The upper triangular group and operations in algebraic K-theory

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1 Introduction

1.1 Let bu and bo denote the stable homotopy spectra representing 2-adically completed unitary and orthogonal connective K-theory respectively. Thus the smash product, $bu \wedge bo$ is a left bu-module spectrum and so we may consider the ring of left bu-module endomorphisms of degree zero in the stable homotopy category of spectra [1], which we shall denote by $End_{left-bu-mod}(bu \wedge bo)$. The group of units in this ring will be denoted by $Aut_{left-bu-mod}(bu \wedge bo)$, the group of homotopy classes of left bu-module homotopy equvialences and let $Aut^0_{left-bu-mod}(bu \wedge bo)$ denote the subgroup of left bu-module homotopy equivalences which induce the identity map on $H_*(bu \wedge bo; \mathbf{Z}/2)$.

Let $U_\infty \mathbf{Z}_2$ denote the group of infinite, invertible upper triangular matrices with entries in the 2-adic integers. That is, $X = (X_{i,j}) \in U_\infty \mathbf{Z}_2$ if $X_{i,j} \in \mathbf{Z}_2$ for each pair of integers $0 \leq i, j$ and $X_{i,j} = 0$ if $j > i$ and $X_{i,i}$ is a 2-adic unit. This upper trianglular group is *not* equal to the direct limit $\lim_{\overrightarrow{n}} U_n \mathbf{Z}_2$ of the finite upper triangular groups.

Our main result (proved in §3.2) is the following:

Theorem 1.2
There is an isomorphism of the form

$$\psi : Aut^0_{left-bu-mod}(bu \wedge bo) \xrightarrow{\cong} U_\infty \mathbf{Z}_2.$$

1.3 There is a similar calculation of the group $Aut^0_{left-bo-mod}(bo \wedge bo)$ which I shall leave to the reader. In fact, the appearance of bo in Theorem 1.2 is just for convenience. The main use of this result will be to realise the 2-adic group-ring, $\mathbf{Z}_2[U_\infty \mathbf{Z}_2]$, as a subring of the left-bu module endomorphisms of $bu \wedge bu \simeq bu \wedge bo \wedge \Sigma^{-2}\mathbf{C}P^2$.

The preference for bu over bo is that, if F is an algebraically closed field of characteristic different from 2, then there is a homotopy equivalence of ring spectra $bu \simeq \underline{KF}\mathbf{Z}_2$ between 2-adic connective K-theory and the algebraic K-theory spectrum of F with coefficients in the 2-adic integers ([17] [18]).

As explained in Remark 4.5, Theorem 1.2 implies that the 2-adic group-ring, $\mathbf{Z}_2[U_\infty \mathbf{Z}_2]$, may be considered as a ring of operations on 2-adic algebraic K-theory. I hope to develop the properties of these operations in a subsequent paper.

The paper is organised in the following manner. In §2 we recall the decomposition of $bu \wedge bu$ and $bu \wedge bo$ together with several related facts about Steenrod algebra structure. In §3 we prove Theorem 1.2. In §4 we explain the application of Theorem 1.2 to the construction of operations on algebraic K-theory and on Chow groups. In §5 we make some remarks about the possible identity of $1 \wedge \psi^3$ in $U_\infty \mathbf{Z}_2$.

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2 Connective K-theory

2.1 In this section we recall the splitting of $bu \wedge bu$ ([1] [8]; see also [9]).

Let X be a finite spectrum and let DX denote its S-dual. Following the account of ([1] pp.190-196) this means that we have an S-map

$$e : DX \wedge X \longrightarrow S^0$$

such that, if W is a finite spectrum and Z is arbitrary, there is an isomorphism ([1] Proposition 5.4 p.195)

$$T : [W, Z \wedge DX]_* \xrightarrow{\cong} [W \wedge X, Z]_*$$

given by $T[f] = [(1_Z \wedge e) \cdot (f \wedge 1_X)]$.

Hence, setting $W = S^0$ and $Z = X$, we have

$$\mu = T^{-1}(1_X) : S^0 \longrightarrow X \wedge DX.$$

Setting $Z = H\mathbf{Z}/2$, the mod 2 Eilenberg-Maclane spectrum, we obtain an isomorphisms

$$U : H^{-*}(DX; \mathbf{Z}/2) \xrightarrow{\cong} H_*(X; \mathbf{Z}/2) \xrightarrow{\cong} Hom(H^*(X; \mathbf{Z}/2), \mathbf{Z}/2)$$

whose composition, U, is given by

$$U(\alpha)(\beta) = \mu^*(\beta \otimes \alpha) \in H^*(S^0; \mathbf{Z}/2) \cong \mathbf{Z}/2$$

for all $\alpha \in H^{-*}(DX; \mathbf{Z}/2)$, $\beta \in H^*(X; \mathbf{Z}/2)$.

Let $\mathcal{A}$ denote the mod 2 Steenrod algebra [16] then, for $m > 0$,

$$U(Sq^m(\alpha))(\beta) = \mu^*(\beta \otimes Sq^m(\alpha)) = \sum_{a=1}^{m} \mu^*(Sq^a(\beta) \otimes Sq^{m-a}(\alpha)),$$

since $Sq^m(\mu^*(\beta \otimes \alpha)) = 0$.

Let χ denote the canonical anti-automorphism ([16] pp.25-26). We have

$$U(Sq^m(\alpha))(\beta) = \sum_{a=1}^{m} U(Sq^{m-a}(\alpha))(Sq^a(\beta)).$$

For $m = 1$, $U(Sq^1(\alpha))(\beta) = U(\alpha)(Sq^1(\beta)) = U(\alpha)(\chi(Sq^1)(\beta))$. If, by induction, we have $U(Sq^n(\alpha))(\beta) = U(\alpha)(\chi(Sq^n)(\beta))$ for all $n < m$ then

$$U(Sq^m(\alpha))(\beta) = \sum_{a=1}^{m} U(\alpha)(\chi(Sq^{m-a})(Sq^a(\beta)) = U(\alpha)(\chi(Sq^m)(\beta)),$$

since $\sum_{a=0}^{m} \chi(Sq^{m-a})Sq^a = 0$.

Therefore, as a left $\mathcal{A}$-module, $H^{-*}(DX; \mathbf{Z}/2)$ is isomorphic to $H_*(X; \mathbf{Z}/2)$ where the left action by Sq^a corresponds to $\chi(Sq^a)_*$, composition with $\chi(Sq^a)$ (cf. [13]). However, $\chi(Sq^1) = Sq^1$ and $\chi(Sq^{01}) = Sq^{01}$, because these are primitives in the Hopf algebra, $\mathcal{A}$. If we set $B = E(Sq^1, Sq^{01})$ then $H^{-*}(DX; \mathbf{Z}/2)$ is isomorphic as a left B-module to $H_*(X; \mathbf{Z}/2)$ on which Sq^1 and Sq^{01} acts via Sq^1_* and Sq^{01}_*, respectively.

2.2 Consider the second loopspace of the 3-sphere, $\Omega^2 S^3$. There is an algebra isomorphism [15] of the form

$$H_*(\Omega^2 S^3; \mathbf{Z}/2) \cong \mathbf{Z}/2[\xi_1, \xi_2, \xi_3, \ldots]$$

where $\xi_t = Q_1^{t-1}(\iota)$ has degree $2^t - 1$ and ξ_t is primitive. Here $\iota = \xi_1$ is the image of the generator of $H_1(S^1; \mathbf{Z}/2)$. The right action of Sq^1 and Sq^{01}, via their duals Sq^1_* and Sq^{01}_*, on $H_*(\Omega^2 S^3; \mathbf{Z}/2)$ is given by [15]

$$(\xi_t)Sq^{01} = \begin{cases} \xi_{t-2}^4 & if\ t \geq 3, \\ 0 & otherwise \end{cases}$$

and

$$(\xi_t)Sq^1 = \begin{cases} \xi_{t-1}^2 & if\ t \geq 2, \\ 0 & otherwise. \end{cases}$$

These are the same formulae which give the right action on $H_*(bu; \mathbf{Z}/2)$ ([1] pp.340-342).

Since $B = E(Sq^1, Sq^{01})$ is a commutative ring, we may consider $H_*(\Omega^2 S^3; \mathbf{Z}/2)$ to be a *left* B-module via the formulae, $Sq^1(\xi_t) = (\xi_t)Sq^1$ and $Sq^{01}(\xi_t) = (\xi_t)Sq^{01}$.

In order to apply these observations to $\Omega^2 S^3$ we would prefer it to be a finite complex. However, there exists a model for $\Omega^2 S^3$ which is filtered by finite complexes ([3] , [14])

$$S^1 = F_1 \subset F_2 \subset F_3 \subset \ldots \subset \Omega^2 S^3 = \bigcup_{k \geq 1} F_k$$

and there is a stable homotopy equivalence, an example of the so-called Snaith splitting, of the form

$$\Omega^2 S^3 \simeq \vee_{k \geq 1} F_k / F_{k-1}.$$

In addition, by ([5]; see also [12]), this stable homotopy equivalence may be assumed to be multiplicative in the sense that the H-space product on $\Omega^2 S^3$ induces a graded homotopy-ring structure

$$\{F_k/F_{k-1} \wedge F_l/F_{l-1} \longrightarrow F_{k+l}/F_{k+l-1}\}$$

on $\vee_{k \geq 1} F_k/F_{k-1}$. To obtain a graded homotopy-ring with identity we add an extra base-point by defining $F_0 = S^0$, $F_j = *$ for $j < 0$ and replacing $\vee_{k \geq 1} F_k/F_{k-1}$ by $\vee_{k \geq 0} F_k/F_{k-1}$.

The geometrical construction of the homology operation, Q_1, shows that $\xi_1 = \iota \in H_1(F_1; \mathbf{Z}/2)$ and that $\xi_t \in H_{2^t-1}(F_{2^{t-1}}/F_{2^{t-1}-1}; \mathbf{Z}/2)$, in terms of the induced splitting of $H_*(\Omega^2 S^3; \mathbf{Z}/2)$ so that there is an algebra isomorphism of B-modules of the form

$$H_*(\vee_{k \geq 0} F_{4k}/F_{4k-1}; \mathbf{Z}/2) \cong \mathbf{Z}/2[\xi_1^4, \xi_2^2, \xi_3, \xi_4, \ldots].$$

Next, write $H_*(\Sigma^{-2}\mathbf{C}P^2; \mathbf{Z}/2) = \mathbf{Z}/2 < 1 > \oplus \mathbf{Z}/2 < x >$ for the mod 2 homology of the double-desuspension of the complex projective plane. Hence Sq^1_* and Sq^{01}_* act trivially on $x \in H_2(\Sigma^{-2}\mathbf{C}P^2; \mathbf{Z}/2)$. Therefore we may define an isomorphism of right B-modules

$$\Phi : H_*(\vee_{k \geq 0}(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2); \mathbf{Z}/2) \xrightarrow{\cong} \mathbf{Z}/2[\xi_1^2, \xi_2^2, \xi_3, \xi_4, \ldots]$$

by the formula, for $\epsilon = 0, 1$,

$$\Phi((\xi_1^4)^{\epsilon_1}(\xi_2^2)^{\epsilon_2}\xi_3^{\epsilon_3}\xi_4^{\epsilon_4}\ldots\xi_t^{\epsilon_t} \otimes x^{\epsilon}) = \xi_1^{4\epsilon_1+2\epsilon}(\xi_2^2)^{\epsilon_2}\xi_3^{\epsilon_3}\xi_4^{\epsilon_4}\ldots\xi_t^{\epsilon_t}.$$

Here the right B-module structure on $\mathbf{Z}/2[\xi_1^2, \xi_2^2, \xi_3, \xi_4, \ldots]$ is that given by the formulae introduced previously.

On the other hand, there is an isomorphism of algebras with right B-module structure ([1] p.340)

$$H_*(bu; \mathbf{Z}/2) \cong \mathbf{Z}/2[\xi_1^2, \xi_2^2, \xi_3, \xi_4, \ldots],$$

where these ξ_i's are the canonical Milnor generators of the dual Steenrod algebra ([16] pp.19-22). Therefore we have a canonical isomorphism of graded, right B-algebras

$$\Phi : H_*(\vee_{k\geq 0}(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2); \mathbf{Z}/2) \xrightarrow{\cong} H_*(bu; \mathbf{Z}/2).$$

If $\lambda = Sq^1$ or Sq^{01}, $f \in H^*(bu; \mathbf{Z}/2)$ and $\alpha \in H_*(\vee_{k\geq 0}(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2); \mathbf{Z}/2)$ then $(\Phi(\alpha))\lambda = \Phi((\alpha)\lambda)$ and

$$< \lambda(f), \Phi(\alpha) >= (\Phi(\alpha))\lambda(f) = \Phi((\alpha)\lambda)(f) =< f, \Phi(\alpha)\lambda) > .$$

On the other hand, if we interpret α as belonging to $H^{-*}(\vee_{k\geq 0}D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2); \mathbf{Z}/2)$ then $(\alpha)\lambda$ becomes the left translate of α by λ, $\lambda(\alpha)$, for $\lambda = Sq^1$ or Sq^{01}. Identifying $H_*(bu; \mathbf{Z}/2)$ with the dual of the left B-module, $H^*(bu; \mathbf{Z}/2)$, we have $f(\Phi(\lambda(\alpha))) = \lambda(f)(\Phi(\alpha))$. This means that the adjoint of Φ,

$$adj(\Phi) \in Hom(H^{-*}(\vee_{k\geq 0}D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2); \mathbf{Z}/2) \otimes H^*(bu; \mathbf{Z}/2), \mathbf{Z}/2)$$

given by $adj(\Phi)(\alpha \otimes f) = f(\Phi(\alpha))$ satisfies, if $\lambda = Sq^1$ or Sq^{01},

$$\begin{aligned} adj(\Phi)(\lambda(\alpha \otimes f)) &= adj(\Phi)(\lambda(\alpha) \otimes f + \alpha \otimes \lambda(f)) \\ &= f(\Phi(\lambda(\alpha))) + \lambda(f)(\Phi(\alpha)) \\ &= 0 \\ &= \lambda(adj(\Phi)(\alpha \otimes f)). \end{aligned}$$

Therefore we have a canonical family of maps. for $k \geq 0$,

$$adj(\Phi)_k \in Hom_B(H^{-*}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2); \mathbf{Z}/2) \otimes H^*(bu; \mathbf{Z}/2), \mathbf{Z}/2)$$

such that $adj(\Phi) = \sum_{k\geq 1}(adj(\Phi)_k$.

The analysis of the right B-module, $H_*(bu; \mathbf{Z}/2)$, in ([1] Proposition 16.4 and pp.340-342) shows that each left B-module of the form $H^{-*}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2); \mathbf{Z}/2)$ satisfies the conditions of ([1] p.353). That is, up to direct sums with projectives these B-modules are equivalent to finite sums of $\Sigma^a I^b$, the a-th suspension of the b-th tensor power of the augmentation ideal, with $a+b$ even. Here a and b may be negative. The same result holds if we smash with a finite number of copies of bu. For such modules as the left variable $Ext_B^{s,t}(-,-)$ vanishes for $s > 0$ and $t-s$ odd and the related Adams spectral sequence collapses. In particular the Adams spectral sequence

$$\begin{aligned} E_2^{s,t} &= Ext_{\mathcal{A}}^{s,t}(H^{-*}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2); \mathbf{Z}/2) \otimes H^*(bu; \mathbf{Z}/2)^{\otimes^2}, \mathbf{Z}/2) \\ &\Longrightarrow \pi_{t-s}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2) \wedge bu \wedge bu) \otimes \mathbf{Z}_2 \end{aligned}$$

has

$$E_2^{s,t} \cong Ext_B^{s,t}(H^{-*}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2); \mathbf{Z}/2) \otimes H^*(bu; \mathbf{Z}/2), \mathbf{Z}/2)$$

and $E_2^{s,t} \cong E_\infty^{s,t}$, by ([1] Lemma 17.12 p.361).

In addition, by the 2-local version of ([1] pp.354-355; see [1] p.358-359), the Hurewicz homomorphisms yield an injection of the form

$$\pi_t(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2) \wedge bu \wedge bu) \otimes \mathbf{Z}_2$$

$$\downarrow$$

$$Ext_B^{0,t}(H^{-*}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2); \mathbf{Z}/2) \otimes H^*(bu; \mathbf{Z}/2), \mathbf{Z}/2)$$

$$\oplus$$

$$H_t(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2) \wedge bu \wedge bu) \otimes \mathbf{Q}_2$$

where $\mathbf{Q}_2$ denotes the field of 2-adic rationals.

The collapsing of the spectral sequence ensures that there exists at least one element

$$adj(\lambda)_k \in \pi_*(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2) \wedge bu \wedge bu) \otimes \mathbf{Z}_2$$

whose mod 2 Hurewicz image is $adj(\Phi)_k$. Such an element corresponds, via S-duality with $W = S^0$, $Z = bu \wedge bu$, to a (2-local) S-map of the form

$$\lambda_k : F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2 \longrightarrow bu \wedge bu$$

whose induced map in mod 2 cohomology is equal to Φ_k^*, the k-component of the dual of Φ.

Now λ_k^* is a left $\mathcal{A}$-module homomorphism

$$\lambda_k^* : H^*(bu \wedge bu; \mathbf{Z}/2) \longrightarrow H^*(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2; \mathbf{Z}/2)$$

while Φ_k^* is a left B-module homomorphism

$$\Phi_k^* : H^*(bu; \mathbf{Z}/2) \longrightarrow H^*(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2; \mathbf{Z}/2)$$

Here we have identified B with its dual Hopf algebra, B_*. The relation between λ_k^* and Φ_k^* is described in the following manner.

There is a left $\mathcal{A}$-module isomorphism, $H^*(bu; \mathbf{Z}/2) \cong \mathcal{A} \otimes_B \mathbf{Z}/2$ ([1] Proposition 16.6 p.335), and an isomorphism ([1] p.338)

$$\psi : \mathcal{A} \otimes_B H^*(bu; \mathbf{Z}/2) \xrightarrow{\cong} H^*(bu; \mathbf{Z}/2) \otimes H^*(bu; \mathbf{Z}/2)$$

given by $\psi(a\otimes_B b)=\sum(a'\otimes_B 1)\otimes a''(b)$ where the diagonal of $a\in\mathcal{A}$ satisfies $\Delta(a)=\sum a'\otimes a''$ and $b\in H^*(bu;\mathbf{Z}/2)$. On the other hand, Φ_k^* induces a left $\mathcal{A}$-module homomorphism

$$\phi_k^*:\mathcal{A}\otimes_B H^*(bu;\mathbf{Z}/2)\longrightarrow H^*(F_{4k}/F_{4k-1}\wedge\Sigma^{-2}\mathbf{C}P^2;\mathbf{Z}/2)$$

given by $\phi_k^*(a\otimes_B q)=a(\Phi_k^*(b))$. These homomorphisms satisfy

$$\phi_k^*=\lambda_k^*:\psi.$$

Now consider the composition

$$L=(m\wedge 1)(\sum_{k\geq 0}1\wedge\lambda_k):\vee_{k\geq 0}bu\wedge(F_{4k}/F_{4k-1}\wedge\Sigma^{-2}\mathbf{C}P^2)\longrightarrow bu\wedge bu\wedge bu\longrightarrow bu\wedge bu$$

where $m:bu\wedge bu\longrightarrow bu$ is the bu-product. This map induces an isomorphism on mod 2 cohomology. To see this it suffices to show that the composition

$$(\sum_{k\geq 0}1\otimes\lambda_k^*)(m^*\otimes 1)\psi:\mathcal{A}\otimes_B H^*(bu;\mathbf{Z}/2)\longrightarrow H^*(\vee_{k\geq 0}F_{4k}/F_{4k-1}\wedge\Sigma^{-2}\mathbf{C}P^2;\mathbf{Z}/2)$$

is an isomorphism. For $a\in\mathcal{A}$ and $b\in H^*(bu;\mathbf{Z}/2)\cong\mathcal{A}\otimes_B\mathbf{Z}/2$ define $\psi'(a\otimes_B b)\in H^*(bu;\mathbf{Z}/2)\otimes(\mathcal{A}\otimes_B H^*(bu;\mathbf{Z}/2))$ by $\psi'(a\otimes_B b)=\sum(a'\otimes_B 1)\otimes(a''\otimes_B b)$ where $\Delta(a)=\sum a'\otimes a''$. We find that

$$(1\otimes\psi)\psi'(a\otimes_B b)$$

$$=(1\otimes\psi)(\sum(a'\otimes_B 1)\otimes(a''\otimes_B b))$$

$$=\sum(a'\otimes_B 1)\otimes(a_1\otimes_B 1)\otimes a_2(b),$$

where $(1\otimes\Delta)\Delta(a)=\sum a'\otimes a_1\otimes a_2$, so that

$$(1\otimes\psi)\psi'=(\Delta\otimes 1)\psi=(m^*\otimes 1)\psi.$$

From this identity we have

$$(\textstyle\sum_{k\geq 0}1\otimes\lambda_k^*)(m^*\otimes 1)\psi(a\otimes_B b)$$

$$=(\textstyle\sum_{k\geq 0}1\otimes\phi_k^*)\psi'(a\otimes_B b)$$

$$=\textstyle\sum(a'\otimes_B 1)\otimes a''(\sum_{k\geq 0}\Phi_k^*(b))$$

$$=\psi(1_{\mathcal{A}}\otimes_B(\textstyle\sum_{k\geq 0}\Phi_k^*))(a\otimes_B b).$$

Since $\sum_{k\geq 0}\Phi_k^*$ is an isomorphism of left B-modules this composition is an isomorphism of left $\mathcal{A}$-modules.

To recapitulate, we have proved the part (i) of the following result, part (ii) being proved in a similar manner.

Theorem 2.3

In the notation of §2.1, there are 2-local homotopy equivalences of left-bu-module spectra of the form

$$(i) \qquad L : \vee_{k\geq 0} bu \wedge (F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2) \longrightarrow bu \wedge bu$$

and

$$(ii) \qquad \hat{L} : \vee_{k\geq 0} bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge bo.$$

Theorem 2.3 should be compared with the odd primary analogue which is described in detail in [6].

2.4 *Comparison with mod 2 cohomology*

It is very easy to compare the 2-local splitting of left *bu*-module spectra

$$L : \vee_{k\geq 0} bu \wedge (F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2) \xrightarrow{\simeq} bu \wedge bu$$

with a corresponding splitting for mod 2 cohomology.

There is a unique, non-trivial map of spectra, $\iota : bu \longrightarrow H\mathbf{Z}/2$, and we wish to construct a homotopy commutative diagram of the form

$$\begin{array}{ccc}
\vee_{k\geq 0} bu \wedge (F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2) & \xrightarrow{\text{L}} & bu \wedge bu \\
\Big\downarrow \vee_{k\geq 0}\iota \wedge 1 \wedge 1 & & \Big\downarrow \iota \wedge 1 \\
\vee_{k\geq 0} H\mathbf{Z}/2 \wedge (F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2) & \xrightarrow{\text{L'}} & H\mathbf{Z}/2 \wedge bu
\end{array}$$

in which L' is a homotopy equivalence.

However, the Adams spectral sequence

$$Ext_{\mathcal{A}}^{s,t}(H^{-*}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2); \mathbf{Z}/2) \otimes H^*(H\mathbf{Z}/2 \wedge bu; \mathbf{Z}/2), \mathbf{Z}/2)$$

$$\Longrightarrow \pi_{t-s}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2) \wedge H\mathbf{Z}/2 \wedge bu)$$

has

$$E_2^{s,t} \cong Ext_B^{s,t}(H^{-*}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2); \mathbf{Z}/2) \otimes \mathcal{A}, \mathbf{Z}/2)$$

which is zero if s is non-zero, since $\mathcal{A}$ is a free B-module [10]. Also, composition with ι corresponds to the canonical map on $E_2^{0,*}$

$$Hom_B(H^{-*}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2); \mathbf{Z}/2) \otimes \mathcal{A} \otimes_B \mathbf{Z}/2, \mathbf{Z}/2)$$

$$\downarrow$$

$$Hom_B(H^{-*}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2); \mathbf{Z}/2) \otimes \mathcal{A}, \mathbf{Z}/2)$$

given by composition with $\iota^* : \mathcal{A} \longrightarrow \mathcal{A} \otimes_B \mathbf{Z}/2$. Hence there exists

$$adj(\lambda')_k \in \pi_0(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2) \wedge H\mathbf{Z}/2 \wedge bu)$$

such that

$$\iota \cdot adj(\lambda)_k \simeq adj(\lambda')_k$$

and therefore

$$\iota \cdot \lambda_k \simeq \lambda'_k : F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2 \longrightarrow H\mathbf{Z}/2 \wedge bu.$$

We see that we may set L' equal to

$$L' = (m' \wedge 1)(\Sigma_{k \geq 0} 1 \wedge \lambda_k) : \vee_{k \geq 0} H\mathbf{Z}/2 \wedge (F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2)$$

$$\longrightarrow H\mathbf{Z}/2 \wedge H\mathbf{Z}/2 \wedge bu \longrightarrow H\mathbf{Z}/2 \wedge bu$$

where $m' : H\mathbf{Z}/2 \wedge H\mathbf{Z}/2 \longrightarrow H\mathbf{Z}/2$ is the cup-product.

Also L' induces an isomorphism in mod 2 homology, since it is a homomorphism between free, graded module of finite type over the polynomial ring, $\mathcal{A}_* \cong \mathbf{Z}/2[\xi_1, \xi_2, \ldots]$, and $\mathbf{Z}/2 \otimes_{\mathcal{A}_*} (L')_* = \mathbf{Z}/2 \otimes_{(\mathcal{A} \otimes_B \mathbf{Z}/2)_*} (L)_*$ is an isomorphism ([7] pp.603-605).

3 The role of the upper triangular group

3.1 In this section I am going to prove Theorem 1.2, which will be accomplished in §3.2 after some prefatory discussion. Let us begin with some motivation from homotopy theory. Let $\psi^3 : bu \longrightarrow bu$ denote the Adams operation. In order to understand the map

$$1 \wedge (\psi^3 - 1) : bu \wedge bu \longrightarrow bu \wedge bu$$

we observe that it is a left bu-module map and therefore we ought to study all such maps. The 2-local splitting of $bu \wedge bu$ of Theorem 2.3(i) implies that we need only study left bu-module maps of the form

$$\phi_{k,l} : bu \wedge (F_{4k}/F_{4k-1}) \wedge \Sigma^{-2}\mathbf{C}P^2 \longrightarrow bu \wedge (F_{4l}/F_{4l-1}) \wedge \Sigma^{-2}\mathbf{C}P^2$$

for each pair, $k, l \geq 0$. In addition, the factor $\Sigma^{-2}\mathbf{C}P^2$ will only be a nuisance so we shall study

$$1 \wedge (\psi^3 - 1) : bu \wedge bo \longrightarrow bu \wedge bo$$

first. By virtue of the 2-local splitting of Theorem 2.3(ii)

$$L'' : bu \wedge bo \xrightarrow{\simeq} \vee_{k \geq 0} bu \wedge (F_{4k}/F_{4k-1})$$

we are led to study the corresponding left bu-module maps, $\{\phi''_{k,l}\}$, of the form

$$\phi''_{k,l} : bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (F_{4l}/F_{4l-1}).$$

A left bu-module map of this form is, in turn, determined by its restriction to $S^0 \wedge (F_{4k}/F_{4k-1})$. This restriction is a homotopy element of the form

$$[\phi''_{k,l}] \in \pi_0(D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu) \otimes \mathbf{Z}_2.$$

This homotopy group is calculated by means of the (collapsed) Adams spectral sequence

$$E_2^{s,t} = Ext_B^{s,t}(H^*(D(F_{4k}/F_{4k-1}); \mathbf{Z}/2) \otimes H^*(F_{4l}/F_{4l-1}; \mathbf{Z}/2), \mathbf{Z}/2)$$

$$\Longrightarrow \pi_{t-s}(D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu) \otimes \mathbf{Z}_2.$$

Recall from ([1] p.332) that Σ^a is the (invertible) B-module given by $\mathbf{Z}/2$ in degree a, $\Sigma^{-a} = Hom(\Sigma^a, \mathbf{Z}/2)$ and I is the augmentation ideal, $I = ker(\epsilon : B \longrightarrow \mathbf{Z}/2)$. Hence, if $b > 0$, $I^{-b} = Hom(I^b, \mathbf{Z}/2)$, where I^b is the b-fold tensor product of I. These duality identifications may be verified using the criteria of ([1] p.334 Theorem 16.3) for identifying $\Sigma^a I^b$.

In ([1] p.341) it is shown that the B-module given by

$$H^{-*}(D(F_{4k}/F_{4k-1}); \mathbf{Z}/2) \cong H_*(F_{4k}/F_{4k-1}; \mathbf{Z}/2)$$

is stably equivalent to $\Sigma^{2^{r-1}+1} I^{2^{r-1}-1}$ when $0 < 4k = 2^r$. Therefore $H^*(D(F_{4k}/F_{4k-1}); \mathbf{Z}/2)$ is stably equivalent to $\Sigma^{-(2^{r-1}+1)} I^{1-2^{r-1}}$ when $0 < 4k = 2^r$. If k is not a power of two we may write $4k = 2^{r_1} + 2^{r_2} + \ldots + 2^{r_t}$ with $2 \leq r_1 < r_2 < \ldots < r_t$. In this case

$$H_*(F_{4k}/F_{4k-1}; \mathbf{Z}/2) \cong \otimes_{j=r_1}^{r_t} H_*(F_{2^j}/F_{2^j-1}; \mathbf{Z}/2)$$

which is stably equivalent to

$$\Sigma^{2^{r_1-1}+1+2^{r_2-1}+1+\ldots+2^{r_t-1}+1} I^{2^{r_1-1}-1+2^{r_2-1}-1+\ldots+2^{r_t-1}-1} = \Sigma^{2k+\alpha(k)} I^{2k-\alpha(k)},$$

where $\alpha(k)$ equals the number of 1's in the dyadic expansion of k. Similarly, $H^*(D(F_{4k}/F_{4k-1}); \mathbf{Z}/2)$ is stably equivalent to $\Sigma^{-2k-\alpha(k)} I^{\alpha(k)-2k}$

Next we observe that $Ext_B^{s,t}(\Sigma^a M, \mathbf{Z}/2) \cong Ext_B^{s,t-a}(M, \mathbf{Z}/2)$. Also the short exact sequence

$$0 \longrightarrow I \otimes M \longrightarrow B \otimes M \longrightarrow M \longrightarrow 0$$

yields a long exact sequence of the form

$$\ldots \longrightarrow Ext_B^{s,t}(B \otimes M, \mathbf{Z}/2) \longrightarrow Ext_B^{s,t}(I \otimes M, \mathbf{Z}/2) \longrightarrow Ext_B^{s+1,t}(M, \mathbf{Z}/2)$$

$$\longrightarrow Ext_B^{s+1,t}(B \otimes M, \mathbf{Z}/2) \longrightarrow \ldots$$

so that, if $s > 0$, there is an isomorphism

$$Ext_B^{s,t}(I \otimes M, \mathbf{Z}/2) \xrightarrow{\cong} Ext_B^{s+1,t}(M, \mathbf{Z}/2).$$

Therefore, for $s > 0$,

$$\begin{aligned} E_2^{s,t} &\cong Ext_B^{s,t}(\Sigma^{2l-2k+\alpha(l)-\alpha(k)} I^{2l-2k-\alpha(l)+\alpha(k)}, \mathbf{Z}/2) \\ &\cong Ext_B^{s+2l-2k-\alpha(l)+\alpha(k), t-2l+2k-\alpha(l)+\alpha(k)}(\mathbf{Z}/2, \mathbf{Z}/2). \end{aligned}$$

Now $Ext_B^{*,*}(\mathbf{Z}/2, \mathbf{Z}/2) \cong \mathbf{Z}/2[a, b]$ where $a \in Ext_B^{1,1}$, $b \in Ext_B^{1,3}$ and the contributions to $\pi_0(D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu) \otimes \mathbf{Z}_2$ come from the groups $\{E_2^{s,s} \mid s \geq 0\}$. This corresponds to $Ext_B^{u,v}(\mathbf{Z}/2, \mathbf{Z}/2)$ when $u = s + 2l - 2k - \alpha(l) + \alpha(k)$ and $v = s - 2l + 2k - \alpha(l) + \alpha(k)$, which implies that $v - u = 4(k - l)$. This implies that this $Ext_B^{u,v}(\mathbf{Z}/2, \mathbf{Z}/2) = 0$ if $l > k$ or, equivalently, that each $E_2^{s,s} = 0$ is zero when $l > k$. Therefore $\pi_0(D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu) \otimes \mathbf{Z}_2 = 0$ if $l > k$.

Now suppose that $l \leq k$. If $Ext_B^{u,v}(\mathbf{Z}/2, \mathbf{Z}/2)$ is non-zero then it is cyclic of order two generated by $a^{(3u-v)/2} b^{(v-u)/2}$ and when $u = s + 2l - 2k - \alpha(l) + \alpha(k), v = s - 2l + 2k - \alpha(l) + \alpha(k)$ this monomial is equal to $a^{s+4l-4k-\alpha(l)+\alpha(k)} b^{2(k-l)}$. Furthermore, in order for this group to be non-zero we must have $s \geq 4(k - l) + \alpha(l) - \alpha(k)$ which implies that $s \geq 0$ if $k = l$ and $s \geq 4(k-l) + \alpha(l) - \alpha(k) \geq 2(k-l) + 1$ if $k > l$. The last inequality is seen by writing $l = 2^{\alpha_1} + \ldots + 2^{\alpha_r}$ with $0 \leq \alpha_1 < \ldots < \alpha_r$ and $k - l = 2^{\epsilon_1} + \ldots + 2^{\epsilon_q}$ with $0 \leq \epsilon_1 < \ldots < \epsilon_q$. Then $\alpha(l) = r$ and $\alpha(l + 2^{\epsilon_q}) \leq r + 1$ so that, by induction, $\alpha(k) \leq r + q$ which yields

$$2(k - l) + \alpha(l) - \alpha(k) \geq 2(k - l) - q \geq \sum_{j=1}^{q} (2^{\epsilon_j + 1} - 1) \geq 1.$$

Suppose now that $k > l$ and consider the non-trivial homotopy classes of left-bu-module maps

$$\phi''_{k,l} : bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (F_{4l}/F_{4l-1})$$

which induce the zero map on mod 2 homology. In the spectral sequence these maps are represented by elements of $E_2^{s,s} = E_\infty^{s,s}$ with $s > 0$ since being represented in $E_\infty^{0,*}$ is equivalent to being detected by the induced map in mod 2 homology. By the preceding discussion, the only other possibility is that $\phi''_{k,l}$ is represented in $E_\infty^{\epsilon+4(k-l)+\alpha(l)-\alpha(k),\epsilon+4(k-l)+\alpha(l)-\alpha(k)}$ for some $\epsilon \geq 0$. Since multiplication by two on $\pi_0(D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu) \otimes \mathbf{Z}_2$ corresponds to multiplication by $a \in Ext_B^{1,1}(\mathbf{Z}/2, \mathbf{Z}/2)$ in the spectral sequence, we see that

$$\phi''_{k,l} = \gamma 2^\epsilon \iota_{k,l},$$

for some 2-adic unit γ and positive integer ϵ, where

$$\iota_{k,l} : bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (F_{4l}/F_{4l-1})$$

represented in the spectral sequence by a generator of $E_2^{4(k-l)+\alpha(l)-\alpha(k),4(k-l)+\alpha(l)-\alpha(k)}$.

When $k = l$ a similar argument shows that

$$\phi''_{k,k} = \gamma 2^\epsilon \iota_{k,k}$$

where $\iota_{k,k}$ denotes the identity map of $bu \wedge (F_{4k}/F_{4k-1})$. In particular, if $\phi''_{k,k}$ induces the identity map on mod 2 homology then $\epsilon = 0$.

3.2 *Proof of Theorem 1.2*

Recall that $Aut^0_{left-bu-mod}(bu \wedge bo)$ is the group, under composition, of homotopy classes of 2-local homotopy equivalences of

$$\vee_{k \geq 0}\, bu \wedge (F_{4k}/F_{4k-1})$$

given by left bu-module maps which induce the identity map on $H_*(-;\mathbf{Z}/2)$. The discussion of §3.1 shows that the elements of this group are in one-one correspondence with the matrices in $U_\infty \mathbf{Z}_2$. More specifically, the discussion shows that there is a bijection

$$\phi : U_\infty(\mathbf{Z}_2) \longrightarrow Aut^0_{left-bu-mod}(bu \wedge bo)$$

given by

$$\phi(X) = \sum_{l \leq k} X_{l,k} \iota_{k,l} : bu \wedge (\vee_{k\geq 0} F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (\vee_{k \geq 0} F_{4k}/F_{4k-1}).$$

Here $\iota_{k,l}$ is chosen as in §3.1.

We shall obtain the isomorphism of Theorem 1.2 by setting $\psi(X) = \phi(X^{-1})$.

However, in order to ensure that ψ is an isomorphism of groups we must choose the $\iota_{k,l}$ more carefully. In fact, I claim that we may choose the $\iota_{k,k}$'s to be the identity maps, choose each $\iota_{k+1,k}$ as in §3.1 and then define

$$\iota_{k,l} = \iota_{l+1,l}\iota_{l+2,l+1} \cdots \iota_{k,k-1}$$

for all $k - l \geq 2$. If this is so then ψ is an isomorphism of groups because ϕ is a bijective anti-homomorphism, since we have

$$\begin{aligned}
&\phi(X) \cdot \phi(Y) \\
&= (\textstyle\sum_{l \leq k} X_{l,k}\iota_{k,l})(\sum_{t \leq s} Y_{t,s}\iota_{s,t}) \\
&= \textstyle\sum_s \sum_{t \leq s \leq k} X_{s,k}Y_{t,s}\iota_{t+1,t} \cdots \iota_{s,s-1}\iota_{s+1,s} \cdots \iota_{k-1,k-2}\iota_{k,k-1} \\
&= \textstyle\sum_{t \leq k} (YX)_{t,k}\iota_{k,t} \\
&= \phi(YX)
\end{aligned}$$

as required.

It remains to verify the claim. For $k > l > m$ we need to know the relation between the composition $\iota_{l,m} \cdot \iota_{k,l}$ and $\iota_{k,m}$. Set $s(k,l) = 4(k-l) + \alpha(l) - \alpha(k)$.

The element, $\iota_{k,l}$ is represented by the generator of $Ext_B^{s(k,l),s(k,l)}(\Sigma^{2l-2k+\alpha(l)-\alpha(k)} I^{2l-2k-\alpha(l)+\alpha(k)}, \mathbf{Z}/2) \cong \mathbf{Z}/2$ and $\iota_{l,m}$ by that of $Ext_B^{s(l,m),s(l,m)}(\Sigma^{2m-2l+\alpha(m)-\alpha(l)} I^{2m-2l-\alpha(m)+\alpha(l)}, \mathbf{Z}/2) \cong \mathbf{Z}/2$ while $\iota_{k,m}$ is represented by a generator of

$$Ext_B^{s(k,m),s(k,m)}(\Sigma^{2m-2k+\alpha(m)-\alpha(k)} I^{2m-2k-\alpha(m)+\alpha(k)}, \mathbf{Z}/2) \cong \mathbf{Z}/2.$$

The composition, $\iota_{l,m} \cdot \iota_{k,l}$, is represented by the product of the representatives under the pairing induced by the tautological B-module isomorphism,

$$\Sigma^a I^b \otimes \Sigma^{a'} I^{b'} \cong \Sigma^{a+a'} I^{b+b'}$$

for suitable positive integers a, a', b, b'. Via the dimension-shifting isomorphisms described in §3.1, the pairing

$$Ext_B^{s,s}(\Sigma^a I^b, \mathbf{Z}/2) \otimes Ext_B^{s',s'}(\Sigma^{a'} I^{b'}, \mathbf{Z}/2) \longrightarrow Ext_B^{s+s',s+s'}(\Sigma^{a+a'} I^{b+b'}, \mathbf{Z}/2)$$

may be identified with the product

$$\begin{aligned}
Ext_B^{s+b,s-a}(\mathbf{Z}/2, \mathbf{Z}/2) &\otimes Ext_B^{s'+b',s'-a'}(\mathbf{Z}/2, \mathbf{Z}/2) \\
&\longrightarrow Ext_B^{s+s'+b+b',s+s'-a-a'}(\mathbf{Z}/2, \mathbf{Z}/2)
\end{aligned}$$

which is an isomorphism whenever both sides are non-trivial. Therefore, since $s(k,l) + s(l,m) = s(k,m)$, this is true in our case and there exists a 2-adic unit $u_{k,l,m} \in \mathbf{Z}_2^*$ such that

$$\iota_{l,m} \cdot \iota_{k,l} = u_{k,l,m} \iota_{k,m}.$$

This relation justifies the choice of $\iota_{k,l}$'s when $k - l \geq 2$ and completes the proof of Theorem 1.2 □

4 An application to algebraic K-theory

4.1 As in §1.1 let bu and bo denote the stable homotopy spectra representing 2-adically completed unitary and orthogonal connective K-theory respectively. Hence bu is a commutative ring spectrum with multiplication and unit maps $m : bu \wedge bu \longrightarrow bu$ and $\eta : S^0 \longrightarrow bu$, respectively. Also bo is a commutative ring spectrum and a two-sided bu-module.

Suppose now that E is a connective, right-bu-module spectrum. Hence we have a multiplication $\mu : E \wedge bu \longrightarrow E$ such that

$$\mu \cdot (1 \wedge m) \simeq \mu \cdot (\mu \wedge 1) : E \wedge bu \wedge bu \longrightarrow E.$$

Form the compositions

$$L_E : E \wedge bu = E \wedge S^0 \wedge bu \stackrel{1\wedge\eta\wedge 1}{\longrightarrow} E \wedge bu \wedge bu$$

$$\stackrel{1\wedge L^{-1}}{\longrightarrow} \vee_{k\geq 0} E \wedge bu \wedge (F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2)$$

$$\stackrel{\vee_{k\geq 0}\mu\wedge 1\wedge 1}{\longrightarrow} \vee_{k\geq 0} E \wedge (F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2)$$

and

$$\hat{L}_E : E \wedge bo = E \wedge S^0 \wedge bo \stackrel{1\wedge\eta\wedge 1}{\longrightarrow} E \wedge bu \wedge bo$$

$$\stackrel{1\wedge \hat{L}^{-1}}{\longrightarrow} \vee_{k\geq 0} E \wedge bu \wedge F_{4k}/F_{4k-1}$$

$$\stackrel{\vee_{k\geq 0}\mu\wedge 1}{\longrightarrow} \vee_{k\geq 0} E \wedge F_{4k}/F_{4k-1}$$

where L and $\hat{L}$ are the 2-local equivalences of Theorem 2.3.

Theorem 4.2

The maps L_E and $\hat{L}_E$ of §4.1 are 2-local homotopy equivalences.

Proof

We must show that L_E and $\hat{L}_E$ induces isomorphisms in mod 2 homology. The two cases are similar. However, this is easily seen for L_E from the discussion of §2.2. Identify $H_*(\vee_{k\geq 0}(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2); \mathbf{Z}/2)$ and $H_*(bu; \mathbf{Z}/2)$ with $\mathbf{Z}/2[\xi_1^2, \xi_2^2, \xi_3, \xi_4, \ldots]$ as in §2.2. Then the construction of L ensures that

$$(L^{-1})_*(1 \otimes z) = 1 \otimes z + \sum_\alpha b_\alpha \otimes c_\alpha$$

where each $b_\alpha \in H_*(bu; \mathbf{Z}/2)$ has strictly positive degree. Hence

$$(L_E)_*(a \otimes z) = a \otimes z + \sum_\alpha \mu_*(a \otimes b_\alpha) \otimes c_\alpha$$

and induction on the degree of z shows that $(L_E)_*$ is an isomorphism. □

4.3 Let F be an algebraically closed field of characteristic different from 2 then there is a homotopy equivalence of ring spectra $bu \simeq \underline{KF}\mathbf{Z}_2$ between 2-adic connective K-theory and the algebraic K-theory spectrum of F with coefficients in the 2-adic integers ([17] [18]). Let X be a scheme over $Spec(F)$ so that the algebraic K-theory spectrum of X with coefficients in the 2-adic integers, $\underline{KX}\mathbf{Z}_2$, is a right-$\underline{KF}\mathbf{Z}_2$-module spectrum. Setting $E = \underline{KX}\mathbf{Z}_2$ in Theorem 4.2 we obtain:

Corollary 4.4

There are 2-local homotopy equivalences of the form

$$L_{KX} : \underline{KX}\mathbf{Z}_2 \wedge \underline{KF}\mathbf{Z}_2 \longrightarrow \vee_{k\geq 0}\underline{KX}\mathbf{Z}_2 \wedge (F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbf{C}P^2)$$

and

$$\hat{L}_{KX} : \underline{KX}\mathbf{Z}_2 \wedge bo \longrightarrow \vee_{k\geq 0}\underline{KX}\mathbf{Z}_2 \wedge F_{4k}/F_{4k-1}.$$

Remark 4.5 The splitting of Theorem 2.3 may be used to give a family of well-behaved operations in connective K-theory. At odd primes this is developed in [6], for example. In a similar manner the splittings of Corollary 4.4 may be used to give a family of operations on the algebraic K-theory of F-schemes.

More precisely, for $k \geq 0$ let

$$\underline{KX}\mathbf{Z}_2(k) = \underline{KX}\mathbf{Z}_2 \wedge F_{4k}/F_{4k-1}$$

so that $\underline{KX}\mathbf{Z}_2(0) = \underline{KX}\mathbf{Z}_2$. Then we may define maps of spectra of the form

$$Q^n : \underline{KX}\mathbf{Z}_2(k) \longrightarrow \underline{KX}\mathbf{Z}_2(k+n)$$

to be given by the components of the composition

$$\vee_{n\geq 0} Q^n : \underline{KX}\mathbf{Z}_2 \wedge F_{4k}/F_{4k-1} \stackrel{1\wedge\eta\wedge 1}{\longrightarrow} \underline{KX}\mathbf{Z}_2 \wedge bo \wedge F_{4k}/F_{4k-1}$$

$$\stackrel{\hat{L}_{KX}\wedge 1}{\longrightarrow} \vee_{n\geq 0} \underline{KX}\mathbf{Z}_2 \wedge F_{4n}/F_{4n-1} \wedge F_{4k}/F_{4k-1}$$

$$\stackrel{1\wedge m}{\longrightarrow} \vee_{n\geq 0} \underline{KX}\mathbf{Z}_2 \wedge F_{4n+4k}/F_{4n+4k-1}.$$

Here $m : F_{4n}/F_{4n-1} \wedge F_{4k}/F_{4k-1} \longrightarrow F_{4n+4k}/F_{4n+4k-1}$, as in §2.2, is induced by the loopspace mulitplication on $\Omega^2 S^3$ via the Snaith splitting. The construction of the Q^n's imitates that of ([6] p.20).

In the case when X is a regular scheme of finite type over F these operations should induce interesting operations on Chow theory by virtue of the isomorphism ([11] Theorem 5.19)

$$H^p(X; \underline{\underline{K}}_p) \cong A^p(X).$$

Operations in connective K-theory have been thoroughly examined before ([1] [6] [8] [9]). The difference between my approach and previous ones is to view the Q^n's as lying in $\mathbf{Z}_2[U_\infty \mathbf{Z}_2]$ in order to control better the relations such as that between $Q^n Q^m$ and Q^{n+m} (cf. [6] p.98). I hope to elaborate on this application elsewhere.

Incidentally, using equivariant intersection cohomology theory, Steenrod operations on Chow theory have been constructed in [2] while similar operations are constructed in [19] using motivic cohomology.

5 Which is the matrix for $1 \wedge \psi^3$?

5.1 If $\psi^3 : bo \longrightarrow bo$ is the Adams operation in 2-local orthogonal connective K-theory then the homotopy class $[1 \wedge \psi^3]$ defines an element of $Aut^0_{left-bu-mod}(bu \wedge bo)$. It would be very interesting to know the identity of the matrix

$$X_{\psi^3} \in U_\infty \mathbf{Z}_2$$

which corresponds to $[1 \wedge \psi^3]$ under the isomorphism of Theorem 1.2. I have not been able to determine X_{ψ^3}. Of course, there is an ambiguity in the definition of the isomorphism of Theorem 1.2 due to the fact that each $\iota_{k+1,k}$

is defined only up to a 2-adic unit. This ambiguity is easy to accommodate. There is a more substantial problem. If

$$X_{\psi^3} = \sum_{0 \leq l \leq k} a_{l,k} \iota_{k,l} \in U_\infty \mathbf{Z}_2$$

the 2-adic integers, $a_{l,k}$, are determined by the effect of the $\iota_{k,l}$'s on $\pi_*(bu \wedge bo) \otimes \mathbf{Z}_2$ modulo torsion. Imitating [1] we study 2-adic homotopy modulo torsion by means of its image in $H_*(bu \wedge bo; \mathbf{Q}_2) \cong \mathbf{Q}_2[u, v^2]$. The Hurewicz image of $\pi_*(bu \wedge bo) \otimes \mathbf{Z}_2$ consists of those polynomials, $f(u, v^2) \in \mathbf{Z}_2[u/2, (v/2)^2]$ such that $f(at, b^2t^2) \in \mathbf{Z}_2[t, t^{-1}]$ for all odd integers a and b. The action of $1 \wedge \psi^3$ is given by the ring endomorphism of $\mathbf{Q}_2[u, v^2]$ which fixes u and sends v^2 to $9v^2$. Modifying the argument of [1] it is possible to determine a free $\mathbf{Z}_2$-basis for the space of such polynomials but we do not know whether it respects the decomposition of $\pi_*(bu \wedge bo) \otimes \mathbf{Z}_2$ into the summands $\pi_*(bu \wedge (F_{4k}/F_{4k-1})) \otimes \mathbf{Z}_2$ given by Theorem 2.3(ii). Having found such a $\mathbf{Z}_2$-basis, it is straightforward to determine the entries in X_{ψ^3} up to 2-adic units.

By way of illustration, consider the coefficient, $a_{0,k}$, of

$$\iota_{k,0} : bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (F_0/F_{-1}) = bu$$

in X_{ψ^3} for $k \geq 1$. This coefficient is determined by the effect on a generator $z_k \in \pi_{4k}(F_{4k}/F_{4k-1})$ represented by the inclusion of the bottom cell. Up to 2-adic units, $(\iota_{k,0})_*(z_k) = 2^{2k-\alpha(k)}u^{2k}$ where $\alpha(k)$ is the number of 1's in the dyadic expansion for k. When $k = 1$ we must have $z_1 = \omega u^2 + (v^2 - u^2)/4$ for some $\omega \in \mathbf{Z}_2$. Hence

$$(1 \wedge \psi^3)_*(z_1) = 9z_1 + (1 - 4\omega)2u^2 = 9(\iota_{1,1})_*(z_1) + (1 - 4\omega)(\iota_{1,0})_*(z_1).$$

This shows that $a_{1,1} = 9$ and $a_{0,1} = 1 - 4\omega \in 1 + 4\mathbf{Z}_2$. In fact ω must be divisible by 2 because $u^2(v^2 - u^2)/4$ is divisible by 2 in $\pi_8(bu \wedge bo) \otimes \mathbf{Z}_2$. Similarly, in order to calculate $a_{1,2}$ one shows that $(\iota_{2,1})_*(z_2) \in \pi_8(bu \wedge (F_4/F_3)) \otimes \mathbf{Z}_2$ is exactly divisible by $2^{2k-\alpha(k)}$. Then one chooses a suitable candidate for z_2, $(9u^4 - 10u^2v^2 + v^4)/16$ for example, determines its image under $(1 \wedge \psi^3)_*$ and reads off $a_{2,2}, a_{1,2}, a_{0,2}$.

Proceeding in this manner the ambiguity in the choice of z_k as a polynomial in u and v^2 soon gets out of hand. The following possibilities for the first six columns of X_{ψ^3} were produced for me by Ian Leary using a Maple programme to choose a $\mathbf{Z}_2$-basis for 2-adic homotopy modulo torsion. The entries in the matrices are only determined up to multiplication by 2-adic units and $e_{s,t} \in \{0, 1\}$.

5.2 *Some possibilities for* X_{ψ^3}[1]

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 9 & 1 & 2^{1-e_{1,3}} & 2^{1-e_{1,4}} & 2^{1-e_{1,5}} \\ 0 & 0 & 9^2 & 2^{1-e_{2,3}} & 1 & 1 \\ 0 & 0 & 0 & 9^3 & 2^{1-e_{3,4}} & 2^{1-e_{3,5}} \\ 0 & 0 & 0 & 0 & 9^4 & 2^{1-e_{4,5}} \\ 0 & 0 & 0 & 0 & 0 & 9^5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 9 & 1 & 1 & 2^{1-e_{1,4}} & 2^{1-e_{1,5}} \\ 0 & 0 & 9^2 & 2^{1-e_{2,3}} & 1 & 1 \\ 0 & 0 & 0 & 9^3 & 2^{1-e_{3,4}} & 2^{1-e_{3,5}} \\ 0 & 0 & 0 & 0 & 9^4 & 2^{1-e_{4,5}} \\ 0 & 0 & 0 & 0 & 0 & 9^5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 9 & 1 & 2^{1-e_{1,3}} & 2^{1-e_{1,4}} & 2^{1-e_{1,5}} \\ 0 & 0 & 9^2 & 2^{1-e_{2,3}} & 1 & 1 \\ 0 & 0 & 0 & 9^3 & 2^{1-e_{3,4}} & 2^{1-e_{3,5}} \\ 0 & 0 & 0 & 0 & 9^4 & 2^{1-e_{4,5}} \\ 0 & 0 & 0 & 0 & 0 & 9^5 \end{pmatrix}$$

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[1]Added January 2002: Using results from [4] Francis Clarke and the author have recently shown that X_{ψ^3} is conjugate in $U_\infty \mathbf{Z}_2$ to the matrix having 9^{i-1} in the (i,i)-entry, 1's in each $(i,i+1)$-entry and zero everywhere else.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 9 & 1 & 2^{1-e_{1,3}} & 2^{1-e_{1,4}} & 1 \\ 0 & 0 & 9^2 & 2^{1-e_{2,3}} & 1 & 1 \\ 0 & 0 & 0 & 9^3 & 2^{1-e_{3,4}} & 2^{1-e_{3,5}} \\ 0 & 0 & 0 & 0 & 9^4 & 2^{1-e_{4,5}} \\ 0 & 0 & 0 & 0 & 0 & 9^5 \end{pmatrix}$$

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