

Around unipotence in groups of finite Morley rank

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1 Introduction

The central question about groups of finite Morley rank is the Cherlin-Zil'ber Conjecture, which states that the infinite simple ones are algebraic. The search for satisfactory analogues in the context of groups of finite Morley rank of key notions in algebraic group theory has therefore been a continuing concern.

Semisimple and *unipotent* elements are fundamental for understanding affine algebraic groups. While *Carter subgroups* of groups of finite Morley rank (Definition 6.5) offer a reasonably well-behaved analogue of maximal tori in algebraic groups (see [12], [14] and [21]), there has been more than one proposition for an analogue of unipotent subgroups of algebraic groups. The most recent and most effective of these has been introduced by Burdges in [7] (see section 2). In this article, following up on Burdges' ideas, we introduce and study *U-groups* (Definition 5.1) as a new analogue of unipotent subgroups for groups of finite Morley rank. Our main result is:

Theorem 6.4. – *Let G be a solvable connected group of finite Morley rank. Then G' is a U -group.*

One of the early and fundamental results on groups of finite Morley rank states that the derived subgroup of a connected solvable group of finite Morley rank is nilpotent (Fact 6.1). This is a weak parallel to the well-known fact about algebraic groups that the derived subgroup of a connected solvable algebraic group is unipotent. The lack of a robust notion of unipotent subgroup in the case of groups of finite Morley rank is an obstacle to a more precise conclusion. By introducing *U-groups*, we try to overcome this obstacle and our main result

provides a closer analogue to the situation in algebraic groups than Fact 6.1. These U -groups are characterized by a structure theorem given as Theorem 5.4, in terms of the fundamental notion of a *homogeneous $U_{0,r}$ -group* (Definition 3.1). From a technical point of view, the critical result is Theorem 4.11, dealing with actions on such groups.

To briefly go over the history around unipotence in groups of finite Morley rank, one can start from Altseimer and Berkman's notion of *quasiunipotent group* [2]. This is a nilpotent definable connected group of finite Morley rank containing no p -torus (a p -torus being an abelian divisible p -subgroup). Quasi-unipotent groups turned out to be a good approximation to unipotent algebraic groups (see Lemma 2.4 of [2]) and they were very useful in [13]. Furthermore we proved that the derived subgroup of a connected solvable group of finite Morley rank is quasiunipotent (Proposition 3.26 of [13]).

The main problem around unipotence is to find a notion which corresponds to the characteristic zero case, since definable connected p -subgroups of bounded exponent in groups of finite Morley rank are good analogues for nonzero characteristic. The notions of *$U_{0,r}$ -groups* and *0-unipotent groups* introduced by Burdges in [7] (see section 2 for the definition) aim at overcoming the complications of the unipotence problem in characteristic zero. Significant recent applications of Burdges' notions can be found in [7], [8], [9] and [14]. Despite its effectiveness, Burdges' notion of unipotence has a weakness: it is not necessarily preserved by passage to definable subgroups. In the present article, in order to remedy this problem, we impose a condition of *homogeneity* on Burdges' $U_{0,r}$ -groups (Definition 3.1).

It should nevertheless be emphasized that neither our notion of U -group nor Burdges' notion of 0-unipotence is the ideal analogue for a unipotent subgroup of an algebraic group. In fact, an ideal analogue should have three essential properties: it should be nilpotent; every solvable group of finite Morley rank should have a unique maximal one; the derived subgroup of a connected solvable group of finite Morley rank should be such a group. In solvable groups, Burdges' 0-unipotent groups enjoy the first two properties but not the last one. Our U -groups satisfy the first and third conditions but are likely never to meet the second one because of the conjectured existence of *bad fields of characteristic zero*. The existence of such hypothetical fields of finite Morley rank with a distinguished proper infinite multiplicative subgroup has been a longstanding open problem in model theory (see for example [20] for more details).

As a result, for the time being we can only approximate unipotence.

The organization of the paper is as follows. In section 2, we go over the definitions of Burdges' notions on unipotence, which are central to this paper as well. In section 3, we introduce and study *homogeneous $U_{0,r}$ -groups*. Section 4 is devoted to the proof of the first and the most technical theorem of this paper (Theorem 4.11). In section 5, we introduce U -groups and obtain their precise structural description in Theorem 5.4. We also show that every definable quotient and every definable subgroup of a U -group is a U -group (Corollary 5.8). Section 6 contains the main result of this article (Theorem 6.8). Afterwards, looking at quotients of solvable connected groups of finite Morley rank, we prove

Theorem 6.20. To state this final result, we introduce a new type of subgroup of a group of finite Morley rank G , namely $C(G)$ (Definition 6.11). Our Theorem 6.20 shows that in a connected solvable group, $C(G)$ is the only place where homogeneity in the sense of this article can be lost, and implies that the *Fitting subgroup* of a centerless connected solvable group is a U -group. These results have recently been applied in the classification of simple groups of finite Morley rank of *odd type* by Adrien Deloro [11].

2 $U_{0,r}$ -groups

For a notion analogous to unipotence in algebraic groups, Burdges [7] introduced the notions of *reduced rank* and $U_{0,r}$ -groups. In this section we recall the definitions and known results.

The notations will be as in [3], which is also our basic reference. Before introducing $U_{0,r}$ -groups we need to recall three facts. The first two of the following facts are well-known consequences of the Zil'ber Indecomposability Theorem [22]. The third is a relatively simple fact about lifting torsion in groups of finite Morley rank. These results will be used in the sequel without mention.

Fact 2.1. – (Zil'ber, [22]) *Let G be a group of finite Morley rank. The subgroup generated by a set of definable connected subgroups of G is definable and connected and it is the setwise product of finitely many of them.*

Fact 2.2. – (Zil'ber, [22]) *Let $H \leq G$ be a definable connected subgroup. Let $X \subseteq G$ be any subset. Then the subgroup $[H, X]$ is definable and connected.*

Fact 2.3. – (Borovik, Nesin, [4]) *Let G be a group of finite Morley rank and let H be a definable normal subgroup of G . If x is an element of G such that \bar{x} is a p -element of $\bar{G} = G/H$ then the coset xH contains a p -element. In particular, if G is torsion-free, then G/H is torsion-free.*

An abelian connected group of finite Morley rank is *indecomposable* if it is not the sum of two proper definable subgroups. The *radical* $J(A)$ of a nontrivial abelian group A of finite Morley rank is the maximal proper definable connected subgroup without a proper definable *supplement*, *i.e.* without a proper definable subgroup B such that $A = J(A)B$ (the existence and the unicity of $J(A)$ is warranted by Lemma 2.6 of [7]). In particular, the radical $J(A)$ of an indecomposable group is its unique maximal proper definable connected subgroup. Also we define $J(1) = 1$.

Fact 2.4. – (Burdges, Lemma 2.4, [7]) *Every connected abelian group of finite Morley rank can be written as a finite sum of indecomposable subgroups.*

If A is an abelian group of finite Morley rank, we define the *reduced rank* as in [7]:

$$\bar{r}(A) = rk(A/J(A)).$$

The most interesting indecomposable groups A are the ones having $A/J(A)$ torsion-free:

Definition 2.5. – *An indecomposable group A is 0-indecomposable if $A/J(A)$ is torsion-free.*

In particular, Fact 2.4 says that any torsion-free group of finite Morley rank is generated by its 0-indecomposable subgroups.

From now on $r \neq 0$ is a fixed natural number.

If G is any group of finite Morley rank, then we also define as in [7]

$$U_{0,r}(G) = \langle A \leq G \mid A \text{ is 0-indecomposable, } \bar{r}(A) = r \rangle$$

We say that G is a $U_{0,r}$ -group whenever $G = U_{0,r}(G)$.

Fact 2.6. – (**Burdges, Lemma 2.11, [7]**) *Let $f : G \rightarrow H$ be a definable homomorphism between two groups of finite Morley rank. Then*

1. *(Push-forward) $f(U_{0,r}(G)) \leq U_{0,r}(H)$ is a $U_{0,r}$ -group.*
2. *(Pull-back) If $U_{0,r}(H) \leq f(G)$ then $f(U_{0,r}(G)) = U_{0,r}(H)$.*

In particular, an extension of a $U_{0,r}$ -group by a $U_{0,r}$ -group is a $U_{0,r}$ -group.

We introduce the *Fitting subgroup*:

Definition 2.7. – *The Fitting subgroup $F(G)$ of a group G is the subgroup generated by all the normal nilpotent subgroups of G .*

Fact 2.8. – (**Nesin, [15]**) *In any group G of finite Morley rank, the Fitting subgroup is definable and nilpotent.*

In any group G of finite Morley rank, the 0-rank of G is defined to be

$$\bar{r}_0(G) = \max \{ \bar{r}(A) \mid A \leq G \text{ is 0-indecomposable} \}$$

We can now state an essential link between $U_{0,r}$ -groups and Fitting subgroup:

Fact 2.9. – (**Burdges, Theorem 2.16, [7]**) *Let G be a connected solvable group of finite Morley rank. Then $F(G)$ contains $U_{0,\bar{r}_0(G)}(G)$.*

Remark 2.10. – In [7], Burdges defines *0-unipotent groups* to be groups G of finite Morley rank such that $U_{0,\bar{r}_0(G)}(G) = G$. This 0-unipotence notion is fundamental in [7] and in [14].

3 Homogeneous $U_{0,r}$ -groups

Recall that r is a fixed nonzero natural number.

Generally, a definable connected subgroup H of a $U_{0,r}$ -group G is not a $U_{0,r}$ -subgroup, even if G is abelian. Thus we are led to introduce a notion of homogeneity.

Definition 3.1. – A $U_{0,r}$ -group H is homogeneous if every definable connected subgroup of H is a $U_{0,r}$ -subgroup.

Remark 3.2. – A definable connected subgroup of a homogeneous $U_{0,r}$ -group is homogeneous.

Lemmas 3.3 and 3.4 are analogous to Fact 2.6:

Lemma 3.3. – A definable quotient of a homogeneous $U_{0,r}$ -group is a homogeneous $U_{0,r}$ -group.

PROOF – Let G be a homogeneous $U_{0,r}$ -group and H be a normal definable subgroup of G . Let K/H be a connected definable subgroup of G/H . By definition of a homogeneous $U_{0,r}$ -group, the connected component K° of K is a $U_{0,r}$ -subgroup. Then $K/H = K^\circ H/H$ is a $U_{0,r}$ -group by fact 2.6 (Push-forward), and finally G/H is a homogeneous $U_{0,r}$ -group. \square

Lemma 3.4. – An extension of a homogeneous $U_{0,r}$ -group by a homogeneous $U_{0,r}$ -group is a homogeneous $U_{0,r}$ -group.

PROOF – Let G be a group of finite Morley rank with H a normal homogeneous $U_{0,r}$ -subgroup such that G/H is a homogeneous $U_{0,r}$ -group. Let L be a definable connected subgroup of G . Then $L/(H \cap L) \cong LH/H$ is a $U_{0,r}$ -group. By Fact 2.6 (Pull-back) we have $L = U_{0,r}(L)(H \cap L) = U_{0,r}(L)(H \cap L)^\circ$, so $L/(H \cap L)^\circ \cong U_{0,r}(L)/U_{0,r}(L) \cap (H \cap L)^\circ$ is a $U_{0,r}$ -group. As H is a homogeneous $U_{0,r}$ -group, $(H \cap L)^\circ$ is a $U_{0,r}$ -group. Now Fact 2.6 (Pull-back) shows that L is a $U_{0,r}$ -group. It follows that G is a homogeneous $U_{0,r}$ -group. \square

Corollary 3.5. – Let G be a group of finite Morley rank with two homogeneous $U_{0,r}$ -subgroups A and B such that $G = AB$. Assume that A is normal in G . Then G is a homogeneous $U_{0,r}$ -group.

We recall the definition of a *bad group*, a hypothetical counterexample to the Cherlin-Zil'ber Conjecture:

Definition 3.6. – A bad group is a nonsolvable connected group G of finite Morley rank all of whose proper definable connected subgroups are nilpotent.

Proposition 3.8 shows that homogeneous $U_{0,r}$ -groups have properties similar to groups of bounded exponent (Fact 3.7):

Fact 3.7. – (Poizat, Proposition 3.23, [19]) *A connected group of finite Morley rank and bounded exponent is either nilpotent or contains a bad group.*

Proposition 3.8. – *A homogeneous $U_{0,r}$ -group is either nilpotent or contains a bad group. In particular, a solvable homogeneous $U_{0,r}$ -group is nilpotent.*

PROOF – Let G be a homogeneous $U_{0,r}$ -group. If $G \neq 1$ and G is solvable, then we have $r = \bar{r}_0(G)$. Thus we obtain $G = U_{0,\bar{r}_0(G)}(G) \leq F(G)$ (Fact 2.9), and G is nilpotent.

If G is nonsolvable, G has a minimal nonsolvable definable connected subgroup H . By minimality of H and by the preceding, H is a bad group. \square

Corollary 3.9. – *A homogeneous $U_{0,r}$ -group not containing a bad group is torsion-free.*

PROOF – Let G be a homogeneous $U_{0,r}$ -group which does not contain a bad group. We may assume that all proper definable connected subgroups of G are torsion-free. By Proposition 3.8, G is nilpotent. Let N be a maximal proper definable connected subgroup of G . Then N is normal in G and N is torsion-free.

But G/N is abelian and, by maximality of N , G/N does not have a nontrivial proper definable connected subgroup, in particular G/N is indecomposable and $J(G/N) = 1$. Moreover G/N is a homogeneous $U_{0,r}$ -group (Lemma 3.3) and so G/N is generated by its 0-indecomposable subgroups. Therefore G/N is a 0-indecomposable group and, since $J(G/N) = 1$, this proves that G/N is torsion-free. As N is torsion-free too, G is torsion-free. \square

On the other hand, we do not know if every homogeneous $U_{0,r}$ -group is torsion-free, or whether it may have an involution. That is why we ask the question:

Question 3.10. – *Let G be a connected group of finite Morley rank whose solvable definable connected subgroups are nilpotent. Is G equal to the union of its nilpotent definable connected subgroups? Can G have an involution?*

Remark 3.11. – By results of [5] and of [10], it is known a group of finite Morley rank whose proper definable connected subgroups are nilpotent, that is a bad group, is equal to the union of its nilpotent definable connected subgroups and that it has no involutions.

We can control the intersections of homogeneous $U_{0,r}$ -subgroups and homogeneous $U_{0,s}$ -subgroups for $s \neq r$:

Corollary 3.12. – *Let G be a group of finite Morley rank with two definable subgroups H and K . Suppose that H is a homogeneous $U_{0,r}$ -subgroup and that K is a homogeneous $U_{0,s}$ -subgroup for $s \neq r$. Then $H \cap K$ is finite. Furthermore, whenever G does not contain a bad group, then $H \cap K = 1$.*

PROOF – The subgroup $(H \cap K)^\circ$ is at the same time a homogeneous $U_{0,r}$ -group and a homogeneous $U_{0,s}$ -group. If $(H \cap K)^\circ \neq 1$, then $(H \cap K)^\circ$ contains a 0-indecomposable subgroup $A \neq 1$. But the reduced rank of A is $r = s$ by homogeneity of H and K . This contradicts $r \neq s$. Hence $H \cap K$ is finite.

Furthermore, whenever G does not contain a bad group, then H is torsion-free (Corollary 3.9), so $H \cap K = 1$. \square

4 Nilpotent $U_{0,r}$ -groups

In this section, using homogeneous $U_{0,r}$ -groups, we wish to elucidate the structure of nilpotent groups. We will prove a key result (Theorem 4.11) for the proof of the principal theorem of this article (Theorem 6.8)

First we recall structural results by Nesin (Fact 4.1) and by Burdges (Fact 4.2).

Fact 4.1. – (Nesin, [17]) *Let G be a nilpotent group of finite Morley rank. Then G is a central product $G = D * C$ where*
 D is definable, connected, characteristic and divisible,
 C is definable, characteristic and of bounded exponent,
the torsion part T of D is divisible and central in D .
Furthermore, $D \cap C$ is finite.

If X is a subset of a group G of finite Morley rank, then the *definable closure* of X , denoted by $d(X)$, is the intersection of all the definable subgroups of G which contain X . By the descending chain condition on definable subgroups, this intersection is definable.

Fact 4.2. – (Burdges, [9]; Theorem 2.31 of [8]) *Let G be a nilpotent group of finite Morley rank. Using the notations of Fact 4.1, the following decomposition of D holds:*

$$D = d(T) * U_{0,1}(G) * U_{0,2}(G) * \cdots * U_{0,\bar{r}_0(G)}(G)$$

Proposition 4.4 is our first result about nilpotent homogeneous $U_{0,r}$ -groups. It is analogous to the result of [17] which says that the commutator subgroup of a divisible nilpotent group of finite Morley rank is torsion-free.

Lemma 4.3. – *Let G be a nilpotent $U_{0,r}$ -group. Then $G/Z(G)^\circ$ is a homogeneous $U_{0,r}$ -group.*

PROOF – By Fact 4.2, G is divisible and, by Fact 4.1, $G/Z(G)^\circ$ is torsion-free. Moreover, by Fact 2.6 (Push-forward), $G/Z(G)^\circ$ is a $U_{0,r}$ -group.

By Fact 4.2 we have $U_{0,s}(G) \leq Z(G)^\circ$ for every $s \neq r$. Hence $G/Z(G)^\circ$ is a homogeneous $U_{0,r}$ -group by Fact 2.6 (Pull-back). \square

Proposition 4.4. – *If G is a nilpotent $U_{0,r}$ -group, then G' is a homogeneous $U_{0,r}$ -subgroup.*

PROOF – We proceed by induction on $rk(G)$. We may assume that G is not abelian. Let $g \in Z_2(G) \setminus Z(G)$. For all $u \in G$ we consider $\gamma(u) = [g, u]$. Then γ is a definable homomorphism of groups and $\gamma(G)$ is isomorphic to $G/C_G(g)$. Thus $\gamma(G)$ is a homogeneous $U_{0,r}$ -subgroup by Lemmas 3.3 and 4.3. But we have $g \in Z_2(G)$, so $\gamma(G)$ is an infinite normal subgroup of G . By induction hypothesis, $G'\gamma(G)/\gamma(G)$ is a homogeneous $U_{0,r}$ -group, and G' is a homogeneous $U_{0,r}$ -subgroup by Lemma 3.4. \square

The rest of this section is devoted to the proof of Theorem 4.11.

Lemma 4.5. – *Let G be a nilpotent group of finite Morley rank. If G has a nontrivial homogeneous $U_{0,r}$ -subgroup, then $Z(G)^\circ$ has a nontrivial homogeneous $U_{0,r}$ -subgroup too.*

PROOF – Proceeding by induction on the Morley rank of G , we may suppose that $Z_2(G)/Z(G)^\circ$ contains a nontrivial homogeneous $U_{0,r}$ -subgroup of the form $V/Z(G)^\circ$ with $Z(G)^\circ \leq V$. Then for $x \in G \setminus C_G(V)$, the commutator map $[x, v]$ induces a homomorphism $\gamma : V/Z(G)^\circ \rightarrow Z(G)$, and by Lemma 3.3 its image is a homogeneous $U_{0,r}$ -subgroup of $Z(G)$. \square

Lemma 4.6. – *Let G be a group of finite Morley rank.*

- (i) *Then G possesses a largest normal homogeneous $U_{0,r}$ -subgroup.*
- (ii) *If G is nilpotent, then G has a unique largest homogeneous $U_{0,r}$ -subgroup.*

PROOF – (i) comes from Corollary 3.5.

If G is nilpotent, then in view of (i) and Lemma 3.4 we may suppose, after passing to a quotient, that G contains no normal homogeneous $U_{0,r}$ -subgroup. Then by Lemma 4.5 G contains no nontrivial homogeneous $U_{0,r}$ -subgroup. \square

Lemma 4.7. – *Let G be a group of finite Morley rank and A an abelian G -minimal $U_{0,r}$ -subgroup of G . Then A is a homogeneous $U_{0,r}$ -subgroup.*

PROOF – Let M be a maximal proper definable connected subgroup of A . By Fact 2.6 (Push-forward), the quotient A/M is a $U_{0,r}$ -group and it is torsion-free by maximality of M . In particular M contains the torsion part T of A , and the G -minimality of A implies $d(T) = 1$. Thus A is torsion-free.

Let B be a minimal infinite definable subgroup of A . Then B is torsion-free and, by minimality of B , it is a homogeneous $U_{0,s}$ -subgroup of A for some s . By G -minimality of A , the G -invariant closure of B is A . Therefore A is a homogeneous $U_{0,s}$ -subgroup (Corollary 3.5), hence $s = r$. \square

Fact 4.8. – **(Zil’ber, [24])** *Let $G = A \rtimes H$ be a group of finite Morley rank where A and H are infinite definable abelian subgroups and A is H -minimal. Assume $C_H(A) = 1$. Then:*

- (i) *The subring $K = \mathbb{Z}[H]/\text{ann}_{\mathbb{Z}[H]}(A)$ of the set $\text{End}(A)$ of endomorphisms of A is a definable algebraically closed field; in fact, there exists an*

integer l such that every element of K can be represented by an endomorphism of the form $\sum_{i=1}^l h_i$, for some elements $h_i \in H$.

(ii) $A \cong K^+$, H is isomorphic to a subgroup T of K^\times and H acts on A by multiplication.

(iii) In particular, H acts freely on A , $K = T + \dots + T$ (l times) and (with additive notation) $A = \{\sum_{i=1}^l h_i a : h_i \in H\}$ for any $a \in A^*$.

Fact 4.9. – (Reineke, Theorem 6.4 of [3]) *In an infinite group of finite Morley rank, a minimal infinite definable subgroup A is abelian. Furthermore, either A is divisible or is an elementary abelian p -group for some prime p .*

Fact 4.10. – (Burdges, [9]; Lemma 2.32 of [8]) *Let G be a solvable group of finite Morley rank, let $S \subseteq G$ be a subset, and let H be a nilpotent $U_{0,r}$ -group which is normal in G . Then $[H, S] \leq H$ is a $U_{0,r}$ -group.*

Theorem 4.11. – *Let G be a connected group of finite Morley rank (not necessarily solvable). Assume that G acts definably by conjugation on H , a nilpotent $U_{0,r}$ -group. Then $[G, H]$ is a homogeneous $U_{0,r}$ -group.*

PROOF – Suppose there is a counterexample G acting by conjugation on H . Let $G_H = H \rtimes G$. We can suppose G_H of minimal Morley rank.

(1) H contains no nontrivial homogeneous $U_{0,r}$ -subgroup and H is abelian.

By Lemma 4.6 (ii), H has a unique largest homogeneous $U_{0,r}$ -subgroup A . If A is nontrivial, then $[G, H]A/A$ is a homogeneous $U_{0,r}$ -group by minimality of G_H , and $[G, H]$ is a homogeneous $U_{0,r}$ -subgroup (Lemma 3.4). This contradicts the choice of G and H . Hence $A = 1$ and H is abelian (Proposition 4.4).

(2) G is abelian.

By Fact 4.9, the group $G/C_G(H)^\circ$ has an infinite abelian definable connected subgroup $A/C_G(H)^\circ$. Assume $rk(G) \neq rk(A/C_G(H)^\circ)$. By minimality of G_H , the subgroup $[A, H]$ is a homogeneous $U_{0,r}$ -subgroup of H . Then A centralizes H by (1), contradicting the choice of A . Therefore $rk(G) = rk(A/C_G(H)^\circ)$, hence $G = A$ and $C_G(H)^\circ = 1$. So G is abelian.

(3) $[G, H] = H$.

Suppose $[G, H] \neq H$. Since $[G, H]$ is a $U_{0,r}$ -group (Fact 4.10), the minimality of G_H and (1) yield $[G, [G, H]] = 1$. Since H is abelian by (1) and $G'_H \leq H$ by (2), the group $[G_H, [G_H, G'_H]]$ is contained in $[G, [G, H]] = 1$ and G_H is nilpotent. Therefore $[G, H]$ is contained in $[G_H, U_{0,r}(G_H)] = U_{0,r}(G_H)'$ (Facts 4.1 and 4.2). Then, by Proposition 4.4, $[G, H]$ is a homogeneous $U_{0,r}$ -subgroup and (1) yields $[G, H] = 1$. This contradicts the choice of G and H .

(4) H has a unique G -minimal subgroup A and H/A is a homogeneous $U_{0,r}$ -group.

Let A and B be G -minimal subgroups of H . By (3) and by minimality of G_H , $H/A = [G, H]A/A$ is a homogeneous $U_{0,r}$ -group and $B/B \cap A \cong BA/A$ is a $U_{0,r}$ -group. Hence, by Fact 2.6 (Pull-back), $B = U_{0,r}(B)(B \cap A)$. If $A \neq B$, then $B \cap A$ is finite and $B = U_{0,r}(B)$ is a $U_{0,r}$ -subgroup, so B is a homogeneous $U_{0,r}$ -subgroup of H (Lemma 4.7), contradicting (1). This proves (4).

$$(5) \quad C_G(A) = C_G(H) = C_G(H/A).$$

Let $x \in G$. By (1) the map $\gamma_x = [x, h]$ is a definable endomorphism of H . Moreover, by (2), the subgroup $I = \text{Im}(\gamma_x)$ is normalized by G .

Assume $x \in C_G(A)$. Since H/A is a homogeneous $U_{0,r}$ -group by (4), the subgroup $I \cong H/C_H(x)$ is a homogeneous $U_{0,r}$ -subgroup (Lemma 3.3). By (1), we obtain $I = 1$ and $x \in C_G(H)$.

Assume $x \in C_G(H/A)$. By G -minimality of A , either I is finite or $I = A$. But $I \cong H/C_H(x)$ is a $U_{0,r}$ -group (Fact 2.6 (Push-forward)), so by (1) and Lemma 4.7, $I = 1$ and $x \in C_G(H)$.

$$(6) \quad H/A \text{ is } G\text{-minimal.}$$

Let B/A be a G -minimal subgroup of H/A . By (4), B/A is a $U_{0,r}$ -group and we obtain $B = U_{0,r}(B)A$ (Fact 2.6 (Pull-back)), in particular $U_{0,r}(B)$ is infinite. Hence, by (4), $U_{0,r}(B)$ contains A , and B is a $U_{0,r}$ -subgroup of H .

Assume $H \neq B$. By (1) and by minimality of G_H , G centralizes B and A . Hence G centralizes H by (5), contradicting the choice of G and H .

$$(7) \quad \text{Final contradiction.}$$

Let R be the (commutative) subring of $\text{End}(H)$ generated by $L = G/C_G(H)$. For all $u \in R$, the kernel $\text{Ker}(u)$ of u and the image $\text{Im}(u)$ of u are definable subgroups of H and they are normalized by G . Thus, by (4) and (6), we have either $\text{Ker}(u)^\circ = 1$ either $\text{Ker}(u)^\circ = A$ or $\text{Ker}(u) = H$.

Suppose $\text{Ker}(u)^\circ = A$. Since H/A is a homogeneous $U_{0,r}$ -group by (4), then $\text{Im}(u) \cong H/\text{Ker}(u)$ is a homogeneous $U_{0,r}$ -group (Lemma 3.3). As $\text{Im}(u)$ is infinite and is normalized by G , we contradict (1).

We have proved that, for all $u \in R$, either $\text{Ker}(u)^\circ = 1$ or $\text{Ker}(u) = H$.

By (5), we can apply Zil'ber's Field Theorem (Fact 4.8) to $A \rtimes L$. Thus there is an integer l such that, for all $u \in R$, the restriction $u|_A$ of u to A can be represented by a sum of l elements g_1, \dots, g_l of L . Let $v = g_1 + \dots + g_l \in R$. Thus $\text{Ker}(u - v)$ contains A by choice of g_1, \dots, g_l , and so $\text{Ker}(u - v) = H$. We deduce from this the equality $u = g_1 + \dots + g_l$. Consequently, all elements of R can be written as a sum of l elements of L . It follows that R has finite Morley rank. Since we proved that, for all $(u, v) \in R \times R$, we have $u = v \Leftrightarrow u|_A = v|_A$, also the ring R is definably isomorphic to the subring $R(A)$ of $\text{End}(A)$ generated by L . By Fact 4.8, $R(A)^+$ is definably isomorphic to A , therefore R^+ is definably isomorphic to A .

By (5) and by (6), we can apply Fact 4.8 to $(H/A) \rtimes L$ too, and the preceding argument applied to $(H/A) \rtimes L$ shows that R^+ is definably isomorphic to H/A .

Thus A and H/A are definably isomorphic. But this contradicts the facts that H/A is a $U_{0,r}$ -group by (4) while A is not by (1) and Lemma 4.7. \square

5 The subgroup $U(G)$

In this section, we attach a subgroup $U(G)$ to each group G of finite Morley rank. This subgroup $U(G)$ is an analogue to the unipotent radical of an algebraic group. We introduce the related notion of a U -group, which is central for the rest of this article.

From now on, let \mathbb{N}^* denote $\mathbb{N} \setminus \{0\}$.

Definition 5.1. – For every group G of finite Morley rank, we denote by $U(G)$ the subgroup of G generated by its normal homogeneous $U_{0,s}$ -subgroups where s covers \mathbb{N}^* and by its normal definable connected subgroups of bounded exponent.

A U -group is a group G of finite Morley rank such that $G = U(G)$.

Remark 5.2. – If G is a group of finite Morley rank, then $U(G)$ is definable and connected. Moreover, we have $U(U(G)) = U(G)$.

Proposition 5.3 shows that U -groups have some similar properties to connected groups of finite Morley rank of bounded exponent (Fact 3.7):

Proposition 5.3. – A U -group is either nilpotent or contains a bad group. In particular, in every solvable group G of finite Morley rank, $U(G)$ is nilpotent.

PROOF – Let G be a U -group which does not contain a bad group. Proposition 3.8 says that, for all $s \in \mathbb{N}^*$, the homogeneous $U_{0,s}$ -subgroups of G are nilpotent. The definable connected subgroups of G of bounded exponent are nilpotent too by Fact 3.7. Thus $G = U(G)$ is generated by its normal nilpotent subgroups. Therefore $G = F(G)$ and G is nilpotent by Fact 2.8. \square

Now we analyze the structure of U -groups:

Theorem 5.4. – Let G be a U -group. Then G has the following decomposition:

$$G = B * U_{0,1}(G) * U_{0,2}(G) * \cdots * U_{0,\bar{r}_0(G)}(G)$$

where B is definable, connected, definably characteristic and of bounded exponent;

$U_{0,s}(G)$ is a homogeneous $U_{0,s}$ -subgroup for each $s \in \{1, 2, \dots, \bar{r}_0(G)\}$;

the intersections of the form $U_{0,s}(G) \cap U_{0,t}(G)$ for $s \neq t$ are finite.

In particular, if G does not contain a bad group, then

$$G = B \times U_{0,1}(G) \times U_{0,2}(G) \times \cdots \times U_{0,\bar{r}_0(G)}(G)$$

PROOF – Let B be the largest normal definable connected subgroup of G of bounded exponent. Then B is definably characteristic in G and we have

$$G = BV_1V_2 \cdots V_{\bar{r}_0(G)}$$

where V_s is the largest normal homogeneous $U_{0,s}$ -subgroup of G for each $s \in \{1, 2, \dots, \bar{r}_0(G)\}$ (Lemma 4.6 (i)).

For each $k \in \{0, \dots, \bar{r}_0(G)\}$, we set $W_k = \langle V_i \mid i \in \{1, \dots, k\} \cup \{s\} \rangle$ and $W_{\bar{r}_0(G)+1} = BW_{\bar{r}_0(G)} = G$. Let A be a 0-indecomposable subgroup of G of reduced rank $s \in \{1, \dots, \bar{r}_0(G)\}$. By Lemma 3.3, for every $k \in \{0, \dots, \bar{r}_0(G)\}$, the group $AW_k/J(A)W_k \cap J(A)W_{k+1}/J(A)W_k$ is trivial. Therefore A is contained in $J(A)W_0 = J(A)V_s$, so $A = J(A)(V_s \cap A)$. Thus $A \leq V_s$ and $V_s = U_{0,s}(G)$. Hence $U_{0,s}(G)$ is a homogeneous $U_{0,s}$ -subgroup.

Let $s \neq t$ be two elements of $\{1, 2, \dots, \bar{r}_0(G)\}$. By Corollary 3.12, $V_s \cap V_t$ is finite. Since $[V_s, V_t]$ is a connected subgroup of $V_s \cap V_t$, we obtain $[V_s, V_t] = 1$.

Moreover V_s contains $[B, V_s]$ which is a definable connected subgroup, so $[B, V_s]$ is a homogeneous $U_{0,s}$ -subgroup. At the same time, B contains $[B, V_s]$, so $[B, V_s]$ is of bounded exponent. This proves that $[B, V_s]$ is trivial. Consequently, G is the central product of $B, V_1, V_2, \dots, V_{\bar{r}_0(G)}$.

If G does not contain a bad group, then the equality follows from Corollaries 3.9 and 3.12 and Proposition 5.3. \square

Corollary 5.5. – *Let G be a U -group. Then $U_{0,r}(G)$ is a homogeneous $U_{0,r}$ -group.*

Corollary 5.6. – *Let G be a group of finite Morley rank. Then $U(G)$ is the largest normal U -subgroup of G . Furthermore, if G is nilpotent, then $U(G)$ is the largest U -subgroup of G .*

PROOF – Let V be a normal U -subgroup of G . By Theorem 5.4, we have $V = B * U_{0,1}(V) * U_{0,2}(V) * \cdots * U_{0,\bar{r}_0(V)}(V)$ where each factor is normal in G and hence contained in $U(G)$. Thus $V \leq U(G)$.

If G is nilpotent, this follows from Lemma 4.6 and the analogous statement for $U_p(G)$. \square

We give a useful characterization of U -groups:

Proposition 5.7. – *Let G be a connected group of finite Morley rank. Then G is a U -group if and only if G has the following properties:*

(i) *there is a normal definable connected subgroup B of G of bounded exponent and such that UB/VB is finite for each abelian torsion section U/V of G with V definable;*

(ii) *for each $s \in \mathbb{N}^*$, $U_{0,s}(G)$ is a homogeneous $U_{0,s}$ -subgroup.*

PROOF – (\implies) Let G be a U -group. Then G satisfies (ii) by Corollary 5.5.

Let U/V be an abelian torsion section of G with V definable. By Theorem 5.4, G has the following decomposition: $G = B * U_{0,1}(G) * U_{0,2}(G) * \cdots *$

$U_{0, \bar{r}_0(G)}(G)$ where B is a definable connected subgroup of bounded exponent. We set $W_0 = B$ and, for each $s \in \{0, 1, \dots, \bar{r}_0(G) - 1\}$, $W_{s+1} = W_s U_{0, s+1}(G)$. Then W_s is a normal definable subgroup of G for each $s \in \{0, 1, \dots, \bar{r}_0(G)\}$. If U/V is finite, then UB/VB too. If U/V is infinite, there is a smallest $s \in \{0, 1, \dots, \bar{r}_0(G)\}$ such that UW_s/VW_s is finite.

Suppose $s \geq 1$. Let $S/VW_{s-1} = (VW_s \cap UW_{s-1})/VW_{s-1}$. By minimality of s , S/VW_{s-1} is infinite. So S/VW_{s-1} is an infinite abelian torsion subgroup of VW_s/VW_{s-1} and $d(S/VW_{s-1})^\circ$ is an infinite abelian definable connected subgroup of VW_s/VW_{s-1} .

Since $U_{0, s}(G)$ is a homogeneous $U_{0, s}$ -subgroup, Lemma 3.3 says that the quotient $VW_s/VW_{s-1} = U_{0, s}(G)VW_{s-1}/VW_{s-1}$ is a homogeneous $U_{0, s}$ -group. Therefore $d(S/VW_{s-1})^\circ$ is a homogeneous $U_{0, s}$ -subgroup. Since $d(S/VW_{s-1})^\circ$ is abelian, $d(S/VW_{s-1})^\circ$ is torsion-free by Corollary 3.9. That is impossible since $S/VW_{s-1} \cap d(S/VW_{s-1})^\circ$ is an infinite torsion subgroup of $d(S/VW_{s-1})^\circ$, proving $s = 0$. Hence $UW_0/VW_0 = UB/VB$ is finite and we obtain (i).

(\Leftarrow) Conversely, let G be a connected group of finite Morley rank satisfying (i) and (ii). Then $U(G)$ contains B and $U_{0, s}(G)$ for each $s \in \{1, 2, \dots, \bar{r}_0(G)\}$. Assume that G is not a U -group, that is $G/U(G) \neq 1$. Let $H/U(G)$ be a minimal infinite definable subgroup of $G/U(G)$. Then $H/U(G)$ is connected and, by Fact 4.9, $H/U(G)$ is abelian.

If $H/U(G)$ has an infinite torsion subgroup $T/U(G)$, then we have $T/U(G) = TB/U(G)B$ and $T/U(G)$ is finite by (i), contradicting the choice of $T/U(G)$. Thus, by Fact 4.9, $H/U(G)$ is divisible, and so $H/U(G)$ is torsion-free. By minimality of $H/U(G)$, $H/U(G)$ is 0-indecomposable. Let s be the reduced rank of $H/U(G)$. By Fact 2.6 (Pull-back), we have $H = U_{0, s}(H)U(G)$. But $U_{0, s}(G)$ contains $U_{0, s}(H)$ and $U_{0, s}(G)$ is a homogeneous $U_{0, s}$ -subgroup by (ii). Consequently, $U(G)$ contains $U_{0, s}(H)$ and we obtain $H = U(G)$, contradicting the choice of $H/U(G)$. This proves that G is a U -group. \square

We derive the following closure properties.

Corollary 5.8. –

- (i) *Every definable quotient of a U -group is a U -group.*
- (ii) *Every definable connected subgroup of a U -group is a U -group.*

PROOF – (i) follows from Theorem 5.4 and Lemma 3.3.

We prove (ii). Let H be a definable connected subgroup of a U -group G . Let B be a subgroup of G satisfying the assertion (i) of the Proposition 5.7. Consider $C = (B \cap H)^\circ$. Thus C is a normal definable connected subgroup of H of bounded exponent. Moreover, if U/V is an abelian torsion section of H with V definable, then UB/VB is finite by choice of B . Therefore $U(B \cap H)/V(B \cap H)$ is finite and also UC/VC since C has finite index in $C \cap H$. Hence H satisfies condition (i) of the Proposition 5.7.

Let $s \in \mathbb{N}^*$. Then by Corollary 5.5, $U_{0, s}(H)$ is a homogeneous $U_{0, r}$ -subgroup, giving condition (ii) of Proposition 5.7 as well. \square

6 U -groups and solvable groups

6.1 Commutator subgroup

Turning to a consideration of the commutator subgroup of a solvable connected group of finite Morley rank, we will prove the principal theorem of this paper.

The following fact of Nesin and Zil'ber is fundamental in the analysis of groups of finite Morley rank. Our Theorem 6.8 will sharpen this result.

Fact 6.1. – (Nesin, [18]; Zil'ber, [23]) *Let G be a solvable connected group of finite Morley rank. Then G' is nilpotent.*

By studying the quotient $G/F(G)^\circ$, Nesin proved a few more:

Fact 6.2. – (Nesin, [16]) *Let G be a connected and solvable group of finite Morley rank. Then $G/F(G)^\circ$ (so also $G/F(G)$) is a divisible abelian group.*

We recall that, for every prime number p , a p -torus is a divisible abelian p -group.

Fact 6.3. – (Borovik, Poizat, [6]) *Let T be a p -torus in a group G of finite Morley rank. Then $[N_G(T) : C_G(T)] < \infty$. Moreover, there exists a natural number c such that $[N_G(T) : C_G(T)] < c$ for any p -torus $T \leq G$.*

Lemma 6.4. – *Let G be a connected group of finite Morley rank. If G acts definably by conjugation on a nilpotent group H of finite Morley rank, then $[G, H]$ is a U -group.*

PROOF – By Fact 4.1, we may assume that H is divisible. By Facts 4.2 and 6.3, we may assume that H is a $U_{0,r}$ -group. Hence Theorem 4.11 proves the lemma. \square

Carter subgroups are crucial in the study of solvable groups of finite Morley rank. We recall their definition and useful properties.

Definition 6.5. – *A subgroup C of a group G of finite Morley rank is a Carter subgroup of G if it is nilpotent and self-normalizing in G .*

By the normalizer condition in nilpotent groups and Fact 6.6, Carter subgroups are definable.

Fact 6.6. – (Corollary 5.38, [3]) *Let G be a group of finite Morley rank and let H be a subgroup of G . If H is solvable (resp. nilpotent) of class n , then $d(H)$ is solvable (resp. nilpotent) of class n .*

Frank Wagner [21] studied Carter subgroups in a more general context than groups of finite Morley rank. In particular, he proved the following.

Fact 6.7. – (Wagner, [21]) *Let G be a solvable connected group of finite Morley rank. Then G has a Carter subgroup C and any two such subgroups are conjugate. Moreover, if L is a normal definable subgroup of G such that G/L is nilpotent, then $G = LC$.*

Theorem 6.8. – *Let G be a solvable connected group of finite Morley rank. Then G' is a U -group.*

PROOF – Let C be a Carter subgroup of G (Fact 6.7). By Lemma 6.4, the subgroups $(C^\circ)'$ and $L = [G, F(G)]$ are some U -groups. It follows from Facts 6.1 and 6.7 that $G = LC^\circ$ and $G' = L(C^\circ)'$. But G' is nilpotent (Fact 6.1), so $U(G')$ contains L and $(C^\circ)'$ by Corollary 5.6. Therefore $U(G')$ contains $G' = L(C^\circ)'$ and G' is a U -group by Corollary 5.8 (ii). \square

Then we obtain an analogue to Fact 6.2.

Corollary 6.9. – *Let G be a connected solvable group of finite Morley rank. Then $G/U(G)$ is a divisible abelian group.*

PROOF – By Fact 4.1, the subgroup $F(G)^\circ$ has a largest definable connected subgroup C of bounded exponent and $F(G)^\circ/C$ is divisible. Then C is contained in $U(G)$ and $F(G)^\circ/U(G)$ is divisible. Hence $G/U(G)$ is divisible-by-divisible (Fact 6.2) and, since this quotient is abelian (Theorem 6.8), it is divisible. \square

In the first version of this article, I asked a question which the referee then answered. Here is his proof.

Theorem 6.10. – (The referee of this paper) *Let G be a connected group of finite Morley rank (not necessarily solvable). Assume that G acts definably by conjugation on H a solvable connected group of finite Morley rank. Then $[G, H]$ is a U -group.*

PROOF – We consider a minimal counterexample G acting on H a solvable connected group of finite Morley rank. We may assume that G is ω -saturated. By minimality of G and by Fact 4.9, $L = G/C_G(H/U(H))$ is abelian, and either L is divisible or L is an elementary abelian p -group for some prime p .

If L is divisible, then by ω -saturation, the group G has an element x such that $\bar{x} = xC_G(H/U(H))$ has infinite order in L . Therefore $D = d(x)^\circ$ is an abelian definable connected subgroup which does not centralize $H/U(H)$. We consider the semidirect product $H \rtimes D$ where D acts by conjugation on H . Then $(H \rtimes D)'$ is a normal U -subgroup of H by Theorem 6.8. Thus we have $[H, D] \leq (H \rtimes D)' \leq U(H)$, contradicting $D \not\leq C_G(H/U(H))$.

If L is an elementary abelian p -group for some prime p , we consider $W = H/U(H) \rtimes L$ where L acts by conjugation on $H/U(H)$. Since $W/F(W)^\circ$ is abelian and divisible (Fact 6.2), the group L is contained in $F(W)^\circ$. But $H/U(H)$ is abelian by Theorem 6.8, so $W = F(W)^\circ$ is nilpotent (Fact 2.8). Since $H/U(H)$ is divisible (Corollary 6.9), Fact 4.1 shows that L centralizes $H/U(H)$, contradicting the choice of L . \square

6.2 Quotient group

In this section, we study the U -group of a quotient of a solvable connected group of finite Morley rank by its center (Theorem 6.20). For this, we introduce a subgroup $C(G)$:

Definition 6.11. – *Let G be a group of finite Morley rank. We denote by $C(G)$ the intersection of all centralizers $C_G(A)$ with A varying over the class of connected groups of finite Morley rank acting definably on G .*

Remark 6.12. – *In every connected group G of finite Morley rank, $C(G)$ is a definable, central and definably characteristic subgroup of G .*

We will need the following known facts and definitions.

If X is a subgroup of a group G of finite Morley rank, the *connected component* of X , denoted by X° , is $X \cap d(X)^\circ$.

Fact 6.13. – (**Borovik, Poizat, [6]**) *Let P be a locally finite p -subgroup of a group G of finite Morley rank. Then the following hold:*

(i) *P° is nilpotent and $P^\circ = B * T$ is a central product of a nilpotent group B of bounded exponent and a p -torus T .*

(ii) *If $P \neq 1$, then $Z(P) \neq 1$ and P satisfies the normalizer condition.*

(iii) *If P is infinite and has a finite exponent then $Z(P)$ has infinitely many elements of order p and P is nilpotent.*

Fact 6.14. – (**Borovik, Nesin, [4]**) *Let G be a connected solvable group of finite Morley rank. Then the Sylow p -subgroups of G are connected.*

Fact 6.15. – (**Altinel, Cherlin, Corredor, Nesin, [1]**) *Let π be a set of primes. Any two maximal π -subgroups of a solvable group of finite Morley rank are conjugate.*

Fact 6.16. – (**Altinel, Cherlin, Corredor, Nesin, [1]**) *Let G be a solvable group of finite Morley rank, N be a normal definable subgroup of G , and let H be a maximal π -subgroup of G for some set π of primes. Then, HN/N is a maximal π -subgroup of G/N , and all maximal π -subgroups of G/N are of this form.*

Fact 6.17. – (**Lemma 4.20, [13]**) *Let G be a solvable connected group of finite Morley rank and T a p -torus of G . Then $T \cap F(G)$ is contained in $Z(G)$.*

Lemma 6.18 improves Fact 6.17:

Lemma 6.18. – *Let G be a solvable connected group of finite Morley rank and T a p -torus of G . Then $T \cap F(G)$ is contained in $C(G)$.*

PROOF – Let A be a connected group of finite Morley rank acting definably by conjugation on G . Consider the semi-direct product $L = G \rtimes A$ induced by this action. Assume that T is a maximal p -torus of G .

By a Frattini argument using Facts 6.13 and 6.15, we obtain $L = N_L(T)G$. By Fact 6.3, $N_L(T)^\circ = C_L(T)^\circ$. Thus $L = C_L(T)G$ and the conclusion now follows from Fact 6.17. \square

Lemma 6.19. – *Let G be a solvable connected group of finite Morley rank. Then $F(G) = C(G)F(G)^\circ$.*

PROOF – Let π be the set of primes dividing the order of $F(G)/F(G)^\circ$. Let $p \in \pi$ and S be a Sylow p -subgroup of G . By Facts 6.13 and 6.14, there exist a definable connected p -subgroup B of bounded exponent and a p -torus T such that $S = B * T$.

Since $G/F(G)^\circ$ is abelian and divisible (Fact 6.2), the group B is contained in $F(G)^\circ$ and all p -elements of $G/F(G)^\circ$ are contained in $TF(G)^\circ/F(G)^\circ$ (Fact 6.16). This proves that all p -elements of $F(G)/F(G)^\circ$ are contained in

$$(F(G)/F(G)^\circ) \cap (TF(G)^\circ/F(G)^\circ) = (F(G) \cap T)F(G)^\circ/F(G)^\circ$$

Moreover Lemma 6.18 proves that $C(G)$ contains $F(G) \cap T$. So all p -elements of $F(G)/F(G)^\circ$ are contained in $C(G)F(G)^\circ/F(G)^\circ$. This is true for each $p \in \pi$ and so, by the choice of π , we obtain $F(G)/F(G)^\circ = C(G)F(G)^\circ/F(G)^\circ$. \square

Theorem 6.20. – *Let G be a solvable connected group of finite Morley rank. Then the Fitting subgroup of $G/C(G)$ is a U -group.*

In particular, if G is centerless, then $F(G)$ is a U -group.

PROOF – As $F(G/C(G)) = F(G)/C(G)$, Lemma 6.19 yields $F(G/C(G)) = C(G)F(G)^\circ/C(G)$ and so $F(G/C(G))$ is connected.

We show that $F(G/C(G))$ satisfies the properties (i) and (ii) of Proposition 5.7. Let T be the maximal divisible torsion subgroup of $F(G)$. If A is a connected group of finite Morley rank acting definably on G , then A centralizes T (Fact 6.3) and so $C(G)$ contains T . Hence $F(G/C(G)) = F(G)/C(G)$ does not have a nontrivial divisible torsion subgroup (Fact 6.16). Therefore, by Fact 4.1, the group $F(G/C(G))$ satisfies condition (i) of Proposition 5.7.

We now show that $F(G/C(G))$ satisfies condition (ii) of the Proposition 5.7. Let $s \in \mathbb{N}^*$. We show that $U_{0,s}(F(G/C(G)))$ is a homogeneous $U_{0,s}$ -group. Since $F(G/C(G))$ does not have a nontrivial divisible torsion subgroup, it suffices to prove that every nontrivial 0-indecomposable subgroup of $U_{0,s}(F(G/C(G)))$ has a reduced rank equal to s . Fact 2.6 (Pull-back) gives:

$$U_{0,s}(F(G/C(G))) = U_{0,s}(F(G)^\circ C(G)/C(G)) = U_{0,s}(F(G)^\circ C(G))C(G)/C(G)$$

But we have $(F(G)^\circ C(G))^\circ = F(G)^\circ$, so we obtain:

$$\begin{aligned} U_{0,s}(F(G/C(G))) &= U_{0,s}(F(G)^\circ C(G)/C(G)) \\ &\cong U_{0,s}(F(G)^\circ)/(U_{0,s}(F(G)^\circ) \cap C(G)) \end{aligned}$$

Assume that $U_{0,s}(F(G/C(G)))$ has a nontrivial 0-indecomposable subgroup having a reduced rank $t \neq s$. Then $U_{0,s}(F(G)^\circ)/(U_{0,s}(F(G)^\circ) \cap C(G))$ has a 0-indecomposable subgroup $A/(U_{0,s}(F(G)^\circ) \cap C(G))$ of reduced rank t .

Let K be a connected group of finite Morley rank acting definably by conjugation on G . Since $U_{0,s}(F(G)^\circ)$ contains A , we have

$$[K, U_{0,t}(A)] \leq [K, U_{0,t}(F(G)^\circ)] \cap [K, U_{0,s}(F(G)^\circ)]$$

Theorem 4.11 proves that $[K, U_{0,t}(F(G)^\circ)]$ is a homogeneous $U_{0,t}$ -subgroup and that $[K, U_{0,s}(F(G)^\circ)]$ is a homogeneous $U_{0,s}$ -subgroup. Since $[K, U_{0,t}(A)]$ is connected, $[K, U_{0,t}(A)] = 1$ by Corollary 3.12. Thus K centralizes $U_{0,t}(A)$, and $C(G)$ contains $U_{0,t}(A)$. We obtain $A = U_{0,t}(A)(U_{0,s}(F(G)^\circ) \cap C(G))$ (Fact 2.6 (Pull-back)), hence $A = U_{0,s}(F(G)^\circ) \cap C(G)$, contradicting the choice of A .

Thus, for each $s \in \mathbb{N}^*$, $U_{0,s}(F(G/C(G)))$ is a homogeneous $U_{0,s}$ -group, proving (ii) of the Proposition 5.7. Hence $F(G/C(G))$ is a U -group. \square

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