Discretionary stopping of one-dimensional Itô diffusions with a staircase payoff function*

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July 22, 2005

Abstract

We consider the problem of optimally stopping a general one-dimensional Itô diffusion X. In particular, we solve the problem that aims at maximising the performance criterion $\mathbb{E}_x[e^{-\int_0^\tau r(X_s)\, ds} f(X_\tau)]$ over all stopping times τ, where the payoff function f can take only a finite number of values and has a "staircase" form. This problem has applications to both asset pricing and managerial decision making. Our results are of an explicit analytic nature and completely characterise the optimal stopping time. Also, it turns out that the problem's value function is not C^1, which is due to the fact that the payoff function f is not continuous.

1   Introduction

This paper is concerned with the problem of optimally stopping the one-dimensional Itô diffusion

$$dX_t = b(X_t)\, dt + \sigma(X_t)\, dW_t, \quad X_0 = x > 0. \tag{1}$$

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Here, W is a standard one-dimensional Brownian motion, and b, σ are deterministic functions such that (1) has a unique weak solution that is non-explosive and assumes values in the interval $]0, \infty[$. The objective of the discretionary stopping problem is to maximise the performance criterion

$$\mathbb{E}_x \left[e^{-\int_0^\tau r(X_s)\, ds} f(X_\tau) \right]$$

over all stopping times τ, where $r > 0$ is a given deterministic function. The payoff function f takes finite values and is increasing and piecewise constant, so its graph looks like a staircase with a finite number of steps.

The simplest version of this problem, which arises when $b \equiv 0$ and $\sigma \equiv 1$, i.e., when X is a standard Brownian motion, and when f can take only two values, was solved by Salminen [S85] using Martin boundary theory. The more general version of Salminen's model that arises when X is a Brownian motion with drift was recently solved by Dayanik and Karatzas [DK03, Section 6.7] using a new methodology for addressing general one-dimensional discretionary stopping problems by means of a new characterisation of excessive functions that they have developed.

The investigations undertaken here have been motivated by two classes of applications. The first of these is concerned with the pricing of digital options of American type. In this context, the stochastic differential equation (1) models the underlying asset price dynamics, and r can be interpreted as the interest rate (i.e., the short rate). The second application arises in scenario-based managerial decision making. In this context, the diffusion X is used to model the evolution of an uncertain economic environment, while the function f models the various discrete payoffs that can be obtained when action is triggered.

We have also been motivated by some general stochastic control theoretic issues; in particular, it is of interest to observe that the problem we study provides an example in which the so-called "principle of smooth fit", which suggests that the value function of an optimal stopping problem should be C^1, does not hold. Indeed, it turns out that the value function is not C^1 at all points that belong to the boundary of the stopping region as well as to the set of points at which f is discontinuous. This phenomenon has been observed by Salminen [S85], and by Dayanik and Karatzas [DK03]. One of the purposes of this paper is to offer a new way of addressing this issue by means of techniques based on the use of local times.

Incidentally, we should mention that we have opted to consider the case in which f takes finite rather than infinite values only to simplify the presentation of our results. Simplicity of exposition has also been behind our assumption that f is increasing. Indeed, our construction of the solution to the problem follows a "stepwise" approach that, at least in principle, can be adapted to account for arbitrary piecewise constant payoff functions.

2 The discretionary stopping problem

We consider a stochastic system, the state process X of which satisfies (1). We impose conditions (ND)$'$ and (LI)$'$ in Karatzas and Shreve [KS88, Section 5.5.C]; these conditions are sufficient for (1) to have a weak solution that is unique in the sense of probability law. In particular, we impose the following assumption.

Assumption 1 The deterministic functions $b, \sigma\ :]0, \infty[\ \rightarrow\ \mathbb{R}$ satisfy the following conditions:

$$\sigma^2(x) > 0, \quad \text{for all } x > 0, \tag{2}$$

and

$$\text{for all } x > 0, \text{ there exists } \varepsilon > 0 \text{ such that } \int_{x-\varepsilon}^{x+\varepsilon} \frac{1 + |b(s)|}{\sigma^2(s)}\, ds < \infty. \tag{3}$$

We also assume that the probability that the diffusion X hits either of the boundaries 0 or ∞ of its state space in finite time is zero.

Assumption 2 The diffusion X is non-explosive.

Feller's test for explosions provides a necessary and sufficient condition for X to be non-explosive (see Karatzas and Shreve [KS88, Theorem 5.5.29]).

We adopt a weak formulation of the optimal stopping problem that we study:

Definition 1 Given an initial condition $x > 0$, a *stopping strategy* is any collection $\mathbb{S}_x = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, W, X, \tau)$, where $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, W, X)$ is a weak solution to (1) and τ is an $(\mathcal{F}_t)$-stopping time. We denote by $\mathcal{S}_x$ the family of all stopping strategies associated with a given initial condition $x > 0$.

With each stopping strategy $\mathbb{S}_x \in \mathcal{S}_x$, we associate the performance criterion

$$J(\mathbb{S}_x) = \mathbb{E}_x \left[e^{-\Lambda_\tau} f(X_\tau) \right], \tag{4}$$

where

$$\Lambda_t = \int_0^t r(X_s)\, ds. \tag{5}$$

The payoff function f appearing here is assumed in the present investigation to have the form of a *finite staircase*, given by

$$f(x) = K_0 \mathbf{1}_{]0, p_1[}(x) + \sum_{j=1}^{N-1} K_j \mathbf{1}_{[p_j, p_{j+1}[}(x) + K_N \mathbf{1}_{[p_N, \infty[},$$

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where $0 < p_1 < \cdots < p_N$ and $K_0 < K_1 < \cdots < K_N$ are given constants. The objective of the discretionary stopping problem is to maximise J over $\mathcal{S}_x$. Accordingly, we define the value function

$$v(x) = \sup_{\mathbb{S}_x \in \mathcal{S}_x} J(\mathbb{S}_x). \tag{6}$$

We shall also need the following additional assumptions.

Assumption 3 σ^2 is locally bounded.

Assumption 4 There exists a constant $r_0 > 0$ such that $r(x) > r_0$, for all $x > 0$.

At this point, we should note that Assumption 4 and the fact that f is bounded imply that (4) is well-defined when the event $\{\tau = \infty\}$ has positive probability. Indeed, in this case, we assume that

$$e^{-\Lambda_\tau} f(X_\tau)\Big|_{\tau=\infty} := \lim_{t\to\infty} e^{-\Lambda_t} f(X_t) = 0.$$

3   The Hamilton-Jacobi-Bellman (HJB) equation

On the basis of standard theory of optimal stopping, we expect that the value function v should satisfy the HJB equation

$$\max\{\mathcal{L}v(x), \ f(x) - v(x)\} = 0, \quad \text{for } x > 0, \tag{7}$$

where the second order elliptic differential operator $\mathcal{L}$ is defined by

$$\mathcal{L}v(x) = \tfrac{1}{2}\sigma^2(x)v''(x) + b(x)v'(x) - r(x)v(x).$$

It turns out that the value function v of our discretionary stopping problem, which is defined by (6), has discontinuities in its first derivative. Therefore, it does not suffice in the present situation merely to consider classical solutions to the HJB equation (7). For this reason, we consider candidates for v that are differences of convex functions; for a survey of the results needed here, see Revuz and Yor [RY94, Appendix 3]. In particular, we consider solutions to (7) in the following sense.

Definition 2 A function $w :]0, \infty[\to \mathbb{R}$ satisfies the HJB equation (7) if it can be expressed as the difference of two convex functions and (7) is true, Lebesgue-a.e., with $\hat{\mathcal{L}}$ in place of $\mathcal{L}$, where the operator $\hat{\mathcal{L}}$ is defined by

$$\hat{\mathcal{L}}w(x) = \tfrac{1}{2}\sigma^2(x)w''_{\text{ac}}(x) + b(x)w'_-(x) - r(x)w(x). \tag{8}$$

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Here, w'_- is the left hand derivative of w. Also,

$$w''(dx) = w''_{\mathrm{ac}}(x)\, dx + w''_{\mathrm{s}}(dx) \tag{9}$$

is the Lebesgue decomposition of the second distributional derivative $w''(dx)$ of w into the measure $w''_{\mathrm{ac}}(x)\, dx$ that is absolutely continuous with respect to the Lebesgue measure and the measure $w''_{\mathrm{s}}(dx)$ which is mutually singular with the Lebesgue measure.

Following Zervos [Z03, Theorem 1], we can now establish conditions that are sufficient for optimality in our problem.

Theorem 1 *In the discretionary stopping problem formulated in Section 2, suppose that Assumptions 1–4 hold, and let $w :]0, \infty[\to \mathbb{R}$ be a solution to the HJB equation (7) in the sense of Definition 2 such that*

$$w \text{ is bounded, } w'_- \text{ is locally bounded,} \tag{10}$$
$$-w''_{\mathrm{s}}(dx) \text{ is a positive measure} \tag{11}$$

and

$$\operatorname{supp} w''_{\mathrm{s}}(dx) \subseteq \mathcal{C}^{\mathrm{c}} := \{x > 0 \mid w(x) = f(x)\}. \tag{12}$$

Then, $v = w$ and, given any initial condition $x > 0$, a stopping strategy

$$\mathbb{S}^*_x = (\Omega^*, \mathcal{F}^*, \mathcal{F}^*_t, \mathbb{P}^*_x, W^*, X^*, \tau^*), \tag{13}$$

where $(\Omega^*, \mathcal{F}^*, \mathcal{F}^*_t, \mathbb{P}^*_x, W^*, X^*)$ is a weak solution to (1) and

$$\tau^* = \inf \{t \geq 0 \mid X^*_t \in \mathcal{C}^{\mathrm{c}}\} \tag{14}$$

is optimal.

Proof. Fix any initial condition $x > 0$ and any weak solution $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, W, X)$ to (1). Using Itô-Tanaka formula (see Revuz and Yor [RY94, Theorem VI.1.5]), we obtain

$$w(X_t) = w(x) + \int_0^t b(X_s) w'_-(X_s)\, ds + \int_0^t \sigma(X_s) w'_-(X_s)\, dW_s + \tfrac{1}{2} \int_0^\infty L^a_t\, w''(da), \tag{15}$$

where L^a is the local time of the process X at level a. With reference to the Lebesgue decomposition (9) and the occupation times formula (see Revuz and Yor [RY94, Corollary VI.1.6]),

$$\int_0^\infty L^a_t w''_{\mathrm{ac}}(a)\, da = \int_0^t \sigma^2(X_s) w''_{\mathrm{ac}}(X_s)\, ds,$$

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so (15) implies

$$w(X_t) = w(x) + \int_0^t \left[\tfrac{1}{2}\sigma^2(X_s)w''_{\mathrm{ac}}(X_s) + b(X_s)w'_-(X_s)\right] ds + \int_0^t \sigma(X_s)w'_-(X_s)\, dW_s + A_t^w,$$

where

$$A_t^w = \tfrac{1}{2} \int_0^\infty L_t^a\, w''_{\mathrm{s}}(da). \tag{16}$$

For future reference, observe that (11) implies

$$-A^w \text{ is a continuous, increasing process,} \tag{17}$$

because such a statement is true for local times. Now, using the integration by parts formula for semimartingales, we obtain

$$e^{-\Lambda_t}w(X_t) = w(x) + \int_0^t e^{-\Lambda_s}\hat{\mathcal{L}}w(X_s)\, ds + M_t + \int_0^t e^{-\Lambda_s}\, dA_s^w, \tag{18}$$

where M is the stochastic integral defined by

$$M_t = \int_0^t e^{-\Lambda_s}\sigma(X_s)w'_-(X_s)\, dW_s. \tag{19}$$

To proceed further, fix any admissible stopping strategy $\mathbb{S}_x \in \mathcal{S}_x$, let (τ_m) be the sequence of $(\mathcal{F}_t)$-stopping times defined by

$$\tau_m = \inf\left\{t \geq 0 \mid X_t \notin [\tfrac{1}{m}, m]\right\}, \quad \text{for } m = 1, 2, \ldots,$$

and note that $\lim_{m\to\infty} \tau_m = \infty$, $\mathbb{P}_x$-a.s., because X is non-explosive. With regard to the local boundedness of σ^2 and w'_- (see Assumption 3 and (10), respectively), and the uniform positivity of the discounting factor r (see Assumption 4), we can see that, given any $m \geq 1$, the stopped process M^{τ_m}, where M is the stochastic integral defined as in (19), has quadratic variation that satisfies

$$\mathbb{E}_x\left[\langle M^{\tau_m}\rangle_\infty\right] = \mathbb{E}_x\left[\int_0^\infty \mathbf{1}_{\{s \leq \tau_m\}}\left[e^{-\Lambda_s}\sigma(X_s)w'_-(X_s)\right]^2 ds\right]$$

$$\leq \frac{1}{2r_0} \sup_{x \in \left[\frac{1}{m}, m\right]}\left[\sigma(x)w'_-(x)\right]^2$$

$$< \infty,$$

which implies that M^{τ_m} is a uniformly square integrable martingale. Therefore, $M_\tau^{\tau_m}$ is well-defined and Doob's optional sampling theorem implies that $\mathbb{E}_x\left[M_\tau^{\tau_m}\right] = 0$. In light of this observation and (18) above, we can see that

$$\mathbb{E}_x\left[e^{-\Lambda_{\tau\wedge\tau_m}}f(X_{\tau\wedge\tau_m})\right] = w(x) + \mathbb{E}_x\left[e^{-\Lambda_{\tau\wedge\tau_m}}\left[f(X_{\tau\wedge\tau_m}) - w(X_{\tau\wedge\tau_m})\right]\right]$$

$$+ \mathbb{E}_x\left[\int_0^{\tau\wedge\tau_m} e^{-\Lambda_s}\hat{\mathcal{L}}w(X_s)\, ds\right] + \mathbb{E}_x\left[\int_0^{\tau\wedge\tau_m} e^{-\Lambda_s}\, dA_s^w\right]. \tag{20}$$

In view of (17) and the fact that w satisfies (7) in the sense of Definition 2, it follows that

$$\mathbb{E}_x \left[e^{-\Lambda_{\tau \wedge \tau_m}} f(X_{\tau \wedge \tau_m}) \right] \leq w(x).$$

However, by passing to the limit $m \to \infty$ in this inequality using the dominated convergence theorem, we can see that $J(\mathbb{S}_x) \leq w(x)$, which proves that $v(x) \leq w(x)$.

Now, let $\mathbb{S}_x^*$ be the stopping strategy given by (13)–(14). Since the measure dL_t^{*a} is supported on the set $\{t \geq 0 \mid X_t^* = a\}$, the definition of τ^* implies

$$L_t^{*a} = 0, \quad \text{for all } t \in [0, \tau^*] \text{ and } a \in \mathcal{C}^{\mathrm{c}},$$

which, in view of (12) and (16), implies $A_t^{w*} = 0$, for all $t \leq \tau^*$. However, combining this observation and the definition of $\mathbb{S}_x^*$ with (20) and the fact that the set $\{x > 0 \mid w(x) = f(x)\}$ is closed, which follows from the upper semicontinuity of f, we can see that

$$\mathbb{E}_x^* \left[e^{-\Lambda_{\tau^* \wedge \tau_m^*}^*} f(X_{\tau^* \wedge \tau_m^*}^*) \right] = \mathbb{E}_x^* \left[e^{-\Lambda_{\tau_m^*}^*} \left[f(X_{\tau_m^*}^*) - w(X_{\tau_m^*}^*) \right] \mathbf{1}_{\{\tau_m^* < \tau^*\}} \right] + w(x).$$

With regard to the boundedness of f and w, and the uniform positivity of the discounting factor r (see Assumption 4), we can pass to the limit $m \to \infty$ using the dominated convergence theorem, to conclude that $J(\mathbb{S}_x^*) = w(x)$, which, combined with the inequality $v(x) \leq w(x)$ that we have established above, proves that $v(x) = w(x)$ and that $\mathbb{S}_x^*$ is an optimal strategy. $\qquad \square$

We shall also need the following result for the construction of an appropriate solution to the HJB equation (7) in the next section.

Lemma 2 *Suppose that Assumptions 1–4 hold, fix two constants $y, z \in [0, \infty]$ such that $y < z$, and suppose that the functions $g, h : [y, z] \to \mathbb{R}$ are differences of two convex functions and satisfy*

$$\hat{\mathcal{L}} g(x) = \hat{\mathcal{L}} h(x) = 0, \quad \text{for all } x \in]y, z[, \tag{21}$$

where $\hat{\mathcal{L}}$ is defined by (8),

$$g(y) \geq h(y) \text{ and } g(z) \geq h(z), \tag{22}$$
$$g'_- \text{ and } h'_- \text{ are both locally bounded}, \tag{23}$$
$$g''_{\mathrm{s}}(dx) \equiv 0 \text{ and } h''_{\mathrm{s}}(dx) \text{ is a positive measure}. \tag{24}$$

Then $h(x) \leq g(x)$, for all $x \in [y, z]$.

Proof. Fix any initial condition $x \in]y, z[$ and any weak solution $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, W, X)$ to (1), and define

$$T = \inf \{t \geq 0 \mid X_t \notin]y, z[\}.$$

Also, to simplify the proof, assume that σ^2, g'_- and h'_- are all bounded rather than just locally bounded: indeed, when $y = 0$ or $z = \infty$, a straightforward adaptation of the "localising" arguments deployed in the proof of Theorem 1 can be used to address the more general case. This assumption implies that the stochastic integral

$$t \mapsto \int_0^{t \wedge T} e^{-\Lambda_s} \sigma(X_s) \left[g'_-(X_s) - h'_-(X_s) \right] dW_s$$

is a uniformly integrable martingale. However, this observation, (21)–(22) and Itô's formula (18) imply

$$0 \le g(x) - h(x) + \mathbb{E}_x \left[\int_0^T e^{-\Lambda_s} dA_s^{g-h} \right], \tag{25}$$

where

$$A_t^{g-h} = \tfrac{1}{2} \int_y^z L_t^a \left[g''_s - h''_s \right] (da)$$

$$= -\tfrac{1}{2} \int_y^z L_t^a h''_s (da), \quad \text{for } t \le T.$$

Since $h''_s(dx)$ is a positive measure, the process $-A^{g-h}$ is increasing, so

$$\mathbb{E}_x \left[\int_0^T e^{-\Lambda_s} dA_s^{g-h} \right] \le 0,$$

which, combined with (25) above, implies that $h(x) \le g(x)$, and the proof is complete. $\quad\square$

4 The solution to the discretionary stopping problem

We will solve the optimal stopping problem that we consider by constructing a solution to the HJB equation (7) that satisfies the requirements of Theorem 1. To this end, we first observe that every solution to the homogeneous ordinary differential equation (ODE)

$$\tfrac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) = 0, \tag{26}$$

which is associated with (7) is given by

$$w(x) = A\varphi(x) + B\psi(x), \tag{27}$$

for some constants $A, B \in \mathbb{R}$. The functions ψ, φ are defined by

$$\psi(x) = \begin{cases} \mathbb{E}_x \left[e^{-\Lambda_{T_y}} \right], & \text{for } x < y, \\ \left(\mathbb{E}_y \left[e^{-\Lambda_{T_x}} \right] \right)^{-1}, & \text{for } x \ge y, \end{cases} \tag{28}$$

$$\varphi(x) = \begin{cases} \left(\mathbb{E}_y \left[e^{-\Lambda_{T_x}} \right] \right)^{-1}, & \text{for } x < y, \\ \mathbb{E}_x \left[e^{-\Lambda_{T_y}} \right], & \text{for } x \ge y, \end{cases} \tag{29}$$

respectively, for a given choice of $y > 0$. Here Λ is defined by (5), while T_x (resp., T_y) is the first hitting time of $\{x\}$ (resp., $\{y\}$). For future reference, we note that

$$\varphi \text{ and } \psi \text{ are both strictly positive and } C^1, \text{ their second derivative exists in the} \qquad (30)$$
$$\text{classical sense, } \varphi \text{ is strictly decreasing and } \psi \text{ is strictly increasing.}$$

Also, the Wronskian $\mathcal{W}$ of φ and ψ, which identifies with the first derivative of the scale function of the diffusion X, is given by

$$\mathcal{W}(x) := \varphi(x)\psi'(x) - \varphi'(x)\psi(x)$$
$$= \mathcal{W}(y) \exp\left(-2 \int_y^x \frac{b(s)}{\sigma^2(s)} \, ds\right), \quad \text{for } x > 0, \qquad (31)$$

for any given choice of $y > 0$. These results are known since several decades and can be found in various forms in several references, including Feller [F52], Breiman [B68], Itô and McKean [IM74], Karlin and Taylor [KT81], and Rogers and Williams [RW00]. Here, we follow the exposition in Johnson and Zervos [JZ05, Appendix], where analytic expressions for the functions φ and ψ are also derived when X is a geometric Brownian motion, a mean-reverting square-root process such as the one used in the Cox-Ingersoll-Ross interest rate model, an exponential Ornstein-Uhlenbeck process such as the one used in the Black-Karasinski interest rate model, or a geometric Ornstein-Uhlenbeck process.

Back to our optimal stopping problem, we conjecture that the value function satisfies the HJB equation (7) in the *classical sense* outside the set of the points at which the discontinuities of f occur, namely, inside the set $]0, \infty[\setminus \{p_1, \ldots, p_N\}$. This conjecture and the intuitive idea that some of the points $p_1, \ldots, p_N$ (e.g., p_N) should belong to the stopping region $\mathcal{C}^c$ of the discretionary stopping problem that we solve motivate a "stepwise" approach, the first objective of which is to solve the following two problems.

Problem 1 Given constants $0 < y < z$ and $K < L$, find a continuous function $\tilde{w} : [y, z] \to \mathbb{R}$ that is a classical solution to (7) with $f(x) = K$, for $x \in]y, z[$, and satisfies the boundary conditions

$$\tilde{w}(y) = K \quad \text{and} \quad \tilde{w}(z) = L.$$

Problem 2 Given constants $z > 0$ and $K < L$, find a continuous, *bounded* function $\tilde{w} : [0, z] \to \mathbb{R}$ that is a classical solution to (7) with $f(x) = K$, for $x \in]0, z[$, and satisfies the boundary conditions

$$\tilde{w}(0) \geq K \quad \text{and} \quad \tilde{w}(z) = L.$$
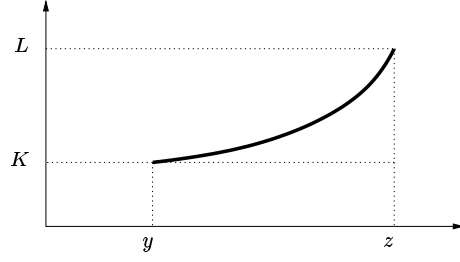
Figure 1: Graph of the first possible solution $\tilde{w}$ to the HJB equation (7) that satisfies the boundary conditions $\tilde{w}(y) = K$ and $\tilde{w}(z) = L > K$ when $f \equiv K$ and the independent variable x takes values in the interval $]y, z[$, for $0 < y < z$ (Problem 1).

The solution to Problem 1, is associated with two qualitatively different possibilities. The first one arises when $\tilde{w}$ satisfies the ODE (26) for all $x \in]y, z[$, in which case, $\tilde{w}$ is given by

$$\tilde{w}(x) = \begin{cases} K, & \text{for } x = y, \\ A\varphi(x) + B\psi(x), & \text{for } x \in]y, z[, \\ L, & \text{for } x = z, \end{cases} \tag{32}$$

where A and B are constants (see Figure 1). The continuity of $\tilde{w}$ at the boundary of $[y, z]$ yields a linear system of two equations for the unknowns A and B, the solution of which is given by

$$A = \left(\frac{L}{\psi(z)} - \frac{K}{\psi(y)} \right) \left(\frac{\varphi(z)}{\psi(z)} - \frac{\varphi(y)}{\psi(y)} \right)^{-1}, \tag{33}$$

$$B = \left(\frac{L}{\varphi(z)} - \frac{K}{\varphi(y)} \right) \left(\frac{\psi(z)}{\varphi(z)} - \frac{\psi(y)}{\varphi(y)} \right)^{-1}. \tag{34}$$

Lemma 3 *The function $\tilde{w}$ defined by (32), where A and B are given by (33) and (34), respectively, provides a solution to Problem 1 if and only if*

$$\frac{\psi'(y)}{\varphi'(y)} \leq \frac{L\psi(y) - K\psi(z)}{L\varphi(y) - K\varphi(z)}. \tag{35}$$

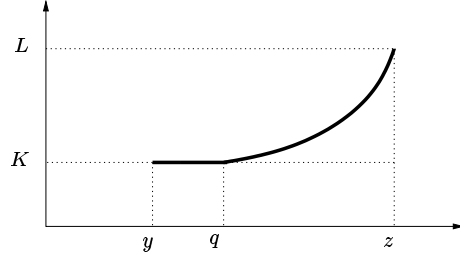We collect in the Appendix the proofs of those results that are not fully developed in the text.

Figure 2: Graph of the second possible solution $\tilde{w}$ to the HJB equation (7) that satisfies the boundary conditions $\tilde{w}(y) = K$ and $\tilde{w}(z) = L > K$ when $f \equiv K$ and the independent variable x takes values in the interval $]y, z[$, for $0 < y < z$ (Problem 1).

The second possibility arises when there is a point $q \in]y, z[$ such that $\tilde{w}(x) = K$ for $x \in [y, q]$, and $\tilde{w}$ satisfies the ODE (26) for $x \in]q, z[$, which is associated with

$$\tilde{w}(x) = \begin{cases} K, & \text{for } x \in [y, q], \\ A\varphi(x) + B\psi(x), & \text{for } x \in]q, z[, \\ L, & \text{for } x = z, \end{cases} \tag{36}$$

where A and B are constants (see Figure 2). To determine A, B and the free boundary point q, we appeal to the requirement that $\tilde{w}$ should satisfy (7) in the classical sense in $]y, z[$, which implies that $\tilde{w}$ should be C^1 at q, and to the boundary condition $\tilde{w}(z) = L$. It is straightforward to see that the resulting system of equations is equivalent to the expressions

$$A = \left(\frac{L}{\psi(z)} - \frac{K}{\psi(q)} \right) \left(\frac{\varphi(z)}{\psi(z)} - \frac{\varphi(q)}{\psi(q)} \right)^{-1}, \tag{37}$$

$$B = \left(\frac{L}{\varphi(z)} - \frac{K}{\varphi(q)} \right) \left(\frac{\psi(z)}{\varphi(z)} - \frac{\psi(q)}{\varphi(q)} \right)^{-1}, \tag{38}$$

and the algebraic equation

$$F(q) = 0, \tag{39}$$

where the function F is defined by

$$F(x) = -[L\psi(x) - K\psi(z)] + [L\varphi(x) - K\varphi(z)]\frac{\psi'(x)}{\varphi'(x)}, \quad \text{for } x \in [y, z[. \tag{40}$$

Lemma 4 *Given any $y > 0$, equation (39) has a solution $q \in]y, z[$ if and only if*

$$\frac{\psi'(y)}{\varphi'(y)} > \frac{L\psi(y) - K\psi(z)}{L\varphi(y) - K\varphi(z)}. \tag{41}$$

If this condition is satisfied, then the solution q to (39) is unique and the function $\tilde{w}$ defined by (36), where A and B are given by (37) and (38), respectively, solves Problem 1.
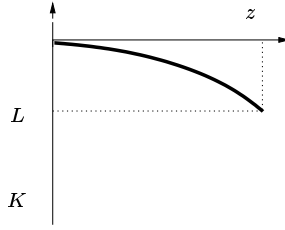
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Figure 3: Graph of the first possible solution $\tilde{w}$ to the HJB equation (7) that satisfies the boundary conditions $\tilde{w}(0) \geq K$ and $\tilde{w}(z) = L > K$ when $f \equiv K$ and the independent variable x takes values in the interval $]0, z[$, for $z > 0$ (Problem 2). Here, we illustrate the case when $K < L < 0$.
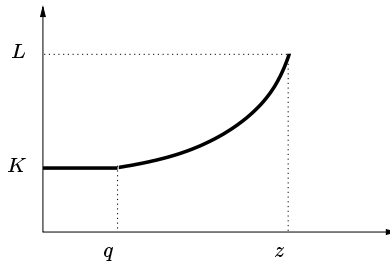

Figure 4: Graph of the second possible solution $\tilde{w}$ to the HJB equation (7) that satisfies the boundary conditions $\tilde{w}(0) \geq K$ and $\tilde{w}(z) = L > K$ when $f \equiv K$ and the independent variable x takes values in the interval $]0, z[$, for $z > 0$ (Problem 2).

Now, let us consider Problem 2, which is again associated with two qualitatively different solutions. Since $\lim_{x \downarrow 0} \varphi(x) = \infty$, which follows from the definition (29) of φ and the assumption that X is non-explosive,

$$\tilde{w}(x) = \frac{L}{\psi(z)} \psi(x), \quad \text{for } x \in [0, z], \tag{42}$$

is the appropriate choice for $\tilde{w}$ that corresponds to Lemma 3 because it is the only bounded solution to the ODE (26) that satisfies the boundary condition $\tilde{w}(z) = L$. With regard to the fact that ψ is strictly increasing and positive, it is straightforward to see that this choice indeed provides the solution to Problem 2 if $L\psi(0) \geq K\psi(z)$, where $\psi(0) := \lim_{x \downarrow 0} \psi(x)$ (see also Figure 3). When the problem's data are such that $L\psi(0) < K\psi(z)$, which can be true only if $K > 0$, we are faced with the possibility for the solution to Problem 2 to be as in Lemma 4 (see also Figure 4).

Lemma 5 *Equation (39) has a unique solution $q \in]0, z[$ if and only if $L\psi(0) < K\psi(z)$. Moreover, the following two statements are true:*

(a) *If $L\psi(0) \geq K\psi(z)$, then (42) provides a solution to Problem 2.*
(b) *If $L\psi(0) < K\psi(z)$, then the function $\tilde{w}$ defined by (36)–(38), where q is the unique solution to (39), with $y = 0$, solves Problem 2.*

We can now construct a solution to the HJB equation (7) in the sense of Definition 2 that identifies with the value function of our discretionary stopping problem using the following algorithm.

Step 1 Set $l = 0$ and define the N-dimensional vectors

$$i^{(l)} = (1, 2, \ldots, N-1, N) \quad \text{and} \quad \rho^{(l)} = (p_1, p_2, \ldots, p_{N-1}, p_N).$$

Step 2 Define the function $w^{(l)} :]0, \infty[\to \mathbb{R}$ by

$$w^{(l)}(x) = w_0^{(l)}(x)\mathbf{1}_{]0,\rho_1^{(l)}[}(x) + \sum_{j=1}^{\dim i^{(l)}-1} w_j^{(l)}(x)\mathbf{1}_{[\rho_j^{(l)},\rho_{j+1}^{(l)}[}(x) + K_N\mathbf{1}_{[p_N,\infty[},$$

where $w_0^{(l)}$ is the solution to Problem 2 with $z = \rho_1^{(l)}$, $K = K_0$ and $L = K_{i_1^{(l)}}$, given by Lemma 5, while, for $j = 1, \ldots, \dim i^{(l)} - 1$, $w_j^{(l)}$ is the solution to Problem 1 with $y = \rho_j^{(l)}$, $z = \rho_{j+1}^{(l)}$, $K = K_{i_j^{(l)}}$ and $L = K_{i_{j+1}^{(l)}}$, given by Lemmas 3 and 4.

Step 3 Let m be index of the first element of the vector $i^{(l)}$ such that

$$\lim_{x\uparrow\rho_m^{(l)}} \frac{d}{dx}w^{(l)}(x) < \lim_{x\downarrow\rho_m^{(l)}} \frac{d}{dx}w^{(l)}(x) \quad \Leftrightarrow \quad \left(w^{(l)}\right)''_{\mathrm{s}}\left(\{\rho_m^{(l)}\}\right) > 0.$$

If no such index exists, then set $w = w^{(l)}$ and STOP. Otherwise, let $i^{(l+1)}$ and $\rho^{(l+1)}$ be the vectors obtained by deleting the m-th entry of the vectors $i^{(l)}$ and $\rho^{(l)}$, respectively, set $l = l + 1$, and go back to Step 2.

Plainly, this algorithm terminates after at most $N-1$ steps and each of the functions $w^{(l)}$ that the algorithm produces is a difference of convex functions. Also, any functions $w^{(l)}$ and $w^{(l+1)}$ produced by two consecutive iterations of the algorithm satisfy $w^{(l)} \leq w^{(l+1)}$, thanks to Lemma 2 (see also Figure 4). Also, we can easily check that the resulting function w satisfies the assumptions of Theorem 1, and, therefore, it identifies with our problem's value function. We conclude with the main result of the paper.

Theorem 6 *The value function of the discretionary stopping problem formulated in Section 2 identifies with the function w resulting from the algorithm above, and an optimal stopping strategy is given by (13)–(14) in Theorem 1.*
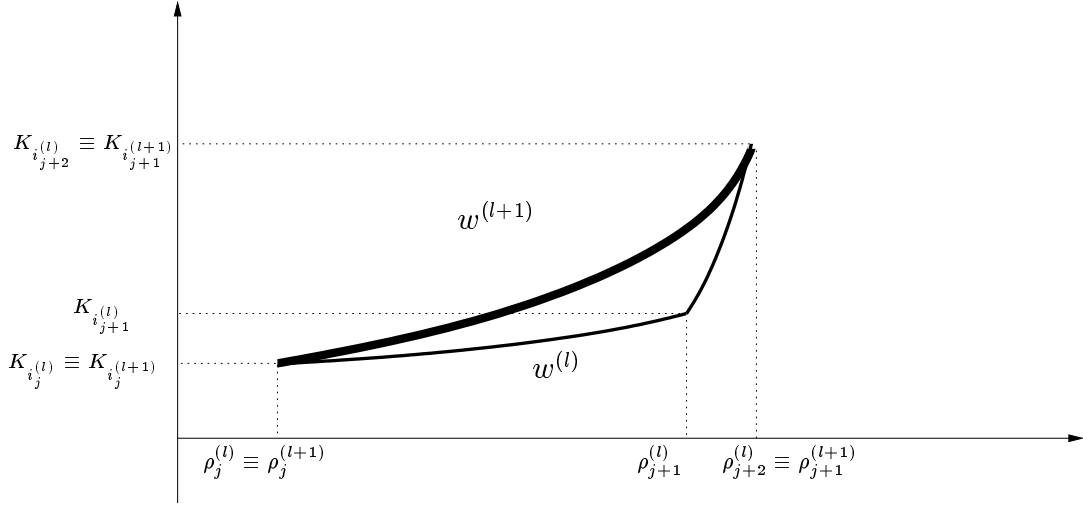
Figure 5: Illustration of two successive iterations of the algorithm that provides the solution to the HJB equation (7).

Appendix

Proof of Lemma 3 By construction, we will show that $\tilde{w}$ satisfies the HJB equation (7) for $x \in \,]y, z[$ if we prove that

$$\tilde{w}(x) \geq K, \quad \text{for all } x \in \,]y, z[. \tag{43}$$

To this end, we first note that the facts that $y < z$ and $K < L$, (30) and the definition of B in (34) imply that $B > 0$. In view of this observation and (30), we can see that

$$\tilde{w}'(x) \equiv A\varphi'(x) + B\psi'(x) \geq 0, \quad \text{for all } x \in \,]y, z[, \tag{44}$$

if and only if

$$-\frac{\psi'(x)}{\varphi'(x)} \geq \frac{A}{B}, \quad \text{for all } x \in \,]y, z[. \tag{45}$$

Now, using the fact that φ, ψ satisfy the ODE (26) and the expression (31) for their Wronskian, we can see that

$$
\begin{aligned}
\frac{d}{dx}\left(-\frac{\psi'(x)}{\varphi'(x)}\right) &= -\frac{\psi''(x)\varphi'(x) - \psi'(x)\varphi''(x)}{[\varphi'(x)]^2} \\
&= \frac{2r(x)\mathcal{W}(x)}{[\sigma(x)\varphi'(x)]^2} \\
&> 0, \qquad \text{for all } x \in \,]y, z[.
\end{aligned}
\tag{46}
$$

This inequality shows that (44)–(45) are both true if and only if

$$-\frac{\psi'(y)}{\varphi'(y)} \geq \frac{A}{B}. \tag{47}$$

Moreover, if (47) is not true, then $\tilde{w}'(x) < 0$ for all x sufficiently close to y, which, combined with the fact that $\tilde{w}(y) = K$, implies that (43) fails to be true. We conclude that (43) is true if and only if (47) holds, which, in view of the definitions of A, B in (33), (34), respectively, is equivalent to (35), and the proof is complete. □

Proof of Lemma 4 In view of (30) and the fact that $K < L$, we can see that

$$F(z) = -\psi(z)\,[L - K] + \varphi(z)\,[L - K]\,\frac{\psi'(z)}{\varphi'(z)} < 0.$$

Also, with reference to (46), we calculate

$$F'(x) = [L\varphi(x) - K\varphi(z)]\,\frac{d}{dx}\left(\frac{\psi'(x)}{\varphi'(x)}\right) < 0, \quad \text{for } x \in]y, z[.$$

It follows that the equation $F(q) = 0$ has a unique solution $q \in]y, z[$ if and only if $F(y) > 0$, which is equivalent to (41).

With regard to its construction, we can see that the function $\tilde{w}$ satisfies the HJB equation (7) for $x \in]y, z[$ if and only if

$$\tilde{w}(x) \geq K, \quad \text{for all } x \in [q, z[. \tag{48}$$

Now, following the same reasoning as in the proof of Lemma 3 above, we obtain

$$\tilde{w}'(x) \geq 0, \text{ for all } x \in]q, z[\quad \Leftrightarrow \quad -\frac{\psi'(q)}{\varphi'(q)} \geq \frac{A}{B}.$$

However, combining this observation with the fact that $\tilde{w}$ is C^1 at q, which implies that

$$\tilde{w}(q) = K \quad \text{and} \quad \tilde{w}'(q) \equiv A\varphi'(q) + B\psi'(q) = 0,$$

we can see that (48) is true , and the proof is complete. □

Proof of Lemma 5 With reference to the proof of Lemma 4, we can see that equation (39) has a unique solution $q \in]0, z[$ if and only if

$$\lim_{x \downarrow 0} F(x) \equiv \lim_{x \downarrow 0} \left[K\psi(z) + L\frac{\mathcal{W}(x)}{\varphi'(x)} - K\varphi(z)\frac{\psi'(x)}{\varphi'(x)} \right] > 0, \tag{49}$$

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where $\mathcal{W}$ is the Wronskian of φ and ψ defined by (31). To establish conditions under which this inequality is true, we calculate

$$\frac{d}{dx}\left(\frac{\mathcal{W}(x)}{\varphi'(x)}\right) = -\frac{2r(x)\mathcal{W}(x)\varphi(x)}{[\sigma(x)\varphi'(x)]^2} < 0,$$

which, combined with the inequality $\mathcal{W}(x)/\varphi'(x) < 0$, which is true for all $x > 0$, implies that $\lim_{x\downarrow 0}\mathcal{W}(x)/\varphi'(x)$ exists in $]-\infty, 0]$. However, this observation, the fact that $\lim_{x\downarrow 0}\psi(x)$ exists in $[0,\infty[$ because ψ is strictly positive and increasing, and the expression

$$\frac{\varphi(x)\psi'(x)}{\varphi'(x)} = \frac{\mathcal{W}(x)}{\varphi'(x)} + \psi(x), \quad \text{for } x > 0, \tag{50}$$

which follows immediately from the definition (31) of $\mathcal{W}$, imply that

$$\lim_{x\downarrow 0}\frac{\varphi(x)\psi'(x)}{\varphi'(x)} \in \,]-\infty, 0].$$

Now, we use a contradiction argument to show that this limit is actually equal to 0. To this end, we suppose that

$$\lim_{x\downarrow 0}\frac{\varphi(x)\psi'(x)}{\varphi'(x)} = -2\varepsilon, \quad \text{for some } \varepsilon > 0. \tag{51}$$

This assumption implies that there exists $x_1 > 0$ such that

$$-\frac{\varphi'(s)}{\varphi(s)} \leq \frac{1}{\varepsilon}\psi'(s), \quad \text{for all } s \in \,]0, x_1].$$

In view of this inequality, we can see that

$$\ln\varphi(x) = \ln\varphi(y) + \int_x^y \left(-\frac{\varphi'(s)}{\varphi(s)}\right)ds$$

$$\leq \ln\varphi(y) + \frac{1}{\varepsilon}\left[\psi(y) - \psi(x)\right], \quad \text{for all } 0 < x < y \leq x_1,$$

which implies

$$\varphi(x) \leq \varphi(y)\exp\left(\frac{1}{\varepsilon}\left[\psi(y) - \psi(x)\right]\right), \quad \text{for all } 0 < x < y \leq x_1. \tag{52}$$

For fixed y, the right hand side of this inequality remains bounded as $x \downarrow 0$ because ψ is positive and increasing, which implies that (52) cannot be true because $\lim_{x\downarrow 0}\varphi(x) = \infty$. It follows that (51) is false, and, therefore,

$$\lim_{x\downarrow 0}\frac{\varphi(x)\psi'(x)}{\varphi'(x)} = 0 \quad \Rightarrow \quad \lim_{x\downarrow 0}\frac{\psi'(x)}{\varphi'(x)} = 0.$$

However, these limits and (50) imply that (49) is equivalent to the inequality $K\psi(z) - L\psi(0) > 0$, which establishes the claim regarding the solvability of (39).

Now, part (a) of the lemma is obvious, while part (b) follows by a straightforward adaptation of the arguments used to establish the corresponding claim in Lemma 4. $\qquad\square$

Acknowledgement

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