ARE MINIMAL DEGREE RATIONAL CURVES DETERMINED BY THEIR TANGENT VECTORS?

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CONTENTS

1. INTRODUCTION

The study of rational curves of minimal degree has proven to be a very useful tool in Fano geometry. The spectrum of application covers topics such as deformation rigidity, stability of the tangent sheaf, classification problems or the existence of non-trivial finite morphisms between Fano manifolds.

In this paper we will consider the situation where X is a projective variety, which is covered by rational curves, e.g. a Fano manifold over $\mathbb{C}$. A typical example of that is $\mathbb{P}^n$, which is covered by lines. The key point of many applications of minimal degree rational curves is showing that they are similar to lines in certain respects. For instance, one may ask:

Question 1.1. *Under what conditions does there exist a unique minimal degree rational curve containing two given points?*

This question found a sharp answer in [Keb02] —see [CMSB00] and [Keb01a] for a number of applications. The argument used there is based on a criterion of Miyaoka, who was the first to observe that if the answer to the question is "No", then a lot of minimal degree curves are singular. We refer to [Kol96, Prop. V.3.7.5] for a precise statement.

As an infinitesimal analogue of this question one may ask the following:

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Question 1.2. *Are there natural conditions which guarantee that a minimal degree rational curve is uniquely determined by a tangent vector?*

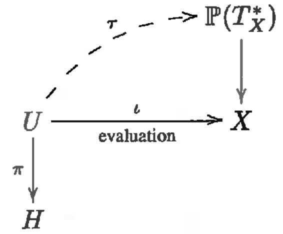
Although a definite answer to the latter question would be as interesting as one to the former, it seems that Question 1.2 has not been studied before. This paper is therefore a first attempt in that direction. We give a criterion which parallels Miyaoka's approach.

To formulate our setup more precisely, let $\mathrm{RatCurves}^n(X)$ be the space of rational curves on X and pick an irreducible component $H \subset \mathrm{RatCurves}^n(X)$ such that the following holds:

(1) the rational curves associated with H dominate X,
(2) the degree of the curves associated with H is minimal among all irreducible components of $\mathrm{RatCurves}^n(X)$ which satisfy (1).

The component H is often called a "maximal dominating family of rational curves of minimal degree". We refer the reader to [Kol96] for a thorough, but rather technical discussion of $\mathrm{RatCurves}^n(X)$.

Let $U \subset H \times X$ be the universal family. By construction, all fibers of the first projection $\pi : U \to H$ are irreducible and generically reduced rational curves. The tangent map of the second projection $\iota : U \to X$, restricted to the relative tangent sheaf $T_{U/H}$, gives rise to a rational map τ:



It has been shown in [Keb02] that τ is generically finite. Examples of rationally connected varieties, however, seem to suggest that the tangent map τ is generically injective for a large class of varieties. Our main result supports this claim.

Theorem 1.1. *Let X be a projective variety over an algebraically closed field k and $H \subset \mathrm{RatCurves}^n(X)$ a proper, covering family of rational curves. Assume that either $\mathrm{char}(k) = 0$, or that there exists a line bundle $L \in \mathrm{Pic}(X)$ such that the intersection number $L.\ell$ of L with a curve $\ell \in H$ does not divide $\mathrm{char}(k)$.*

Then τ is generically injective, unless H contains a cuspidal curve.

For complex projective manifolds we have a stronger statement.

Theorem 1.2. *Let X be a smooth projective variety over the field of complex numbers and let $H \subset \mathrm{RatCurves}^n(X)$ be a dominating family of rational curves such that the subfamily*

$$H_x := \{\ell \in H \mid x \in \ell\} \subset H$$

is proper for all points $x \in X$, outside a subvariety $S \subset X$ of codimension at least 2.

Then τ is generically injective, unless the curves associated with the closed subfamily $H^{cusp} \subset H$ of cuspidal curves dominate X, and the subvariety

$$D := \{x \in X \mid \exists \ell \in H^{cusp} : \ell \text{ has a cuspidal singularity at } x\},$$

where curves have cuspidal singularities, has codimension 1.

Remark 1.3. It is known ([Kol96, Chap. II, Prop. 2.14]) that the family H_x is proper if H is a maximal dominating family of rational curves of minimal degree and if x is a general point. The assumption that H_x is proper for all points outside a set of codimension 2, however, is restrictive.

It is also known that H is proper, e.g., if there exists a line bundle $L \in \mathrm{Pic}(X)$ that intersects a curve $\ell \in H$ with multiplicity $L.\ell = 1$.

Remark 1.4. Let X be a projective contact manifold over $\mathbb{C}$, different from the projective space. It has been shown in [Keb01b] that X is covered by a compact family of rational curves H such that for a general point x, all curves associated with points in H_x are smooth. Thus, the assumptions of Theorem 1.2 are satisfied, and τ is known to be generically injective. This has been shown previously in [Keb01c] using rather involved argument which heavily relies on the contact geometry.

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After the main part of this paper was written, J.-M. Hwang has informed the authors that he together with N. Mok has shown a similar, but stronger, statement using entirely different methods.

2. Dubbies

Throughout the proofs of Theorems 1.1 and 1.2, which we give in Section 3, we will assume that the tangent map τ is not generically injective. This is to say that we are assuming that X contains a large number of pairs (ℓ_t, ℓ_s) of minimal rational curves which intersect tangentially in at least one point. A detailed study of these pairs will be given in the present chapter.

Definition 2.1. *A* dubby *is a reduced, reducible curve, isomorphic to the union of a line and smooth conic in $\mathbb{P}^2$ intersecting tangentially in a single point.*

Remark 2.1. The definition may suggest at first glance that one component of a dubby is special in that it has a higher degree than the other. We remark that this is not so. A dubby does not come with a natural polarization. In fact, there exists an involution in the automorphism group that swaps the irreducible components.

Later we will need the following estimate for the dimension of the space of global sections of a line bundle on a dubby. Let $\ell = \ell_1 \cup \ell_2$ be a dubby and $L \in \mathrm{Pic}(\ell)$ a line bundle. We say that L has type (d_1, d_2) if the restriction of L to the irreducible components ℓ_1 and ℓ_2 has degree d_1 and d_2, respectively.

Lemma 2.2. *Let ℓ be a dubby and $L \in \mathrm{Pic}(\ell)$ a line bundle of type (d_1, d_2). Then $h^0(\ell, L) \geq d_1 + d_2$.*

Proof. By assumption, we have that $L|_{\ell_i} \simeq \mathcal{O}_{\mathbb{P}^1}(d_i)$. Let $\ell_1.\ell_2$ be the scheme theoretic intersection of ℓ_1 and ℓ_2, $\iota^i : \ell_i \to \ell$ the natural embedding, and $L_i = \iota^i_*(L|_{\ell_i})$ for $i = 1, 2$. Then one has the following short exact sequence:

$$0 \to L \to L_1 \oplus L_2 \to \mathcal{O}_{\ell_1.\ell_2} \to 0.$$

This implies that $h^0(\ell, L) \geq \chi(L) = \chi(L_1) + \chi(L_2) - \chi(\mathcal{O}_{\ell_1.\ell_2}) = d_1 + d_2$. □

2.1. **The identification of the components of a dubby.** In the situation which is of interest to us, let $L \in \text{Pic}(X)$ be an ample line bundle, and assume that $\ell = \ell_1 \cup \ell_2 \subset X$ is a dubby where both components are members of the same connected family H of minimal rational curves. In particular, $L|_\ell$ will be of type (m, m), where $m > 0$. Remarkably, the line bundle L induces a canonical identification of the two components ℓ_1 and ℓ_2, at least when m is coprime to the characteristic of the base field k. Over the field of complex numbers, the idea of construction is the following: Fix a trivialization $t : L|_U \to \mathcal{O}_U$ of L on an open neighborhood U of the intersection point $\{z\} = \ell_1 \cap \ell_2$. Given a point $x \in \ell_1 \setminus \ell_2$, let $\sigma_1 \in H^0(\ell_1, L|_{\ell_1})$ be a non-zero section which vanishes in x with multiplicity m. Then there exists a *unique* section $\sigma_2 \in H^0(\ell_2, L|_{\ell_2})$ with the following properties:

(1) The section σ_2 vanishes on exactly one point $y \in \ell_2$.
(2) The sections σ_1 and σ_2 agree on the intersection of the components:

$$\sigma_1(z) = \sigma_2(z)$$

(3) The differentials of σ_1 and σ_2 agree on z:

$$\vec{v}(t \circ \sigma_1) = \vec{v}(t \circ \sigma_2)$$

for all non-vanishing tangent vectors $\vec{v} \in T_{\ell_1} \cap T_{\ell_2}$.

The map that associates x to y gives the identification of the components and does not depend on the choice of t.

In the remaining part of the present chapter, we will give a proper construction of the identification morphism which also works in the relative setup and in arbitrary characteristic.

2.2. **Bundles of dubbies.** For the proof of the main theorems we will need to consider bundles of dubbies, i.e. morphisms where each scheme-theoretic fiber is isomorphic to a dubby. On first reading, the reader might want to skip the rather lengthy proof and go directly to Chapter 3.

Proposition 2.3. *Let $\lambda : \Lambda \to B$ be a family of dubbies over a normal base B and assume that Λ is not irreducible. Then it has exactly two irreducible components Λ_1 and Λ_2, both $\mathbb{P}_1$-bundles over B.*

Assume further that there exists a line bundle $L \in \text{Pic}(\Lambda)$ whose restriction to a λ-fiber has type (m, m), where m is positive and relatively prime to $\text{char}(k)$. Then there exists an identification of Λ_1 and Λ_2. More precisely, there exists a morphism γ from Λ to a $\mathbb{P}_1$-bundle over B whose restriction to each of the two components Λ_1 and Λ_2 is an isomorphism.

Proof of Proposition 2.3, the irreducible components of Λ. The map λ is flat because all its scheme-theoretic fibers are isomorphic. Let $\Lambda_1 \subset \Lambda$ be one of the irreducible components. It is easy to see that if $x \in \Lambda_1$ is a general point, then Λ_1 contains the (unique) irreducible component of $\ell_{\lambda(x)} := \lambda^{-1}\lambda(x)$ which contains x. Since λ is proper and flat, $\lambda(\Lambda_1) = B$. Hence Λ_1 contains one of the irreducible components of ℓ_b for all $b \in B$. Repeating the same argument with another irreducible component, Λ_2, one finds that it also contains one of the irreducible components of ℓ_b for all $b \in B$. However, they cannot contain the same irreducible component for any $b \in B$: In fact, if they contained the same component of ℓ_b for infinitely many points $b \in B$, then they would agree. On the other hand, if they contained the same component of ℓ_b for finitely many points $b \in B$, then Λ would have an irreducible component that does not dominate B. This, however, would contradict the flatness of λ. Hence $\Lambda_1 \cup \Lambda_2 = \Lambda$. They are both $\mathbb{P}_1$-bundles over B by [Kol96, Thm. II.2.8.1]. □

Proof of Proposition 2.3, the intersection of Λ_1 *and* Λ_2. We have seen above that Λ_1 and Λ_2 are $\mathbb{P}_1$-bundles over B. Let $\Sigma = \Lambda_1 \cap \Lambda_2 \subset \Lambda_1$ be the scheme-theoretic intersection, and $\Sigma_{\text{red}} \subset \Sigma$ the associated reduced subscheme. By assumption, the subscheme Σ is isomorphic to the first infinitesimal neighborhood of Σ_{red}. Thus, if $\mathcal{J}_1 \subset \mathcal{O}_{\Lambda_1}$ is the ideal sheaf of $\Sigma_{\text{red}} \subset \Lambda_1$, then $\mathcal{J}_1^2$ is the ideal sheaf of $\Sigma \subset \Lambda_1$.

In order to study Pic(Λ), it will be necessary to express the sheaf $\mathcal{O}_\Sigma^*$ of invertible functions on Σ purely in terms of the reduced subvariety Σ_{red}. Recall from [Har77, III. Ex.4.6] that there exists a short exact sequence of sheaves of Abelian groups [1],

$$0 \longrightarrow \underbrace{\mathcal{J}_1/\mathcal{J}_1^2}_{\mathcal{N}^\vee_{\Sigma_{\text{red}}|\Lambda_1}} \xrightarrow{\alpha} \mathcal{O}_\Sigma^* \xrightarrow{\beta} \mathcal{O}_{\Sigma_{\text{red}}}^* \longrightarrow 1. \tag{2.1}$$

where β is the canonical restriction map and α is given by

$$\begin{array}{rccc} \alpha: & (\mathcal{J}_1/\mathcal{J}_1^2, +) & \to & \mathcal{O}_\Sigma^* \\ & f & \mapsto & 1+f. \end{array}$$

□

Proof of Proposition 2.3, Mayer-Vietoris sequence for $\mathcal{O}^*$. Let $\pi : \tilde{\Lambda} \to \Lambda$ be the normalization morphism. The variety $\tilde{\Lambda}$ is isomorphic to the disjoint union of the two $\mathbb{P}_1$-bundles, Λ_1 and Λ_2. The aim of this subsection is to establish the existence of the following morphism between short exact sequences of sheaves of Abelian groups.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}_\Lambda^* & \longrightarrow & \pi_*\mathcal{O}_{\tilde{\Lambda}}^* & \longrightarrow & \mathcal{O}_\Sigma^* & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{O}_{\Sigma_{\text{red}}}^* & \xrightarrow{\text{diagonal}} & \mathcal{O}_{\Sigma_{\text{red}}}^* \times \mathcal{O}_{\Sigma_{\text{red}}}^* & \xrightarrow{\text{quotient}} & \mathcal{O}_{\Sigma_{\text{red}}}^* & \longrightarrow & 1. \end{array} \tag{2.2}$$

Although the first line is classically known as the Mayer-Vietoris sequence for $\mathcal{O}^*$ (see e.g. [Eis95, Chap. 11, Ex. 11.15]), we discuss it here briefly for the reader's convenience.

Recall that $\pi_*\mathcal{O}_{\tilde{\Lambda}}^* \cong \mathcal{O}_{\Lambda_1}^* \times \mathcal{O}_{\Lambda_2}^*$. Thus, if an open set $U \subset \Lambda$ and a section $s \in H^0(U, \pi_*\mathcal{O}_{\tilde{\Lambda}}^*)$, $s = (s_1, s_2) \in H^0(U, \mathcal{O}_{\Lambda_1}^*) \times H^0(U, \mathcal{O}_{\Lambda_2}^*)$ is given, then s comes from a section in $H^0(U, \mathcal{O}_\Lambda^*)$ if and only if s_1 and s_2 agree on the scheme-theoretic intersection $\Sigma = \Lambda_1 \cap \Lambda_2$. Consequently $\mathcal{O}_\Lambda^*$ is exactly the kernel of the map

$$\begin{array}{ccc} \pi_*\mathcal{O}_{\tilde{\Lambda}}^* & \to & \mathcal{O}_\Sigma^* \\ (s_1, s_2) & \mapsto & {}^{s_1}/_{s_2}|_\Sigma. \end{array}$$

The existence of the first short exact sequence is thus shown. The vertical arrows are simply the natural restriction morphisms. □

Proof of Proposition 2.3, existence of a line bundle of type (1,1). As the next step, we need to find a line bundle $\tilde{L} \in \text{Pic}(\Lambda)$ whose restriction to the fibers of λ is of type $(1,1)$. First note that Σ_{red} gives a section of λ, $\sigma\colon B \to \Lambda$, such that $\lambda \circ \sigma = \text{id}_B$

[1] The sequence (2.1) is actually split, but we do not use this fact here.

and $\sigma \circ \lambda|_{\Sigma_{\rm red}} = \mathrm{id}_{\Sigma_{\rm red}}$. Then the long exact sequences associated with the short exact sequences (2.2) can be written the following way:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \mathrm{Pic}(\Lambda) & \longrightarrow & H^1(\Lambda, \pi_* \mathcal{O}^*_{\tilde\Lambda}) & \longrightarrow & H^1(\Sigma, \mathcal{O}^*_\Sigma) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle \sigma^*} & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & \mathrm{Pic}(B) & \xrightarrow{\text{diagonal}} & \mathrm{Pic}(B) \times \mathrm{Pic}(B) & \xrightarrow{\text{quotient}} & H^1(\Sigma_{\rm red}, \mathcal{O}^*_{\Sigma_{\rm red}}) & \longrightarrow \cdots \end{array}$$

Since π is finite, we have that

$$H^1(\Lambda, \pi_* \mathcal{O}^*_{\tilde\Lambda}) \simeq \mathrm{Pic}(\tilde\Lambda) \simeq \mathrm{Pic}(\Lambda_1) \times \mathrm{Pic}(\Lambda_2) \simeq \mathrm{Pic}(B) \times \mathrm{Pic}(B) \times \mathbb{Z}^2$$

Using this, and sequence (2.1) from above, we obtain an exact sequence of morphisms between the kernels as follows:

$$\cdots \longrightarrow \ker(\sigma^*) \xrightarrow{\ \phi\ } \mathbb{Z}^2 \xrightarrow{\ \psi\ } H^1(\Sigma_{\rm red}, \mathcal{N}^\vee_{\Sigma_{\rm red}|\Lambda_1}) \longrightarrow \cdots$$

In order to find $\bar L$, set

$$L' := L \otimes \lambda^* \sigma^* L^{-1}$$

It follows immediately from the construction that $L' \in \ker(\sigma^*)$ and that $\phi(L') = (m, m)$. In particular, we have that $\psi(m,m) = m \cdot \psi(1,1) = 0$. Since $H^1(\Sigma_{\rm red}, \mathcal{N}^\vee_{\Sigma_{\rm red}|\Lambda_1})$ is the additive group of a k-vector space and since m and $\mathrm{char}(k)$ are coprime, this implies that $(1,1) \in \ker(\psi) = \mathrm{im}(\phi)$. The existence of $\bar L$ is therefore established. □

Proof of Proposition 2.3, the identification of Λ_1 *and* Λ_2. Let $b \in B$ be any (closed) point and $\ell_b = \ell_b^1 \cup \ell_b^2$ the associated λ-fiber. Consider the restriction morphisms

$$r_b^i : H^0(\ell_b, \bar L|_{\ell_b}) \to H^0(\ell_b^i, \bar L|_{\ell_b^i}) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)).$$

We claim that the morphism r_b^i is an isomorphism for all $b \in B$ and $i \in \{1,2\}$. If this claim holds, then it implies that $\bar L|_{\ell_b}$ is generated by global sections, and thus gives a morphism $\gamma_b : \ell_b \to \mathbb{P}^1$, whose restriction $\gamma_b|_{\ell_b^i}$ to any of the two components is an isomorphism. Then the natural morphism

$$\gamma : \Lambda \to \mathbb{P}(\lambda_* \bar L)$$

restricted to either Λ_1 or Λ_2 is an isomorphism giving the required identification between Λ_1 and Λ_2.

To finish the proof of Proposition 2.3, let us prove the above claim. The roles of r_b^1 and r_b^2 are symmetric, so it is enough to prove the claim for r_b^1. First note that $h^0(\ell_b, \bar L|_{\ell_b}) \geq 2$ by Lemma 2.2. It is then sufficient to prove that r_b^1 is injective. Let $s \in \ker(r_b^1) \subset H^0(\ell_b, \bar L|_{\ell_b})$. In order to show that $s = 0$ it is enough to show that $r_b^2(s) = 0$. But $r_b^2(s)$ is a section in $H^0(\ell_b^2, \bar L|_{\ell_b^2})$ that vanishes on the scheme-theoretic intersection $\ell_b^1 \cap \ell_b^2$. The length of this intersection is two, but any non-zero section in $H^0(\ell_b^2, \bar L|_{\ell_b^2}) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ has a unique zero of order one, hence $r_b^2(s)$ must be zero, and so the proof of Proposition 2.3 is finished. □

Corollary 2.4. *Let ℓ be a dubby. Then its Picard group fits into the following short exact sequence:*

$$0 \longrightarrow \mathbb{G}_a \longrightarrow \mathrm{Pic}(\ell) \xrightarrow{\text{type}} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0.$$

In particular, line bundles of type $(0,0)$ *on* ℓ *are parametrized by* $\mathbb{G}_a$.

Proof. Let $B = \operatorname{Spec} k$ and $\Lambda = \ell$. Then the long exact sequence associated to (2.2) implies the statement. Note that this part of Proposition 2.3 does not use the existence of L, so it holds for all dubbies. □

3. Proof of the Main Theorem

3.1. **Setup.** Throughout the present section we maintain the notation of the introduction and suppose that either the assumptions of Theorem 1.1 or those of Theorem 1.2 hold. Furthermore, we will assume throughout that the tangent map τ is not generically injective. We will fix a curve $\ell \subset X$ that corresponds to a general point in H.

Lemma 3.1. *For each point $x \in \ell$, the subfamily*

$$H_x := \{\ell' \in H \mid x \in \ell'\} \subset H$$

is proper.

Proof. There is nothing to prove if we are in the setup of Theorem 1.1 where H is assumed to be proper.

In the setup of Theorem 1.2, where the subfamily H_x is assumed to be proper only for those points $x \in X$ that are not contained in a subvariety S of codimension two, the assertion follows from the fact that on a projective manifold over the complex number field a general curve does not intersect a set S of codimension 2 [Kol96, Chapt. II, Prop. 3.7 and Thm. 3.11]. □

Corollary 3.2. *If ℓ is a general member of H, then the subfamily,*

$$H_\ell := \{\ell' \in H \mid \ell \cap \ell' \neq \emptyset\} \subset H$$

is proper.

Proof. Let $Z_\ell = \pi^{-1}(H_\ell) \cap \iota^{-1}(\ell) \subset U$. The restriction of ι gives a morphism $\zeta : Z_\ell \to \ell$. For any $x \in \ell$, $\zeta^{-1}(x) = \iota^{-1}(x) \cap \pi^{-1}(H_x)$ is a closed subset of $\pi^{-1}(H_x)$, and since $\pi^{-1}(H_x)$ is proper by Lemma 3.1, so is $\zeta^{-1}(x)$. Therefore we conclude that Z_ℓ is proper, since ℓ is. Finally, $H_\ell = \pi(Z_\ell)$, so the statement is proven. □

Next, consider the subvariety $\mathbb{P}(T_\ell^*) \subset \mathbb{P}(T_X^*)$ and let $\hat{B}_0, \ldots, \hat{B}_k \subset U$ be the curves that dominate $\mathbb{P}(T_\ell^*)$ via the generically finite tangent morphism τ. One of these curves, say $\hat{B}_0$, is the fiber of the natural projection $\pi : U \to H$. Since we assume that τ is not generically injective and since the restriction $\iota|_{\hat{B}_0}$ is injective, there exists a further component $\hat{B}_1$, which is not a fiber of π. Set

$$B := \pi(\hat{B}_1).$$

By Corollary 3.2 the curve B is proper in H.

The assumptions imply that for every general point $x \in \ell$, there exists a point $b \in B$ with fiber $\ell_b := \pi^{-1}(b)$ and a point $s \in \tau^{-1}(\mathbb{P}(T_\ell^*)) \cap \ell_b$ which is not a fundamental point of τ, such that

1. The curve ℓ_b is smooth at s.
2. The scheme-theoretic preimage $(\iota|_{\ell_b})^{-1}(\ell)$ contains the point s with multiplicity at least two.

Formulated in more geometric terms, items (1) and (2) guarantee that the rational curves ℓ_b and ℓ intersect tangentially in x. Figure 3.1 depicts the setup.

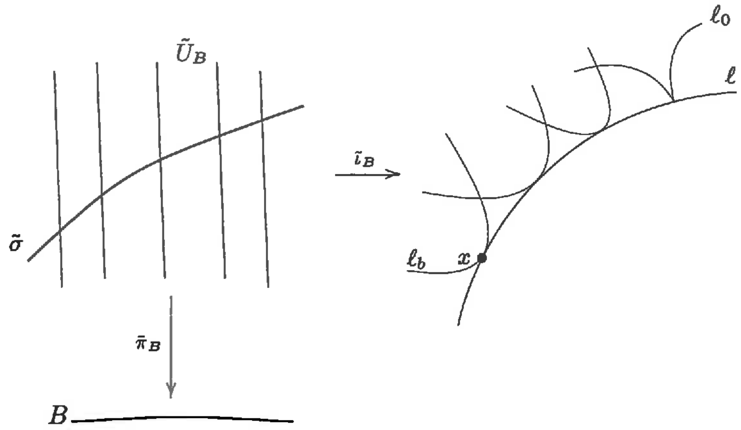
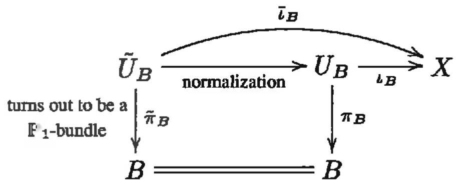


FIGURE 3.1. Setup of the proof

Let $U_B := \pi^{-1}(B)$ be the universal family over B, and let $\tilde U_B$ be its normalization. We obtain a diagram as follows.



After a base change, replacing B by a finite cover if necessary, we can (and will henceforth) assume that the following holds:

(3) The curve B is normal and therefore smooth.
(4) The bundle space $\tilde U_B$ is a $\mathbb{P}_1$-bundle over B in the sense that for every closed point $b \in B$, the scheme-theoretic fiber $\tilde\pi_B^{-1}(b)$ is isomorphic to the projective line. We refer to [Kol96, Thm. II.2.8.1] for a proof of this fact. Note that even though the proof is straightforward in characteristic zero, it is somewhat involved in finite characteristic. See [Keb01d, Sect. 1.5] for an elementary worked example.
(5) The scheme-theoretic preimage $\tilde\iota_B^{-1}(\ell)$ contains a section $\tilde\sigma \subset \tilde U_B$ over B such that
  (a) the differential of the morphism $(\tilde\pi_B \times \tilde\iota_B) : \tilde U_B \to B \times X$ has rank two generically along $\tilde\sigma$,
  (b) the scheme-theoretic preimage $\tilde\iota_B^{-1}(\ell)$ is not reduced along $\tilde\sigma$.

Of course, items (5a) and (5b) parallel (1) and (2) above. Note that by the choice of B, the image $\tilde\iota_B(\sigma)$ is the entire curve ℓ, and not just a point on ℓ.

Finally, let $\rho \subset B \times \ell$ be the image $\rho = (\tilde\pi_B \times \tilde\iota_B)(\tilde\sigma)$, $\nu : \tilde\ell \to \ell$ the normalization of ℓ, and $\tilde\rho \subset B \times \tilde\ell$ the strict transform ρ.

3.2. **The triviality of $\tilde U_{B^0}$.** Let $B^0 \subset B$ be the maximal, Zariski-open subset such that the following holds;

(1) The differential of the morphism $(\tilde\pi_B \times \tilde\iota_B) : \tilde U_B \to B \times X$ has rank two along $\tilde\sigma$ over B^0.

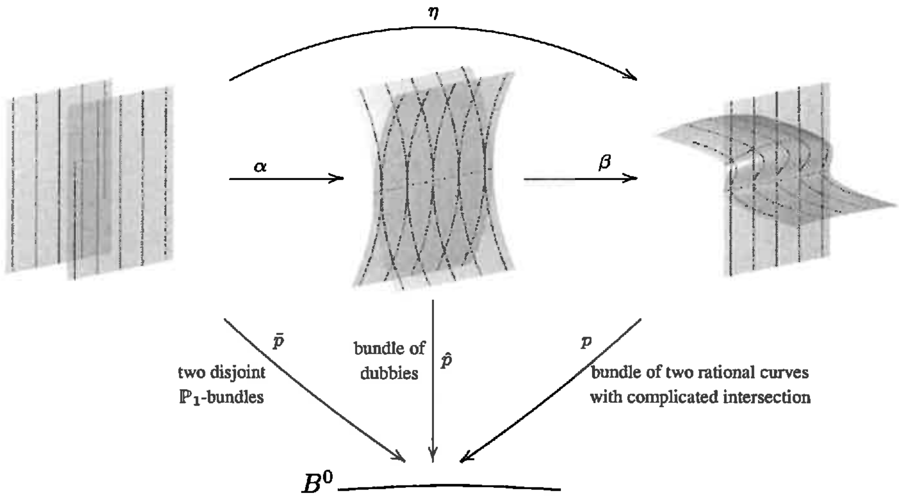


FIGURE 3.2. A partial resolution of singularities

(2) The differential of the morphism $(\mathrm{id}_B \times \nu) : B \times \tilde{\ell} \to B \times \ell$ has rank two along $\tilde{\rho}$ over B^0.

Roughly speaking, $B^0 \subset B$ is the set where the cuspidal singular points of the π_B-fibers stay off the distinguished section $\tilde{\sigma}$.

The aim of this section is to show that there exists a trivialization of the $\mathbb{P}_1$-bundle $\tilde{U}^0_B := \tilde{\pi}_B^{-1}(B^0)$. The strategy of the proof is the following: Consider the reducible space

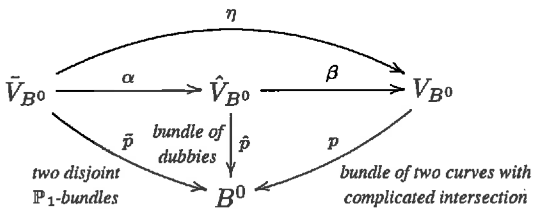
$$V_{B^0} := U_{B^0} \cup \{B^0 \times \ell\}$$

and let $p : V_{B^0} \to B^0$ be the natural projection. If $b \in B^0$ is any point, then the fiber $p^{-1}(b) \subset V_{B^0}$ is a union of ℓ and another rational curve, ℓ_b. These two curves intersect tangentially in one point and may have complicated intersection otherwise.

In the simplest possible case, where the two curves intersect tangentially in exactly one point, the fiber $p^{-1}(b)$ would be isomorphic to a dubby. The space V_{B^0} would be a bundle of dubbies, and we could employ Proposition 2.3 from page 4 in order to identify the two components U_{B^0} and $\{B^0 \times \ell\}$ of V_{B^0}.

In practice, however, we cannot assume that V_{B^0} is a bundle of dubbies. In this situation we note that the normalization $\tilde{V}_{B^0}$ of V_{B^0} is a disjoint union of two $\mathbb{P}_1$-bundles over B^0 which are isomorphic to $\tilde{U}_{B^0}$ and to $\{B^0 \times \tilde{\ell}\}$, respectively. The core idea is to construct a partial resolution of singularities, $\hat{V}_{B^0}$, which factors the normalization map and has the structure of a bundle of dubbies. The technically correct formulation of this approach is the following:

Proposition 3.3. *After changing the base, if necessary, the normalization morphism* η *factors as follows:*



where

(1) *The variety* $\hat{V}_{B^0}$ *is a bundle of dubbies in the sense that for every closed point* $b \in B^0$, *the fiber* $\hat{p}^{-1}(b)$ *is isomorphic to a dubby.*
(2) *The bundle space* $\hat{V}_{B^0}$ *decomposes into two irreducible components* $\hat{V}_{B^0} = \hat{V}_{B^0,1} \cup \hat{V}_{B^0,2}$ *where* $\hat{V}_{B^0,1}$ *is canonically isomorphic to* $\tilde{U}_{B^0}$ *and* $\hat{V}_{B^0,2} = \beta^{-1}(B^0 \times \ell)$ *is isomorphic to the trivial* $\mathbb{P}_1$*-bundle* $B^0 \times \tilde{\ell}$.
(3) *The singular locus of* $\hat{V}_{B^0}$ *satisfies:* $(\hat{V}_{B^0})_{\mathrm{Sing}} = \hat{V}_{B^0,1} \cap \hat{V}_{B^0,2}$, *a section of* $\hat{p}$.

Proof of Proposition 3.3, construction of $\hat{V}_{B^0}$ *and* $\tilde{V}_{B^0}$. First we will construct the space $\hat{V}_{B^0}$. Because the normalization map η is an affine morphism, it seems appropriate to construct a suitable subsheaf $\mathcal{A} \subset \eta_* \mathcal{O}_{\tilde{V}_{B^0}}$, which is a coherent sheaf of $\mathcal{O}_{V_{B^0}}$-modules and set $\hat{V}_{B^0} := \mathbf{Spec}(\mathcal{A})$.

To this end, we will find an identification of the first infinitesimal neighborhoods $\tilde{\rho}^1$ of $\tilde{\rho} \subset B^0 \times \tilde{\ell}$ and $\tilde{\sigma}^1$ of $\tilde{\sigma} \subset \tilde{U}_{B^0}$. In fact, since η is a separable morphism, it follows directly from the choice of B^0 that both $\tilde{\sigma}^1$ and $\tilde{\rho}^1$ map isomorphically onto their scheme-theoretic images $\eta(\tilde{\sigma}^1)$ and $\eta(\tilde{\rho}^1)$, which are both supported on ρ, have relative length 2 over B^0 and are subschemes of $B \times \ell \subset B \times X$. Furthermore, since $B^0 \times \ell$ is of relative dimension one, it follows that the images $\eta(\tilde{\sigma}^1)$ and $\eta(\tilde{\rho}^1)$ agree along the open set $(B^0 \times \ell_{\mathrm{Reg}}) \cap \rho \subset \rho$. Thus, the separatedness of the relative Hilbert-scheme implies that $\eta(\tilde{\sigma}^1) \cong \eta(\tilde{\rho}^1)$ are isomorphic over B^0. We obtain the desired identification of first infinitesimal neighborhoods

$$\gamma^1 : \tilde{\rho}^1 \to \tilde{\sigma}^1.$$

Let

$$i_\rho : \tilde{\rho}^1 \to B^0 \times \tilde{\ell} \subset \tilde{V}_{B^0} \quad \text{and} \quad i_\sigma : \tilde{\sigma}^1 \to \tilde{U}_{B^0} \subset \tilde{V}_{B^0}$$

be the inclusion maps and consider the sheaf morphism

$$\varphi := \gamma^{1\#} \circ i_\sigma^\# - i_\rho^\# : \mathcal{O}_{\tilde{V}_{B^0}} \to \mathcal{O}_{\tilde{\rho}^1}.$$

The sheaf

$$\mathcal{A} := \eta_* \ker(\varphi)$$

is thus a coherent sheaf of $\mathcal{O}_{V_{B^0}}$-modules. As mentioned above, define $\hat{V}_{B^0} := \mathbf{Spec}(\mathcal{A})$ and let $\tilde{V}_{B^0}$ be the normalization of $\hat{V}_{B^0}$. The existence of the morphisms α and β follow from the construction and we let $\hat{p} = p \circ \beta$ and $\tilde{p} = p \circ \beta \circ \alpha$. It remains to show that $\hat{V}_{B^0}$ satisfies properties (1)–(3). □

Proof of Proposition 3.3, property (1). Let $b_0 \in B^0$ be any closed point. After shrinking B^0, if necessary, we can assume that

(1) the curve B^0 is affine, say $B^0 \cong \operatorname{Spec} R$, and
(2) the $\mathbb{P}_1$-bundle $\tilde{U}_{B^0}$ is trivial. To see this, note that $\eta_* \mathcal{O}_{\tilde{U}_{B^0}}(\sigma)$ is locally free and $\tilde{U}_{B^0}$ is naturally identified with $\mathbb{P}\left(\eta_* \mathcal{O}_{\tilde{U}_{B^0}}(\sigma)\right)$.

Shrinking B^0 further, if necessary, we can thus find two sections, $\tilde{\sigma}_\infty \subset \tilde{U}_{B^0}$ and $\tilde{\rho}_\infty \subset \{B^0 \times \tilde{\ell}\}$ which are disjoint from both $\tilde{\sigma}$, $\tilde{\rho}$, and from the preimage $\eta^{-1}(V_{B^0,\mathrm{Sing}})$ of the singular locus. We can thus find homogeneous bundle coordinates $[x_0 : x_1]$ on $\tilde{U}_{B^0}$ and $[y_0 : y_1]$ on $B^0 \times \tilde{\ell}$ such that

$$\tilde{\sigma} = \{([x_0 : x_1], b) \in \tilde{U}_{B^0} \mid x_0 = 0\} \qquad \tilde{\sigma}_\infty = \{([x_0 : x_1], b) \in \tilde{U}_{B^0} \mid x_1 = 0\}$$
$$\tilde{\rho} = \{([y_0 : y_1], b) \in \{B^0 \times \tilde{\ell}\} \mid y_0 = 0\} \quad \tilde{\rho}_\infty = \{([y_0 : y_1], b) \in \{B^0 \times \tilde{\ell}\} \mid y_1 = 0\}.$$

If we set

$$\tilde{U}_0 := \tilde{V}_{B^0} \setminus (\tilde{\sigma}_\infty \cup \tilde{\rho}_\infty),$$

then the image $U_0 := \eta(\tilde{U}_0)$ is affine, and we can write the relevant modules as

$$\mathcal{O}_{\tilde{V}_{B^0}}(\tilde{U}_0) \cong R \otimes (k[x_0] \oplus k[y_0])$$
$$\mathcal{O}_{\tilde{\rho}^1}(\tilde{U}_0) \cong R \otimes k[x_0, y_0]/(x_0 - y_0, y_0^2).$$

Adjusting the bundle coordinates, if necessary, we can assume that the identification morphism $\gamma^{1\#} : \mathcal{O}_{\tilde{\sigma}^1}(\tilde{U}_0) \to \mathcal{O}_{\tilde{\rho}^1}(\tilde{U}_0)$ is written as

$$\begin{array}{rccc} \gamma^{1\#}: & R \otimes k[x_0]/(x_0^2) & \to & R \otimes k[y_0]/(y_0^2) \\ & r \otimes x_0 & \mapsto & r \otimes y_0. \end{array}$$

In this setup, we can find the morphism φ explicitly:

$$\begin{array}{rccc} \varphi: & R \otimes (k[x_0] \oplus k[y_0]) & \to & R \otimes k[x_0, y_0]/(x_0 - y_0, y_0^2) \\ & r \otimes (f, g) & \mapsto & r \otimes (f - g). \end{array}$$

As an R-algebra, $\ker(\varphi)(\tilde{U}_0)$ is therefore generated by the two elements $u := 1_R \otimes (x_0, y_0)$ and $v := 1_R \otimes (x_0^2, 0)$ which satisfy the relation $v(u^2 - v) = 0$. Thus

$$\ker(\varphi)(\tilde{U}_0) = R \otimes k[u, v]/(v(u^2 - v)).$$

In other words, $\beta^{-1}(U_0)$ is a bundle of two affine lines, meeting tangentially in a single point.

It follows directly from the construction of the sheaf $\mathcal{A}$ that α is isomorphic away from $\tilde{\sigma} \cup \tilde{\rho}$. The curve $\hat{p}^{-1}(b_0)$ is therefore smooth outside of $\hat{p}^{-1}(b_0) \cap \beta^{-1}(U_0)$, and it follows from Zariski's main theorem that $\hat{p}^{-1}(b_0)$ is a dubby indeed. This ends the proof of property (1). □

Proof of Proposition 3.3, property (2). By Proposition 2.3, the bundle $\hat{V}_{B^0}$ of dubbies decomposes as $\hat{V}_{B^0} = \hat{V}_{B^0,1} \cup \hat{V}_{B^0,2}$, where $\hat{V}_{B^0,i} \to B^0$ is a $\mathbb{P}_1$-bundle for $i = 1, 2$. In particular, $\hat{V}_{B^0,1}$ and $\hat{V}_{B^0,2}$ are smooth. The restriction of α to the components of $\tilde{V}_{B^0}$ gives an isomorphism onto respective components of $\hat{V}_{B^0}$ outside $\tilde{\sigma}$ and $\tilde{\rho}$. Since the targets are smooth over B and α is a B morphism, this means that the restriction of α to both components is an isomorphism. □

Proof of Proposition 3.3, property (3). On one hand, it is clear that α is isomorphic away from $\tilde{\sigma} \cup \tilde{\rho}$. On the other hand, the two (smooth) components $\hat{V}_{B^0,1}$ and $\hat{V}_{B^0,2}$ meet in a section over B^0. Together, these yield the claim. □

Corollary 3.4. *After a finite base change, if necessary, the bundle $\tilde{U}_{B^0}$ is trivial. There exists a canonical identification $\tilde{U}_{B^0} \cong B^0 \times \tilde{\ell}$ such that the second projection $\pi_2 : \tilde{U}_{B^0} \to \tilde{\ell}$ satisfies the following equality:*

$$\nu \circ \pi_2|_{\tilde{\sigma}} = \tilde{\iota}_{B^0}|_{\tilde{\sigma}}.$$

Proof. The identification of $\tilde{U}_{B^0}$ and $B^0 \times \tilde{\ell}$ is an immediate consequence of Proposition 3.3 and of the identification Proposition 2.3 (see page 4). For this, recall the assumption that either $\mathrm{char}(k) = 0$, or that there exists an ample line bundle $L \in \mathrm{Pic}(X)$ such that $\mathrm{char}(k)$ does not divide the degree $L.\ell$. It is clear from the construction of the maps and the bundles that $\nu \circ \pi_2|_{\tilde{\sigma}} = \tilde{\iota}_{B^0}|_{\tilde{\sigma}}$. □

3.3. **End of proof.** To end the proof we will now consider the cases $B = B^0$ and $B \neq B^0$ separately.

3.3.1. *The case $B = B^0$.* In this case we will derive a contradiction by calculating certain intersection numbers on $\tilde{U}_{B^0}$. Since $\tilde{U}_{B^0}$ is isomorphic to the trivial bundle $B \times \tilde{\ell}$, we have a decomposition of the numerical classes as follows:

$$\mathrm{Num}(\tilde{U}_{B^0}) \simeq \mathbb{Z} \cdot F_H \oplus \mathbb{Z} \cdot F_V,$$

where F_H is the class of a fiber of the map $\tilde{U}_{B^0} \to \tilde{\ell}$, and F_V that of a fiber of the other projection $\tilde{U}_{B^0} \to B$. We let d be the degree of the (finite, surjective) morphism $\tilde{\iota}_B|_{\tilde{\sigma}} : \tilde{\sigma} \to \ell$.

It follows directly from Corollary 3.4 that $\tilde{\sigma}$ intersects a curve of type F_H with multiplicity d. Since $\tilde{\sigma}$ is a section over B, we obtain that

$$\tilde{\sigma} \equiv F_H + d \cdot F_V,$$

where $\equiv$ denotes numerical equivalence. On the other hand, if $L \in \mathrm{Pic}(X)$ is any ample line bundle which intersects the curves associated with B with multiplicity m, then we obtain that

$$\begin{aligned} \tilde{\iota}_B^*(L).F_V &= m \\ \tilde{\iota}_B^*(L).\tilde{\sigma} &= d \cdot m \\ &= \tilde{\iota}_B^*(L).(F_H + d \cdot F_V). \end{aligned}$$

It follows that $\tilde{\iota}_B^*(L)$ intersects all fibers of the projection $\tilde{U}_{B^0} \to \ell$ with multiplicity 0. This is absurd since $\tilde{\iota}_B$ is finite and does not contract a curve. A contradiction is thus reached, and the proof of Theorems 1.1 and 1.2 is finished in this case.

3.3.2. *The case $B \neq B^0$.* Using the results of the previous Section 3.3.1, we may now assume that $B \neq B^0$. If $b \in B \setminus B^0$ is any point, and $\ell_b \subset X$ the associated rational curve, then it follows from the definition of B^0 that either ℓ has a cuspidal singularity, or that ℓ_b has a cuspidal singularity which is contained in ℓ. This shows that H contains a cuspidal curve and ends the proof if we are in the setup of Theorem 1.1.

It remains to consider the case where X is a smooth variety over $\mathbb{C}$. In order to show that

$$D := \{y \in X \mid \exists \ell \in H^{\mathrm{cusp}} : \ell \text{ has a cuspidal singularity at } y\},$$

has codimension 1 in X, recall the fact that we used already: if D had codimension ≥ 2, then the general curve $\ell \in H$ would be disjoint from D. As a consequence, B^0 would be equal to B, which would violate the assumption that $B \neq B^0$.

In order to show that X is covered by cuspidal rational curves, we will again argue by contradiction: we may assume that all cuspidal curves are contained in a divisor. The total space of the family of cuspidal curves is at least $(\dim D + 1)$-dimensional, so for a general point $x \in D$ there exists a positive dimensional family of cuspidal curves that contain x and are contained in D. That, however, is impossible: it has been shown in [Keb02, Thm. 3.3] that in the projective variety D, a general point is contained in no more then finitely many cuspidal curves. This finishes the proof of Theorem 1.2.

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