A discretionary stopping problem with applications to the optimal timing of investment decisions*

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Abstract

We consider the discretionary stopping problem that aims at maximising the performance criterion $\mathbb{E}_x\left[e^{-\int_0^\tau r(X_s)\,ds} g(X_\tau) \mathbf{1}_{\{\tau<\infty\}}\right]$ over all stopping times τ, where X is a general one-dimensional positive Itô diffusion, r is a strictly positive function and g is a given payoff function. This optimal stopping problem has several applications in mathematical finance and economics. These include the pricing of perpetual American options as well as the optimal timing to invest in a project or capitalising an asset, which are fundamental issues in the theory of real options. We develop a set of sufficient conditions on the problem's data under which this optimal stopping problem admits a solution that conforms with standard financial and economic intuition. Our analysis leads to results of an explicit analytic nature and is illustrated by a number of special cases that are of interest in applications, and aspects of which have been considered in the literature. In the course of our analysis we also establish a range of results that can provide useful tools for developing the solution to other stochastic control problems.

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1 Introduction

A fundamental problem in the theory of real options is concerned with determining the optimal time to invest in a given project. The results on this problem, which was introduced by McDonald and Siegel [MS86], as well as the majority of real options models in the current literature, assume that the underlying asset's value dynamics are modelled by a geometric Brownian motion, the associated payoff function is affine and the discounting rate is constant. The objective of this paper is to significantly relax all of these assumptions and to provide a much more realistic modelling framework within which results of an explicit nature can be obtained.

Apart from offering modellers additional flexibility, developing the existing theory so that it can account for asset price dynamics driven by general Itô diffusions becomes essential once one recognises that assets that exist in equilibrium market conditions tend to fluctuate about some long-term mean level, rather than, on average, grow or fall exponentially, as modelled by a geometric Brownian motion. This observation, which is supported by empirical evidence (e.g., see Metcalf and Hassett [MH95] and Sarkar [Sar03]), suggests that real asset dynamics should be modelled by mean-reverting diffusions rather than by a geometric Brownian motion. The mean reversion property of real asset price dynamics is suggestive also of the commonly accepted hypothesis that the short interest rate is mean-reverting, which has been factored into the majority of fixed income models.

Introducing state dependent discounting enables a more realistic modelling framework for investment decisions in the presence of default risk. In practice, investment decision-making involves the choice of a discounting rate that accounts for the time-value of money and the associated investment's depreciation rate as well as for the likelihood of the investment's default. In view of this observation, discounting should reflect the dependence of default likelihood of an investment project on the economic environment affecting the project, which is stochastically related to the underlying asset's value or demand.

Considering general payoff functions, rather than affine ones, plainly provides significant additional modelling flexibility, which allows, for example, the incorporation of tax effects on payoffs. Also, it enables utility based decision making, which, apart from the work of Henderson and Hobson [HH02], and despite its fundamental importance, has hardly found its way into real options theory. Indeed, the introduction of utility functions into real option models is a major economic contribution of the paper.

With regard to the discussion above and with a view to a number of applications such as the ones discussed below, we consider a stochastic system, the state process X of which satisfies the one-dimensional Itô diffusion

$$dX_t = b(X_t)\, dt \;+\; \sigma(X_t)\, dW_t, \quad X_0 = x > 0,$$

where W is a standard, one-dimensional Brownian motion, and b, σ are given deterministic functions such that $X_t > 0$, for all $t > 0$, with probability 1. The objective is to solve the discretionary stopping problem that aims to maximise

$$\mathbb{E}_x\left[e^{-\Lambda_\tau} g(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right], \quad \text{where} \quad \Lambda_t = \int_0^t r(X_s)ds,$$

over all stopping times τ, where r is strictly positive and g is a given payoff function.

The theory of discretionary stopping has numerous applications and has attracted the interest of numerous researchers. Important, older accounts of this theory include Shiryaev [Shi78], El-Karoui [EK79] and Krylov [Kry80] , while more recent contributions include Salminen [Sal85], Davis and Karatzas [DK94], Øksendal and Reikvam [ØR98], Guo and Shepp [GS01] and Dayanik and Karatzas [DK03]. The special structure of the problem considered here allows us to elaborate on the existing theory in an explicit, analytic nature.

The discretionary stopping problem that we solve has a range of applications in finance and economics. The most fundamental one aims to maximise

$$\mathbb{E}_x\left[e^{-r\tau} g(X_\tau - K)\mathbf{1}_{\{\tau<\infty\}}\right]$$

where g is a utility function, and addresses the question of when is it optimal to sell an asset, the price of which is modelled by the state process X, and selling incurs a cost K, which may be the purchase cost at time 0, while r is a discounting rate. In this context, if an agent has been endowed with an asset, such as an equity or a quantity of gold, then $K = 0$ and the question is when to optimally dispose of, i.e., capitalise, the asset.

A second application arises in the field of real options (e.g., see Dixit and Pindyck [DP94] and Trigeorgis [Tri96]). As discussed above, a fundamental issue in real options is concerned with determining the optimal time to invest in a project, within a random economic environment, such as the development of an offshore oil production facility. In this context, X_t models the expected, given the information available at time t, discounted cash-flow that the project will yield if developed at time t, while $K > 0$ models the cost of developing the project. Alternatively we can relate the mathematical problem that we solve with answering the question when is it optimal to abandon an economic activity, such as the management of an offshore oil production facility. In this case, the state process X can be used to model the value of capitalising the underlying asset, while K is used to model costs associated with capitalisation.

A further application arises in the context of pricing perpetual American call options, a problem initially studied by Merton [Mer73]. One of the attractive features of perpetual options is that one can obtain explicit analytic expressions for their values. However, perpetual options are important in the theory of finance because their prices provide upper bounds for the corresponding finite maturity ones. In addition, our analysis provides the prices of perpetual American "power" options, which have been studied in discrete time by Novikov and Shiryaev [NS04], for a range of underlying asset price dynamics. We should also note that, although our analysis considers payoff functions generalising the ones associated with a call option, it can easily be modified to account for the symmetric cases arising when, for example, g is the payoff function of a put option.

We solve the optimal stopping problem under consideration under general assumptions on the underlying state process X, the payoff function g and the discounting rate r. To illustrate how our results can be used to develop specific models we analyse a number of special cases. These cases involve a number of choices for the underlying sate process X that have been considered in the literature. In particular we consider the cases that arise when X is a geometric Brownian motion, a square-root mean-reverting process as in the Cox-Ingersoll-Ross interest rate model, an exponential Ornstein-Uhlenbeck process as in the Black-Karasinski interest rate model, and a geometric Ornstein-Uhlenbeck process, which has been proposed by Cortazar and Schwartz as a model for a commodity's price and has been used in population modelling.

The paper is organised as follows. Section 2 is concerned with a rigorous formulation of the optimal stopping problem that we solve. In this section we also develop a set of assumptions that are sufficient for our problem to admit a solution, the structure of which conforms with the applications in finance and economics discussed above. Although all of the special cases of interest that we are aware of are associated with SDEs that have unique, strong solutions, we adopt a weak formulation. Adopting this more general framework, which involves no additional technicalities, has been motivated by the extra degrees of freedom that it offers relative to modelling, and has a view to a wider range of applications. In Section 3, we solve the optimal stopping problem under consideration, while, in Section 4, we address a number of special cases of interest. Part of the results presented in this section can be found in Watson [Wat03]. Finally, the Appendix is concerned with a study of an ODE that plays a fundamental role in our analysis. Most of the results presented there were established by Feller [Fel52] and can be found in various forms in several references that include Breiman [Bre68], Mandl [Man68], Itô and McKean [IM74], Karlin and Taylor [KT81], and Rogers and Williams [RW94]. Our presentation, which is based on modern probabilistic techniques, has largely been inspired by Rogers and Williams [RW94, Sections V.3, V.5, V.7] and includes ramifications not found in the literature.

2 Problem formulation, assumptions and preliminary estimates

We consider a stochastic system, the state process X of which satisfies the one-dimensional Itô diffusion

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_0 = x > 0, \tag{1}$$

where W is a one-dimensional, standard Brownian motion and $b, \sigma :]0,\infty[\to \mathbb{R}$ are given deterministic functions satisfying conditions (ND)$'$ and (LI)$'$ in Karatzas and Shreve [KS91, Section 5.5C].

Assumption 2.1 The functions $b, \sigma :]0,\infty[\to \mathbb{R}$ satisfy the following conditions:

$$\sigma^2(x) > 0, \text{ for all } x \in]0,\infty[,$$
$$\text{for all } x \in]0,\infty[, \text{ there exists } \varepsilon > 0 \text{ such that } \int_{x-\varepsilon}^{x+\varepsilon} \frac{1+|b(s)|}{\sigma^2(s)}\,ds < \infty.$$

This assumption guarantees the existence of a unique, in the sense of probability law, solution to (1) up to an explosion time. In particular, given $x_0 > 0$, the *scale function* p_{x_0} and the *speed measure* $m_{x_0}(dx)$, given by

$$p_{x_0}(x) = \int_{x_0}^{x} \exp\left(-2\int_{x_0}^{s} \frac{b(u)}{\sigma^2(u)}\,du\right) ds, \quad \text{for } x > 0, \tag{2}$$

$$m_{x_0}(dx) = \frac{2}{\sigma^2(x) p'_{x_0}(x)}\,dx, \tag{3}$$

are well-defined.

We also assume that the diffusion X is *non-explosive*. In particular, we impose the following assumption (see Karatzas and Shreve [KS91, Theorem 5.5.29]).

Assumption 2.2 If we define

$$u_{x_0}(x) = \int_{x_0}^{x} \Big[p_{x_0}(x) - p_{x_0}(y)\Big]\, m(dy), \tag{4}$$

then $\lim_{x\downarrow 0} u_{x_0}(x) = \lim_{x\to\infty} u_{x_0}(x) = \infty$.

We adopt a weak formulation of the optimal stopping problem that we solve.

Definition 2.1 Given an initial condition $x > 0$, a *stopping strategy* is any pair $(\mathbb{S}_x, \tau)$ such that $\mathbb{S}_x = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, X, W)$ is a weak solution to (1) and τ is an $(\mathcal{F}_t)$-stopping-time. We denote by $\mathcal{S}_x$ the set of all such stopping strategies.

The objective is to maximise the performance criterion

$$J(\mathbb{S}_x, \tau) = \mathbb{E}_x\left[e^{-\Lambda_\tau} g(X_\tau) \mathbf{1}_{\{\tau<\infty\}}\right],$$

where $\Lambda_t = \int_0^t r(X_s)\, ds$ and $g :]\,0, \infty[\to \mathbb{R}$ and $r :]0, \infty[\to]0, \infty[$ are given deterministic functions, over all stopping strategies $(\mathbb{S}_x, \tau) \in \mathcal{S}_x$. Accordingly we define the value function v by

$$v(x) = \sup_{(\mathbb{S}_x,\tau)\in\mathcal{S}_x} J(\mathbb{S}_x, \tau), \quad \text{for } x > 0.$$

Now, with a view to deriving a set of additional assumptions that are suitable for this problem not to have a trivial solution and to provide a realistic model for the applications discussed in the introduction, we consider the case of a perpetual American call option written on an underlying asset, the stochastic dynamics of which are modelled by a geometric Brownian motion.

Lemma 2.1 *Suppose that X is a geometric Brownian motion, so that $b(x) = bx$ and $\sigma(x) = \sigma x$, for some constants b and σ, and $r(x) \equiv r > 0$, for some constant r. Suppose also that the payoff function is given by $g(x) = x - K$, where $K \geq 0$ is a constant. If $b > r$ (resp., $b < r$), then the process $(e^{-rt}X_t,\ t \geq 0)$ is a submartingale (resp., supermartingale) and $v(x) = \infty$ (resp., if $K = 0$, then $v(x) = x$).*

Proof. Given any initial condition $x > 0$,

$$e^{-rt}X_t = xe^{(b-r)t}e^{-\frac{1}{2}\sigma^2 t+\sigma W_t}, \quad \text{for } t \geq 0.$$

Combining this observation with the fact that the process $(e^{-\frac{1}{2}\sigma^2 t+\sigma W_t},\ t \geq 0)$ is a martingale, we can see that all of the claims made are true. □

In the context of this lemma, we can see that $(\mathbb{S}_x, 0)$ is an optimal strategy if $K = 0$ and $b < r$. Given any $K \geq 0$, if $b - r > \frac{1}{2}\sigma^2$, then the stopping strategy $(\mathbb{S}_x^*, \tau^*)$, where $\mathbb{S}_x^*$ is a weak solution to (1) and

$$\tau^* = \inf\{t \geq 0 |\ W_t = -a\},$$

where $a > 0$ is any constant, provides an optimal strategy. Indeed, since $\tau^* < \infty$, $\mathbb{P}_x$-a.s., and $\mathbb{E}_x[\tau^*] = \infty$, this claim follows from the calculation

$$\begin{aligned}\mathbb{E}_x[e^{-r\tau^*}(X_{\tau^*} - K)] &\geq xe^{-a\sigma}\mathbb{E}_x\left[e^{\left(b-r-\frac{1}{2}\sigma^2\right)\tau^*}\right] - K \\ &> xe^{-a\sigma}\left[1 + \left(b - r - \frac{1}{2}\sigma^2\right)\mathbb{E}_x[\tau^*]\right] - K \\ &= \infty.\end{aligned}$$

When $b > r$ and $b - r < \frac{1}{2}\sigma^2$, we have not been able to find an optimal stopping strategy. As a matter of fact, we have been tempted to conjecture that there is no optimal stopping strategy in this case.

We note that, when $b > r$, which is associated with $v \equiv \infty$, and when $b = r$, which is a case that we have not associated with a conclusion,

$$\lim_{t\to\infty} \mathbb{E}_x \left[e^{-\Lambda_t} g(X_t) \right] \equiv \lim_{t\to\infty} \mathbb{E}_x \left[e^{-rt} X_t \right] > 0,$$

for all initial conditions $x > 0$. This observation gives rise to the requirement that the problem's data should satisfy the so-called *transversality* condition

$$\lim_{t\to\infty} \mathbb{E}_x \left[e^{-\Lambda_t} g(X_t) \right] \equiv \lim_{t\to\infty} \mathbb{E}_x \left[e^{-rt} X_t \right] = 0.$$

Such a condition has a natural economic interpretation because it reflects the idea that one should expect that the present value of any asset should be equal to zero at the end of time, given that nobody can benefit by holding the asset after the end of time. This transversality condition is incorporated in (5) in Assumption 2.3a below (see also Remark 2.1a).

To proceed further, we note the following obvious generalisation of Lemma 2.1.

Lemma 2.2 *Given an initial condition $x > 0$ and a solution $\mathbb{S}_x$ to (1), if the process $(e^{-\Lambda_t} g(X_t), t \geq 0)$ is a supermartingale (resp., submartingale), then an optimal stopping strategy is given by $(\mathbb{S}_x, 0)$ (resp., the performance of the stopping strategy $(\mathbb{S}_x, t)$ converges to $v(x)$ as t tends to ∞).*

Now, assuming that the associated stochastic integral is a martingale, we can use Itô's formula to calculate

$$\mathbb{E}_x \left[e^{-\Lambda_t} g(X_t) \right] = g(x) + \mathbb{E}_x \left[\int_0^t e^{-\Lambda_s} \left(\frac{1}{2}\sigma^2 g'' + bg' - rg \right) (X_s)\, ds \right].$$

In light of this calculation, we can see that condition (10) in Assumption 2.3b below is sufficient to rule out trivial optimal strategies such as the ones appearing in Lemma 2.2, and it is satisfied in all cases of practical interest that we consider.

Apart from the general Assumptions 2.1 and 2.2, we impose the following assumption. Some of the conditions appearing here have been motivated by the analysis above, while others are of a technical nature. We impose these technical ones based on hindsight relative to our subsequent analysis.

Assumption 2.3 The function $g :]0, \infty[\to \mathbb{R}$ is C^1, has absolutely continuous first derivative, and there exist constants $x_2 > x_1 > \varepsilon_0 > 0$ and $C > 0$ such that the following conditions are satisfied:

(a) The *Lisbon condition* is satisfied, namely, there exists a function $l :]0,\infty[\to [1,\infty[$ such that $\lim_{x\to\infty} l(x) = \infty$,

$$\lim_{t\to\infty} \mathbb{E}_x \left[e^{-\Lambda_t} l(X_t) \big| g(X_t) \big| \right] = 0, \tag{5}$$

and

$$\frac{1}{2}\sigma^2(x)(lg)''(x) + b(x)(lg)'(x) - r(x)(lg)(x) \le 0, \quad \text{for } x > x_2. \tag{6}$$

Also,

$$g(x) \ge -C, \quad \text{for all } x > 0, \tag{7}$$

$$g(x) > 0, \quad \text{for all } x > x_2, \tag{8}$$

$$g(x) < C \quad \text{and} \quad g'(x) > -C, \quad \text{for all } x < \varepsilon_0. \tag{9}$$

(b) The following inequalities hold:

$$\frac{1}{2}\sigma^2(x)g''(x) + b(x)g'(x) - r(x)g(x) \begin{cases} > 0, & \text{for } x < x_1, \\ < 0, & \text{for } x > x_1. \end{cases} \tag{10}$$

(c) There exists a constant $j \ge 1$ such that

$$\sigma^2(x) \le C(1 + x^j), \quad \text{for all } x > 0, \tag{11}$$

$$\left[\sigma(x)g'(x)\right]^2 \le C(1 + x^j), \quad \text{for all } x > x_1, \tag{12}$$

and

$$\int_0^t \mathbb{E}\left[X_s^j\right] ds < \infty, \quad \text{for all } t \ge 0. \tag{13}$$

(d) There exists a constant $r_0 > 0$ such that

$$r(x) \ge r_0, \quad \text{for all } x > 0.$$

Example 2.1 We give special emphasis to the choices

$$g(x) = \xi x^\eta - K \quad \text{and} \quad g(x) = \left(\xi x^\eta - K\right)^+, \tag{14}$$

$$g(x) = \xi \ln(x + \eta) - K, \tag{15}$$

$$g(x) = \gamma\big(1 - \xi e^{-\eta x}\big), \tag{16}$$

where $\xi, \eta, \gamma > 0$ and $K \in \mathbb{R}$ are constants. For $\eta \in]0,1[$ and $K = 0$, the choice of g as in (14) identifies with a power utility function, while for $\eta \ge 1$, such a choice is associated with a perpetual American power option, discussed in the introduction. Choices of g as in (15) and (16) are associated with logarithmic utility and exponential utility functions, repectively.

The following remark is concerned with a discussion of Assumption 2.3 and with a number of sufficient conditions that are simple to check.

Remark 2.1 We can make the following comments corresponding to the conditions (a)–(d) in Assumption 2.3:

(a1) Since the function l takes on values in $[1, \infty[$, (5) implies that the transversality condition

$$\lim_{t\to\infty} \mathbb{E}_x \left[e^{-\Lambda_t} \big| g(X_t) \big| \right] = 0 \tag{17}$$

is satisfied, which conforms with the spirit of the discussion following Lemma 2.1.

(a2) Plainly, all of the choices of g as in (14)–(16) satisfy (7)–(9) in Assumption 2.3.

(a3) We can derive conditions that are sufficient for the Lisbon condition, namely (5)–(6) in Assumption 2.3a, to hold true as follows. For g as in (14), let us consider the choice

$$l(x) = 1 \vee \ln(x), \quad \text{for } x > 0.$$

It is a matter of straightforward calculations to verify that (6) is implied by

$$\frac{\eta(\eta-1)}{2}\sigma^2(x) + \eta x b(x) - x^2 r(x) < 0, \quad \text{for } x > x_2, \tag{18}$$

provided that x_2 is chosen sufficiently large, depending on σ, b and r. In this case

$$\lim_{t\to\infty} \mathbb{E}_x \left[e^{-\Lambda_t} X_t^{\eta+1} \right] = 0, \tag{19}$$

is plainly a sufficient condition for (5) to be true.

For g given by (15), we can see that, if we choose

$$l(x) = 1 \vee \ln(x+\eta), \quad \text{for } x > 0,$$

then, for x_2 sufficiently large, (6) is implied by

$$-\sigma^2(x) + 2xb(x) - x^2 \ln(x+\eta) r(x) < 0, \quad \text{for } x > x_2, \tag{20}$$

while, either of

$$\lim_{t\to\infty} \mathbb{E}_x \left[e^{-\Lambda_t} \ln^2 \left(X_t + \eta \right) \right] = 0 \quad \text{or} \quad \lim_{t\to\infty} \mathbb{E}_x \left[e^{-\Lambda_t} X_t \right] = 0 \tag{21}$$

imply (5).

If g is given by (16), then we can see that if we choose

$$l(x) = 1 \vee x, \quad \text{for } x > 0,$$

then, for x_2 sufficiently large, (6) is implied by

$$-\frac{\eta^2 \xi}{2} \sigma^2(x) + \eta \xi b(x) - r(x) < 0, \quad \text{for } x > x_2, \tag{22}$$

and (5) is implied by

$$\lim_{t \to \infty} \mathbb{E}_x \left[e^{-\Lambda_t} X_t \right] = 0. \tag{23}$$

(b) Verifying the associated condition in Assumption 2.3 is a matter of simple calculation given specific choices of b and σ.

(c) Plainly, any choice of g as in (14)–(16) satisfies (12) in Assumption 2.3 provided σ^2 is of polynomial growth. Furthermore, if there exists a constant $C_1 > 0$ such that

$$b^2(x) + \sigma^2(x) \le C_1(1 + x^2),$$

then the estimates in Karatzas and Shreve [KS91, Problem 5.3.15] imply that the solution to (1) has finite moments of all orders and (13) in Assumption 2.3 is satisfied for all $j \ge 1$.

(d) This condition is a most mild one, and it is needed to guarantee the convergence of several integrals.

The following lemma will play a fundamental role in proving our main result in the next section.

Lemma 2.3 *Suppose that Assumptions 2.1 and 2.2 hold, that the Lisbon condition (5)–(6) and (8) in (a) as well as conditions (c) and (d) of Assumption 2.3 are satisfied. If ψ is the strictly increasing function defined by (66) in the Appendix, then*

$$\lim_{x \to \infty} \frac{\psi(x)}{g(x)} = \infty. \tag{24}$$

Proof. We prove the result by contradiction. To this end, we assume that (24) is not true, in which case there exists $C_2 > 0$ such that

$$\psi(x) \le C_2 \big(1 + g(x)\big), \quad \text{for all } x > x_2, \tag{25}$$

where x_2 is as in (6) of Assumption 2.3a.

Now, given $x > x_2$, let $\mathbb{S}_x$ be a weak solution to (1), and define the stopping times

$$\upsilon = \inf\left\{t \geq 0 \,|\, X_t \leq x_2\right\} \quad \text{and} \quad \tau_m = \inf\left\{t \geq 0 \,|\, X_t \geq x_2 + m\right\}, \quad \text{for } m \geq 1.$$

Using Itô's formula, we calculate

$$e^{-\Lambda_{t\wedge\tau_m\wedge\upsilon}}\psi(X_{t\wedge\tau_m\wedge\upsilon}) = \psi(x) + \int_0^{t\wedge\tau_m\wedge\upsilon} e^{-\Lambda_s}\left[\frac{1}{2}\sigma^2\psi'' + b\psi' - r\psi\right](X_s)\,ds + M_t^m, \tag{26}$$

where

$$M_t^m = \int_0^{t\wedge\tau_m\wedge\upsilon} e^{-\Lambda_s}\sigma(X_s)\psi'(X_s)\,dW_s.$$

With reference to Itô's isometry, the continuity of ψ', (11) and (13) in Assumption 2.3c, we can see that

$$\begin{aligned}\mathbb{E}_x\left[(M_t^m)^2\right] &= \mathbb{E}_x\left[\int_0^t \mathbf{1}_{\{s\leq\tau_m\wedge\upsilon\}}\left[e^{-\Lambda_s}\sigma(X_s)\psi'(X_s)\right]^2 ds\right] \\ &\leq C \sup_{x\in[x_2,x_2+m]}\left[\psi'(x)\right]^2\left(t + \int_0^t \mathbb{E}\left[X_s^j\right] ds\right) \\ &< \infty, \quad \text{for all } t \geq 0,\end{aligned}$$

This calculation shows that M^m is a square-integrable martingale, therefore, $\mathbb{E}_x\left[M_t^m\right] = 0$. Combining this observation with the fact that ψ satisfies the ODE (64), we can see that (26) implies

$$0 < \psi(x) = \mathbb{E}_x\left[e^{-\Lambda_{t\wedge\tau_m\wedge\upsilon}}\psi(X_{t\wedge\tau_m\wedge\upsilon})\right]. \tag{27}$$

With regard to (25), we observe that

$$\begin{aligned}\mathbb{E}_x\left[e^{-\Lambda_{t\wedge\tau_m\wedge\upsilon}}\psi(X_{t\wedge\tau_m\wedge\upsilon})\right] \leq\ & \psi(x_2)\mathbb{E}_x\left[e^{-\Lambda_\upsilon}\mathbf{1}_{\{\upsilon\leq t\wedge\tau_m\}}\right] \\ &+ C_2\left(\mathbb{E}_x\left[e^{-\Lambda_{\tau_m}}\mathbf{1}_{\{\tau_m\leq t\wedge\upsilon\}}\right] + g(x_2+m)\mathbb{E}_x\left[e^{-\Lambda_{\tau_m}}\mathbf{1}_{\{\tau_m\leq t\wedge\upsilon\}}\right]\right) \\ &+ C_2\left(\mathbb{E}_x\left[e^{-\Lambda_t}\mathbf{1}_{\{t\leq\tau_m\wedge\upsilon\}}\right] + \mathbb{E}_x\left[e^{-\Lambda_t}g(X_t)\mathbf{1}_{\{t\leq\tau_m\wedge\upsilon\}}\right]\right).\end{aligned} \tag{28}$$

In view of Assumption 2.3d we have

$$\lim_{m\to\infty}\mathbb{E}_x\left[e^{-\Lambda_{\tau_m}}\right] = \lim_{t\to\infty}\mathbb{E}_x\left[e^{-\Lambda_t}\right] = 0. \tag{29}$$

Now arguing the same way as in establishing (27), using (6) in Assumption 2.3a, we obtain

$$\begin{aligned} l(x)g(x) &\geq \mathbb{E}_x\left[e^{-\Lambda_{t\wedge\tau_m\wedge v}} l(X_{t\wedge\tau_m\wedge v})g(X_{t\wedge\tau_m\wedge v})\right] \\ &\geq l(x_2+m)g(x_2+m)\mathbb{E}_x\left[e^{-\Lambda_{\tau_m}}\mathbf{1}_{\{\tau_m\leq t\wedge v\}}\right], \end{aligned}$$

the inequality following because l and g are positive in $[x_2, \infty[$ (see Assumption 2.3a). This calculation and the assumption that $\lim_{x\to\infty} l(x) = \infty$ imply

$$\lim_{m\to\infty} g(x_2+m)\mathbb{E}_x\left[e^{-\Lambda_{\tau_m}}\mathbf{1}_{\{\tau_m\leq t\wedge v\}}\right] = 0. \tag{30}$$

Furthermore, (5) and (8) in Assumption 2.3a (see also Remark 2.1a) imply

$$0 \leq \lim_{t\to\infty}\lim_{m\to\infty}\mathbb{E}_x\left[e^{-\Lambda_t}g(X_t)\mathbf{1}_{\{t\leq\tau_m\wedge v\}}\right] \leq \lim_{t\to\infty}\mathbb{E}_x\left[e^{-\Lambda_t}\left|g(X_t)\right|\right] = 0.$$

However, combining this with (28)–(30), we can see that (27) implies

$$\begin{aligned} \psi(x) &= \lim_{t\to\infty}\lim_{m\to\infty}\mathbb{E}_x\left[e^{-\Lambda_{t\wedge\tau_m\wedge v}}\psi(X_{t\wedge\tau_m\wedge v})\right] \\ &\leq \psi(x_2)\mathbb{E}_x\left[e^{-\Lambda_v}\right] \\ &= \psi(x_2)\frac{\phi(x)}{\phi(x_2)} \end{aligned}$$

where ϕ is the function defined by (67) in the Appendix. However, recalling that $x_2 < x$, this inequality contradicts the fact that ψ and ϕ are strictly increasing and strictly decreasing, respectively. □

We shall also need the following technical result.

Lemma 2.4 *Suppose that Assumptions 2.1 and 2.2 hold. Let p_{x_0} be the scale function defined by (2) and let ψ be the strictly increasing function defined by (66) in the Appendix. Then, given any $x_0 > 0$,*

$$\lim_{x\downarrow 0}\frac{\psi'(x)}{p'_{x_0}(x)} = \lim_{x\downarrow 0}\frac{\psi(x)}{p'_{x_0}(x)} = 0. \tag{31}$$

Proof. Let ϕ be the strictly decreasing function defined by (67) in the Appendix. Since ψ and ϕ are independent solutions of the homogeneous ODE (64) in the Appendix, their Wronskian $\mathcal{W}$, satisfies

$$\mathcal{W}(x) := \phi(x)\psi'(x) - \phi'(x)\psi(x) = \mathcal{W}(x_0)p'_{x_0}(x) > 0, \quad \text{for all } x > 0.$$

Since $\phi, \psi > 0$ and $\phi' \leq 0 \leq \psi'$, this expression implies

$$0 < \frac{\phi(x)\psi'(x)}{\mathcal{W}(x_0)p'_{x_0}(x)} < 1 \quad \text{and} \quad 0 < -\frac{\phi'(x)\psi(x)}{\mathcal{W}(x_0)p'_{x_0}(x)} < 1, \quad \text{for all } x > 0. \tag{32}$$

Also, the fact that $\lim_{x\downarrow 0}\phi(x) = \infty$ implies that $\lim_{x\downarrow 0}\phi'(x) = \infty$. However, these limits imply that (32) can be true only if the equalities in (31) hold, and the proof is complete. □

3 The solution to the optimal stopping problem

With regard to standard theory of optimal stopping, we expect that the value function v identifies with a solution w to the Hamilton-Jacobi-Bellman (HJB) equation

$$\max\left\{\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x),\ g(x) - w(x)\right\} = 0, \quad x > 0. \tag{33}$$

In view of the structure of the optimal stopping problem under consideration, we postulate that there is a critical point x^* such that it is optimal to wait for as long as the state process X assumes values less than x^* and stop as soon as X hits the set $[x^*, \infty[$. With reference to standard heuristic arguments that explain the structure of (33), we therefore look for a solution w to (33) that satisfies

$$\frac{1}{2}\sigma^2(x)\, w''(x) + b(x)w'(x) - r(x)w(x) = 0, \quad \text{for } x < x^*, \tag{34}$$

$$g(x) - w(x) = 0, \quad \text{for } x \geq x^*. \tag{35}$$

Such a solution is given by

$$w(x) = \begin{cases} A\psi(x) + B\phi(x), & \text{if } x < x^*, \\ g(x), & \text{if } x \geq x^*, \end{cases} \tag{36}$$

where ψ (resp., ϕ) is the strictly increasing (resp., decreasing) function given by (66) (resp., (67)) in the Appendix. Since the payoff function g is bounded for small x and is positive for large x, we expect that the value function should be positive and remains bounded as x tends to zero. This observation suggests that we must have $B = 0$. To specify the parameter A and x^*, we appeal to the so-called "smooth-pasting" condition of optimal stopping that requires the value function to be C^1, in particular, at the free boundary point x^*. This requirement yields the system of equations

$$A\psi(x^*) = g(x^*) \quad \text{and} \quad A\psi'(x^*) = g'(x^*),$$

which is equivalent to

$$A = \frac{g(x^*)}{\psi(x^*)} = \frac{g'(x^*)}{\psi'(x^*)} \quad \text{and} \quad q(x^*) = 0, \tag{37}$$

where q is defined by

$$q(x) = g(x)\psi'(x) - g'(x)\psi(x), \quad x > 0. \tag{38}$$

We can now prove our main result.

Theorem 3.1 *Consider the optimal stopping problem formulated in Section 2 and suppose that Assumptions 2.1, 2.2 and 2.3 hold. The value function v identifies with the function w defined by (36) with $B = 0$, $A > 0$ being given by (37), and $x^* > 0$ being the unique solution to $q(x) = 0$, where q is defined by (38). Furthermore, given any initial condition $x > 0$, the stopping strategy $(\mathbb{S}_x^*, \tau^*) \in \mathcal{S}_x$, where $\mathbb{S}_x^*$ is a weak solution to (1) and*

$$\tau^* = \inf\{t \geq 0 \mid X_t \geq x^*\},$$

is optimal.

Proof. We first prove that the equation $q(x) = 0$ has a unique solution $x^* > 0$. To this end, we combine the calculation $q'(x) = g(x)\psi''(x) - g''(x)\psi(x)$ with the fact that ψ satisfies (64) to calculate

$$q'(x) = -\frac{2b(x)}{\sigma^2(x)} q(x) - \frac{2}{\sigma^2(x)} \left(\frac{1}{2}\sigma^2(x) g''(x) + b(x) g'(x) - r(x) g(x) \right) \psi(x). \tag{39}$$

This calculation implies

$$\frac{d}{dx}\left(\frac{q(x)}{p'_{x_1}(x)}\right) = -\frac{2\psi(x)}{\sigma^2(x) p'_{x_1}(x)} \left[\frac{1}{2}\sigma^2(x) g''(x) + b(x) g'(x) - r(x) g(x) \right] \tag{40}$$

where $p_{x_1}(x)$ is the scale function defined by (2) and x_1 is as in Assumption 2.3b. This implies that the function q/p'_{x_1} is strictly decreasing in $]0, x_1[$. However, combining this with (9) in Assumption 2.3a and Lemma 2.4, we can see that

$$\frac{q(x_1)}{p'_{x_1}(x_1)} < \frac{q(x)}{p'_{x_1}(x)} < \lim_{x \downarrow 0} \frac{q(x)}{p'_{x_1}(x)} \leq 0, \quad \text{for all } x \in]0, x_1[. \tag{41}$$

Now, with regard to (40) and Assumption 2.3b, it follows that the equation $q(x) = 0$ has a unique solution $x^* > 0$ if and only if $\limsup_{x\to\infty} q(x) > 0$. To see that this inequality is true, we observe that

$$q(x) = g^2(x) \frac{d}{dx}\left(\frac{\psi(x)}{g(x)}\right), \quad \text{for } x > 0. \tag{42}$$

Now, if $\limsup_{x\to\infty} q(x) \leq 0$, then this calculation implies

$$\limsup_{x\to\infty} \frac{d}{dx}\left(\frac{\psi(x)}{g(x)}\right) \leq 0.$$

However, this inequality, (8) in Assumption 2.3a and the continuity of ψ and g imply that

$$\limsup_{x\to\infty} \frac{\psi(x)}{g(x)} < \infty,$$

which contradicts Lemma 2.3. For future reference, we note that these arguments also establish that

$$x^* > x_1 \quad \text{and} \quad q(x) < 0, \quad \text{for all } x \in]0, x_1[. \tag{43}$$

Now, to prove that w given by (36) satisfies the HJB equation (33), we need to show that

$$\frac{1}{2}\sigma^2(x)\, w''(x) + b(x)w'(x) - r(x)w(x) \leq 0, \quad \text{for } x > x^*, \tag{44}$$

$$w(x) - g(x) \leq 0, \quad \text{for } x \leq x^*. \tag{45}$$

With regard to the structure of w, (44) is equivalent to

$$\frac{1}{2}\sigma^2(x)g''(x) + b(x)g'(x) - r(x)g(x) \leq 0, \quad \text{for all } x > x^*,$$

which is implied by the first inequality in (43) and (10) in Assumption 2.3b. Also, using the first expression for A in (37), we can see that (45) is equivalent to

$$A = \frac{g(x^*)}{\psi(x^*)} \geq \frac{g(x)}{\psi(x)}, \quad \text{for all } x \leq x^*.$$

However, this inequality follows immediately once we observe that

$$\frac{d}{dx}\left(\frac{g(x)}{\psi(x)}\right) = -\frac{q(x)}{\psi^2(x)}, \quad \text{for all } x > 0,$$

and the second inequality in (43) imply that the function $x \mapsto g(x)/\psi(x)$ is strictly increasing in $]0, x^*[$.

To prove that the solution w to the HJB equation (33) that we have constructed identifies with the value function v of the optimal stopping problem, we fix any initial condition $x > 0$ and any stopping strategy $(\mathbb{S}_x, \tau) \in \mathcal{S}_x$, and we define

$$\tau_n = \inf\left\{t \geq 0 \,\middle|\, X_t \leq 1/n\right\}, \quad \text{for } n \geq 1.$$

Given any $T > 0$, since $w \in C^1(]0,\infty[) \cap C^2(]0,\infty[\setminus \{x^*\})$, we can use Itô's formula to calculate

$$e^{-\Lambda_{\tau\wedge\tau_n\wedge T}} w(X_{\tau\wedge\tau_n\wedge T}) = w(x) + \int_0^{\tau\wedge\tau_n\wedge T} e^{-\Lambda_s}\left[\frac{1}{2}\sigma^2 w'' + bw' - rw\right](X_s)\, ds + M_\tau^{n,T}, \tag{46}$$

where

$$M_t^{n,T} = \int_0^{t\wedge\tau_n\wedge T} e^{-\Lambda_s}\sigma(X_s)w'(X_s)\, dW_s.$$

With reference to Itô's isometry, (12) and (13) in Assumption 2.3c, the C^1 continuity of w and (43), we can see that

$$\begin{aligned}\mathbb{E}_x\left[\left(M_T^{n,T}\right)^2\right] &= \mathbb{E}_x\left[\int_0^T \left[e^{-\Lambda_s}\sigma(X_s)w'(X_s)\right]^2 \mathbf{1}_{\{s\le\tau_n\}}ds\right]\\ &\le \sup_{x\in[1/n,x^*]}[\sigma(x)w'(x)]^2T + \mathbb{E}_x\left[\int_0^T \left[\sigma(X_s)w'(X_s)\right]^2 \mathbf{1}_{\{X_s>x^*\}}ds\right]\\ &\le \sup_{x\in[1/n,x^*]}[\sigma(x)w'(x)]^2T + C\left(T+\int_0^T \mathbb{E}_x\left[X_t^j\right]dt\right)\\ &<\infty,\end{aligned}$$

which proves that $M^{n,T}$ is a square-integrable martingale. Therefore, by appealing to Doob's optional sampling theorem, it follows that $\mathbb{E}_x\left[M_\tau^{n,T}\right]=0$. In view of this observation, we can add the term $e^{-\Lambda_\tau}g(X_\tau)\mathbf{1}_{\{\tau\le\tau_n\wedge T\}}$ to both sides of (46), take expectations and note that w satisfies (33) to calculate

$$\mathbb{E}_x\left[e^{-\Lambda_\tau}g(X_\tau)\mathbf{1}_{\{\tau\le\tau_n\wedge T\}}\right] \le w(x) - w(1/n))\mathbb{E}_x\left[e^{-\Lambda_{\tau_n}}\mathbf{1}_{\{\tau_n\le T\le\tau\}}\right] - \mathbb{E}_x\left[e^{-\Lambda_T}w(X_T)\mathbf{1}_{\{T<\tau_n<\tau\}}\right]. \tag{47}$$

Now, since g is bounded from below, by (7) in Assumption 2.3a, we can use the dominated and the monotone convergence theorems to obtain

$$\lim_{T\to\infty}\lim_{n\to\infty}\mathbb{E}_x\left[e^{-\Lambda_\tau}g(X_\tau)\mathbf{1}_{\{\tau\le\tau_n\wedge T\}}\right] = \mathbb{E}_x\left[e^{-\Lambda_\tau}g(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right]. \tag{48}$$

The fact that w remains bounded as x tends to 0 together with the dominated convergence theorem imply

$$\lim_{n\to\infty}\mathbb{E}_x\left[e^{-\Lambda_{\tau_n}}w(1/n)\mathbf{1}_{\{\tau_n\le T<\tau\}}\right]=0. \tag{49}$$

Furthermore, since there exists a constant $C>0$ such that $0\le w(x)\le C\big(1+|g(x)|\big)$, for all $x>0$,

$$0\le\mathbb{E}_x\left[e^{-\Lambda_T}w(X_T)\right]\le C(\mathbb{E}_x[e^{-\Lambda_T}]+\mathbb{E}_x[e^{-\Lambda_T}|g(X_T)|]) \xrightarrow[T\to\infty]{} 0,$$

with the limit following thanks to (5) in Assumption 2.3a and Assumption 2.3d (see also Remark 2.1a). However, this shows that

$$\lim_{T\to\infty}\lim_{n\to\infty}\mathbb{E}_x\left[e^{-\Lambda_T}w(X_T)\mathbf{1}_{\{T<\tau_n<\tau\}}\right]=0. \tag{50}$$

In view of (48)–(50), (47) implies

$$\mathbb{E}_x \left[e^{-\Lambda_\tau} g(X_\tau) \mathbf{1}_{\{\tau \le \infty\}} \right] \le w(x),$$

which proves $v(x) \le w(x)$.

To prove the reverse inequality, let $(\mathbb{S}_x^*, \tau^*)$ be the strategy considered in the statement of the theorem. By following the arguments that lead to (47) we can see that

$$\mathbb{E}_x \left[e^{-\Lambda^*_{\tau^*}} g(X^*_{\tau^*}) \mathbf{1}_{\{\tau^* \le \tau_n^* \wedge T\}} \right] = w(x) - \mathbb{E}_x \left[e^{-\Lambda^*_{\tau_n^*}} w(1/n) \mathbf{1}_{\{\tau_n^* \le T < \tau^*\}} \right] - \mathbb{E}_x \left[e^{-\Lambda^*_T} w(X^*_T) \mathbf{1}_{\{T < \tau_n^* < \tau^*\}} \right].$$

This calculation and (48)–(50) imply

$$\mathbb{E}_x \left[e^{-\Lambda^*_{\tau^*}} g(X^*_{\tau^*}) \mathbf{1}_{\{\tau^* \le \infty\}} \right] = w(x),$$

which proves $v(x) \ge w(x)$, and establishes the optimality of $(\mathbb{S}_x^*, \tau^*)$, and the proof is complete.

Finally, to see that $A > 0$, fix any $\hat{x} > x_2$, where x_2 is as in Assumption 2.3. Given any initial condition $x > 0$, let $(\hat{\mathbb{S}}_x, \hat{\tau})$ be a stopping strategy such that $\hat{\mathbb{S}}_x$ is a weak solution to (1) and $\hat{\tau}$ is the associated hitting time of $\hat{x}$. With regard to (8) in 2.3a,

$$J(\hat{\mathbb{S}}_x, \hat{\tau}) > 0, \quad \text{for all } x > 0.$$

However, this inequality implies that $v(x) > 0$, for all $x > 0$, which, in view of the construction of v, implies $A > 0$, and the proof is complete. □

4 Special cases

We now consider a number of special cases of the general discretionary stopping problem that we studied in the previous section. These cases are differentiated by the choice of the underlying state process dynamics. In particular, we investigate the situation when X is a *geometric Brownian motion*, in which case

$$b(x) = bx \quad \text{and} \quad \sigma(x) = \sigma x, \quad \text{for all } x > 0,$$

a *square-root mean-reverting* process, which arises when

$$b(x) = \kappa(\theta - x) \quad \text{and} \quad \sigma(x) = \sigma\sqrt{x}, \quad \text{for all } x > 0,$$

an *exponential Ornstein-Uhlenbeck* process, where

$$b(x) = \left(\kappa(\theta - \ln x) + \frac{1}{2}\sigma^2\right) x \quad \text{and} \quad \sigma(x) = \sigma x, \quad \text{for all } x > 0,$$

or a *geometric Ornstein-Uhlenbeck* process, in which case

$$b(x) = \kappa(\theta - x)x \quad \text{and} \quad \sigma(x) = \sigma x, \quad \text{for all } x > 0.$$

These Itô diffusions have been well studied in the literature, and they all satisfy Assumptions 2.1, 2.2 and 2.3c. In all cases, we assume that $r(x) \equiv r$, for some constant $r > 0$, so that Assumption 2.3d is satisfied. We also take g to be as in (14)–(16) in Example 2.1.

For X being a geometric Brownian motion and g being given by (14), we can see that the conditions (18) and (19) in Remark 2.1a3, which are sufficient for the Lisbon condition (5)–(6) in Assumption 2.3a to be true, are satisfied if

$$r > \eta b + \frac{1}{2}\eta(\eta - 1)\sigma^2, \;\; K > 0 \quad \text{and} \quad r > (\eta + 1)b + \frac{1}{2}\eta(1 + \eta)\sigma^2, \tag{51}$$

respectively. Also, for X being a geometric Brownian motion and g being given by (15), the sufficient conditions (20) and (21) are satisfied if

$$r > b. \tag{52}$$

In all other cases, we can appeal to standard theory to conclude that the sufficient conditions (18)–(23) in Remark 2.1a3 are satisfied. Assuming that (51) or (52) hold, where relevant, depending on the case, and in view of Remark 2.1a2, it follows that all our assumptions are satisfied if

$$\text{(10) in Assumption 2.3b holds true.} \tag{53}$$

It turns out that a number of the cases considered are related to Kummer's ordinary differential equation

$$zu''(z) + (\beta - z)u'(z) - \alpha u(z) = 0, \tag{54}$$

where $\alpha, \beta > 0$ are constants. Independent solutions to this ordinary differential equation can be expressed in terms of the confluent hypergeometric function ${}_1F_1$, defined by

$${}_1F_1(\alpha, \beta; z) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{(\alpha)_m}{(\beta)_m} z^m,$$

where $(\alpha)_0 = 1$ and $(\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1)$, and the function U, which is defined by

$$U(\alpha, \beta; z) = \frac{\pi}{\sin \pi\beta} \left[\frac{{}_1F_1(\alpha, \beta; z)}{\Gamma(1+\alpha-\beta)\Gamma(\beta)} - z^{1-\beta} \frac{{}_1F_1(\alpha+1-\beta, 2-\beta; z)}{\Gamma(\alpha)\Gamma(2-\beta)} \right]$$

(see Magnus, Oberhettinger and Soni [MOS66, Chapter VI] or Abramowitz and Stegun [AS72, Chapter 13]).

For future reference, observe that for $\alpha, \beta > 0$, ${}_1F_1(\alpha, \beta; \cdot)$ is positive and strictly increasing on $]0, \infty[$, ${}_1F_1(\alpha, \beta; 0) = 1$ and $\lim_{z\to\infty} {}_1F_1(\alpha, \beta; z) = \infty$. Also, recalling the identity

$$\frac{\pi}{\sin \pi\beta} = \Gamma(\beta)\Gamma(1-\beta),$$

(see Magnus, Oberhettinger and Soni [MOS66, Chapter I] or Abramowitz and Stegun [AS72, 6.1.7]), we can see that

$$U(\alpha, \beta; z) = \frac{\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)} {}_1F_1(\alpha, \beta; z) - \frac{\Gamma(\beta)}{(1-\beta)\Gamma(\alpha)} z^{1-\beta} {}_1F_1(\alpha+1-\beta, 2-\beta; z).$$

With regard to this expression, it is worth noting that, although the gamma function $x \mapsto \Gamma(x)$ has simple poles at $x = -m$, $m \in \mathbb{N}^*$, U is well defined and finite for $\beta = 2, 3, 4, \ldots$. Although we do not need this result in our analysis, it is worth noting that $\lim_{z\downarrow 0} U(\alpha, \beta; z) = \infty$ if $\beta > 1$. Also, for $\alpha > 0$ and $\beta > 1$, $U(\alpha, \beta; \cdot)$ is positive, strictly decreasing in $]0, \infty[$ and $\lim_{z\to\infty} U(\alpha, \beta; z) = 0$ (see Magnus, Oberhettinger and Soni [MOS66, Chapter VI] or Abramowitz and Stegun [AS72, Chapter 13]).

4.1 Geometric Brownian motion

Geometric Brownian motion is the most commonly used model in finance for the value of an asset. In this case, the state process dynamics are given by

$$dX_t = bX_t\,dt + \sigma X_t\,dW_t, \quad X_0 = x > 0,$$

where b, σ are constants and the ODE associated with (33) is given by

$$\frac{1}{2}\sigma^2 x^2 w''(x) + bxw'(x) - rw(x) = 0, \quad \text{for } x > 0. \tag{55}$$

The proof of the following well-known result is straightforward and omitted.

Lemma 4.1 *The increasing function ψ and the decreasing function ϕ spanning the solution set to (55) are given by*

$$\psi(x) = x^n \quad \textit{and} \quad \phi(x) = x^m,$$

where the constants $m < 0 < n$ are defined by

$$m, n = \frac{1}{2} - \frac{b}{\sigma^2} \pm \sqrt{\left(\frac{b}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}\ .$$

It is a straightforward, all be it tedious, exercise to verify that (10) in Assumption 2.3b is satisfied when

$$\begin{aligned}
&g \text{ is given by (14), } K > 0 \text{ and } r > \eta b + \frac{1}{2}\eta(\eta - 1)\sigma^2, \\
&g \text{ is given by (15) and } K > \xi \ln \eta, \\
&g \text{ is given by (16) and } \xi > 1,
\end{aligned}$$

which addresses (53).

4.2 Square-root mean-reverting process

The diffusion X defined by

$$dX_t = \kappa(\theta - X_t)\, dt \ + \ \sigma\sqrt{X_t}\, dW_t, \quad X_0 = x > 0,$$

where κ, θ and σ are positive constants satisfying $\kappa\theta - \frac{1}{2}\sigma^2 > 0$ models the short rate in the Cox-Ingersoll-Ross interest rate model, and has attracted considerable interest in the theory of finance. Note that the assumption that $\kappa\theta - \frac{1}{2}\sigma^2 > 0$ is imposed because it is necessary and sufficient for X to be non-explosive, in particular for the hitting time of 0 to be infinite with probability 1. Also, the ODE associated with (33) takes the form

$$\frac{1}{2}\sigma^2 x w''(x) + \kappa(\theta - x)w'(x) - rw(x) = 0, \quad \text{for } x > 0. \tag{56}$$

Lemma 4.2 *The increasing function ψ and the decreasing function ϕ spanning the solution set to (56) are given by*

$$\psi(x) = {}_1F_1\left(\frac{r}{\kappa}, \frac{2\kappa\theta}{\sigma^2}; \frac{2\kappa}{\sigma^2}x\right) \qquad \text{and} \qquad \phi(x) = U\left(\frac{r}{\kappa}, \frac{2\kappa\theta}{\sigma^2}; \frac{2\kappa}{\sigma^2}x\right).$$

Proof. Setting $y = 2\kappa x/\sigma^2$ and $h(y) = w(x)$, the ODE (56) becomes

$$yh''(y) + \left(\frac{2\kappa\theta}{\sigma^2} - y\right) h'(y) - \frac{r}{\kappa}h(y) = 0,$$

which is Kummer's equation for $\alpha = r/\kappa > 0$ and $\beta = 2\kappa\theta/\sigma^2 > 1$, the inequality as a consequence of the assumption that $\kappa\theta - \frac{1}{2}\sigma^2 > 0$, and the result follows. □

With regard to (53), we can verify that (10) in Assumption 2.3b is satisfied when

$$\begin{aligned} &g \text{ is given by (14) and } K > 0, \\ &g \text{ is given by (15) and } K > \xi\left(\ln\eta - \frac{\kappa\theta}{\eta r}\right), \\ &g \text{ is given by (16) and } \xi > \frac{r}{r + \eta\kappa\theta}. \end{aligned}$$

4.3 Exponential Ornstein-Uhlenbeck process

The diffusion $X := e^Y$, where Y is the Ornstein-Uhlenbeck process given by

$$dY_t = \kappa(\theta - Y_t)\,dt + \sigma\,dW_t, \quad Y_0 = y \in \mathbb{R}, \tag{57}$$

for some constants $\kappa, \theta, \sigma > 0$ models the short rate in the Black-Karasinski interest rate model. Using Itô's formula we can verify that X satisfies

$$dX_t = \left[\kappa(\theta - \ln X_t) + \frac{1}{2}\sigma^2\right] X_t\,dt + \sigma X_t\,dW_t, \quad X_0 = x > 0, \tag{58}$$

while the ODE associated with (33) is given by

$$\frac{1}{2}\sigma^2 x^2 w''(x) + \left[\kappa(\theta - \ln x) + \frac{1}{2}\sigma^2\right] xw'(x) - rw(x) = 0, \quad \text{for } x > 0. \tag{59}$$

Lemma 4.3 *The increasing function ψ and the decreasing function ϕ spanning the solution set to (59) are given by*

$$\psi(x) = \begin{cases} \frac{\Gamma(\frac{r+\kappa}{2\kappa})}{\pi} U\left(\frac{r}{2\kappa}, \frac{1}{2}; \frac{\kappa}{\sigma^2}(\theta - \ln x)^2\right), & \text{for } x \le e^\theta, \\ {}_1F_1\left(\frac{r}{2\kappa}, \frac{1}{2}; \frac{\kappa}{\sigma^2}(\theta - \ln x)^2\right), & \text{for } x > e^\theta, \end{cases}$$

and

$$\phi(x) = \begin{cases} {}_1F_1\left(\frac{r}{2\kappa}, \frac{1}{2}; \frac{\kappa}{\sigma^2}(\theta - \ln x)^2\right), & \textit{for } x \le e^\theta, \\ \frac{\Gamma(\frac{r+\kappa}{2\kappa})}{\pi} U\left(\frac{r}{2\kappa}, \frac{1}{2}; \frac{\kappa}{\sigma^2}(\theta - \ln x)^2\right), & \textit{for } x > e^\theta. \end{cases}$$

Proof. Setting $w(x) = u\left(\kappa(\theta - \ln x)^2/\sigma^2\right)$ and $z = \kappa(\theta - \ln x)^2/\sigma^2$, we can see that (59) becomes

$$zu''(z) + \left(\frac{1}{2} - z\right)u'(z) - \frac{r}{2\kappa}u(z) = 0, \quad z > 0,$$

which is Kummer's equation with $\alpha = r/(2\kappa)$ and $\beta = \frac{1}{2}$. The functions ${}_1F_1(\alpha, \beta; z)$ and $U(\alpha, \beta; z)$ for $z = \kappa(\theta - \ln x)^2/\sigma^2$ are not monotone in x. For this reason the functions ψ and ϕ have to be defined in a piecewise manner. However, combining this observation with the requirement that ψ and ϕ should be C^1 with absolutely continuous first derivatives and that

$$\frac{\Gamma(\alpha + 1 - \beta)}{\Gamma(1 - \beta)} U(\alpha, \beta; \frac{\kappa}{\sigma^2}(\theta - \ln x)^2)\bigg|_{x=e^\theta} = {}_1F_1(\alpha, \beta; \frac{\kappa}{\sigma^2}(\theta - \ln x)^2)\bigg|_{x=e^\theta},$$

$$\frac{d}{dx} U(\alpha, \beta; \frac{\kappa}{\sigma^2}(\theta - \ln x)^2)\bigg|_{x=e^\theta} = 0 = \frac{d}{dx}\, {}_1F_1(\alpha, \beta; \frac{\kappa}{\sigma^2}(\theta - \ln x)^2)\bigg|_{x=e^\theta},$$

the results follow. □

It is a straightforward, all be it tedious, exercise to verify that (10) in Assumption 2.3b is satisfied when

g is given by (14) and $K > 0$,
g is given by (16) and $\xi > 1$,

which addresses (53). When g is given by (15) we have not found conditions under which (10) is satisfied that are as simple to state as the ones above. However it is straightforward to check whether (10) is satisfied for specific parameter values.

4.4 Geometric Ornstein-Uhlenbeck process

The diffusion X defined by

$$dX_t = \kappa(\theta - X_t)X_t\,dt + \sigma X_t\,dW_t, \quad X_0 = x > 0, \tag{60}$$

where κ, θ and σ are positive constants, has been proposed by Cortazar and Schwartz [CS97] as a model for a commodity's price and has played a role in population modelling. The ordinary differential equation associated with (33) for this diffusion takes the form

$$\frac{1}{2}\sigma^2 x^2 w''(x) + \kappa(\theta - x)xw'(x) - rw(x) = 0, \quad \text{for } x > 0. \tag{61}$$

Lemma 4.4 *The increasing function ψ and the decreasing function ϕ spanning the solution set to (61) are given by*

$$\psi(x) = x^n \,{}_1F_1\left(n, 2n + \frac{2\kappa\theta}{\sigma^2}; \frac{2\kappa x}{\sigma^2}\right) \qquad \text{and} \qquad \phi(x) = x^n U\left(n, 2n + \frac{2\kappa\theta}{\sigma^2}; \frac{2\kappa x}{\sigma^2}\right),$$

where

$$n = \frac{1}{2} - \frac{\kappa\theta}{\sigma^2} + \sqrt{\left(\frac{\kappa\theta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}\ .$$

Proof. Motivated by Dixit and Pindyck [DP94, Chapter 5, Section 5A], we consider a candidate for the solution to (61) of the form

$$w(x) = Ax^n f(x)$$

which results in

$$x^n f(x)\left[\frac{1}{2}\sigma^2 n(n-1) + \kappa\theta n - r\right] \\ + x^{n+1}\left[\frac{1}{2}\sigma^2 x f''(x)\big(\sigma^2 n + \kappa[\theta - x]\big)f'(x) - \kappa n f(x)\right] = 0.$$

This can be true for all $x > 0$ only if

$$\frac{1}{2}\sigma^2 n(n-1) + \kappa\theta n - r = 0, \tag{62}$$

and

$$\frac{1}{2}\sigma^2 x f''(x) + (\sigma^2 n + \kappa\theta - \kappa x)f'(x) - \kappa n f(x) = 0. \tag{63}$$

We note that the negative solution to (62) would result in choices for ψ and ϕ not having the required monotonicity properties. Choosing n to be the positive solution to (62), and setting $x = \sigma^2 y/(2\kappa)$ and $g(y) = f(x)$, we can see that (63) becomes

$$yg''(y) + \left(2n + \frac{2\kappa\theta}{\sigma^2} - y\right)g'(y) - ng(y) = 0,$$

which is Kummer's equation with $\alpha = n > 0$ and $\beta = 2n + 2\kappa\theta/\sigma^2 > 0$ and the expressions for ψ and ϕ in the statement follow.

Since x^n and ${}_1F_1$ are both increasing functions, the function ψ is plainly increasing. To see that ϕ is decreasing, we recall that

$$zU(a, b+1; z) = U(a-1, b; z) + (b-a)U(a, b; z)$$

(see Magnus, Oberhettinger and Soni [MOS66, Section 6.2]). Using this result, we calculate,

$$\frac{d}{dx}\phi(x) = \; -(\beta - \alpha + 1)nx^{n-1}U(\alpha + 1, \beta; x)$$

which is negative for if and only if $\beta > \alpha - 1$. However, we can see that $\beta > \alpha - 1$ if and only if

$$\frac{3}{2}\frac{\kappa\theta}{\sigma^2} + \sqrt{\left(\frac{\kappa\theta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > \frac{1}{2}\,,$$

which is true for all $\kappa, \theta, \sigma, r > 0$. □

With reference to (53), we can verify that (10) in Assumption 2.3b is satisfied when

g is given by (14) and $K > 0$,
g is given by (15) and $K > \xi \ln \eta$,
g is given by (16) and $\xi > 1$.

A Study of a homogeneous ordinary differential equation

We now study the ordinary differential equation

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) = 0, \quad x \in \mathcal{I}, \tag{64}$$

which has played a fundamental role in our analysis above. Here $\mathcal{I} \subseteq \mathbb{R}$ is a given open interval. We impose conditions (ND)′ and (LI)′ in Karatzas and Shreve [KS91, Section 5.5 C], as well as Feller's condition for no explosions (see Karatzas and Shreve [KS91, Theorem 5.5.29]), which are Assumptions 2.1 and 2.2 with $\mathcal{I}$ in place of $]0, \infty[$,

Assumption A.1 The functions $b, \sigma : \mathcal{I} \to \mathbb{R}$ satisfy the following conditions:

$$\sigma^2(x) > 0 \text{ for all } x \in \mathcal{I},$$

$$\text{for all } x \in \mathcal{I}, \text{ there exists } \varepsilon > 0 \text{ such that } \int_{x-\varepsilon}^{x+\varepsilon} \frac{1 + |b(s)|}{\sigma^2(s)} ds < \infty.$$

Also,

$$\lim_{x \downarrow \inf \mathcal{I}} u_z(x) = \lim_{x \uparrow \sup \mathcal{I}} u_z(x) = 0,$$

where the function u_z is defined as in (4), for some $z \in \mathcal{I}$.

Under this assumption, the Itô diffusion given by (1) has a non-explosive weak solution $\mathbb{S}_x$, namely a collection $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, W, X)$, where $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x)$ is a filtered probability space satisfying the usual conditions, W is a standard, one-dimensional $(\mathcal{F}_t)$-Brownian motion and X is a continuous $(\mathcal{F}_t)$-adapted process with values in $\mathcal{I}$, such that (1) is well defined and satisfied. Moreover, this assumption guarantees that such a solution is unique in the sense of probability law and X is a strong Markov process as well as a regular diffusion.

We also need the following assumption.

Assumption A.2 *The function $r : \mathcal{I} \to]0, \infty[$ is locally bounded.*

The objective is to show that the general solution to (64) is given by

$$w(x) = A\psi(x) + B\phi(x), \tag{65}$$

where $A, B \in \mathbb{R}$ are constants and the functions ψ, ϕ are defined by

$$\psi(x) = \begin{cases} \mathbb{E}_x[e^{-\Lambda_{\tau_z}}] & \text{for} \quad x < z, \\ 1/\mathbb{E}_z[e^{-\Lambda_{\tau_x}}] & \text{for} \quad x \geq z, \end{cases} \tag{66}$$

$$\phi(x) = \begin{cases} 1/\mathbb{E}_z[e^{-\Lambda_{\tau_x}}] & \text{for} \quad x < z, \\ \mathbb{E}_x[e^{-\Lambda_{\tau_z}}] & \text{for} \quad x \geq z, \end{cases} \tag{67}$$

respectively, for a given choice of $z \in \mathfrak{I}$. In these definitions, as well as in what follows, given a weak solution to (1) and any $a \in \mathfrak{I}$, we denote by τ_a the first hitting time of $\{a\}$, i.e., $\tau_a = \inf\{t \geq 0 |\, X_t = a\}$, with the usual convention that $\inf \emptyset = \infty$. Since X is continuous, a simple inspection of these definitions reveals that ψ (resp., ϕ) is strictly increasing (resp., decreasing). Also, since X is non-explosive, these definitions imply

$$\lim_{x \uparrow \sup \mathfrak{I}} \psi(x) = \lim_{x \downarrow \inf \mathfrak{I}} \phi(x) = \infty.$$

One purpose of the following result is to show that the definitions of ψ, ϕ in (66), (67), respectively, do not depend, in a non-trivial way, on the choice of $z \in \mathfrak{I}$.

Lemma A.1 *Given any $x, y \in \mathfrak{I}$ the functions ψ, ϕ defined by (66), (67), respectively, satisfy*

$$\psi(x) = \psi(y)\mathbb{E}_x[e^{-\Lambda_{\tau_y}}] \quad \textit{and} \quad \phi(y) = \phi(x)\mathbb{E}_y[e^{-\Lambda_{\tau_x}}], \quad \textit{for all } x < y. \tag{68}$$

Moreover, the processes $(e^{-\Lambda_t}\psi(X_t),\ t \geq 0)$ and $(e^{-\Lambda_t}\phi(X_t),\ t \geq 0)$ are both local martingales.

Proof. Given any points $a, b, c \in \mathfrak{I}$ such that $a < b < c$, we calculate

$$\begin{aligned}\mathbb{E}_a[e^{-\Lambda_{\tau_c}}] &= \mathbb{E}_a\left[e^{\Lambda_{\tau_b}}\, \mathbb{E}_a[e^{-(\Lambda_{\tau_c} - \Lambda_{\tau_b})} | \mathcal{F}_{\tau_b}]\right] \\ &= \mathbb{E}_a[e^{-\Lambda_{\tau_b}}]\mathbb{E}_b[e^{-\Lambda_{\tau_c}}],\end{aligned}$$

where the second equality follows thanks to the strong Markov property of X. In view of this result, given any $x < z < y$, the choice $a = x$, $b = z$ and $c = y$ yields

$$\mathbb{E}_x[e^{-\Lambda_{\tau_y}}] = \mathbb{E}_x[e^{-\Lambda_{\tau_z}}]\mathbb{E}_z[e^{-\Lambda_{\tau_y}}],$$

which, combined with the definition of ψ, implies the first identity in (68). We can verify the first identity in (68) when $x < y < z$ or $z < x < y$ as well as the second identity in (68) by appealing to similar arguments.

Now, given any initial condition x and any sequence (x_n) such that $x < x_1$ and $\lim_{n \to \infty} x_n = \sup \mathfrak{I}$, we observe that the first identity in (68) implies

$$\psi(X_t)\mathbf{1}_{\{t \leq \tau_{x_n}\}} = \psi(x_n)\mathbb{E}_{X_t}[e^{-\Lambda_{\tau_{x_n}}}]\mathbf{1}_{\{t \leq \tau_{x_n}\}}, \quad \text{for all } t \geq 0.$$

In view of this identity, we appeal to the strong Markov property of X, once again to calculate

$$\begin{aligned}\mathbb{E}_x\left[e^{-\Lambda_{\tau_{x_n}}}\psi(X_{\tau_{x_n}}) \middle| \mathcal{F}_t\right] &= e^{-\Lambda_t}\psi(x_n)\mathbb{E}_x\left[e^{-(\Lambda_{\tau_{x_n}} - \Lambda_t)} \middle| \mathcal{F}_t\right] \mathbf{1}_{\{t < \tau_{x_n}\}} + e^{-\Lambda_{\tau_{x_n}}}\psi(x_n)\mathbf{1}_{\{\tau_{x_n} \leq t\}} \\ &= e^{-\Lambda_t}\psi(X_t)\mathbf{1}_{\{t < \tau_{x_n}\}} + e^{-\Lambda_{\tau_{x_n}}}\psi(x_n)\mathbf{1}_{\{\tau_{x_n} \leq t\}} \\ &= e^{-\Lambda_{(t \wedge \tau_{x_n})}}\psi(X_{t \wedge \tau_{x_n}}).\end{aligned}$$

However, this calculation and the tower property of conditional expectation implies that, given any times $s < t$,

$$\begin{aligned}\mathbb{E}_x\left[e^{-\Lambda_{t\wedge\tau_{x_n}}}\psi(X_{t\wedge\tau_{x_n}})|\mathcal{F}_s\right] &= \mathbb{E}_x\left[\mathbb{E}_x\left[e^{-\Lambda_{t\wedge\tau_{x_n}}}\psi(X_{t\wedge\tau_{x_n}})\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_s\right] \\ &= e^{-\Lambda_{s\wedge\tau_{x_n}}}\psi(X_{s\wedge\tau_{x_n}}),\end{aligned}$$

which proves that $\left(e^{-\Lambda_t}\psi(X_t),\, t \geq 0\right)$ is a local-martingale. Proving that $\left(e^{-\Lambda_t}\phi(X_t),\, t \geq 0\right)$ is a local-martingale follows similar arguments. □

We can now prove the following result.

Theorem A.1 *Suppose that Assumption A.1 holds. The general solution to the ordinary differential equation (64) exists in the* classical sense*, namely there exists a two dimensional space of functions that are* C^1 *with absolutely continuous first derivatives, and that satisfy (64) Lebesgue-a.e.. This solution is given by (65), where* $A, B \in \mathbb{R}$ *are constants and the functions* ψ*,* ϕ *are given by (66), (67), respectively. Moreover,* ψ *is strictly increasing,* ϕ *is strictly decreasing, and, if the drift* $b \equiv 0$*, then both* ψ *and* ϕ *are strictly convex.*

Proof. First, we recall that, given $l < x < m$,

$$\mathbb{P}_x(\tau_l < \tau_m) = \frac{p_{x_0}(x) - p_{x_0}(m)}{p_{x_0}(l) - p_{x_0}(m)} \tag{69}$$

(e.g., see Karatzas and Shreve [KS91, Proposition 5.5.22] or Rogers and Williams [RW94, Definition V.46.10]). Also in view of the first identity in (68), we can see that

$$\begin{aligned}\psi(x) &< \psi(m)\mathbb{E}_x\left[\mathbf{1}_{\{\tau_m<\tau_l\}}\right] \;+\; \psi(m)\mathbb{E}_x\left[e^{-\Lambda_{\tau_m}}\,\mathbf{1}_{\{\tau_l<\tau_m\}}\right] \\ &= \psi(m)\mathbb{P}_x(\tau_m < \tau_l) + \psi(m)\mathbb{E}_x\left[\mathbb{E}_x\left[e^{-\Lambda_{\tau_m}}\middle|\mathcal{F}_{\tau_l}\right]\,\mathbf{1}_{\{\tau_l<\tau_m\}}\right].\end{aligned}$$

Now, since X has the strong Markov property we can see that

$$\begin{aligned}\mathbb{E}_x\left[e^{-\Lambda_{\tau_m}}\middle|\mathcal{F}_{\tau_l}\right]\mathbf{1}_{\{\tau_l<\tau_m\}} &= \; e^{-\Lambda_{\tau_l}}\mathbb{E}_x\left[e^{-\Lambda_{(\tau_m-\tau_l)}}\middle|\mathcal{F}_{\tau_l}\right]\mathbf{1}_{\{\tau_l<\tau_m\}} \\ &= \; e^{-\Lambda_{\tau_l}}\frac{\psi(l)}{\psi(m)}\mathbf{1}_{\{\tau_l<\tau_m\}},\end{aligned}$$

with the last equality following thanks to (68). Combining these calculations we can see that

$$\begin{aligned}\psi(x) &< \psi(m)\mathbb{P}_x(\tau_m < \tau_l) \;+\; \psi(l)\mathbb{E}_x\left[e^{-\Lambda_{\tau_l}}\,\mathbf{1}_{\{\tau_l<\tau_m\}}\right] \\ &< \psi(m)\mathbb{P}_x(\tau_m < \tau_l) \;+\; \psi(l)\mathbb{P}_x(\tau_l < \tau_m).\end{aligned} \tag{70}$$

Now, let us assume that $b \equiv 0$, so that the diffusion X defined by (1) is in *natural scale*, in which case $p_{x_0}(x) = x - x_0$. Combining this fact with (69), it is straightforward to verify that

$$x = l\mathbb{P}_x(\tau_l < \tau_m) + m\mathbb{P}_x(\tau_m < \tau_l).$$

However, this calculation and (70) imply that ψ is strictly convex. In this case, we also have

$$\mathbb{P}_x(\tau_l < \tau_m) = \frac{x - m}{l - m}. \tag{71}$$

Under the assumption that $b \equiv 0$, which implies that ψ is strictly convex, we can use the Itô-Tanaka and the occupation times formulae to calculate

$$\psi(X_t) - \int_0^t r(X_s)\psi(X_s)\,ds = \psi(x) + \int_{\mathcal{I}} L_t^a \frac{1}{\sigma^2(a)} \left[\frac{1}{2}\sigma^2(a)\,\mu''(da) - r(a)\psi(a)\,da\right] + \int_0^t \psi'_-(X_s)\sigma(X_s)\,dW_s,$$

where ψ'_- is the left-hand-side first derivative of ψ, $\mu''(da)$ is the distributional second derivative of ψ, and L^a is the local time process of X at level a. With regard to the integration by parts formula, this implies

$$e^{-\Lambda_t}\psi(X_t) = \psi(x) + \int_0^t e^{-\Lambda_s}\,d\int_{\mathcal{I}} L_s^a \frac{1}{\sigma^2(a)} \left[\frac{1}{2}\sigma^2(a)\,\mu''(da) - r(a)\psi(a)\,da\right] + \int_0^t e^{-\Lambda_s}\psi'_-(X_s)\,\sigma(X_s)\,dW_s.$$

Since $(e^{-\Lambda_t}\psi(X_t),\ t \geq 0)$ is a local-martingale (see Lemma A.1), this identity implies that the finite variation process Q defined by

$$Q_t = \int_0^t e^{-\Lambda_s} d\int_{\mathcal{I}} L_s^a \frac{1}{\sigma^2(a)} \left[\frac{1}{2}\sigma^2(a)\,\mu''(da) - r(a)\psi(a)\,da\right], \quad \text{for } t \geq 0,$$

is a local martingale. Since finite-variation local martingales are constant, it follows that $Q \equiv 0$, which implies

$$\int_{\mathcal{I}} L_t^a\,\nu(da) = 0, \quad \text{for all } t \geq 0, \tag{72}$$

where the measure ν is defined by

$$\nu(da) = \frac{1}{2}\mu''(da) - \frac{r(a)\psi(a)}{\sigma^2(a)\psi''(a)}. \tag{73}$$

To proceed further, fix any points $l < a < m$, define

$$\tau_{l,m} = \inf \{t \geq 0 \mid X_t \notin]l, m[\} ,$$

and let (T_j) be a localising sequence for the local martingale $\int_0^\cdot \operatorname{sgn}(X_s - a) dX_s$. With regard to the definition of local times and Doob's optional sampling theorem, we can see that

$$\begin{aligned} \mathbb{E}_x\Big[\big|X_{\tau_{l,m}\wedge T_j} - a\big| \Big] &= |x-a| + \mathbb{E}_x\bigg[\int_0^{\tau_{l,m}\wedge T_j} \operatorname{sgn}(X_s - a)\, dX_s \bigg] + \mathbb{E}_x\Big[L^a_{\tau_{l,m}\wedge T_j} \Big] \\ &= |x-a| + \mathbb{E}_x\Big[L^a_{\tau_{l,m}\wedge T_j} \Big]. \end{aligned}$$

However, passing to the limit using the dominated convergence theorem on the left hand side and the monotone convergence theorem on the right hand side, we can see that this identity implies

$$\begin{aligned} \mathbb{E}_x\Big[L^a_{\tau_{l,m}} \Big] &= \mathbb{E}_x\Big[\big|X_{\tau_{l,m}} - a\big| \Big] - |x-a| \\ &= \frac{(m-a)(x-l)}{m-l} + \frac{(a-l)(m-x)}{m-l} - |x-a| \,, \end{aligned} \tag{74}$$

the second equality following thanks to (71). Now, (72), the fact that $t \mapsto L^a_t$ increases on the set $\{t \geq 0 \mid X_t = a\}$ and Fubini's theorem, imply

$$\begin{aligned} 0 = \mathbb{E}_x\Big[\int_{\mathfrak{I}} L^a_{\tau_{l,m}}\, \nu(da) \Big] &= \mathbb{E}_x\Big[\int_{]l,m[} L^a_{\tau_{l,m}}\, \nu(da) \Big] \\ &= \int_{]l,m[} \mathbb{E}_x\big[L^a_{\tau_{l,m}} \big]\, \nu(da). \end{aligned}$$

Combining this calculation with (74), it is a matter of algebraic calculation to verify that

$$\int_l^m h(a; l, x, m)\, \nu(da) = 0, \tag{75}$$

where $h(\cdot; l, x, m)$ is the *tent-like* function of height 1 defined by

$$h(a; l, x, m) = \begin{cases} (a-l)/(x-l), & \text{for } a \in [l, x], \\ (m-a)/(m-x), & \text{for } a \in [x, m]. \end{cases}$$

Now, fix any points $x_l < x_m$ in $\mathfrak{I}$ and let (l_j) and (m_j) be strictly decreasing and strictly increasing, respectively, sequences such that

$$l_1 < \frac{x_l + x_m}{2} < m_1, \quad \lim_{j\to\infty} l_j = x_l \quad \text{and} \quad \lim_{j\to\infty} m_j = x_m.$$

We can see that

$$\mathbf{1}_{]x_l,x_m[}(a) = \lim_{j\to\infty} H_j(a), \quad \text{for all } a \in \mathfrak{I},$$

where the increasing sequence of functions (H_j) is defined by

$$\begin{aligned} H_j(a) = {} & h\left(a; x_l, \frac{x_l + x_m}{2}, x_m\right) \\ & + \frac{x_l + x_m - 2l_j}{x_m - x_l}\, h\left(a; x_l, l_j, \frac{x_l + x_m}{2}\right) \\ & + \frac{2m_j - (x_l + x_m)}{x_m - x_l}\, h\left(a; \frac{x_l + x_m}{2}, m_j, x_m\right), \quad \text{for } a \in \mathfrak{I} \text{ and } j \geq 1. \end{aligned}$$

Using the monotone convergence theorem and (75), it follows that

$$\nu(]x_l, x_m[) = \lim_{j\to\infty} \int_{x_l}^{x_m} H_j(a)\, \nu(da) = 0,$$

which proves that the signed measure ν assigns measure 0 to every open subset of $\mathfrak{I}$. However, this observation and the definition of ν in (73) imply that the total variation of ν is zero, and, therefore, $\mu''(da)$ is an absolutely continuous measure. It follows that there exists a function ψ'' such that

$$\mu''(da) = \psi''(a)\, da \quad \text{and} \quad \frac{1}{2}\sigma^2(a)\psi''(a) = r(a)\psi(a), \quad \text{Lebesgue-a.e..}$$

However, the second identity here shows that ψ is a classical solution to (64).

Now, let us consider the general case where the drift b does not vanish. In this case, we use Itô's formula to verify that, if $\bar{X} = p_{x_0}(X)$, then

$$d\bar{X}_t = \bar{\sigma}(\bar{X}_t)dW_t, \quad \bar{X}_0 = p_{x_0}(x),$$

where

$$\bar{\sigma}(\bar{x}) = p'_{x_0}\left(p_{x_0}^{-1}(\bar{x})\right) \sigma\left(p_{x_0}^{-1}(\bar{x})\right), \quad \text{for } \bar{x} \in]\inf \mathfrak{I}, \sup \mathfrak{I}[.$$

Since $\bar{X}$ is a diffusion in natural scale, the associated function $\bar{\psi}$ defined as in (66) is a classical solution of

$$\frac{1}{2}\sigma^2(a)\bar{\psi}''(a) - r(a)\bar{\psi}(a) = 0. \tag{76}$$

Now, recalling that p_{x_0} is twice differentiable in the classical sense, we can see that if we define $\tilde{\psi}(x) = \bar{\psi}(p_{x_0}(x))$ then

$$\tilde{\psi}'(x) = \bar{\psi}'\big(p_{x_0}(x)\big)p'_{x_0}(x),$$
$$\tilde{\psi}''(x) = \bar{\psi}''\big(p_{x_0}(x)\big)\big[p'_{x_0}(x)\big]^2 \;+\; \bar{\psi}'\big(p_{x_0}(x)\big)p''_{x_0}(x).$$

However, combining these calculations with (76), we can see that $\tilde{\psi}$ satisfies the ODE (64).

To prove that $\tilde{\psi}$, namely the classical solution to (64), as constructed above, identifies with ψ defined by (66), we apply Itô's formula to $e^{-\Lambda(\tau_y \wedge T)}\tilde{\psi}(X_{\tau_y \wedge T})$, where $T > 0$ is a constant, and we use arguments similar to the ones employed in the proof of Theorem 3.1, to show that

$$\mathbb{E}_x\big[e^{-\Lambda_{\tau_y \wedge T}}\tilde{\psi}(X_{\tau_y \wedge T})\big] = \tilde{\psi}(x), \qquad \text{for all } x < y.$$

Since $\psi > 0$ is increasing, the monotone and the dominated convergence theorems imply

$$\lim_{T \to \infty} \mathbb{E}_x\big[e^{-\Lambda_{\tau_y \wedge T}}\tilde{\psi}(X_{\tau_y \wedge T})\big] = \tilde{\psi}(y)\mathbb{E}_x\big[e^{-\Lambda_{\tau_y}}\big], \qquad \text{for all } x < y.$$

However, these calculations, show that $\tilde{\psi}$ satisfies the first identity in (68) and therefore identifies with ψ defined by (66). Proving all of the associated claims for ϕ follows similar reasoning. □

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