^p(X, \Omega_{X/S}^q(\log D) \otimes f^* \omega_S(\Delta) \otimes \mathcal{L} \otimes f^* \mathcal{E}) \neq 0$$

for some p, q , such that $p + q \geq n$. Then by (2.6) there exist $\mathcal{E}_1 \in \mathfrak{T}(\mathfrak{A})$ and $q_1 > q$ such that

$$H^{p-1}(X, \Omega_{X/S}^{q_1}(\log D) \otimes f^* \omega_S(\Delta) \otimes \mathcal{L} \otimes f^* \mathcal{E}_1) \neq 0.$$

Now $p - 1 + q_1 \geq p + q \geq n$, so this step can be repeated. I.e., for $j = 2, \dots, n - k + 1$, there exist $\mathcal{E}_j \in \mathfrak{T}^j(\mathfrak{A})$ and $q_j > q_{j-1}$ such that

$$H^{p-j}(X, \Omega_{X/S}^{q_j}(\log D) \otimes f^* \omega_S(\Delta) \otimes \mathcal{L} \otimes f^* \mathcal{E}_1) \neq 0.$$

However, $q_{n-k+1} > n - k$, so $\Omega_{X/S}^{q_{n-k+1}}(\log D) = 0$, a contradiction. \square

2.6.1 REMARK. Note that the ‘‘extra’’ positivity assumption of (2.5) is asking positivity in a horizontal direction. In other words, if \mathcal{L} is relatively ample, then the assumption asks, that \mathcal{L} be more positive than certain given sheaves coming from S . Therefore the applicability of the theorem does not depend so much on the family than on the base. In particular, if one studies families over a fixed base, with fixed degeneracy locus, then the assumptions provide a fixed threshold for the required positivity of \mathcal{L} for all families over this base. This should be understood as a kind of boundedness statement for families over a fixed based with fixed degeneracy locus (or smooth families over $S \setminus \Delta$) cf. §4.

2.7 Corollary. *Let $f : X \rightarrow S$ be as in (2.1). Let \mathcal{M} be a line bundle on X . Assume that \mathcal{M} contains another line bundle \mathcal{L} such that for a suitable choice of filtrations on S , $\mathcal{L}(D_{\text{red}}) \otimes f^* \mathcal{E}$ is semi-ample (with respect to X), and big and 1-ample with respect to $X \setminus D$ for all $\mathcal{E} \in \mathfrak{T}^i(\mathcal{O}_S)$, $i = 0, \dots, n - k$. Then*

$$H^n(X, f^* \omega_S(\Delta) \otimes \mathcal{M}) = 0.$$

Proof. $H^n(X, f^* \omega_S(\Delta) \otimes \mathcal{L}) = 0$ by (2.5). The cokernel of the embedding $\mathcal{L} \subseteq \mathcal{M}$ is supported in codimension 1, so the map $0 = H^n(X, f^* \omega_S(\Delta) \otimes \mathcal{L}) \rightarrow H^n(X, f^* \omega_S(\Delta) \otimes \mathcal{M})$ is surjective. \square

2.8 Corollary. *If in addition f is smooth and S is an abelian variety, then $H^n(X, \mathcal{K}) = 0$ for every line bundle \mathcal{K} on X , that contains a big and 1-ample line bundle. \square*

§3. WEAK POSITIVITY

§§3.1 Weakly positive sheaves. The following notation will be fixed for the rest of (3.1). Let S be a quasi-projective variety, and $U \subseteq S$ an open subset.

3.1.1 Definition. [Viehweg83a] Let \mathcal{F} be a coherent torsion free sheaf on S and $\iota : V \hookrightarrow S$ the maximal open subscheme such that $\mathcal{F}|_V$ is locally free. We define $S^r(\mathcal{F}) = \iota_* \text{Sym}^r(\mathcal{F}|_V)$. Let \mathcal{L} be an ample line bundle on S . Then \mathcal{F} is called *weakly positive* over U if $\mathcal{F}|_U$ is locally free and for all $a > 0$ there exists some $b > 0$ such that $S^{ab}(\mathcal{F}) \otimes \mathcal{L}^b$ is generated by global sections over U , i.e., the natural map $H^0(S, S^{ab}(\mathcal{F}) \otimes \mathcal{L}^b) \otimes \mathcal{O}_U \rightarrow S^{ab}(\mathcal{F}) \otimes \mathcal{L}^b|_U$ is surjective.

3.1.2 Lemma. *Let \mathcal{E} be a coherent sheaf on S . Assume that either \mathcal{E} is an ample locally free sheaf or $\mathcal{E} = \mathcal{F} \otimes \mathcal{L}$, where \mathcal{F} is a weakly positive sheaf over U and \mathcal{L} is an ample line bundle. Then there exists a $\lambda_0 > 0$, such that for all λ , divisible by λ_0 , and for any $t_1, t_2 \in U, t_1 \neq t_2$, the natural map $\rho_\lambda : H^0(S, S^\lambda(\mathcal{E})) \rightarrow (S^\lambda(\mathcal{E}) \otimes k(t_1)) \oplus (S^\lambda(\mathcal{E}) \otimes k(t_2))$ is surjective.*

Proof. Let \mathfrak{m}_i be the ideal sheaf of $\{t_i\}$ for $i = 1, 2$ and $\mathfrak{I}_{1,2} = \mathfrak{m}_1 \cap \mathfrak{m}_2$ that of $\{t_1, t_2\}$.

Let \mathcal{E} be an ample locally free sheaf. Then $H^1(S, S^\lambda(\mathcal{E}) \otimes \mathfrak{I}_{1,2}) = 0$ for $\lambda \gg 0$. In fact by semi-continuity there is a λ_0 such that this vanishing holds for all $\lambda \geq \lambda_0$ and for all pairs of points on S , so the statement follows.

Let $\mathcal{E} = \mathcal{F} \otimes \mathcal{L}$ as above. Since \mathcal{L} is ample, there exists an $n > 0$ such that \mathcal{L}^{bn} is very ample for all $b > 0$. Hence there exist sections $\sigma_1, \sigma_2 \in H^0(S, \mathcal{L}^{bn})$ such that $\sigma_i \in \mathfrak{m}_i \mathcal{L}^{bn}$, but $\sigma_i \notin \mathfrak{m}_{3-i} \mathcal{L}^{bn}$ for $i = 1, 2$. Now let $a = n + 1$. Since \mathcal{F} is weakly positive, there exists a $b_0 > 0$, such that $S^{ab}(\mathcal{F}) \otimes \mathcal{L}^b$ is globally generated over U for all $b \geq b_0$ [Viehweg83b, 3.2]. Then $S^{ab}(\mathcal{E}) \simeq S^{ab}(\mathcal{F}) \otimes \mathcal{L}^b \otimes \mathcal{L}^{bn}$, and using σ_i one sees easily, that $\text{im } \rho_{ab}$ contains $(S^{ab}(\mathcal{E}) \otimes k(t_1)) \oplus 0$ and $0 \oplus (S^{ab}(\mathcal{E}) \otimes k(t_2))$. Therefore ρ_{ab} is surjective, so the statement holds with $\lambda_0 = ab_0$. \square

3.1.3 Lemma. *Let $f : X \rightarrow S$ be a surjective morphism, \mathcal{M} a line bundle on X , and \mathcal{N} a line bundle on S . Assume, that $\mathcal{M}_t = \mathcal{M}|_{X_t}$ is generated by global sections for all $t \in U$, $f_* \mathcal{M}$ is locally free on U , and $f_* \mathcal{M}^m$ is reflexive on S for all m large and divisible enough. Assume further, that for some $r > 0$ there exists a subsheaf $\mathcal{E} \subseteq S^r(f_* \mathcal{M} \otimes \mathcal{N})$, such that $\text{supp}(S^r(f_* \mathcal{M} \otimes \mathcal{N})/\mathcal{E}) \subseteq S \setminus U$, and either \mathcal{E} is an ample locally free sheaf or $\mathcal{E} = \mathcal{F} \otimes \mathcal{L}$, where \mathcal{F} is a weakly positive sheaf over U and \mathcal{L} is an ample line bundle on S .*

Then $\mathcal{M} \otimes f^ \mathcal{N}$ is big and semi-ample with respect to $X \setminus f^{-1}U$ and if m is large and divisible enough, then the global sections of $\mathcal{M}^m \otimes f^* \mathcal{N}^m$ separate the fibers over U and on X_t induce the same map as do the sections of \mathcal{M}_t^m .*

Proof. Since \mathcal{M}_t is generated by global sections for $t \in U$, $f^* S^m(f_* \mathcal{M}) \rightarrow \mathcal{M}^m$ is surjective on $X \setminus f^{-1}U$. In particular $\alpha : f^* S^m(f_* \mathcal{M}) \otimes (\mathcal{O}_{X_t} \oplus \mathcal{O}_{X_s}) \rightarrow \mathcal{M}_t^m \oplus \mathcal{M}_s^m$ is surjective for $t, s \in U$. Now one has the following commutative diagram:

$$\begin{array}{ccc} H^0(S, S^m(f_* \mathcal{M} \otimes \mathcal{N})) \otimes \mathcal{O}_X & \longrightarrow & H^0(S, f_* \mathcal{M}^m \otimes \mathcal{N}^m) \otimes \mathcal{O}_X \xrightarrow{\cong} H^0(X, \mathcal{M}^m \otimes f^* \mathcal{N}^m) \otimes \mathcal{O}_X \\ \beta \downarrow & & \downarrow \gamma \\ f^*(S^m(f_* \mathcal{M}) \otimes (k(t) \oplus k(s))) & \xrightarrow{\cong} & f^* S^m(f_* \mathcal{M}) \otimes (\mathcal{O}_{X_t} \oplus \mathcal{O}_{X_s}) \xrightarrow{\alpha} \mathcal{M}_t^m \oplus \mathcal{M}_s^m \end{array}$$

If m is large and divisible enough, then β is surjective by (3.1.2), hence $\alpha \circ \beta$ is surjective, and then so is γ . This proves the statement. \square

3.1.4 Corollary.

(3.1.4.1) *If \mathcal{M}_t is l -big for general $t \in S$, then $\mathcal{M} \otimes f^*\mathcal{N}$ is l -big.*

(3.1.4.2) *If \mathcal{M}_t is l -ample for $t \in U$, then $\mathcal{M} \otimes f^*\mathcal{N}$ is l -ample with respect to $X \setminus f^{-1}U$.*

§3.2 Weak positivity of push-forwards of powers of dualizing sheaves.

3.2.1 Let $f : X \rightarrow S$ be a surjective morphism between smooth projective varieties with connected general fibre. Let $n = \dim X$, $k = \dim S$, and $\Delta \subset S$ denote the subset of S such that $X \setminus f^{-1}\Delta \rightarrow S \setminus \Delta$ is smooth. Assume, that $\omega_{X_{\text{gen}}}$ is nef and big, $\text{Var}(f) = \dim S$, and if m is large and divisible enough, then $f_*\omega_{X/S}^m$ is locally free on S .

3.2.1.1 REMARK. The last technical assumption is likely avoidable. The rest of the assumptions imply that $f_*\omega_{X/S}^m$ is locally free on $S \setminus \Delta$. A possible way to try to avoid the extra assumption, at least for families of canonically polarized manifolds, is to assume that the minimal model program works in dimension n and use the results of [Karu00] on the existence of compactified moduli. Those allow one to find a birational model with the required property. Another possibility is to use the weak semistable reduction of [Abramovich-Karu00] as it is done in [Viehweg-Zuo01bc]. For details please consult the latter articles.

3.2.2 Lemma. *Let $f : X \rightarrow S$ be as in (3.2.1). Then $\omega_{X/S}$ is big and if $\phi : X \dashrightarrow Z$ is the birational map induced by $\omega_{X/S}^m$, then ϕ separates the fibers of f over an open dense subset of S and $\phi|_{X_{\text{gen}}}$ is equal to the map induced by $\omega_{X_{\text{gen}}}^m$ for m large and divisible enough.*

Proof. Let $l > 0$ be divisible enough and such that $\omega_{X_{\text{gen}}}^l$ is generated by global sections. Then by [Viehweg83b, Thm II] there exists an ample locally free sheaf \mathcal{A} on S and an injective morphism $\alpha : \mathcal{A} \rightarrow S^r(f_*\omega_{X/S}^l)$ for some $r > 0$, such that α is an isomorphism over a dense open subvariety, $U \subseteq S$. The statement follows by applying (3.1.3) with $\mathcal{M} = \omega_{X_{\text{gen}}}^l$. \square

3.2.3 Definition. [Esnault-Viehweg90] Let F be a normal Gorenstein variety with rational singularities, \mathcal{L} a line bundle on F and Γ an effective divisor such that $\mathcal{L} = \mathcal{O}_F(\Gamma)$. Let

- (1) $\mathcal{C}(\Gamma, N) = \text{coker} \left(\tau_*\omega_{\tilde{F}} \left(- \left[\frac{\tilde{\Gamma}}{N} \right] \right) \rightarrow \omega_F \right)$ where $\tau : \tilde{F} \rightarrow F$ is a resolution of singularities such that $\tilde{\Gamma} = \tau^*\Gamma$ is a normal crossing divisor.
- (2) $e(\Gamma) = \min\{N \in \mathbb{N}_+ \mid \mathcal{C}(\Gamma, N) = 0\}$
- (3) $e(\mathcal{L}) = \sup\{e(\Gamma) \mid \exists \lambda \in H^0(F, \mathcal{L}) \text{ such that } \Gamma = (\lambda = 0)\}$

$e(\mathcal{L})$ will be called the *Esnault-Viehweg threshold* of \mathcal{L} . For properties of $e(\mathcal{L})$ the reader should consult [Viehweg95, §5.3-4]

Using the notation of (2.1), $r(m)$ will denote the rank of $f_*\omega_{X/S}^m$. This is equal to the m -th plurigenus of the general fiber of f , $P_m(X_{\text{gen}})$. $e(m)$ will denote the Esnault-Viehweg threshold of $\omega_{X_{\text{gen}}}^m$. If $\omega_{X_{\text{gen}}}$ is ample, then $e(m) \leq m^n K_{X_{\text{gen}}}^n + 1$ for $m \gg 0$ by [Viehweg95, 5.12].

The following statement gives an effective measure of the positivity of $f_*\omega_{X/S}^m$. The main ideas of the proof originates in the works of Kawamata, Kollár and Viehweg. However, there are some essential differences compared to the many variations that have appeared in the literature so far. Since the base is higher dimensional the statement is somewhat weaker than in the case when the base is a curve. It turns out that for the applications here this statement is sufficient. Another difference is a trivial, but crucial improvement: it is not necessary to start the proof by passing to a finite cover as it has been customary.

3.2.4 Proposition. *Let $f : X \rightarrow S$ be as in (3.2.1), and m sufficiently large and divisible. Then there exists a sheaf \mathcal{F} weakly positive over $S \setminus \Delta$ and an embedding $\iota : \mathcal{F} \hookrightarrow S^{e(m)r(m)}(f_*\omega_{X/S}^m) \otimes (\det f_*\omega_{X/S}^m)^{-1}$ such that ι is an isomorphism on $S \setminus \Delta$.*

Proof. Let $\mathcal{D} = \det f_*\omega_{X/S}^m$, $q = e(m)r(m)$, $\pi : Z \rightarrow X^q = X \times_S X \times_S \cdots \times_S X$ a resolution of singularities, $\rho = f^q \circ \pi : Z \rightarrow S$, and $\mathcal{M} = \pi^*\omega_{X^q/S} = \pi^*(\otimes \text{pr}_i^*\omega_{X/S})$. By the assumptions $\rho_*\mathcal{M}^m$ is locally free on S , hence \mathcal{M} is big by (3.2.2). f^q is a Gorenstein morphism and the fibres over $S \setminus \Delta$ have rational singularities, so there are natural injective maps:

$$\mathcal{D}^{e(m)} \hookrightarrow (f_*\omega_{X/S}^m)^{\otimes q} \simeq f_*^q\omega_{X^q/S}^m, \quad (3.2.4.1)$$

$$f_*^q\omega_{X^q/S}^m \hookrightarrow \rho_*\mathcal{M}^m \quad (3.2.4.2)$$

$$(\rho_*(\mathcal{M}^{m-1} \otimes \omega_{Z/S}))^{**} \hookrightarrow f_*^q\omega_{X^q/S}^m \quad (3.2.4.3)$$

where (3.2.4.2) and (3.2.4.3) are isomorphisms over $S \setminus \Delta$. The composition of (3.2.4.1) and (3.2.4.2) gives an injective map $\rho^*\mathcal{D}^{e(m)} \rightarrow \rho^*\rho_*\mathcal{M}^m \rightarrow \mathcal{M}^m$ and hence a section $\sigma \in H^0(Z, \mathcal{M}^m \otimes \rho^*\mathcal{D}^{-e(m)})$. Let $A = (\sigma = 0)$. Since π was an arbitrary resolution of singularities, one may replace it by further blow-ups. In particular one may assume that A is a normal crossing divisor.

Let $\mathcal{J} \subseteq \mathcal{O}_Z$ be the ideal sheaf defined as $\text{im}[\rho^*f_*^q\omega_{X^q/S}^m \rightarrow \mathcal{M}^m] = \mathcal{M}^m \otimes \mathcal{J}$. Note that $\text{supp } \mathcal{O}_X/\mathcal{J} \subseteq \text{supp } f^{-1}\Delta$. By blowing up \mathcal{J} one can assume that it is a line bundle, trivial over $X \setminus f^{-1}\Delta$. By [Viehweg89, II, 2.7(ii)] $f_*\omega_{X/S}^m$ is weakly positive over $S \setminus \Delta$, so $\mathcal{M}^m \otimes \mathcal{J}$ is weakly positive over $X \setminus f^{-1}\Delta$. Since \mathcal{J} is trivial over $X \setminus f^{-1}\Delta$, one obtains that \mathcal{M} itself is weakly positive over $X \setminus f^{-1}\Delta$.

Let \mathcal{L} be an ample line bundle on X and $\mathcal{K} = \mathcal{M}^{m-1} \otimes \rho^*\mathcal{D}^{-1}$. Since $\mathcal{N} = \mathcal{K}^{e(m)}(-A) = \mathcal{M}^{e(m)(m-1)-m}$ is big, there exists an $a > 0$ such that $\mathcal{N}^a \otimes \mathcal{L}^{-1} \simeq \mathcal{O}_X(B)$ for an effective divisor B and by further blowing up Z we may assume that $A+B$ is a normal crossing divisor. Since \mathcal{N} is weakly positive over $X \setminus f^{-1}\Delta$, $\mathcal{N}^N(-B) = \mathcal{N}^{N-a} \otimes \mathcal{L}$ is semi-ample with respect to $X \setminus f^{-1}\Delta$ for all $N > a$. For $N \gg 0$, $\left[\frac{N \cdot A + B}{N \cdot e(m)}\right] = \left[\frac{A}{e(m)}\right]$, so by [Viehweg89, II, 4.3 and III, 2.6], $\rho_*(\mathcal{K} \otimes \omega_{Z/S}(-[A/e(m)]))$ is weakly positive over $S \setminus \Delta$ and by [Viehweg95, 5.14, 5.21], $\rho_*(\mathcal{K} \otimes \omega_{Z/S}(-[A/e(m)])) \hookrightarrow \rho_*(\mathcal{K} \otimes \omega_{Z/S})$ is an isomorphism on $S \setminus \Delta$. On the other hand $(\rho_*(\mathcal{K} \otimes \omega_{Z/S}))^{**} \simeq (\rho_*(\mathcal{M}^{m-1} \otimes \omega_{Z/S}))^{**} \otimes \mathcal{D}^{-1} \hookrightarrow (f_*\omega_{X/S}^m)^{\otimes q} \otimes \mathcal{D}^{-1}$, is also an isomorphism on $S \setminus \Delta$ by (3.2.4.3). Let \mathcal{F} be the image of $\rho_*(\mathcal{K} \otimes \omega_{Z/S}(-[A/e(m)]))$ in $S^q(f_*\omega_{X/S}^m) \otimes \mathcal{D}^{-1}$ via the natural map $(f_*\omega_{X/S}^m)^{\otimes q} \otimes \mathcal{D}^{-1} \rightarrow S^q(f_*\omega_{X/S}^m) \otimes \mathcal{D}^{-1}$. This \mathcal{F} has all the desired properties. \square

3.2.5 Corollary. *Using the notation and assumptions of (3.2.4), let \mathcal{N} be a line bundle on S such that $\det f_*\omega_{X/S}^m \otimes \mathcal{N}^{me(m)r(m)}$ is ample. Assume that ω_{X_t} is big and 1-ample for $t \in S \setminus \Delta$. Then $\omega_{X/S} \otimes f^*\mathcal{N}$ is big and 1-ample with respect to $X \setminus f^{-1}\Delta$.*

Proof. By (3.2.4), $S^{e(m)r(m)}(f_*\omega_{X/S}^m \otimes \mathcal{N}^m)$ contains a weakly positive sheaf that is equal to it on $S \setminus \Delta$. Then by (3.1.3) $\omega_{X/S}^m \otimes f^*\mathcal{N}^m$ is big and 1-ample with respect to $X \setminus f^{-1}\Delta$. \square

§4. BOUNDEDNESS

4.1 Theorem. *Let $f : X \rightarrow S$ be a smooth morphism of smooth projective varieties. Assume that $\text{Var}(f) = \dim S$ and X_{gen} is either a minimal surface of general type or a canonically po-*

larized variety. Then for every ample line bundle \mathcal{K} on S there exists a constant $\nu = \nu(n, S, \mathcal{K})$ depending only on n, S , and \mathcal{K} such that $\det f_* \omega_{X/S}^m \otimes \mathcal{L}^{-me(m)r(m)}$ is not ample for any line bundle \mathcal{L} that contains \mathcal{K}^ν . In particular $\det f_* \omega_{X/S}^m \otimes \mathcal{K}^{-\nu me(m)r(m)}$ is not ample.

Proof. Let $n = \dim X$. For a line bundle \mathcal{L} on S such that $\det f_* \omega_{X/S}^m \otimes \mathcal{L}^{-me(m)r(m)}$ is ample, $\omega_{X/S} \otimes f^* \mathcal{L}^{-1}$ is big and 1-ample by (3.2.5). Let \mathcal{K} be an ample line bundle on S and let $\nu = \nu(n, S, \mathcal{K})$ be such that $\mathcal{K}^\nu \otimes \mathcal{E}$ is ample for all $\mathcal{E} \in \mathfrak{I}^i(\mathcal{O}_S)$, $i = 0, \dots, n - k$. Finally let $\mathcal{L}_\nu = \omega_{X/S} \otimes f^*(\mathcal{L}^{-1} \otimes \mathcal{K}^\nu)$. Then $\mathcal{L}_\nu \otimes f^* \mathcal{E}$ is big and 1-ample for all $\mathcal{E} \in \mathfrak{I}^i(\mathcal{O}_S)$, $i = 0, \dots, n - k$. Now if $\mathcal{K}^\nu \subseteq \mathcal{L}$, then $\mathcal{L}_\nu \subseteq \omega_{X/S}$ so

$$H^n(X, \omega_X) = H^n(X, f^* \omega_S \otimes \mathcal{M}) = 0$$

follows by (2.7). Since $H^n(X, \omega_X) \neq 0$, this finishes the proof. \square

4.2 Corollary. *If in addition $\rho(S) = 1$, then there exists a constant $\nu(n, S)$ such that*

$$\deg f_* \omega_{X/S}^m \leq \nu(n, S) me(m)r(m).$$

§5. HYPERBOLICITY

The vanishing theorems obtained in §2 put restrictive constraints on the positivity of ω_X if $\omega_S(\Delta)^{-1}$ is nef. In particular if S has non-positive Kodaira dimension, then one expects that there will be no semi-stable families with small Δ . A more precise formulation of this follows.

§§5.1 S is an abelian variety. This case was already studied in [Kovács97a]. A slight generalization of the results of that article is presented here.

5.1.1 Theorem. *Let A be an abelian variety and $f : X \rightarrow A$ a smooth morphism. Assume that for all $t \in A$, X_t is either a minimal surface of general type or a canonically polarized variety. Then $\text{Var}(f) < \dim A$.*

Proof. Suppose that $\text{Var}(f) = \dim A$. Then ω_X is big and 1-ample by (3.2.2). Since A is an abelian variety, there exists a choice of filtrations on A , such that $\mathfrak{I}^i(\mathcal{O}_A) = \{\mathcal{O}_A\}$. Then by (2.8), $H^n(X, \omega_X) = 0$, a contradiction. \square

From this, using the technique of [Kovács97a, §2], one can easily derive the following:

5.1.2 Corollary. *Let A be an abelian variety and $f : X \rightarrow A$ a smooth morphism. Assume that for all $t \in A$, X_t is either a minimal surface of general type or a canonically polarized variety. Then f is isotrivial.*

§§5.2 S is a projective space or a quadric.

5.2.1 Conjecture. (i) *Let $f : X \rightarrow \mathbb{P}^k$ be a family of varieties of general type and $\Delta \subset \mathbb{P}^k$ an effective divisor such that $X \setminus f^* \Delta \rightarrow \mathbb{P}^k \setminus \Delta$ is smooth. Assume that $\text{Var}(f) = k$. Then $\deg \Delta > k + 1$. Or more generally,*

(ii) *Let $f : X \rightarrow S$ be a family of varieties of general type and $\Delta \subset S$ an effective divisor such that $X \setminus f^* \Delta \rightarrow S \setminus \Delta$ is smooth. Assume that $\text{Var}(f) = \dim S$ (and perhaps that $\rho(S) = 1$ if needed). Then $\omega_S(\Delta)$ is big.*

5.2.1(i) holds for $k = 1$ by [Beauville81], [Migliorini95], [Kovács96], [Kovács97b], [Bedulev-Viehweg00], [Kovács00a], [Kovács00b], and [Viehweg-Zuo01a]. Here we prove this conjecture for semi-stable families of curves and $k \leq 3$ cf. [Viehweg00, 2.6].

5.2.2 Theorem. *Let $f : X \rightarrow \mathbb{P}^k$ be a semi-stable family of curves of genus at least 2 and $\Delta \subset \mathbb{P}^k$ a normal crossing divisor such that $X \setminus f^*\Delta \rightarrow \mathbb{P}^k \setminus \Delta$ is smooth. Assume that $k \leq 3$ and that there exists an effective divisor E on X such that $\omega_X(-E)$ is ample. Then $\deg \Delta > k + 1$.*

Proof. Suppose that $\deg \Delta \leq k + 1$. Then $\omega_{\mathbb{P}^k}(\Delta)^{-1}$ is nef, so $\mathcal{M} = f^*\omega_{\mathbb{P}^k}(\Delta)^{-1} \otimes \omega_X$ contains an ample line bundle, $\mathcal{L} = f^*\omega_{\mathbb{P}^k}(\Delta)^{-1} \otimes \omega_X(-E)$. It is easy to see that $\Omega_{\mathbb{P}^k}^q(\log \Delta) \otimes \omega_{\mathbb{P}^k}^{-1}$ is semipositive for $q = 0, \dots, k$. Since $k \leq 3$, all of these sheaves have rank at most k , so if one chooses the trivial filtration for all locally free sheaves on S , then $\mathcal{L}(f^*\Delta) \otimes f^*\mathcal{E}$ is ample for all $\mathcal{E} \in \mathfrak{X}^i(\mathcal{O}_S)$, $i = 0, 1$. Then by (2.7), $H^n(X, \omega_X) = 0$, a contradiction. \square

5.2.3 REMARK. The reason for the restriction $k \leq 3$ is because this way $\Omega_{\mathbb{P}^k}^q(\log \Delta) \otimes \omega_{\mathbb{P}^k}^{-1}$ has at most rank k for any q . If one finds a filtration of these sheaves such that the associated quotients are semi-positive, then this assumption can be removed.

The restriction to families of curves is for the same reason. To allow higher dimensional fibers one has to find suitable filtrations of tensor products of the sheaves of logarithmic forms.

An argument, similar to the above one, proves 5.2.1(ii) for a smooth quadric hypersurface of dimension at most 3.

5.2.4 Theorem. *Let $Q \subset \mathbb{P}^{k+1}$ be a smooth quadric hypersurface, $f : X \rightarrow Q$ a semi-stable family of curves of genus at least 2 and $\Delta \subset Q$ a normal crossing divisor such that $X \setminus f^*\Delta \rightarrow Q \setminus \Delta$ is smooth. Assume that $k \leq 3$ and that there exists an effective divisor E supported on $\text{supp } f^*\Delta$ such that $\omega_X(-E)$ is ample. Then $\omega_Q(\Delta)$ is ample, i.e., $\deg_{\mathbb{P}^{k+1}} \Delta > 2k$.*

Proof. Suppose that $\omega_Q(\Delta)^{-1}$ is nef or has a non-zero global section. Then $\mathcal{M} = f^*\omega_Q(\Delta)^{-1} \otimes \omega_X$ contains an ample line bundle, $\mathcal{L} = f^*\omega_Q(\Delta)^{-1} \otimes \omega_X(-E)$. The rest of the proof works the same way as above. \square

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