

Hasimoto Transformation and Vortex Soliton Motion Driven by Fluid Helicity

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Abstract

Vorticity filament motions with respect to the Dirac bracket of Rasetti and Regge [1975] are known to be related to the nonlinear Schrödinger equation by the Hasimoto transformation (HT), when the Hamiltonian is the Local Induction Approximation (LIA) of the kinetic energy. We show that when the Hamiltonian is the LIA of Euler-fluid helicity $\int \mathbf{u} \cdot \text{curl } \mathbf{u}$, the vorticity filament equation of motion under the Rasetti-Regge Dirac bracket is mapped by HT to the integrable complex modified Korteweg-de Vries (cmKdV) equation, the second equation in the nonlinear Schrödinger hierarchy.

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1 Introduction

Hasimoto [1972] discovered the fascinating map from vortex filament solutions of Euler's equations for ideal incompressible fluids in the Local Induction Approximation (LIA) to soliton solutions of the NonLinear Schrödinger (NLS) equation. Langer & Perline [1991] showed the Hasimoto transformation (HT) is a momentum map, which takes the Marsden and Weinstein [1983] Lie-Poisson bracket for vortex filaments as space curves, to the (fourth) Poisson bracket for NLS. (Marsden and Weinstein [1983] had also conjectured that such a momentum map might exist.) Langer & Perline [1991] also found a recursion relation which generates the hierarchy of space curve equations which maps by HT to the NLS hierarchy. An outstanding problem is to determine the corresponding fluid Hamiltonians whose LIA vortex dynamics produce the entire hierarchy of NLS equations. We began our inquiry into this question by considering the helicity of an ideal fluid.

Vortex filaments for Euler fluids can be knotted, and their *helicity*, defined by

$$\Lambda = \int \mathbf{u} \cdot \text{curl } \mathbf{u} \, d^3\mathbf{x} = \int \boldsymbol{\omega} \cdot \text{curl}^{-1} \boldsymbol{\omega} \, d^3\mathbf{x}, \quad (1.1)$$

measures the linkage number of $\boldsymbol{\omega} = \text{curl } \mathbf{u}$ in three dimensions, according to a formula due to Gauss. See, e.g., Moffatt [1969] for an early discussion of why fluid dynamicists are interested in helicity. For recent reviews of vortex filament dynamics from the viewpoint of helicity, see Ricca [1996]; Ricca & Berger [1996].

This paper starts in the framework of the Dirac bracket of Rasetti and Regge [1975] for Hamiltonian dynamics for vorticity filaments. Within this framework, we find that using the LIA helicity $\Lambda^{(LIA)}$ as the Hamiltonian (instead of using the LIA kinetic energy) produces dynamics for vortex filaments defined on space curves, in terms of the Marsden-Weinstein bracket $\{\cdot, \cdot\}_{MW}$ as

$$\mathbf{X}_t(t, s) = \{\mathbf{X}, \Lambda^{(LIA)}\}_{MW}, \quad (1.2)$$

where subscript t is partial time derivative at fixed arclength s along $\mathbf{X} \in \mathbb{R}^3$ and $\Lambda^{(LIA)}$ is helicity Λ of the filament (1.1) evaluated using the LIA. We apply the Hasimoto transformation to the space curve equation (1.2), whose explicit form is given in equation (4.9). This Hasimoto transformation recovers the complex modified Kortweg-de Vries equation (cmKdV), which is the *second* equation in the NLS hierarchy, for $\psi \in \mathbb{C}$,

$$\psi_t = \psi_{sss} + \frac{3}{2}|\psi|^2\psi_s. \quad (1.3)$$

Thus, soliton solutions of (1.3) yield LIA vorticity-filament solutions of (1.2) and *vice versa*. This connection between cmKdV and fluid helicity is the main result of the paper.

Recovery of the remaining equations in the NLS hierarchy from fluid Hamiltonians is possible, in principle, by adapting the recursion relation for space curves due

to Langer & Perline [1991] to the case of fluids. Unfortunately, the corresponding fluid Hamiltonians for these higher order space curve equations do not seem to be physically significant. The obstacle is that the reparameterization of the arclength variable which appears in the recursion relation for space curves due to Langer & Perline [1991] does not seem to yield Hamiltonians which have a fluid dynamical significance, except for two cases: the classical case of kinetic energy and the case of helicity treated here.

1.1 Outline of the paper

The main contents of the present paper are as follows:

1. In §2 we review Hamilton’s principle for vortex filaments.
2. In §3 we study the Hamiltonian dynamics of singular vorticity filaments supported on space curves in 3D. We show the Hamiltonian dynamics of these vortex solutions defined on filaments are governed by the Dirac-constrained Poisson bracket of Rasetti and Regge [1975]. We also relate the Rasetti-Regge Dirac bracket (RRDB) to the Lie-Poisson bracket developed in Marsden and Weinstein [1983] for vortex solutions defined on space curves in 3D, parameterized by arclength. These two Poisson brackets for vorticity filament dynamics are found to be equivalent up to a time-dependent reparameterization of coordinates along the filament.
3. As shown by Langer & Perline [1991], the Hasimoto transformation (HT) from the Frenet-Serret equations for 3D space curves to the nonlinear Schrödinger (NLS) equation defines an equivariant momentum map. An interesting question is to determine the *fluid* Hamiltonians which produce the entire hierarchy of NLS equations. In §4 we study filaments of $\boldsymbol{\omega} = \text{curl } \mathbf{u}$ and their space curve dynamics with respect to the Rasetti-Regge Dirac (RRD) bracket. When the Hamiltonian is $\Lambda^{(LIA)}$, the LIA version of helicity $\int \mathbf{u} \cdot \text{curl } \mathbf{u}$, we show that the Hasimoto transformation of the $\boldsymbol{\omega}$ -filament equation (4.13) arising in this approximation produces the integrable cmKdV equation in the NLS hierarchy. This is the main result of the paper.
4. In §5 we briefly explain the role of Langer-Perline reparameterization in generating other singular vortex solutions supported on filaments, whose LIA dynamics on space curves would be mapped by the Hasimoto transformation to members of the NLS hierarchy.
5. Finally, in §6, we discuss some of the remaining challenges and speculate on some of the possible future directions for this work.

Disclaimer of rigor Analytical issues (e.g., existence and uniqueness of weak solutions, etc.) will be neglected in this paper. Instead, we are primarily interested in exploring the formal properties of the singular vortex filament solutions. In some of these situations the Hamiltonian is given by a functional which is not even a

norm, thereby introducing many analytical issues beyond the scope of the present work. We shall also assume homogeneous boundary conditions everywhere, so we may freely perform integrations by parts.

2 Hamilton's principle for vortex filaments

Definition of vortex filament A *vortex filament* is a distribution of vorticity $\boldsymbol{\omega} = \text{curl } \mathbf{u}$ supported on a curve $\mathbf{R}(\sigma, t) \in \mathbb{R}^3$, as

$$\boldsymbol{\omega}(\mathbf{x}, t) = \int \mathbf{R}_\sigma \delta(\mathbf{x} - \mathbf{R}(\sigma, t)) d\sigma, \quad (2.1)$$

where $\mathbf{R}_\sigma = \partial \mathbf{R} / \partial \sigma$ is the vector tangent to the curve and equal to the vorticity at that point, $\delta(\mathbf{x} - \mathbf{R}(\sigma, t)) d\sigma$ is the Dirac measure along the curve, and σ is a *fixed* parametrization of the curve, say $\sigma \in [0, 1]$. The dynamics of this curve will depend on the choice of Hamiltonian, so in general $\boldsymbol{\omega}$ will *not* be frozen into the fluid motion. In fact, these filaments do *not* satisfy $\dot{\mathbf{R}}(\sigma, t) = \mathbf{u}(\mathbf{R}(\sigma, t), t)$, unless the Hamiltonian is taken to be the kinetic energy of the fluid. However, the velocity of the fluid which is induced by the filament vorticity is always given by the Biot–Savart law, which expresses $\mathbf{u} = \text{curl}^{-1} \boldsymbol{\omega}$ as,

$$\mathbf{u}(\mathbf{x}, t) = \frac{1}{4\pi} \int \mathbf{R}_\sigma \times \frac{\mathbf{x} - \mathbf{R}(\sigma, t)}{|\mathbf{x} - \mathbf{R}(\sigma, t)|^3} d\sigma.$$

Gauge freedom associated with time-dependent reparameterization When the curve is parametrized by arclength s , we denote it by $\mathbf{X}(s(\sigma, t), t) = \mathbf{R}(\sigma, t)$, and we will often distinguish between the time derivative $\dot{\mathbf{R}}(\sigma, t)$ at constant σ , and the time derivative at constant arclength, $\mathbf{X}_t(s, t)$. This time-dependent reparameterization of its fixed σ coordinate corresponds to fluid motion along the vortex filament. Such a flow leaves the filament configuration invariant, so it may be regarded as the *gauge freedom* in vortex filament dynamics.

Hamilton's principle for vortex filaments The Lagrangian in Hamilton's principle for the self-induced motion of vortex filaments given in [Rasetti and Regge \[1975\]](#) involves the difference between a purely geometric term and a dynamical term expressing the vortex filament energy,

$$S = \int \mathcal{L}[\mathbf{R}, \dot{\mathbf{R}}] dt = \int \left\{ \frac{1}{3} \int \dot{\mathbf{R}} \cdot \mathbf{R} \times \mathbf{R}_\sigma d\sigma - H[\mathbf{R}] \right\} dt. \quad (2.2)$$

The equations of motion for the filament follow from Hamilton's principle, as

$$0 = \delta S = - \int \delta \mathbf{R} \cdot \left\{ \dot{\mathbf{R}} \times \mathbf{R}_\sigma + \frac{\delta H}{\delta \mathbf{R}} \right\} d\sigma dt. \quad (2.3)$$

Consistency requires $\mathbf{R}_\sigma \cdot \delta H / \delta \mathbf{R} = 0$. Hamiltonians H for which $\mathbf{R}_\sigma \cdot \delta H / \delta \mathbf{R} = 0$ are called *gauge invariant* in [Rasetti and Regge \[1975\]](#). In particular, [Holm \[2003\]](#)

shows that vorticity Hamiltonians are gauge invariant. We write $\delta H/\delta\boldsymbol{\omega}(\mathbf{R})$ as the variation of $H(\boldsymbol{\omega})$ with respect to the Eulerian vorticity, *evaluated on the filament*. This yields the following relations among functional derivatives [Holm \[2003\]](#),

$$\frac{\delta H(\boldsymbol{\omega})}{\delta \mathbf{R}} = \mathbf{R}_\sigma \times \text{curl}_R \frac{\delta H}{\delta \boldsymbol{\omega}} = \mathbf{t} \times \mathbf{t} \times \frac{\partial}{\partial \sigma} \frac{\delta H}{\delta \boldsymbol{\omega}}(\mathbf{R}(\sigma, t), t) \equiv -\hat{P} \frac{\partial}{\partial \sigma} \frac{\delta H}{\delta \boldsymbol{\omega}}(\mathbf{R}(\sigma, t), t).$$

Here we have introduced the operator $\hat{P} \equiv -\mathbf{t} \times \mathbf{t} \times$ where $\mathbf{t} = \mathbf{R}_\sigma/R_\sigma$ is the unit tangent vector and $R_\sigma = |\mathbf{R}_\sigma|$. The operator \hat{P} projects any vector at a point on the vortex filament onto the transverse plane normal to the filament at that point. Taking the dot product recovers the result in [Holm \[2003\]](#) that $\mathbf{R}_\sigma \cdot \delta H(\boldsymbol{\omega})/\delta \mathbf{R} = 0$ for vorticity Hamiltonians.

Hence, the result of Hamilton's principle in equation (2.3) may be written as

$$0 = \delta S = - \int \mathbf{R}_\sigma \times \delta \mathbf{R} \cdot \left\{ \dot{\mathbf{R}} + \frac{1}{R_\sigma^2} \mathbf{R}_\sigma \times \frac{\delta H}{\delta \mathbf{R}} \right\} d\sigma dt. \quad (2.4)$$

This yields the motion equation,

$$\hat{P} \dot{\mathbf{R}} = - \frac{1}{R_\sigma^2} \mathbf{R}_\sigma \times \frac{\delta H}{\delta \mathbf{R}} \quad \text{where} \quad \hat{P} \dot{\mathbf{R}} = -\mathbf{t} \times \mathbf{t} \times \dot{\mathbf{R}}, \quad (2.5)$$

which is the same equation as found in [Rasetti and Regge \[1975\]](#) when using their Poisson bracket for gauge invariant Hamiltonians.

The vortex filament Lagrangian is singular Being linear in $\dot{\mathbf{R}}$, the Lagrangian in (2.2) is *singular*. That is, the canonical momentum for the filament obtained from Hamilton's principle, may be expressed as a function of \mathbf{R} ,

$$\mathbf{P} = \frac{\delta S}{\delta \dot{\mathbf{R}}} = \frac{1}{3} \mathbf{R} \times \mathbf{R}_\sigma. \quad (2.6)$$

This, of course, imposes a relation between the canonical variables \mathbf{P} and \mathbf{R} . This functional dependence between momentum \mathbf{P} and position \mathbf{R} was addressed in [Rasetti and Regge \[1975\]](#) by using the Dirac constraint procedure in their derivation of the Poisson bracket for the Hamiltonian formulation of vortex filament dynamics. The vortex filament Hamiltonian is found from the Legendre transformation,

$$\mathcal{H}[\mathbf{R}, \mathbf{P}] = \int \mathbf{P} \cdot \dot{\mathbf{R}} d\sigma - \mathcal{L}[\mathbf{R}, \dot{\mathbf{R}}] = H[\mathbf{R}]. \quad (2.7)$$

So the filament Hamiltonian depends only on \mathbf{R} , and this dependence is determined entirely by the second term $H[\mathbf{R}]$ in the singular Lagrangian in (2.2). This means Hamilton's principle for vorticity filaments (2.3) applies generally, even for Lagrangians in which the second term $H[\mathbf{R}]$ is *not* the kinetic energy, as in the Euler case.

Vortex impulse Geometrically, the *vortex impulse* $\mathbf{I} = \oint_{c(\mathbf{R})} \mathbf{P} d\sigma$ for a closed vortex filament loop $c(\mathbf{R})$ is two-thirds its projected area. Namely,

$$\mathbf{I} = \oint_{c(\mathbf{R})} \mathbf{P} d\sigma = \frac{1}{3} \oint_{c(\mathbf{R})} \mathbf{R} \times d\mathbf{R} = \frac{2}{3} \int d\mathbf{S}. \quad (2.8)$$

Thus, the vortex impulse is a geometrical quantity associated with the shape of the filament loop. Its preservation for Euler vortex filament loops is a well known property, as discussed in [Newton \[2001\]](#); [Saffman \[1992\]](#). Remarkably, vortex impulse preservation holds for all Hamiltonian functionals $H[\mathbf{R}_\sigma]$ (See equation (3.3).) That is, vortex impulse is preserved for all Hamiltonian functionals that depend on $\mathbf{R}(\sigma, t)$ only through its derivatives with respect to the coordinate σ .

Action integral The geometric term in Hamilton's principle turns out to be the *action integral* for the filament,

$$\oint_{c(\mathbf{R})} \mathbf{P} \cdot \dot{\mathbf{R}} d\sigma dt = \frac{1}{3} \iint \mathbf{R} \cdot \mathbf{R}_\sigma \times \dot{\mathbf{R}} d\sigma dt \equiv \frac{1}{3} \text{Tr} \iint \mathbf{R} \cdot d\mathbf{R} \wedge d\mathbf{R}. \quad (2.9)$$

This strongly geometrical object resembles a differential form in Chern-Simons string theory, defined over a space-time surface $\mathbf{S}(\sigma, t)$ whose spatial boundary $\partial\mathbf{S}$ is the vortex filament, regarded as the circuit $c(\mathbf{R}(\sigma, t))$ at time t . For more discussion and references about this viewpoint, see [Speliotopoulos \[2002\]](#). For an interesting discussion of topological invariants of space curves, see [Thurston \[1999\]](#).

3 Equivalence of Rasetti & Regge and Marsden & Weinstein Poisson brackets

Rasetti & Regge Dirac constrained bracket

By using the Dirac constraint procedure, [Rasetti and Regge \[1975\]](#) derived the Poisson bracket which gives the dynamics for a vorticity filament, parameterized by an arbitrary coordinate, σ . This bracket takes the form

$$\dot{\mathbf{R}}(\sigma, t) = \{\mathbf{R}, H\}_{RR} = -\frac{1}{R_\sigma^2} \mathbf{R}_\sigma \times \frac{\delta H}{\delta \mathbf{R}}, \quad (3.1)$$

where, as before, the subscript σ denotes differentiation and $R_\sigma = \sqrt{\mathbf{R}_\sigma \cdot \mathbf{R}_\sigma}$ is the magnitude of the tangent vector. The form of the bracket given in equation (3.1) is valid only for *gauge invariant* Hamiltonians. These include Hamiltonians which can be written as functionals of the vorticity, $H[\mathbf{R}_\sigma]$, as shown in [Holm \[2003\]](#), and this will cover all the Hamiltonians we will consider. For Hamiltonians which are not gauge invariant, additional terms would appear in the RRDB (3.1). However, we shall not need them in what follows.

Impulse conservation For *gauge invariant* Hamiltonians $H[\mathbf{R}_\sigma]$, the vortex impulse \mathbf{I} in equation (2.8) is conserved under the vortex filament evolution. Using incompressibility, one computes the time derivative $d\mathbf{I}/dt$ as

$$\begin{aligned} \frac{d\mathbf{I}}{dt} &= \frac{d}{dt} \oint_{c(\mathbf{R})} \mathbf{R} \times \mathbf{R}_\sigma d\sigma = 2 \oint_{c(\mathbf{R})} \dot{\mathbf{R}} \times \mathbf{R}_\sigma d\sigma + \underbrace{\oint_{c(\mathbf{R})} \partial_\sigma (\mathbf{R} \times \dot{\mathbf{R}}) d\sigma}_{\text{vanishes}} \\ &= \{\mathbf{I}, H\}_{RR} = 2 \oint_{c(\mathbf{R})} \frac{1}{R_\sigma^2} \mathbf{R}_\sigma \times \left(\mathbf{R}_\sigma \times \frac{\delta H}{\delta \mathbf{R}} \right) d\sigma \\ &= -2 \oint_{c(\mathbf{R})} \frac{\delta H}{\delta \mathbf{R}} d\sigma = 2 \oint_{c(\mathbf{R})} \partial_\sigma \left(\frac{\delta H}{\delta \mathbf{R}_\sigma} \right) d\sigma = 0. \end{aligned} \quad (3.2)$$

This conservation law holds for *every* Hamiltonian $H[\mathbf{R}_\sigma]$. Thus, the vortex impulse $\mathbf{I} = \oint_{c(\mathbf{R})} \mathbf{P} d\sigma$ for a closed vortex filament loop $c(\mathbf{R})$ is a *Casimir* of the Rasetti-Regge Dirac bracket for vortex filament dynamics. That is, vortex filament dynamics under the Rasetti-Regge Dirac bracket preserves the projected area of a closed vortex loop for any vorticity Hamiltonian, $H[\mathbf{R}_\sigma]$.

Marsden & Weinstein symmetry reduction bracket

Marsden and Weinstein [1983] applied their method of reduction by symmetry to the study of Clebsch variables and vortices for the incompressible motion of ideal fluids. The Eulerian fluid velocity for such motions is in the Lie algebra of divergence-free vector fields \mathbf{X} on an n -dimensional manifold M (such as \mathbb{R}^n). Reduction by invariance of the kinetic energy of the fluid under Lagrangian relabeling symmetry induces a Lie-Poisson Hamiltonian structure on the dual Lie algebra \mathfrak{X}^* , as found in Kuznetsov & Mikhailov [1980]. Marsden and Weinstein [1983] identified \mathfrak{X}^* with the space of Eulerian vorticities and interpreted their Helmholtz dynamics,¹

$$\frac{\partial}{\partial t} F[\boldsymbol{\omega}] = \{F, H\}[\boldsymbol{\omega}] = \left\langle \boldsymbol{\omega}, \left[\frac{\delta F}{\delta \boldsymbol{\omega}}, \frac{\delta H}{\delta \boldsymbol{\omega}} \right] \right\rangle = \int \boldsymbol{\omega} \cdot \text{curl} \frac{\delta F}{\delta \boldsymbol{\omega}} \times \text{curl} \frac{\delta H}{\delta \boldsymbol{\omega}} d^3x, \quad (3.3)$$

as preservation of coadjoint orbits. (An equivalent dual interpretation of vorticity dynamics as preservation of adjoint orbits was also available to them, but was not discussed.) Clebsch variables were considered as momentum maps. In their discussions of vortex filaments, they parametrized the vortex filament as a space curve with the arclength s , and their equation for the reduced Poisson bracket with the vortex filament position was

$$\dot{\mathbf{R}}(\sigma, t) = \{\mathbf{R}, H\}_{MW} = -\mathbf{t} \times \frac{\delta H}{\delta \mathbf{R}}, \quad (3.4)$$

¹The helicity $\Lambda(\boldsymbol{\omega}) = \int \boldsymbol{\omega} \cdot \text{curl}^{-1} \boldsymbol{\omega} d^3x$ is a Casimir for this Lie-Poisson vorticity bracket. Thus, as is well known, vorticity dynamics for $\boldsymbol{\omega} = \text{curl} \mathbf{u}$ in three dimensions under the Lie-Poisson vorticity bracket preserves the linkage number of the vorticity distribution for any vorticity Hamiltonian, $H(\boldsymbol{\omega})$. In particular, the flow induced by the helicity $\Lambda(\boldsymbol{\omega})$ under the Lie-Poisson vorticity bracket (3.3) Poisson-commutes with the Euler flow, induced by the kinetic energy. As shown in Holm, Marsden and Ratiu [1987] the Lagrangian relabeling symmetry induced by the helicity $\Lambda(\boldsymbol{\omega})$ under the Lie-Poisson vorticity bracket shifts Lagrangian fluid parcels along the streamlines of steady Beltrami flows.

where $\mathbf{t} = \mathbf{R}_\sigma/R_\sigma$ is the unit tangent vector of the vortex filament.

Equivalence of Rasetti & Regge and Marsden & Weinstein brackets

At first sight, the two brackets in equations (3.1) and (3.4) seem to differ, but the difference is only in the interpretation of the variational derivative, $\delta H/\delta \mathbf{R}$. Rasetti and Regge [1975] used a fixed parametrization σ of the space curve, so their variational derivative would be defined by

$$\delta H = \int \frac{\delta H^{(\sigma)}}{\delta \mathbf{R}} \cdot \delta \mathbf{R} d\sigma. \quad (3.5)$$

On the other hand, Marsden and Weinstein [1983] worked with the arclength parametrization, so their variational derivative would be defined by

$$\delta H = \int \frac{\delta H^{(s)}}{\delta \mathbf{R}} \cdot \delta \mathbf{R} ds. \quad (3.6)$$

The time-dependent change of independent variables between σ and $s(\sigma, t)$ leads to $ds = R_\sigma d\sigma$. Consequently, equations (3.5) and (3.6) show that the two variational derivatives are related by

$$\frac{\delta H^{(\sigma)}}{\delta \mathbf{R}} = \frac{\delta H^{(s)}}{\delta \mathbf{R}} R_\sigma. \quad (3.7)$$

Taking this into account, it is clear that the two brackets in equations (3.1) and (3.4) are the same. We also note here some other equivalences (in a slight abuse of notation)

$$-\frac{1}{R_\sigma} \frac{\delta H^{(\sigma)}}{\delta \mathbf{R}} = \partial_s \frac{\delta H^{(\sigma)}}{\delta \mathbf{R}_\sigma} = \partial_s \frac{\delta H^{(s)}}{\delta \mathbf{t}}, \quad (3.8)$$

which can be shown in a similar fashion with an integration by parts.

Consequently, we have made the following observation.

Lemma 3.1 (Gauge equivalence of the two vortex Poisson brackets). *The two Poisson brackets for ω -vortex filament dynamics due separately to Rasetti and Regge [1975] and Marsden and Weinstein [1983] are equivalent under a time-dependent reparameterization of coordinates along the filament. That is, the Poisson bracket $\{\mathbf{X}, H\}_{MW}$ in equation (3.4) is a reparameterization of $\{\mathbf{R}, H\}_{RR}$ in equation (3.1). In this sense, the two Poisson brackets are gauge equivalent.*

4 Helicity filament dynamics and complex mKdV

In the rest of the paper, we will examine the dynamics of filaments of $\boldsymbol{\omega} = \text{curl } \mathbf{u}$ with respect to the RRD bracket. In particular, in this section we study the case of such $\boldsymbol{\omega}$ -filaments when the Euler-fluid helicity $\int \mathbf{u} \cdot \text{curl } \mathbf{u}$ is used as the Hamiltonian for the RRD bracket. We will show that under the Local Induction Approximation, the corresponding equation of motion is mapped to the complex modified KdV equation by the Hasimoto transformation.

4.1 The Helicity-Driven Filament Equation

For our vortex filament in (2.1), the vorticity $\boldsymbol{\omega} = \text{curl}\mathbf{u}$ takes the value $\mathbf{R}_\sigma(\sigma, t)$ for points on the filament and it vanishes for points not on the filament. The helicity can then be written in terms of the filament locus $\mathbf{R}(\sigma, t)$ as

$$\begin{aligned} \Lambda &= \int \boldsymbol{\omega} \cdot \text{curl}^{-1}\boldsymbol{\omega} \, d^3x \\ &= \frac{1}{4\pi} \int \mathbf{R}_\sigma(\sigma, t) \cdot \left(\int \frac{\mathbf{R}_{\sigma'}(\sigma') \times (\mathbf{R}(\sigma) - \mathbf{R}(\sigma'))}{|\mathbf{R}(\sigma) - \mathbf{R}(\sigma')|^3} d\sigma' \right) d\sigma \\ &\equiv \frac{1}{4\pi} \text{Tr} \iint \frac{(\mathbf{R} - \mathbf{R}') \cdot d\mathbf{R} \wedge d\mathbf{R}'}{|\mathbf{R} - \mathbf{R}'|^3}. \end{aligned} \quad (4.1)$$

The last expression emphasizes the similarity between the numerator of the helicity integrand and the space-time geometrical quantity of the action in (2.9)

In order to compute the RRDB $\{\mathbf{R}, \Lambda\}_{RR}$, we rewrite its definition from equation (3.1) using equation (3.8):

$$\{\mathbf{R}, \Lambda\}_{RR} = -\frac{1}{R_\sigma^2} \mathbf{R}_\sigma \times \frac{\delta\Lambda}{\delta\mathbf{R}} = \mathbf{t} \times \partial_s \frac{\delta\Lambda}{\delta\mathbf{t}}. \quad (4.2)$$

Since the helicity in equation (4.1) is symmetric in \mathbf{R} and \mathbf{R}' , we may easily use the form of the RRDB in equation (4.2) to obtain the exact helicity-driven filament dynamics

$$\dot{\mathbf{R}}(\sigma, t) = \{\mathbf{R}, \Lambda\}_{RR} = \frac{1}{2\pi} \mathbf{t} \times \partial_s \left(\int \mathbf{R}_{\sigma'} \times \frac{\mathbf{R}(\sigma) - \mathbf{R}(\sigma')}{|\mathbf{R}(\sigma) - \mathbf{R}(\sigma')|^3} d\sigma' \right). \quad (4.3)$$

The Local Induction Approximation implies (for details, see [Arms and Hama \[1965\]](#)),

$$\int \mathbf{R}_{\sigma'} \times \frac{\mathbf{R}(\sigma) - \mathbf{R}(\sigma')}{|\mathbf{R}(\sigma) - \mathbf{R}(\sigma')|^3} d\sigma' \approx \log(\epsilon^{-1}) \kappa \mathbf{b}, \quad (4.4)$$

where ϵ is the radius of the thin vortex tube which our filament approximates. Consequently, we obtain helicity-driven filament dynamics in the LIA,

$$\dot{\mathbf{R}}(\sigma, t) \approx \frac{\log \epsilon^{-1}}{2\pi} \mathbf{t} \times \partial_s (\kappa \mathbf{b}). \quad (4.5)$$

Using the Serret-Frenet relations,

$$\mathbf{t}_s = \kappa \mathbf{n}, \quad \mathbf{n}_s = \tau \mathbf{b} - \kappa \mathbf{t}, \quad \mathbf{b}_s = -\tau \mathbf{n}, \quad (4.6)$$

where \mathbf{n} and \mathbf{b} are the unit normal and unit binormal vectors, we write equation (4.5) as

$$\dot{\mathbf{R}}(\sigma, t) = \kappa_s \mathbf{n} + \kappa \tau \mathbf{b}, \quad (4.7)$$

where we have rescaled time to absorb $-\log \epsilon^{-1}/2\pi$ and assumed $\log \epsilon^{-1}$ is approximately constant.

4.2 Hasimoto Transformation

In order to apply the Hasimoto transformation, we need to write equation (4.5) in terms of a time derivative with arclength s held constant, $\mathbf{X}_t(s, t)$. Using the chain rule, we see that

$$\dot{\mathbf{R}}(\sigma, t) = \frac{d\mathbf{X}}{dt}(s(\sigma, t), t) = \mathbf{X}_t(s, t) + \frac{\partial s}{\partial t} \mathbf{t}.$$

For a vector field

$$\dot{\mathbf{R}}(\sigma, t) = \mathbf{W} = g\mathbf{n} + h\mathbf{b},$$

we thus have

$$\frac{\partial s}{\partial t} = \int^\sigma \frac{\partial R_{\sigma'}}{\partial t}(\sigma', t) d\sigma' = \int^\sigma \mathbf{t} \cdot (\partial_{s'} \dot{\mathbf{R}}) R_{\sigma'} d\sigma' = - \int^s g\kappa ds'. \quad (4.8)$$

Hence, equation (4.7) reparameterizes to

$$\mathbf{X}_t(s, t) = \frac{1}{2}\kappa^2 \mathbf{t} + \kappa_s \mathbf{n} + \kappa \tau \mathbf{b}, \quad (4.9)$$

in the arclength representation. This equation was shown in Langer & Perline [1991] to map into the complex mKdV equation (4.11) under the Hasimoto transformation.

For the Hasimoto transformation (see Hasimoto [1972] for details), one encodes the geometric information of the curve in a new complex variable

$$\psi(s, t) = \kappa(s, t) \exp\left(i \int^s \tau(s') ds'\right). \quad (4.10)$$

In terms of this variable, the helicity-driven filament equation (4.9) becomes the complex mKdV equation

$$\psi_t = \psi_{sss} + \frac{3}{2}|\psi|^2 \psi_s. \quad (4.11)$$

Hence, we have proven our main result.

Theorem 4.1 (Complex modified KdV soliton vortex arises from helicity).

*The vortex filament dynamics (4.7) of $\boldsymbol{\omega} = \text{curl } \mathbf{u}$ driven by helicity (4.1) in LIA may be reparameterized as the space curve equation (4.9) which maps via the Hasimoto transformation into the **complex modified Kortweg-de Vries equation** for $\psi(s, t) \in \mathbb{C}$,*

$$\psi_t = \psi_{sss} + \frac{3}{2}|\psi|^2 \psi_s, \quad (4.12)$$

where s is arclength. Thus, soliton solutions of (4.12) yield LIA $\boldsymbol{\omega}$ -filament solutions driven by helicity, and vice versa.

Remarks

- Because of the equivalence between Poisson brackets shown in Lemma 3.1, we may write the space curve equation (4.9) in terms of the Marsden-Weinstein bracket as

$$\mathbf{X}_t(t, s) = \{\mathbf{X}, \Lambda^{(LIA)}\}_{MW} = \frac{1}{2}\kappa^2\mathbf{t} + \kappa_s\mathbf{n} + \kappa\tau\mathbf{b}. \quad (4.13)$$

By using the Serret-Frenet relations (4.6) and the definition of the tangent vector $\mathbf{t} = \mathbf{X}_s/X_s$, one finds $\mathbf{X}_s \cdot \mathbf{X}_{st} = 0$; so, as expected, equation (4.13) preserves the magnitude of the vortex strength $X_s = |\mathbf{X}_s|$. Hence, if it is initially constant along the filament, the vortex strength will remain constant under the helicity LIA dynamics of (4.13).

- Perhaps surprisingly, the complex modified KdV equation was also found from the Hasimoto transformation for a *different* vortex Hamiltonian in Kuznetsov & Ruban [2000]. Namely, Kuznetsov & Ruban [2000] state that the complex modified KdV equation (4.12) is found (they say, up to a gauge transformation) via the Hasimoto transformation of the space curve equation resulting from the vortex Hamiltonian expressed in a mixed representation as $\mathcal{H} = \int |\text{curl } \mathbf{u}| \chi \, d^3x$, where χ is the torsion of the vortex line. The present result is obtained by using the helicity (4.1) as the vortex Hamiltonian, instead.
- Being a Casimir, the vorticity dynamics induced by the helicity under the Lie-Poisson bracket for vorticity (3.3) is compatible with (leaves invariant, or commutes with) the corresponding vorticity dynamics induced by the Euler kinetic energy. The LIA and HT each preserve this compatibility. Langer & Perline [1991] show that the LIA space curve dynamics (4.13) is compatible with the da Rios-Betchov equation for space curve motion of vorticity filaments induced by the Euler kinetic energy. And of course by mapping these to the NLS isospectral hierarchy of integrable equations, the HT preserves compatibility.

Simple Solution Behavior

The behavior of some simple solutions to the filament equation (4.7),

$$\dot{\mathbf{R}}(\sigma, t) = \kappa_s\mathbf{n} + \kappa\tau\mathbf{b}, \quad (4.14)$$

can immediately be seen.

Circles A circle has constant curvature and zero torsion. Thus the filament equation above with a circular filament as the initial condition has a simple solution: the circular filament remains where it is. The velocity field it generates is then steady and given by the Biot-Savart law.

In contrast, a circular filament for the standard da Rios-Betchov LIA equation,

$$\dot{\mathbf{R}}(\sigma, t) = \kappa\mathbf{b}, \quad (4.15)$$

behaves in a different manner. This is the well-known LIA vortex filament solution for $\ell[\mathbf{u}] = \frac{1}{2} \int \mathbf{u} \cdot \mathbf{u}$, the Euler fluid. The circular filament moves with constant velocity along the axis through its center. The velocity field it generates is then a Galilean shift of the velocity field generated by the circular filament solution to equation (4.14). For this case of the Euler-fluid, the vortex filament moves with the fluid. By the way, note that reparameterization $\sigma \rightarrow s(\sigma, t)$ leaves the form of equation (4.15) invariant. That is, in the da Rios-Betchov case, $\dot{\mathbf{R}}(\sigma, t) = \mathbf{X}_t(s(\sigma, t), t)$.

Filaments Lying in a Plane Since any space curve which lies in a plane has zero torsion, any planar filament evolving under equation (4.14) will remain in the plane. This may not be surprising, because the helicity for a flow in the plane vanishes identically.

5 Langer-Perline reparameterization and mapping to the NLS hierarchy

Langer & Perline [1991] found the hierarchy of compatible filament equations which are mapped to the NLS hierarchy under the Hasimoto transformation. In fact, they showed that the Hasimoto transformation is a Poisson map with respect to the Marsden-Weinstein bracket and the fourth NLS bracket. The recursion operator \mathcal{R} that generates the hierarchy of filament equations from the da Rios-Betchov local induction equation $\mathbf{X}_t = \kappa \mathbf{b}$ is given by

$$\mathcal{R}\mathbf{W} = \mathcal{P}(\mathbf{t} \times \partial_s \mathbf{W}). \quad (5.1)$$

The operator \mathcal{P} acts on a vector field $\mathbf{W} = g\mathbf{n} + h\mathbf{b}$ by

$$\mathcal{P}\mathbf{W} = \left(\int^s g \kappa ds' \right) \mathbf{t} + g\mathbf{n} + h\mathbf{b}. \quad (5.2)$$

Thus, \mathcal{P} reparametrizes a vector field $\dot{\mathbf{R}}(\sigma, t) = \mathbf{W}$ by writing it in the equivalent form $\mathbf{X}_t(s, t) = \mathcal{P}\mathbf{W}$, as is seen from equation (4.8) above.

With the result of Langer & Perline [1991] showing the correspondence between filament equations and the NLS hierarchy, a natural question arises when we take one step back: Which fluid Hamiltonians lead to those filament equations? We showed in §4 that using the helicity $\Lambda = \int \mathbf{u} \cdot \text{curl } \mathbf{u} d^3x$ in the RRD bracket generates the the filament equation $\mathbf{X}_t = \frac{1}{2}\kappa^2 \mathbf{t} + \kappa_s \mathbf{n} + \kappa \tau \mathbf{b}$, which maps to the second equation (complex mKdV) in the NLS hierarchy under the Hasimoto transformation. The filament equations we seek which map to the higher NLS flows are given by

$$\mathbf{X}_t(s, t) = [\mathcal{P}(\mathbf{t} \times \partial_s)]^n(\kappa \mathbf{b}), \quad n = 0, 1, \dots \quad (5.3)$$

As a step in the direction of finding the fluid Hamiltonians needed to obtain equation (5.3), we note the correspondence

$$\text{curl} \quad \leftrightarrow \quad \mathbf{t} \times \partial_s. \quad (5.4)$$

But the Hamiltonians

$$\int \mathbf{u} \cdot \text{curl}^n \mathbf{u} \quad (5.5)$$

will lead ² to the $\boldsymbol{\omega}$ -filament equations

$$\mathbf{X}_t(s, t) = \mathcal{P}(\mathbf{t} \times \partial_s)^n(\kappa \mathbf{b}),$$

which are slightly different from those in equation (5.3). The other part of the recursion operator \mathcal{R} which must be accounted for is the operator \mathcal{P} given in equation (5.2).

The correspondence (5.4) does not provide the desired recursion relation linking other fluid properties in (5.5) beyond the kinetic energy ($n = 0$) and the helicity ($n = 1$) to higher NLS flows via LIA and the Hasimoto transformation. In addition, the higher degree curls in (5.5) do not produce vorticity functionals which mutually commute under the Lie-Poisson vorticity bracket. To account for the integral in equation (5.2), we can speculate that the relation (see also equation (4.8))

$$\int^{\mathbf{x}} \boldsymbol{\omega} \cdot \nabla_{\mathbf{x}'} \mathbf{W} d^3x \quad \leftrightarrow \quad \int^s \mathbf{t} \cdot \partial_s \mathbf{W} ds',$$

would indicate the correspondence

$$(1 - \int^{\mathbf{x}} d^3x' \boldsymbol{\omega} \cdot \nabla_{\mathbf{x}'}) \text{curl} \mathbf{W} \quad \leftrightarrow \quad \mathcal{P}(\mathbf{t} \times \partial_s \mathbf{W}),$$

which includes the required reparameterization. However, iterating this correspondence produces a sequence of Hamiltonians which apparently have no fluid dynamical significance. Kuznetsov & Ruban [2000] reach a similar conclusion for a different set of Hamiltonians defined using a mixture of vorticity and filament properties such as curvature and torsion.

6 Outlook

Brief summary A time-dependent reparameterization of coordinates along a vortex filament corresponds to a collinear flow along the filament. This collinear flow is a gauge symmetry which has no physical significance, but it facilitates the application of the Hasimoto transformation through the use of the Serret-Frenet equations, when the reparameterization is chosen to be the arclength coordinate on the filament. The helicity produces the LIA filament equation (4.7) which may be reparameterized into (4.9), whose space curve dynamics in the arclength representation was shown in Langer & Perline [1991] to map into the complex mKdV equation (4.11) under the Hasimoto transformation. Further applications of the Langer & Perline [1991] recursion to obtain space curve dynamics corresponding to higher order equations in the NLS hierarchy seem not to correspond to physically interesting fluid Hamiltonians.

²We note that the Local Induction Approximation $\text{curl}^{-1} \boldsymbol{\omega} \approx \kappa \mathbf{b}$ must be used even if it is not needed. For instance, when the Hamiltonian $\int \boldsymbol{\omega} \cdot \text{curl}^2 \boldsymbol{\omega}$ is used with the bracket of equation (4.2), we would obtain $\mathbf{X}_t(s, t) = \mathcal{P}(\mathbf{t} \times \partial_s)^4(\kappa \mathbf{b})$ by writing $\text{curl}^2 \mathbf{R}_\sigma = \text{curl}^3 \text{curl}^{-1} \mathbf{R}_\sigma \approx \text{curl}^3(\kappa \mathbf{b})$.

Other Issues Of course there are many other issues remaining to explore that are suggested by the above setting. These include investigating,

- Typical motions of space curves according to the dynamics of the space curve equation (4.9) for vortex filaments driven by helicity in the LIA.
- Vortex solitons, that is, the map from the solitons of the complex modified KdV equation (4.12) to filament motions.

We shall, however, leave these issues for other publications and other researchers.

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