On the motion of a compact elastic body

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Abstract

We study the problem of motion of a relativistic, ideal elastic solid with free surface boundary by casting the equations in material form ("Lagrangian coordinates"). By applying a basic theorem due to Koch, we prove short-time existence and uniqueness for solutions close to a trivial solution. This trivial, or natural, solution corresponds to a stress-free body in rigid motion.

1 Introduction

In the problem of motion for classical continua with free surface boundary, despite its obvious physical relevance, there is a surprising scarcity of rigorous results. A recent review of known results with and without gravity can be found in [13]. In the case of a nonrelativistic, non-gravitating perfect fluid local wellposedness has only been proved very recently by Lindblad [11]. In the present paper we consider the analogous problem for relativistic elastic solids without self-gravity. The nonrelativistic case of our results (which is in principle known - see [14] - although even there a detailed treatment seems to be lacking) can be proved by similar methods. The nonrelativistic case for incompressible materials has, by different methods, been treated in [6].
The plan of this paper is as follows. In Sect.2 we describe elasticity theory on an arbitrary curved background as a Lagrangian field theory. Sect.3 is devoted to the notion of "natural state", that-is-to-say a solution to the elastic equations corresponding to a configuration with zero stress. The existence of such

a solution requires the elastic body to move along a geodesic, timelike Killing vector. We require the Killing vector to be also hypersurface-orthogonal. (In Special Relativity a geodesic, timelike Killing motion has to be inertial, so the above assumption is superfluous in that case.) When the metric on the space orthogonal to the Killing vector is flat, our background spacetime is Minkowski. In the case where the metric on the space orthogonal to the Killing vector is flat, we are in Special Relativity. In Sect.4 we perform a 3+1 decomposition of the elastic equations corresponding to the natural space-time splitting afforded by the Killing vector. We then write the elastic equations in "material" form (often called "Lagrangian representation"). This means that the maps making up the elastic configuration space - which go from spacetime to the 3 dimensional "material space" - are replaced by time dependent maps ("deformations") from material space into physical space. In the material representation the boundary of the body is fixed, namely the boundary of material space. In Sect.5 we state a corollary to the theorem of Koch [10], which is the version of existence theorem we are using. In Sect.6 we state our basic constitutive assumption. This assumption, which is satisfied by elastic materials occurring in practice, implies the validity of the conditions in the Koch theorem. From this one concludes the main result, which is stated in Sect.6. In the appendix we show that the corner conditions on the boundary, which initial data have to satisfy in order for the time evolved solution to be a classical solution, can be satisfied for a large class of initial data. We can state the central result of this paper as follows:
For initial data close to initial data for the natural deformation in the appropriate function space, there exists, for sufficiently short times, a unique solution to the elastic equations. This solution depends continuously on initial data, in particular tends to the natural deformation, when the initial data tend to that of the natural deformation.

2 The theory

We treat elasticity as a Lagrangian field theory in the manner of [2] or [4], see also [9]. In the language of standard elasticity this means that the material in question is "hyperelastic". The dynamical objects of the theory are furnished by sufficiently regular maps f sending a closed region $\bar{\mathcal{S}} = \mathcal{S} \cup \partial\mathcal{S}$ of spacetime $\mathcal{M}$ onto $\bar{\mathcal{B}} = \mathcal{B} \cup \partial\mathcal{B}$, with $\mathcal{B}$ a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\mathcal{B}$ called the "body" or material space. The domain $\mathcal{B}$ is to be thought of as an abstract collection of points ("labels") making up the elastic continuum, and f is to be thought of as the back-to-labels map sending spacetime events to the particles by which they are occupied. We endow $\mathcal{B}$ with a volume form Ω which is smooth on $\bar{\mathcal{B}}$. Coordinates on $\mathcal{M}$ are written as x^μ, with μ, ν etc ranging from 0 to 3 and coordinates on $\mathcal{B}$ as X^A with A, B etc going from 1 to 3. The metric $g_{\mu\nu}$ is also taken to be smooth. There are additional requirements on the maps

f. Namely the equation

$$f^A{}_{,\mu} u^\mu = 0, \qquad g_{\mu\nu} u^\mu u^\nu = -1 \tag{1}$$

should have a solution u^μ which is unique up to sign. Thus particles move along trajectories in $\mathcal{M}$ with unit tangent u^μ. Furthermore the function n on $\bar{\mathcal{S}}$ defined by the equation

$$\Omega_{ABC}(f(x)) f^A{}_{,\mu}(x) f^B{}_{,\nu}(x) f^C{}_{,\lambda}(x) = n(x) \epsilon_{\mu\nu\lambda\rho}(x) u^\rho(x) \tag{2}$$

should be everywhere positive on $\bar{\mathcal{S}}$. It is well known that (2) implies that the continuity equation

$$\nabla_\mu (n u^\mu) = 0 \tag{3}$$

is identically satisfied. The equations of motion are the Euler-Lagrange equations of the action

$$S[f, \partial f] = \int_{\mathcal{S}} \rho(f, \partial f, x)\, (-g)^{\frac{1}{2}}\, d^4x. \tag{4}$$

The function ρ is required to be smooth in all its arguments. It plays the double role of being the Lagrangian as well as energy (i.e. rest mass plus internal elastic energy) density of the material. Diffeomorphism invariance demands that ρ be of the form

$$\rho(f, \partial f, x) = \hat{\rho}(f, H^{AB})|_{H^{AB} = h^{AB}(\partial f, x)}, \tag{5}$$

where $h^{AB}(\partial f, x) = f^A{}_{,\mu}(x) f^B{}_{,\nu}(x) g^{\mu\nu}(x)$. Thus ρ depends only on points X on $\mathcal{B}$ and positive definite contravariant tensors H^{AB} (see [2]). (In nonrelativistic elasticity materials governed by such a Lagrangian are said to satisfy the condition of "material frame indifference".) By abuse of notation we henceforth omit the hat from $\hat{\rho}$. It is useful to know that the Euler-Lagrange equations are equivalent to divergence-lessness of the symmetric energy momentum tensor. More precisely, there holds the identity (see [2])

$$-\nabla_\nu T_\mu{}^\nu = \mathcal{E}_A f^A{}_{,\mu}, \tag{6}$$

where

$$T_{\mu\nu} = 2 \frac{\partial \rho}{\partial g^{\mu\nu}} - \rho g_{\mu\nu} \tag{7}$$

and

$$-\mathcal{E}_A = (-g)^{-\frac{1}{2}} \partial_\mu \left((-g)^{\frac{1}{2}} \frac{\partial \rho}{\partial (f^A{}_{,\mu})} \right) - \frac{\partial \rho}{\partial f^A} \tag{8}$$

We now turn to boundary conditions. The boundary conditions usually considered in nonrelativistic elasticity are either the so-called "boundary conditions of place" - where, in our language, the set $f^{-1}(\partial \mathcal{B})$ is prescribed, or "traction boundary conditions" - where the normal traction, i.e. the components of the stress tensor normal to this surface are prescribed. When the normal traction is zero, one speaks of "natural" boundary conditions: these are the boundary conditions appropriate for a freely floating elastic body considered in the present work. They are employed e.g. in geophysics for the elastic motion corresponding to seismic waves, the free boundary in question corresponding to the surface of the earth (see [1] [1]).While one could in the present framework in principle consider all the above boundary conditions, it is interesting to observe that these conditions - except for the natural condition - become inconsistent once one couples to the Einstein equations. Namely the standard junction conditions, together with the Einstein equations, imply that $T_\mu{}^\nu n_\nu|_{f^{-1}(\partial\mathcal{B})}$ be zero, where n_μ is the conormal of $f^{-1}(\partial\mathcal{B})$. However these conditions are precisely the natural boundary conditions. To see this, we have to first compute the right hand side of Eq.(7). The result is

$$T_{\mu\nu} = \rho u_\mu u_\nu + t_{\mu\nu}, \tag{9}$$

where $t_{\mu\nu}$ is the (negative) Cauchy stress tensor given by

$$t_{\mu\nu} = n\tau_{AB} f^A{}_{,\mu} f^B{}_{,\nu} \tag{10}$$

and we have written ρ as

$$\rho = n\epsilon \tag{11}$$

and $\tau_{AB} = 2\frac{\partial\epsilon}{\partial H^{AB}}$. Note this makes sense, since n - whence ϵ - is a function purely of f and H^{AB}, as is apparent from the identity

$$6n^2 = H^{AA'}H^{BB'}H^{CC'}\Omega_{ABC}\Omega_{A'B'C'}. \tag{12}$$

The function ϵ is the relativistic version of the "stored-energy-function" of standard elasticity. The quantity τ_{AB} corresponds to the negative of the "second Piola-Kirchhoff stress" of nonrelativistic elasticity. Now to the boundary conditions, namely

$$T_\mu{}^\nu n_\nu|_{f^{-1}(\partial\mathcal{B})} = 0. \tag{13}$$

It follows from the tangency of u^μ to the inverse images of points of $\mathcal{B}$ under f, that $(u, n) = 0$. Consequently the contraction of Eq.(13) with u^μ, using (9), is identically satisfied. The remaining components yield

$$\tau_{AB} f^B{}_{,\mu} g^{\mu\nu} n_\nu|_{f^{-1}(\partial\mathcal{B})} = 0 \tag{14}$$

Equation (14) will turn out to be a Neumann-type boundary condition, but it has a free (i.e. determined-by-f) boundary.

[1] Needless to say, these studies are confined to the linear approximation in which the problem of the free boundary disappears.

3 Natural configuration

We assume there exists a contravariant metric $H_0^{AB}(X)$, smooth and positive definite on $\bar{\mathcal{B}}$, such that

$$\epsilon|_{(X,H^{AB}=H_0^{AB}(X))} = \epsilon_0(X) > 0, \qquad \frac{\partial \epsilon}{\partial H^{CD}}|_{(X,H^{AB}=H_0^{AB}(X))} = 0 \tag{15}$$

on $\bar{\mathcal{B}}$. The quantity H_0^{AB} will play the role of "stressfree strain". The function $\epsilon_0(X)$ is the density of rest mass in the stressfree configuration.
If there exists a configuration f_0 such that

$$H_0^{AB}(f_0(x)) = h^{AB}(\partial f_0(x), x) = f_0{}^A{}_{,\mu}(x) f_0{}^B{}_{,\nu}(x) g^{\mu\nu}(x) =: h_0^{AB}(x), \tag{16}$$

this map f_0 is called a natural or relaxed configuration. Note that spacetime has to be special in order for a natural configuration to exist. Namely it follows from (16) that h_0^{AB} has to be constant along u_0^μ, the four velocity associated with f_0, from which one infers that u_0^μ is a Born rigid motion (see [15]), i.e.

$$\mathcal{L}_{u_0}(g_{\mu\nu} + u_{0\,\mu} u_{0\,\nu}) = 0 \tag{17}$$

One deduces from (6,7,9) and (15) that f_0 solves the equations of motion $\mathcal{E}_A = 0$ if and only if this rigid motion is geodesic. This in turn implies that u_0^μ is Killing. In this work we will assume that u_0^μ is in addition irrotational. Using coordinates in which $u_0^\mu \partial_\mu = \partial_t$ and $u_{0\,\mu} dx^\mu = -dt$, the spacetime metric has thus to be of the form $(i, j = 1, 2, 3)$

$$g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + g_{ij}(x^k) dx^i dx^j. \tag{18}$$

(In particular: when the positive definite three metric g_{ij} is flat, we are in Minkowski spacetime.) With this choice of coordinates, the natural configuration f_0 is of the form $f_0(t, x) = f_0(x)$ with $f_0(x)$ a smooth, invertible function $\bar{\mathcal{S}} \cap \{t = 0\} \to \bar{\mathcal{B}}$ where, using coordinates on M and $\bar{\mathcal{B}}$ in which ϵ_{ijk} and Ω_{ABC} are both positive, $det(f_0{}^A{}_{,i})$ is positive on $\bar{\mathcal{S}} \cap \{t = 0\}$, the latter condition guaranteeing that n_0 is positive and h_0^{AB} is positive definite on $\bar{\mathcal{S}} \cap \{t = 0\}$. The inverse of $f_0(x)$ will be denoted by $\Phi_0(X)$.

4 3 + 1 and the material picture

In this section we switch to the material description of the elastic medium by changing to "Lagrangian" coordinates. In a first step we replace the four velocity u^μ corresponding to the configuration f by the coordinate velocity $v^\mu \partial_\mu = \partial_t + V^i \partial_i$, which is a multiple of u^μ. Using (1) we find that V^i is given in terms of f^A by (note that $f^A{}_{,i}$ is non-degenerate by construction)

$$\dot{f}^A + f^A{}_{,i} V^i = 0, \tag{19}$$

where a dot denotes partial differentiation w.r. to t. It follows from (18) that

$$h^{AB} = f^A{}_{,\mu} f^B{}_{,\nu} g^{\mu\nu} = f^A{}_{,i} f^B{}_{,j} \, (g^{ij} - V^i V^j). \tag{20}$$

The timelike nature of u^μ (equivalently: the positive definiteness of h^{AB}) requires that

$$|V|^2 = g_{ij} V^i V^j < 1 \tag{21}$$

We also see that

$$n = k \, (1 - |V|^2)^{1/2}, \tag{22}$$

where k is defined by

$$k \, \epsilon_{ijk} = f^A{}_{,i} f^B{}_{,j} f^C{}_{,k} \, \Omega_{ABC} \tag{23}$$

with ϵ_{ijk} being the volume element of g_{ij}. Consequently the action (4) takes the form

$$S = \int_{\mathcal{S}} (1 - |V|^2)^{1/2} \, \epsilon \, k \, (\det \, g_{ij})^{1/2} dt \, d^3x \tag{24}$$

We now pass over to the material representation. By this we mean that configurations f^A are replaced by "deformations" $x^i = \Phi^i(t, X)$ defined by

$$f^A(t, \Phi(t, X))) = X^A \tag{25}$$

with $det(\Phi^i{}_{,A})$ positive in $\bar{\mathcal{B}}$. It follows from (25) that

$$f^A{}_{,i}(t, \Phi(t, X)) \, \Phi^i{}_{,B}(t, X) = \delta^A{}_B, \quad \Phi^i{}_{,A}(t, X) \, f^A{}_{,j}(t, \Phi(t, X)) \, = \delta^i{}_j \tag{26}$$

and

$$\dot{f}^A(t, \Phi(t, X)) + f^A{}_{,i}(t, \Phi(t, X)) \dot{\Phi}^i(t, X) = 0 \tag{27}$$

and, from (27) and (19), that

$$V^i(t, \Phi(t, X)) = \dot{\Phi}^i(t, X) \tag{28}$$

In order to study the field equations in the material representation it will be extremely convenient to change representation directly in the action (24). Using (23) there holds

$$S = \int_{\{t\} \times \mathcal{B}} \epsilon \, \gamma^{-1} \, dt \, d^3\Omega \tag{29}$$

with

$$\gamma^{-1} = (1 - |V|^2)^{\frac{1}{2}} \tag{30}$$

and $d^3\Omega = \Omega_{ABC}(X) \, dX^A \wedge dX^B \wedge dX^C = \Omega^{\frac{1}{2}} \, d^3X$. Now the field equations take the form

$$\partial_t \left(\frac{\partial L}{\partial \dot{\Phi}^i} \right) + \partial_A \left(\frac{\partial L}{\partial \Phi^i{}_{,A}} \right) - \frac{\partial L}{\partial \Phi^i} = 0 \quad \text{in } \mathcal{B} \tag{31}$$

where $L = \epsilon\,\gamma^{-1}\,\Omega^{\frac{1}{2}}$ for functions Φ such that $det(\Phi^i{}_A)$ is positive in $\bar{\mathcal{B}}$. Note that Ω depends only on X^A, g_{ij} depends only on Φ^i and γ^{-1} depends only on $(\Phi^i, \dot{\Phi}^j)$. Finally h^{AB}, i.e.

$$h^{AB} = F^A{}_i(\Phi^k{}_{,C})F^B{}_j(\Phi^l{}_{,D})[g^{ij}(\Phi) - \dot{\Phi}^i\dot{\Phi}^j], \tag{32}$$

where $F^A{}_i(\Phi^j{}_{,B})$ is the inverse of $\Phi^i{}_{,A}$, depends on $(\Phi^i, \dot{\Phi}^j, \Phi^k{}_{,A})$, so that ϵ depends on $(X^A, \Phi^i, \dot{\Phi}^j, \Phi^k{}_{,B})$. Using these facts, together with the identity

$$\frac{\partial h^{AB}}{\partial \Phi^i{}_{,C}} = -2h^{C(A}F^{B)}{}_i, \tag{33}$$

we find that

$$\frac{\partial L}{\partial \Phi^i{}_{,A}} = -2\,\Omega^{\frac{1}{2}}N\gamma^{-1}\frac{\partial\epsilon}{\partial H^{BC}}|_{H^{DE}=h^{DE}}\,h^{AB}F^C{}_i \tag{34}$$

and

$$\frac{\partial L}{\partial \dot{\Phi}^i} = -\Omega^{\frac{1}{2}}[\epsilon\,\gamma\,g_{ij} + 2\left(\frac{\partial\epsilon}{\partial H^{AB}}\right)|_{H^{CD}=h^{CD}}F^A{}_iF^B{}_j]\dot{\Phi}^j. \tag{35}$$

From (34) we infer that the boundary conditions (14) are simply equivalent to

$$\left(\frac{\partial L}{\partial \Phi^i{}_{,A}}\right)n_A|_{\{t\}\times\partial\mathcal{B}} = 0, \tag{36}$$

where n_A is a conormal of $\partial\mathcal{B}$.
We now turn to the natural deformation, which is the material version of the natural configuration described in Sect.3. In the chosen foliation it is time-independent, namely of the form

$$\Phi^i(t, X) = \Phi^i_0(X), \tag{37}$$

where $\Phi_0(f_0(x)) = x$. Furthermore we have that

$$F^A{}_i(\Phi_0{}^k{}_{,C})F^B{}_j(\Phi_0{}^l{}_{,D})g^{ij}(\Phi_0(X)) = H_0^{AB}(X), \tag{38}$$

in short: $F_0{}^A{}_i\,F_0{}^B{}_j\,g_0{}^{ij} = H_0^{AB}$. We know from Sect.3 that the field equations and the boundary conditions are identically satisfied in the stress-free state. In the present context this fact takes the form

$$\left(\frac{\partial L}{\partial \dot{\Phi}^i}\right)_0 = 0, \qquad \left(\frac{\partial L}{\partial \Phi^i{}_{,A}}\right)_0 = 0, \qquad \left(\frac{\partial L}{\partial \Phi^i}\right)_0 = 0. \tag{39}$$

To derive Eq.(39) we have used (15,34,35).
The causality properties of this system are essentially governed by the nature of the coefficients $M^{tt}{}_{ij}$ of $\ddot{\Phi}^j$ and $M^{AB}{}_{ij}$ of $\Phi^j{}_{,AB}$. In order to be able to apply

the theorem of Koch [10], we will need a "negativity" condition on ${M^{tt}}_{ij}$ and a certain positivity ("coerciveness") condition on ${M^{AB}}_{ij}$. We easily find that

$$M_0^{tt}{}_{ij} = \left(\frac{\partial^2 L}{\partial \dot{\Phi}^i \partial \dot{\Phi}^j} \right)_0 = -\Omega^{\frac{1}{2}}\, \epsilon_0\, g_{0\,ij} \tag{40}$$

For the coefficient of ${\Phi^j}_{,AB}$, using (34), observe that

$$M_0^{AB}{}_{ij} = \left(\frac{\partial^2 L}{\partial {\Phi^i}_{,A} \partial {\Phi^j}_{,B}} \right)_0 = 4\,\Omega^{\frac{1}{2}}\, L_{0\,CEDF}\, H_0^{AE} H_0^{BF} {F_0{}^C}_i\, {F_0{}^D}_j, \tag{41}$$

where

$$L_{0\,ABCD} := \left(\frac{\partial^2 \epsilon}{\partial H^{AB} \partial H^{CD}} \right) |_{H^{EF}=H_0^{EF}} \tag{42}$$

5 Koch theorem

For convenience we state here a corollary of the theorem in [10], which is the precise statement we need.

Let $\Phi^i(t,X)$ be maps from $\{t\} \times \mathcal{B}$ to $\mathbb{R}^3$, where $\mathcal{B}$ is an open set in $\mathbb{R}^3$ with smooth boundary $\partial\mathcal{B}$. We are given a system of the form of

$$\partial_\alpha {\mathfrak{F}^\alpha}_i = w_i \quad \text{in } \{t\} \times \mathcal{B}, \tag{43}$$

where α runs from 0 to 3 and with ${\mathfrak{F}^\alpha}_i$ and w_j all being smooth functions of $(X^A, \Phi^i, {\Phi^j}_{,\alpha})$ on $\bar{\mathcal{B}} \times \mathbb{R}^3 \times \mathbb{R}^{12}$, together with the boundary conditions

$${\mathfrak{F}^\alpha}_i n_\alpha|_{\{t\}\times\partial\mathcal{B}} = {\mathfrak{F}^A}_i n_A|_{\{t\}\times\partial\mathcal{B}} = 0. \tag{44}$$

We make the following further assumptions:
(i) Symmetry: there hold the symmetries ${M^{\alpha\beta}}_{ij} = {M^{\beta\alpha}}_{ji}$ where ${M^{\alpha\beta}}_{ij} = \frac{\partial {\mathfrak{F}^\alpha}_i}{\partial {\Phi^j}_{,\beta}}$.
(ii) Static solution: There is given a time independent function $\Phi_0^i(t,X) = \Phi_0^i(X) \in C^\infty(\bar{\mathcal{B}})$ satisfying

$$\mathfrak{F}^\alpha_{0\,i} = 0, \qquad w_{0\,i} = 0 \tag{45}$$

and the estimates
(iii) Time components:

$$M_0^{tt}{}_{ij} \eta^i \eta^j \leq -\kappa |\eta|^2 \quad \text{in } \mathcal{B} \tag{46}$$

for a positive constant κ,
(iv) Space components:

$$\int_{\mathcal{B}} M_0^{AB}{}_{ij}\ \delta{\Phi^i}_{,A}\, \delta{\Phi^j}_{,B}\, d^3X + ||\delta\Phi||^2_{L^2(\mathcal{B})} \geq \sigma ||\delta\Phi||^2_{H^1(\mathcal{B})} \tag{47}$$

for $\sigma > 0$ and all $\delta\Phi \in C^\infty(\bar{\mathcal{B}})$, where H^s denotes the Sobolev space $H^{s,2}$. We remark that condition (47) implies the "strong ellipticity" or "Legendre-Hadamard" condition, namely that $M_0^{AB}{}_{ij} N_A N_B k^i k^j > 0$ in $\bar{\mathcal{B}}$ for non-zero N_A, k^i. This latter condition is the relevant one for wellposedness in the pure initial value problem (see [8]).

Theorem: Let $(\Phi(0), \dot{\Phi}(0))$ lie in a small neighborhood of $(\Phi_0, 0)$ in $H^{s+1}(\mathcal{B}) \times H^s(\mathcal{B}), s \geq 3$ and satisfy the corner conditions of order s in the sense that $\partial_t^r \mathcal{F}_i^A n_A|_{\{t=0\}\times\partial\mathcal{B}}$ is in $H^{s-r}(\mathcal{B}) \cap H_0^1(\mathcal{B})$ for $0 \leq r \leq s$ [2]. Then there exists, for sufficiently small t_0, a unique classical solution Φ of (43,44) in $C^2([0,t_0) \times \bar{\mathcal{B}})$ with $(\Phi(0), \dot{\Phi}(0))$ as initial data. Furthermore $\partial^r \Phi(t) \in L^2(\mathcal{B})$ for $0 \leq r \leq s+1$. In the last expression $\partial^r \Phi$ denotes all partial derivatives of order r. The evolved solution stays close to the static solution Φ_0 in the sense that $(\Phi(t), \dot{\Phi}(t))$ remains close to $(\Phi_0, 0)$ in $H^{s+1}(\mathcal{B}) \times H^s(\mathcal{B})$ and $\partial^r \Phi(t)$ remains close to $\partial^r \Phi_0 \in L^2(\mathcal{B})$ for $0 \leq r \leq s+1$. (This last statement is the "stability" part of the theorem.)

6 Main result

We now add our final constitutive assumption, sometimes called "pointwise stability" see [12], namely that

$$L_{0\,ABCD}\, N^{AB} N^{CD} \geq \sigma \,(H_{0\,CA} H_{0\,BD} + H_{0\,CB} H_{0\,AD})\, N^{AB} N^{CD} \quad \text{in } \mathcal{B} \tag{48}$$

where σ is a positive constant which only depends on the choice of coordinates and with $H_{0\,AB}$ being the inverse of H_0^{AB}. An important special case is where

$$4\epsilon_0 L_{0\,ABCD} = \lambda H_{0\,AB} H_{0\,CD} + \mu (H_{0\,CA} H_{0\,BD} + H_{0\,CB} H_{0\,AD}) \tag{49}$$

and

$$\mu(X) > 0, \quad 3\lambda(X) + 2\mu(X) > 0 \quad \text{in } \bar{\mathcal{B}} \tag{50}$$

The quantities μ and λ, when they are independent of X, are the Lamé constants of homogeneous, isotropic materials. We now invoke the Korn inequality (see e.g. [7] or [5]) of which we need a slight generalization due to [3], in the following form: Let Ψ^A be vector field on $(\mathcal{B}, H_{0\,AB})$ in some chart. Let $L(\Psi)$ be defined by $L(\Psi)_{AB} = 2H_{0\,C(A}\Psi^C{}_{,B)}$. Then there is a positive constant σ' such that

$$\|L(\Psi)\|^2_{L^2(\mathcal{B})} + \|\Psi\|^2_{L^2(\mathcal{B})} \geq \sigma' \|\Psi\|^2_{H^1(\mathcal{B})}. \tag{51}$$

We now assume coordinates X in $\mathcal{B}$ to be chosen so that Φ_0 is the identity map. Combining (41) with (48) and using (51), there follows condition (iv), i.e. that

$$\int_{\mathcal{B}} M_0^{AB}{}_{ij}\, \delta\Phi^i{}_{,A} \delta\Phi^j{}_{,B}\, d^3X + \|\delta\Phi\|^2_{L^2(\mathcal{B})} \geq \sigma'' \|\delta\Phi\|^2_{H^1(\mathcal{B})} \tag{52}$$

[2]This condition says that $(\partial_t^r \mathcal{F}^A{}_i)\, n_A|_{\{t=0\}\times\partial\mathcal{B}}$ is zero for $0 \leq r \leq s$ In the appendix we show that it can be met by a suitable choice of odd-order normal derivatives of $(\Phi(0), \dot{\Phi}(0))$ on $\partial\mathcal{B}$.

for positive σ''. The validity of the condition (iii) in the Koch theorem is immediate from Eq.(40). Furthermore the validity of (ii) has been checked in Sect.4. The validity of (i) is obvious from the variational character of the equations. Thus all assumptions of the Koch theorem are met.
It remains to check the determinant condition $det(\Phi^i{}_{,A}) > 0$ in $\bar{\mathcal{B}}$. But, when $(\Phi(0), \dot{\Phi}(0))$ is close to $(\Phi_0, 0)$ in $H^{s+1}(\mathcal{B}) \times H^s(\mathcal{B})$, $s \geq 3$, this immediately follows by Sobolev embedding and the positivity of $det(\Phi^i_{0,A})$ in $\bar{\mathcal{B}}$. Now the precise form of our final statement can be read off from the Koch theorem in Sect.5.
Stated somewhat informally, our final result is as follows:

Theorem: Let there be given a volume form $\Omega(X^A)$, a spacetime metric $g_{\mu\nu}(x^\lambda)$ of the form (18) and an internal energy $\epsilon(X^A, H^{BC})$ satisfying (15), all smooth functions of their respective arguments. Suppose there exists a smooth natural (i.e. stress-less) configuration such that the corresponding vector field u_0^μ is a static Killing field. Also suppose that the elasticity tensor for this natural configuration satisfies the pointwise stability condition (48). Let the initial data $(\Phi(0), \dot{\Phi}(0))$ for the deformation be close to those for the natural deformation and satisfy the corner condition of the appropriate order. Then there exists, for these initial data, a solution $\Phi(t)$ of the dynamical equations (31) with boundary conditions (36) for small times. This solution remains close to the natural deformation.

We end with two remarks on possible generalizations of the results presented here. The first remark concerns our assumption of having initial data which are close to a stress-free state: this is not essential, but was made because of its physical importance and to simplify the statement of the theorem. In particular the assumption that the background spacetime have a geodesic, static Killing vector ("ultrastatic spacetime") could be removed at the expense of having to introduce more complicated or less explicit assumptions. Secondly, and more importantly, one should extend the results above to the case of an elastic body (or bodies, if there are several) which are self-gravitating. The presence in G.R. of constraints and gauge freedom will make things more complicated. Furthermore the fact that the elastic equations are written in material form, but the Einstein equations are equations on spacetime, means that one is now not dealing with a system of partial differential equations in the strict sense.

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7 Appendix: corner conditions

We study corner conditions for the system (43) with the boundary conditions (44). Suppose that $(\Phi(0)|_{\partial\mathcal{B}}, \dot{\Phi}(0)|_{\partial\mathcal{B}})$ is sufficiently close to $(\Phi_0|_{\partial\mathcal{B}}, 0)$. In order for obtaining a solution of the equations of motion, one has to be able to satisfy conditions on the behaviour of certain normal derivatives of $(\Phi(0), \dot{\Phi}(0))$ on $\partial\mathcal{B}$, which guarantee that, not only $\mathcal{F}^A{}_i\, n_A|_{\partial\mathcal{B}} = 0$ at $t = 0$, but also a sufficient number of time derivatives at $t = 0$ of that condition is satisfied. (It is understood that higher-than-second-order time derivatives are eliminated in terms of spatial and lower-order time derivatives, using the equations motion: this by virtue of the negativity of $M_0^{tt}{}_{ij}$ is always possible for $(\Phi(0)|_{\partial\mathcal{B}}, \dot{\Phi}(0)|_{\partial\mathcal{B}})$ close to $(\Phi_0|_{\partial\mathcal{B}}, 0)$.)
First recall that $\Phi_0(t, X) = \Phi_0(X)$ identically satisfies the equations of motion and the boundary condition. Now we check that the corner condition of order 0, i.e. the undifferentiated-in-time Eq.(36) can be satisfied by a choice of $\partial_n\Phi(0)|_{\partial\mathcal{B}}$ (where ∂_n denotes any derivative transversal to $\partial\mathcal{B}$). To see this notice first that, by the above observation, $(\Phi_0|_{\partial\mathcal{B}}, 0)$ solves the order-0 corner condition. Furthermore

$$\frac{\partial^2 L}{\partial\partial_n\Phi^j\partial\Phi^i{}_{,A}} n_A|_{\partial\mathcal{B}} = \frac{\partial^2 L}{\partial\Phi^j{}_{,B}\partial\Phi^i{}_{,A}} n_A n_B|_{\partial\mathcal{B}}. \tag{53}$$

Then the result follows from the (finite-dimensional) implicit function theorem using that, by virtue of (48), the quadratic form

$$M_0^{AB}{}_{ij}\; n_A n_B|_{\partial\mathcal{B}} \tag{54}$$

is nonsingular, which in turn follows from the strong ellipticity condition (see Sect.5). One now performs the process of taking ($s > 1$ say) time derivatives of the boundary condition and eliminating $\overset{(m)}{\Phi}(0)|_{\partial\mathcal{B}}$ for $s \geq m > 1$. Using that $\mathcal{F}^0{}_i$ is zero for $(\Phi(0)|_{\partial\mathcal{B}}, \dot{\Phi}(0)|_{\partial\mathcal{B}}) = (\Phi_0|_{\partial\mathcal{B}}, 0)$, the s-th order corner condition becomes an equation of the form

$$M_0^{AB}{}_{ij}\; n_A n_B\; \partial_n^s \dot{\Phi}^j(0)|_{\partial\mathcal{B}} \,\hat{=}\, \text{lower order} \tag{55}$$

for odd s and

$$M_0^{AB}{}_{ij}\; n_A n_B\; \partial_n^{s+1} \Phi^j(0)|_{\partial\mathcal{B}} \,\hat{=}\, \text{lower order} \tag{56}$$

for even s, where "$\hat{=}$ lower order" means expressions which depend on lower-order, even-numbered normal derivatives of $(\Phi(0)|_{\partial\mathcal{B}}, \dot{\Phi}(0)|_{\partial\mathcal{B}})$, modulo terms which depend on normal derivatives of the same order, but which are zero when $(\Phi(0)|_{\partial\mathcal{B}}, \dot{\Phi}(0)|_{\partial\mathcal{B}}) = (\Phi_0|_{\partial\mathcal{B}}, 0)$. The equations (55,56) are identically satisfied if $(\Phi(0)|_{\partial\mathcal{B}}, \dot{\Phi}(0)|_{\partial\mathcal{B}})$ and their normal derivatives appearing on both sides are replaced by those of $(\Phi_0|_{\partial\mathcal{B}}, 0)$. Using the fact that $M_0^{AB}{}_{ij}\; n_A n_B$ depends only on first derivatives of Φ, we are now able to recursively solve the corner conditions, with $\Phi(0)|_{\partial\mathcal{B}}$ and $\dot{\Phi}(0)|_{\partial\mathcal{B}}$ and their even normal derivatives given arbitrarily, provided that $(\Phi(0)|_{\partial\mathcal{B}}, \dot{\Phi}(0)|_{\partial\mathcal{B}})$ is close to $(\Phi_0|_{\partial\mathcal{B}}, 0)$. We have to - and by the above can - choose a large class of initial data $(\Phi(0), \dot{\Phi}(0))$ close to $(\Phi_0, 0)$ so that the corner conditions are fulfilled for arbitrary order and the negativity of ${M^{tt}}_{ij}$ and the coerciveness of ${M^{AB}}_{ij}$ are satisfied.

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