The Ernst equation and ergosurfaces

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Abstract

We show that solutions $\mathscr{E}$ of the Ernst equation with a smooth, non-degenerate zero-level-set of $\Re\mathscr{E}$ lead to smooth ergosurfaces. Some partial results on critical zeros are obtained.

1 Introduction

A standard procedure for constructing stationary axi-symmetric solutions of the Einstein equations proceeds by a reduction of the Einstein equations to a 1+1 nonlinear equation — the Ernst equation [2] — using the asymptotically timelike Killing vector field X as the starting point of the reduction. One then finds a complex valued field $\mathscr{E} = f + ib$, by e.g. solving a boundary-value problem [10]. The space-time metric is then obtained by solving ODEs for the metric functions. Those ODE's are singular at the zero-level-set

$$E_f := \{f = 0\}$$

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of $f := \Re\mathscr{E}$; we will refer to E_f as the *$\mathscr{E}$-ergosurface.* It is the object of this note to show that the singularities of the solutions of those ODEs conspire to produce a smooth space-time metric, provided that the gradient $D(\Re\mathscr{E})$ is nowhere vanishing on the $\mathscr{E}$–ergosurface.

We expect that the above non-degeneracy condition is not necessary, but we have not been able to find a general argument for that. In Section 4 we report some preliminary results on critical zeros of $\Re\mathscr{E}$, with non-vanishing Hessian, based on computer algebra.

The results presented here originated in numerical experiments by PCh and SSz, together with previous unpublished analytical results by RM.

2 The field equations and ergosurfaces

We consider a vacuum gravitational field in Weyl-Lewis-Papapetrou coordinates

$$ds^2 = f^{-1}\left[e^{2k}\left(d\rho^2 + d\zeta^2\right) + \rho^2 d\phi^2\right] - f\left(dt + ad\phi\right)^2 \tag{2.1}$$

with all functions depending only upon ρ and ζ. The vacuum Einstein equations for the metric functions k, f, and a are equivalent to the Ernst equation

$$(\Re\mathscr{E})\left(\mathscr{E}_{,\rho\rho} + \mathscr{E}_{,\zeta\zeta} + \frac{1}{\rho}\mathscr{E}_{,\rho}\right) = \mathscr{E}_{,\rho}^2 + \mathscr{E}_{,\zeta}^2 \tag{2.2}$$

for the complex function $\mathscr{E}(\rho,\zeta)$

$$\mathscr{E} = f + \mathrm{i}b, \tag{2.3}$$

where b replaces a via

$$a_{,\rho} = \rho f^{-2} b_{,\zeta}, \quad a_{,\zeta} = -\rho f^{-2} b_{,\rho} \tag{2.4}$$

and k can be calculated from

$$k_{,\rho} = \frac{\rho}{4f^2}\left[f_{,\rho}^2 - f_{,\zeta}^2 + b_{,\rho}^2 - b_{,\zeta}^2\right], \quad k_{,\zeta} = \frac{\rho}{2f^2}\left[f_{,\rho}f_{,\zeta} + b_{,\rho}b_{,\zeta}\right]. \tag{2.5}$$

We will think of ρ and ζ as being cylindrical coordinates in $\mathbb{R}^3$ equipped with the flat metric

$$\mathring{g} = d\rho^2 + \rho^2 d\varphi^2 + d\zeta^2 ,$$

with all the above functions being φ–independent functions on $\mathbb{R}^3$. Then (2.2) can be rewritten as

$$f\Delta f = |Df|^2 - |Db|^2 , \tag{2.6}$$

$$f\Delta b = 2(Df, Db) . \tag{2.7}$$

where Δ is the flat Laplace operator of the metric $\mathring{g}$, and $(\cdot,\cdot)$ denotes the $\mathring{g}$-scalar product, similarly the norm $|\cdot|$ is the one associated with $\mathring{g}$.

The equations (2.6)-(2.7) degenerate at $\{f=0\}$, and it is not clear that f or b will smoothly extend across $\{f=0\}$, if at all. However, there are large classes of solutions which do have this property. Examples can be obtained as follows: First, every space-time obtained from an Ernst map $\mathscr{E}'$ associated to the reduction that uses the axial Killing vector ∂_φ (see, e.g., [1, 11]) will lead to a solution $\mathscr{E}$ as considered here that extends smoothly across the *space-time ergosurfaces* (if any; recall that an ergosurface is defined to be a *timelike* hypersurface where the Killing vector X, which asymptotes a time translation in the asymptotic region, becomes null. Those ergosurfaces correspond then to $\mathscr{E}$-ergosurfaces across which f does indeed extend smoothly. However, we emphasise that we are interested in the construction of a space-time starting from $\mathscr{E}$, and we have no *a priori* reason to expect that an $\mathscr{E}$–ergosurface, defined as smooth zero-level set of $\Re\mathscr{E}$, will lead to a smooth space-time ergosurface; it could lead e.g. to space-time curvature singularities.

Next, large classes of further examples are given in [3, 6–10, 12][1]. Some of the solutions in those references have non-trivial zero-level sets of $\Re\mathscr{E}$, with $g_{\rho\rho}=g_{zz}$ and $g_{t\varphi}$ smooth across E_f (see, e.g., [6]), but the smoothness of $g_{\varphi\varphi}$ is not manifest.

3 Non-degenerate zeros of f

We claim the following:

Theorem 3.1 *For $\rho>0$ consider a smooth solution $f+\mathrm{i}b$ of (2.6)-(2.7) such that $|Df|$ has no zeros at the $\mathscr{E}$–ergosurface $E_f:=\{f=0\}$. Then the metric (2.1) constructed by solving (2.4)-(2.5) is smooth and has Lorentzian signature.*

Proof: We need to show that the functions

$$\alpha := g_{\varphi t} = af\ , \quad \beta := \ln g_{\zeta\zeta} = \ln g_{\rho\rho} = 2k - \ln f^{-1}\ ,$$

as well as

$$g_{\varphi\varphi} = \frac{\rho^2-(af)^2}{f}$$

[1]The solutions we are referring to here are not necessarily vacuum everywhere, and some of them have a function $\mathscr{E}$ which is singular somewhere in the (ρ,ζ) plane. Our analysis applies to the vacuum region, away from the rotation axis, and away from the singularities of the Ernst map $f+\mathrm{i}b$.

are smooth across $\{f=0\}$, and that $g_{\varphi t}$ does *not* vanish whenever $g_{tt}=f$ does.

We start by Taylor-expanding f and b to order two near any point (ρ_0,ζ_0) such that $f(\rho_0,\zeta_0)=0$:

$$\begin{aligned}
f(\rho,\zeta) &= \mathring{f}_{,\rho}(\rho-\rho_0)+\mathring{f}_{,\zeta}(\zeta-\zeta_0)\\
&\quad+\frac{1}{2}\mathring{f}_{,\rho\rho}(\rho-\rho_0)^2+\frac{1}{2}\mathring{f}_{,\zeta\zeta}(\zeta-\zeta_0)^2+\mathring{f}_{,\rho\zeta}(\rho-\rho_0)(\zeta-\zeta_0)+\dots,\\
b(\rho,\zeta) &= \mathring{b}+\mathring{b}_{,\rho}(\rho-\rho_0)+\mathring{b}_{,\zeta}(\zeta-\zeta_0)\\
&\quad+\frac{1}{2}\mathring{b}_{,\rho\rho}(\rho-\rho_0)^2+\frac{1}{2}\mathring{b}_{,\zeta\zeta}(\zeta-\zeta_0)^2+\mathring{b}_{,\rho\zeta}(\rho-\rho_0)(\zeta-\zeta_0)+\dots,
\end{aligned}$$

where a circle over a function indicates that the value at ρ_0 and ζ_0 is taken. Inserting these expansions into (2.6)-(2.7), after tedious but elementary algebra one obtains either

$$\begin{gathered}
\mathring{b}_{\rho}=\mp\mathring{f}_{\zeta},\qquad \mathring{b}_{\zeta}=\pm\mathring{f}_{\rho},\\
\mathring{f}_{,\rho\rho}+\mathring{f}_{,\zeta\zeta}=\frac{\mathring{f}_{,\rho}}{\rho_0},\quad \mathring{b}_{,\rho\rho}+\mathring{b}_{,\zeta\zeta}=\frac{\mathring{f}_{,\zeta}}{\rho_0},\quad \mathring{b}_{,\rho\zeta}=\mathring{f}_{,\zeta\zeta},\quad \mathring{f}_{,\rho\zeta}=\mathring{b}_{,\rho\rho},
\end{gathered}\tag{3.1}$$

or

$$\mathring{b}_{\rho}=\mathring{f}_{\zeta}=\mathring{b}_{\zeta}=\mathring{f}_{\rho}=0\;.\tag{3.2}$$

The second possibility is excluded by our hypothesis that $Df\neq 0$ on E_f.

Suppose, first, that the lower signs arise in the first line of (3.1). From (2.4) we obtain

$$\alpha_{,\rho} = \frac{f_{,\rho}}{f}\alpha+\frac{\rho}{f}b_{,\zeta}\;,\tag{3.3}$$

$$\alpha_{,\zeta} = \frac{f_{,\zeta}}{f}\alpha-\frac{\rho}{f}b_{,\rho}\;,\tag{3.4}$$

so that

$$\left(\frac{\alpha-\rho}{f}\right)_{,\rho} = \underbrace{[\rho(b_{,\zeta}+f_{,\rho})-f]}_{=:\sigma_\rho}f^{-2}\;,\tag{3.5}$$

$$\left(\frac{\alpha-\rho}{f}\right)_{,\zeta} = \underbrace{\rho(f_{,\zeta}-b_{,\rho})}_{=:\sigma_\zeta}f^{-2}\;.\tag{3.6}$$

Inserting (3.1) into the definitions of σ_ρ and σ_ζ we find

$$\sigma_\rho=\sigma_\zeta=0=d\sigma_\rho=d\sigma_\zeta$$

at every point (ρ_0, ζ_0) lying on the $\mathscr{E}$-ergosurface. Here, as elsewhere, dh denotes the differential of a function h.

Recall that Df does not vanish on $E_f = \{f = 0\}$. We can thus introduce coordinates (x, y) near each connected component of E_f so that $f = x$. Since the σ_a's are smooth we have the Taylor expansions

$$\sigma_a = \sigma_a|_{E_f} + (\partial_x \sigma_a)|_{E_f} x + r_a x^2 \ ,$$

for some remainder terms r_a which are smooth functions on space-time. But we have shown that $\sigma_a|_{E_f} = (\partial_x \sigma_a)|_{E_f} = 0$. Hence

$$\sigma_a = r_a x^2 = r_a f^2 \ ,$$

It follows that the right-hand-sides of (3.5)-(3.6) extend by continuity across E_f to smooth functions. Hence the derivatives of $(\alpha - \rho)/f$ extend by continuity to smooth functions, and by integration

$$\alpha - \rho = f\hat{\alpha} \ , \tag{3.7}$$

for some smooth function $\hat{\alpha}(\rho, \zeta)$. This proves smoothness both of $g_{t\varphi}$ and of $g_{\varphi\varphi}$. We also obtain that $g_{t\varphi} = \rho$ when $f = 0$, and since $\rho > 0$ by assumption we obtain non-vanishing of $g_{t\varphi}$ on that part of the $\mathscr{E}$–ergosurface which does not intersect the rotation axis $\{\rho = 0\}$.

In the case where the upper choice of sign in (3.1) occurs, instead of (3.5)-(3.6) we write equations for $(\alpha + \rho)/f$, and an identical argument applies.

We pass now to the analysis of $g_{\rho\rho} = g_{zz}$. From (2.5),

$$(k - \frac{1}{2}\ln f)_{,\rho} = \underbrace{\frac{1}{4}\left[\rho(f_{,\rho}^2 - f_{,\zeta}^2 + b_{,\rho}^2 - b_{,\zeta}^2) - 2ff_{,\rho}\right]}_{=:\kappa_\rho} f^{-2}, \tag{3.8}$$

$$(k - \frac{1}{2}\ln f)_{,\zeta} = \underbrace{\frac{1}{2}\left[\rho(f_{,\rho}f_{,\zeta} + b_{,\rho}b_{,\zeta}) - ff_{,\zeta}\right]}_{=:\kappa_\zeta} f^{-2}. \tag{3.9}$$

Evaluating κ_a and its derivatives on E_f and using (3.1) one obtains again

$$\kappa_a = d\kappa_a = 0$$

on E_f. As before we conclude that $g_{\rho\rho}$ and $g_{\zeta\zeta}$ are smooth across E_f.

□

4 Non-degenerate critical zeros of f

In this section we wish to examine zeros of f at points p such that $f(p) = Df(p) = 0$, with non-vanishing Hessian $DDf(p) \neq 0$ of f. Points with this property are necessarily isolated. The analysis proceeds as in the previous section: we Taylor expand f and b to order n,

$$f(\rho,\zeta) \quad = \sum_{0\le i+j\le n} \frac{\mathring{f}_{i,j}}{i!j!}(\rho-\rho_0)^i(\zeta-\zeta_0)^j + r_n \ , \tag{4.1}$$

where

$$\mathring{f}_{i,j} := \frac{\partial^{i+j} f}{\partial^i\rho\partial^j\zeta}(\rho_0,\zeta_0) \ .$$

Similarly we denote the Taylor coefficients of b by $\mathring{b}_{i,j}$. We insert the resulting expansions in the Ernst equations, obtaining relations between the Taylor coefficients. The algebra has been done using MAPLE, and crosschecked with MATHEMATICA, the interested reader can download the worksheets from `http://cornus.if.uj.edu.pl/~szybka/CMS`. The results in this section have a preliminary character, as we have not carefully crosschecked all special cases which might have remained unnoticed by the computer algebra systems.

By inspection of the equations involved one finds that the knowledge of the Taylor coefficients up to $n = 4$ is necessary for the analysis of $g_{\rho\rho}$, while $n = 5$ is needed for that of $g_{\varphi\varphi}$.

Perhaps the most significant result of this calculation is the following:

PROPOSITION 4.1 *There exist no points such that $f = Df = 0$, and with DDf – either positive definite, or negative definite.*

In other words, every critical point of f on E_f with non-vanishing Hessian is necessarily a saddle point. This implies immediately that $\{f = 0\}\cap\{Df = 0\}$ *has no isolated points at which* $DDf \not\equiv 0$. Unfortunately, this does not exclude isolated points of E_f at which some high order derivative would produce a Taylor polynomial with a definite sign; similarly isolated zeros of infinite order are not excluded; but we find both those possibilities rather unlikely. It should be kept in mind that solutions of the Ernst equation are necessarily real analytic functions away from E_f, but analyticity could fail at E_f.

In any case, the result of Proposition 4.1 is not welcome for the analysis of the metric functions at critical zeros of f. Indeed, quotients involving f

are easy to analyse near points at which the Hessian DDf is either strictly positive or strictly negative. On the other hand, the analysis near saddle points is more delicate. It is conceivable that the results in [4, 5] can be used to settle this, but this does not seem to be staightforward. The results that follow show that there are *no obvious obstructions to smoothness of* $g_{\mu\nu}$ *near saddle points*, but they do *not* establish smoothness.

The results of our calculations are summarised as follows: consider the polynomials W_a, $a = 1, 2$, obtained by inserting the Taylor expansion of f and b, with $\mathring{f} = D\mathring{f} = 0$, into (2.6) and (2.7). The requirement that those polynomials vanish up-to-and-including order two imposes the following alternative sets of conditions:

$$\text{I.} \qquad \mathring{b}_{2,0} = \mathring{b}_{1,1} = \mathring{b}_{0,2} = \mathring{f}_{1,2} = 0 \;, \quad \mathring{f}_{1,1} = \mathring{f}_{2,2} \in \mathbb{R} \;, \tag{4.2}$$

$$\text{II.} \quad \mathring{f}_{2,0} = -\mathring{f}_{0,2} = -\mathring{b}_{1,1} \in \mathbb{R} \;, \quad \mathring{b}_{0,2} = -\mathring{f}_{1,1} = -\mathring{b}_{2,0} \in \mathbb{R} \;, \tag{4.3}$$

as well as a set which is related to II. above by exchanging b with $-b$. The first set leads to $\mathring{f}_{1,1} = 0$ when requiring that the polynomials W_a just defined vanish to one order higher, which proves Proposition 4.1. On the other hand, the set II. and the requirement of vanishing of the third-order coefficients of W leads to the conditions

$$\mathring{f}_{3,0} + \tfrac{2}{\rho_0}\mathring{b}_{1,1} = \tfrac{1}{\rho_0}\mathring{b}_{1,1} - \mathring{b}_{2,1} = \mathring{b}_{0,3} \in \mathbb{R} \;, \quad \mathring{f}_{1,2} = \mathring{b}_{2,1} \;, \tag{4.4}$$

$$\mathring{b}_{3,0} - \tfrac{1}{\rho_0}\mathring{f}_{1,1} = -\mathring{b}_{1,2} = -\mathring{f}_{0,3} \in \mathbb{R} \;, \quad \mathring{f}_{2,1} = \mathring{b}_{3,0} \;. \tag{4.5}$$

Inserting (4.3)-(4.5) into the definitions (3.8)-(3.9) of κ, and (3.5)-(3.6) of σ, gives

$$\kappa = \partial_i\kappa = \partial_i\partial_j\kappa = \partial_i\partial_j\partial_k\kappa\sigma = \partial_i\sigma = \partial_i\partial_j\sigma = \partial_i\partial_j\partial_k\sigma = 0$$

at (ρ_0, ζ_0). In fact,

$$\kappa_\rho = -\tfrac{1}{4\rho_0}f^2 + O(|\vec{x} - \vec{x}_o|^5) \;, \quad \text{where } \vec{x} = (\rho, \zeta) \;, \tag{4.6}$$

$$\kappa_\zeta = O(|\vec{x} - \vec{x}_o|^5) \;. \tag{4.7}$$

This is obtained by the requirement of vanishing of the fourth-order coefficients of W, which leads to the equations

$$\mathring{f}_{2,2} = -\mathring{b}_{1,3} - \tfrac{1}{\rho_0}\mathring{b}_{0,3} = \mathring{b}_{3,1} \in \mathbb{R} \;, \quad \mathring{f}_{3,1} = \mathring{b}_{4,0} \in \mathbb{R} \;, \tag{4.8}$$

$$\mathring{b}_{2,2} = \mathring{f}_{1,3} = -\tfrac{1}{\rho_0^2}\mathring{f}_{1,1} - \mathring{b}_{4,0} - \tfrac{1}{\rho_0}\mathring{b}_{1,2} \;, \quad \mathring{f}_{0,4} = \mathring{b}_{1,3} \in \mathbb{R} \;, \tag{4.9}$$

$$\mathring{f}_{4,0} = \mathring{b}_{1,3} + \tfrac{1}{\rho_0^2}\mathring{b}_{1,1} + \tfrac{2}{\rho_0}\mathring{b}_{0,3} \;, \quad \mathring{b}_{2,2} + \mathring{b}_{0,4} = \tfrac{1}{\rho_0}\mathring{b}_{1,2} \;. \tag{4.10}$$

One also has, assuming that neither $\mathring{b}_{1,1}$ nor $\mathring{f}_{1,1}$ vanish,

$$\sigma_\rho = \frac{1}{6\mathring{b}_{1,1}^2}\left(-\mathring{b}_{1,3} + (\mathring{b}_{0,5} + \mathring{f}_{1,4})\rho_0\right) f^2 + O(|\vec{x} - \vec{x}_o|^5) \,, \tag{4.11}$$

$$\sigma_\zeta = \frac{\rho_0(-\mathring{f}_{3,2} + \mathring{b}_{4,1})}{6\mathring{f}_{1,1}\mathring{b}_{1,1}} f^2 + O(|\vec{x} - \vec{x}_o|^5) \,. \tag{4.12}$$

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