On the identification of quasiprimary scaling operators in local scale-invariance

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Abstract. In non-equilibrium systems far from a stationary state the relationship between physical observables defined in lattice models and the associated (quasi-)primary scaling operators of the underlying field-theory is in general only defined up to a time-dependent amplitude. The associated exponent enters into the time-dependent scaling functions. This leads to a generalization of predictions of the theory of local scale-invariance for two-time response and correlation functions. Applications to non-equilibrium critical dynamics are discussed.

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The analysis of the collective behaviour of many-body systems is greatly helped in situations where some scale-invariance allows an efficient description through field-theoretical methods. A necessary requirement for the application of these is the possibility to identify the physical observables typically defined in terms of a lattice model, e.g. $\sigma_{\boldsymbol{r}}$ for the order-parameter at the site $\boldsymbol{r}$, with a continuum field $\phi(\boldsymbol{r})$ (called a *scaling operator* [1]) with well-defined scaling properties $\phi(\boldsymbol{r}) = \mathfrak{b}^{-x}\phi(\boldsymbol{r}/\mathfrak{b})$. In other words, one generally expects that the correspondence ($\mathfrak{a}$ is the lattice constant)

$$\sigma_{\boldsymbol{r}} \to \mathfrak{a}^{-x}\phi(\boldsymbol{r}) \tag{1}$$

can be defined in equilibrium systems or more generally steady-states of non-equilibrium systems, see e.g. [2, 1, 3]. In addition, in equilibrium systems one expects the same sort of relationship to hold true where $\phi(\boldsymbol{r})$ is now a primary scaling operator of a conformal field-theory and allows space-dependent rescaling factors $\mathfrak{b} = \mathfrak{b}(\boldsymbol{r})$ [1].

In this letter, we reconsider this correspondence for systems with dynamical scaling and far from equilibrium, as it occurs for example in ageing phenomena. Concrete examples are phase-ordering kinetics or non-equilibrium critical dynamics, see [4, 5, 6] for reviews. Among the main quantities of interest are the two-time autocorrelation function $C(t,s)$ and the autoresponse function $R(t,s)$

$$\begin{aligned} C(t,s) &= \langle \phi(t,\boldsymbol{r})\phi(s,\boldsymbol{r})\rangle \;=\; s^{-b} f_C(t/s) \\ R(t,s) &= \left.\frac{\delta\langle\phi(t,\boldsymbol{r})\rangle}{\delta h(s,\boldsymbol{r})}\right|_{h=0} \;=\; \left\langle \phi(t,\boldsymbol{r})\widetilde{\phi}(s,\boldsymbol{r})\right\rangle \;=\; s^{-1-a} f_R(t/s) \end{aligned} \tag{2}$$

where $\widetilde{\phi}$ is the response field in the Janssen-de Dominicis formalism [7, 8], a and b are ageing exponents and f_C and f_R are scaling functions such that $f_{C,R}(y) \sim y^{-\lambda_{C,R}/z}$ for $y \gg 1$. These scaling forms are only valid in the scaling regime where $t,s \to \infty$ *and* $y = t/s > 1$ fixed. We stress that in the kind of system under consideration invariance under time-translations is broken. In an attempt to try to derive the form of the scaling functions in a model-independent way it has been argued [9] that the scaling operators ϕ and $\widetilde{\phi}$ should transform covariantly under a larger group than mere dynamical scale-transformations. If such an invariance exists, one may call it a *local scale-invariance (LSI)*. The infinitesimal generators of local scale-invariance read [9, 10, 11]

$$X_0 = -t\partial_t - \frac{x}{z} \;\;,\;\; X_1 = -t^2\partial_t - \frac{2}{z}\,(x+\xi)\,t \tag{3}$$

where for simplicity we have suppressed the terms acting on the space coordinates which are not important for what follows. We have also not written down the further generators of LSI which do not modify the time t but only act on the space coordinates $\boldsymbol{r}$. Here x is the scaling dimension of the scaling operator $\phi(t,\boldsymbol{r}) = \mathfrak{b}^{-x/z}\phi(t/\mathfrak{b}^z, \boldsymbol{r}/\mathfrak{b})$ where z is the dynamical exponent and ξ is a constant. It is the purpose of this letter to clarify the meaning of this constant ξ.

Motivated by the analogy with two-dimensional conformal invariance, we generalize the dilatation generator X_0 and the generator X_1 of 'special' transformations as follows to all $n \geq 0$

$$X_n = -t^{n+1}\partial_t - \frac{x}{z}(n+1)t^n - \frac{2\xi}{z}nt^n \tag{4}$$

such that the commutator $[X_n, X_m] = (n-m)X_{n+m}$ holds for all $n, m \in \mathbb{N}_0$ (with the convention $0 \in \mathbb{N}_0$).† Next, the global form of these transformations reads as follows. If $t = \beta(t')$ such that $\beta(0) = 0$, then $\phi(t)$ transforms as

$$\phi(t) = \dot{\beta}(t')^{-x/z} \left(\frac{t'\dot{\beta}(t')}{\beta(t')}\right)^{-2\xi/z} \phi'(t') \tag{5}$$

where again the space-dependence of ϕ was suppressed. The infinitesimal generators X_n are recovered for $\beta(t) = t + \epsilon t^{n+1}$, with $|\epsilon| \ll 1$. From this, it is clear that ϕ is not transforming as an usual primary scaling operator. But if one defines $\Phi(t) := t^{-2\xi/z}\phi(t)$ the scaling operator $\Phi(t)$ becomes a conventional primary scaling operator of LSI, viz.

$$\Phi(t) = \dot{\beta}(t')^{-(x+2\xi)/z}\ \Phi'(t') \tag{6}$$

but with a modified scaling dimension $x \to x + 2\xi$. In other words, if time-dependent observables of lattice models $\sigma_{\boldsymbol{r}}(t)$ can be related to a primary scaling operator $\Phi(t)$ at all, it should be via the relation

$$\sigma_{\boldsymbol{r}}(t) \to \mathfrak{a}^{-x}\ \phi(t) = \mathfrak{a}^{-x}\ t^{2\xi/z}\ \Phi(t) \tag{7}$$

rather than by eq. (1). Of course, (7) is only possible because of the absence of time-translation invariance. We emphasize that the scaling of ϕ is unusual in that under a dilatation $t \to \mathfrak{b}^z t$ the scaling dimension remains x but for more general scale transformations a new effective scaling dimension $x + 2\xi$ appears.

As a simple application, consider the two-time autoresponse function. For quasiprimary scaling operators $\Phi(t)$ and $\widetilde{\Phi}(s)$ with scaling dimensions x and $\widetilde{x}$, respectively, local scale-invariance predicts $\langle\Phi(t)\widetilde{\Phi}(s)\rangle = (t/s)^{(\widetilde{x}-x)/z}(t-s)^{-(x+\widetilde{x})/z}$, up to normalization [9]. In view of (7), the physical autoresponse function rather reads

$$\begin{aligned} R(t,s) &= \left\langle \phi(t)\widetilde{\phi}(s)\right\rangle = \left\langle t^{2\xi/z}\Phi(t)s^{2\widetilde{\xi}/z}\widetilde{\Phi}(s)\right\rangle \\ &= s^{-(x+\widetilde{x})/z}\left(\frac{t}{s}\right)^{(2\widetilde{\xi}+\widetilde{x}-x)/z}\left(\frac{t}{s}-1\right)^{-(x+\widetilde{x}+2\xi+2\widetilde{\xi})/z} \\ &= s^{-1-a}\left(\frac{t}{s}\right)^{1+a'-\lambda_R/z}\left(\frac{t}{s}-1\right)^{-1-a'} \end{aligned} \tag{8}$$

(up to normalization) and where the effective scaling dimensions of $\Phi(t)$ and $\widetilde{\Phi}(s)$ as read off from eq. (6) must be used. In the last line, we have reintroduced the standard

† This is the unique semi-infinite extension of the algebra $\langle X_0, X_1\rangle$ which does not introduce further differential operators into X_n and is compatible with eq. (3).

Table 1. Values of the exponents a, a' and λ_R/z in several non-disordered and a few glassy systems which are at a critical point of their stationary state. If a numerical result is quoted without an error bar it is taken form the literature, otherwise the numbers in brackets give our estimate of the uncertainty in the last digit(s). FA stands for the Frederikson-Andersen model. The methods of calculation of the two-time autoresponse are D: direct space, P: momentum space, A: alternating external field; E refers to an exact solution and N to a numerical study.

model	a	$a'-a$	λ_R/z	Method	Ref.
OJK-model	$(d-1)/2$	$-1/2$	$d/4$	D,E	[12, 13, 11]
$1D$ Ising	0	$-1/2$	$1/2$	D,E	[14, 10]
$2D$ Ising	0.115	$-0.187(20)$	0.732(5)	P,N	[17]
$3D$ Ising	0.506	$-0.022(5)$	1.36	P,N	[17]
$1D$ contact process	-0.681	$+0.270(10)$	1.76(5)	D,N	[21, 22]
FA, $d>2$	$1+d/2$	-2	$2+d/2$	P,E	[15]
FA, $d=1$	1	$-3/2$	2	P,E	[15, 16]
$3D$ Ising spin glass	0.060(4)	$-0.76(3)$	0.38(2)	A,N	[11]

exponents a, a' and λ_R and hence reproduce the result quoted in [11]. Early discussions of local scale-invariance had assumed $a'=a$ from the outset. In the appendix, we discuss the scaling form of the autocorrelator $C(t,s)$ in those cases where $z=2$.

It appears that the more general correspondence (7) and consequently the response (8) with $a'\neq a$ actually occurs in non-equilibrium critical dynamics, as we shall now illustrate in a few examples. In table 1 we collect results on the exponents a, a' and λ_R/z in some models with a critical stationary state and where $a'\neq a$.‡ In several cases, these exponents can be read off from the exact solution, i.e., for the magnetic response in the OJK-model [12, 13] and the $1D$ Glauber-Ising model at zero temperature [14] or else the energy response in the zero-temperature Frederikson-Andersen model [15, 16].

Another interesting test case is provided by the critical Ising model in $2D$ and $3D$. Indeed, it was pointed out some time ago that the numerical calculation of the two-time response function $\widehat{R}_{\boldsymbol{q}}(t,s)=\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{r}\, R(t,s;\boldsymbol{r})e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}$ in *momentum* space provides a more sensitive test on the form of its scaling function than in direct space [17]. The methods of LSI can be readily adapted to momentum space and the analogue of (8) is, again up to normalization,

$$\widehat{R}_{\boldsymbol{0}}(t,s) = s^{-1-a+d/z}\left(\frac{t}{s}\right)^{1+a'-\lambda_R/z}\left(\frac{t}{s}-1\right)^{-1-a'+d/z} \tag{9}$$

Since measurements of autoresponse functions are much affected by statistical noise,

‡ In table 1, D,E means that the exact response agrees with (8) with the given values of the exponents, while P,E means that there is exact agreement with (9).

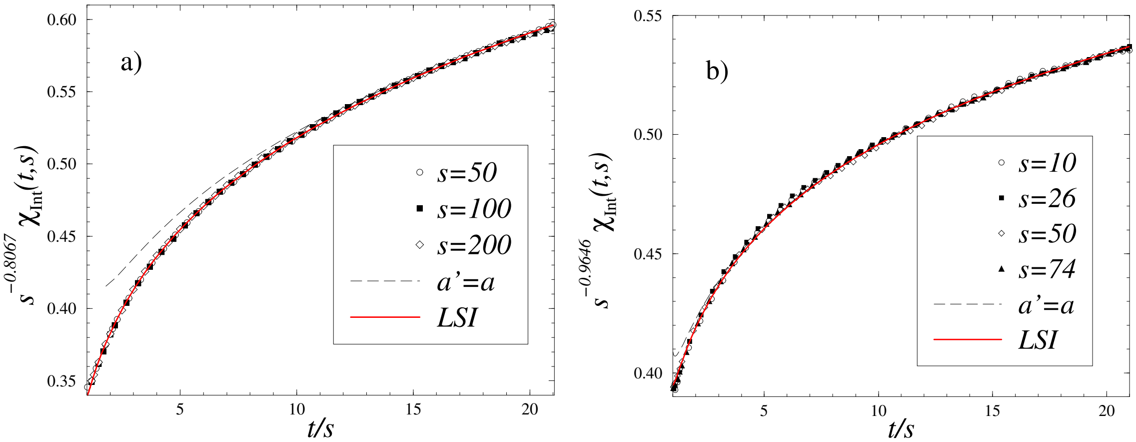

Figure 1. Intermediate susceptibility $\chi_{\rm Int}(t,s)$ in momentum space in the (a) $2D$ and (b) $3D$ critical Ising model, for several values of the waiting time s. The full curve is the LSI prediction eq. (10,11) with the exponents as listed in table 1. The dashed line corresponds to the case $a' = a$.

one often rather studies integrated response functions. Here we consider

$$\chi_{\rm Int}(t,s) := \int_{s/2}^{s} {\rm d}u\, \widehat{R}_{\mathbf{0}}(t,u) = \chi_0 s^{-a+d/z} f_\chi(t/s) \tag{10}$$

which is free from effects which mask the true scaling behaviour in several other variants of integrated responses [17]. The scaling function $f_\chi(y)$ follows from LSI, eq. (9):

$$f_\chi(y) = y^{(d-\lambda_R)/z} \left[{}_2F_1\left(1+a'-\frac{d}{z}, \frac{\lambda_R}{z}-a; 1+\frac{\lambda_R}{z}-a; \frac{1}{y}\right) \right.$$
$$\left. -2^{a-\lambda_R/z} {}_2F_1\left(1+a'-\frac{d}{z}, \frac{\lambda_R}{z}-a; 1+\frac{\lambda_R}{z}-a; \frac{1}{2y}\right)\right] \tag{11}$$

and where ${}_2F_1$ is Gauss' hypergeometric function. In figure 1 we compare simulational data [17] with this prediction for both the $2D$ and $3D$ critical Ising model with non-conserved heat-bath dynamics. It had already been observed before [17] that local scale-invariance with the additional assumption $a' = a$ does not agree with the numerical data in $2D$ and only marginally so in $3D$ and we confirm this finding. However, we also see that the data can be perfectly matched by LSI, within the numerical precision, if a and a' are allowed to be different. We did check that the integrated TRM response functions in direct space as studied in [18] do not change appreciably with $a' - a$.

A similar conclusion can also be drawn for the $1D$ critical contact process. It has been shown recently that the phenomenology of ageing can also be found in critical stochastic processes although these do *not* satisfy detailed balance and have a non-equilibrium steady-state [19, 20, 21]. In figure 2a we compare the numerical data obtained directly for $R(t,s)$ either from the LCTMRG [21] or Monte Carlo simulations [22]. It is satisfying that the data from both methods are consistent with each other

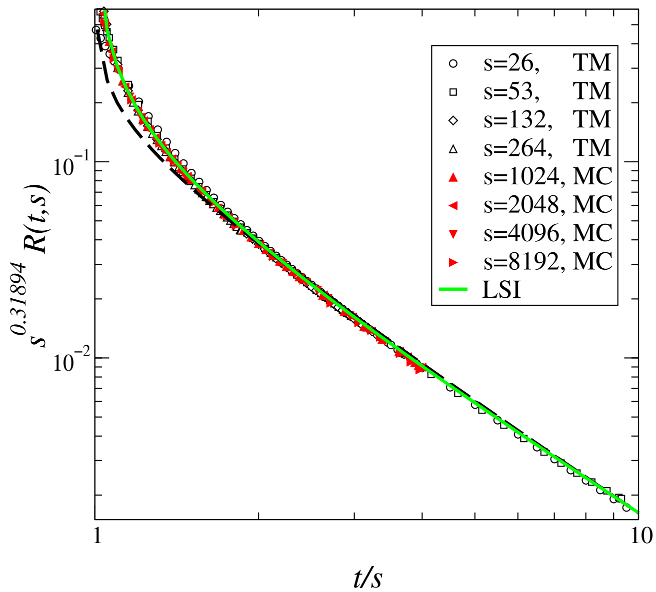

Figure 2. Autoresponse function for the critical $1D$ contact process for several waiting times s. The data labelled TM come from the transfer matrix renormalization group [21] and MC denotes Hinrichsen's Monte Carlo data [22]. The dashed line corresponds to the case $a' = a$ and the full curve gives the LSI prediction eq. (8) with the exponents as listed in table 1.

in the scaling regime, where s and $t - s$ are both large enough. Again, we observe an almost perfect agreement with eq. (8), provided $a' \neq a$.§

One the other hand, when one looks closer at the region where $t/s \lessapprox 1.1$, one does observe deviations of the data from (8) [22]. In trying to analyze this, recall that non-equilibrium *critical* dynamics is special in the sense that both the ageing regime (where $t - s \sim \mathrm{O}(s)$) and the quasistationary regime (where $t - s \ll s$) display dynamical scaling with the same length scale $L(t) \sim t^{1/z}$, where z is the equilibrium dynamical critical exponent. Hence one usually expects some crossover to occur. In terms of the response function, this might be formalized by writing $R = \mathcal{R}(s/\tau_*, (t-s)/\tau_*, s)$ where τ_* is some reference time scale such that, with $(t - s)/\tau_* = \mathrm{O}(1)$

$$\lim_{s\to\infty} R = \begin{cases} R_{\rm eq}(t-s) & ; \text{ for } s/\tau_* \to \infty \\ s^{-1-a} f_R(t/s) & ; \text{ for } s/\tau_* = \mathrm{O}(1) \end{cases}$$

Since in lattice calculations, s is always finite, the crossover can be illustrated by studying $Q := R(t,s)/R_{\rm eq}(t-s) \sim R(t,s)(t-s)^{1+a}$. As long as LSI still describes the data, one expects $Q \sim (y-1)^{a-a'}$ for $y = t/s \gtrapprox 1$ and deviations from it should signal the presumed crossover to the quasistationary regime. Instead we find for the critical contact process that for $y = t/s \lessapprox 1.1$, $Q(y) \sim (y-1)^{-0.15}$ and this

§ Hinrichsen quotes $\lambda_R/z \approx 1.75$ and $1 + a' \approx 0.59$ [22] in good agreement with our estimates. The contact process is the only known example where $a' - a > 0$, which might be related to the fact that $z < 2$ there.

behaviour continues at least down to $t/s-1 \approx 10^{-3}$. For smaller values of t/s, which correspond to $t-s=\mathrm{O}(1)$, strong finite-time effects occur and the change towards a quasistationary behaviour, where $Q(y)$ would become y-independent, could not be observed. In comparison, unpublished data for the $2D$ Ising model [22] show convergence towards $Q(y) \sim (y-1)^{0.187}$ as s increases before finite-time effects destroy scaling. We conclude that LSI does accurately describe the data as long as t/s is large enough such that the effects of the crossover are not yet notable. A quantitative analysis of data from the region $t/s \lessapprox 1.1$ would require a precise theory of the cross-over between the ageing regime and the region $t-s \ll s$ and/or the rôle of finite-time effects. In the absence of such a theory, much larger values of s would presumably be needed to really carry out a test of LSI for values of t/s closer to unity than it is possible with the data of figure 2.

Finally, we recall that studying the scaling behaviour of an alternating susceptibility gives yet another direct access to the exponent $a'-a$. This was applied to the critical $3D$ Ising spin glass [11], with a binary distribution of the couplings $J_{i,j}=\pm J$.

In summary, we have reconsidered the way how observables defined in non-equilibrium lattice models might be related to (quasi-)primary scaling operators of field-theory. Our result eq. (7) points to a so far overlooked subtlety which might be of relevance in the discussion of the functional form of non-equilibrium scaling functions, for example in ageing phenomena. The results on $R(t,s)$ as collected in table 1 of some models with non-equilibrium critical dynamics appear to be compatible with the predictions eqs. (8,9) of local scale-invariance, provided cross-over effects to non-ageing regimes are negligible. The multitude of examples in table 1 suggests that rather being a kind of exotic exception (a belief implicit in [9, 10, 11]), the case $a' \neq a$ might turn out to be the generic situation. Having seen that the same mechanism also explains the exact autocorrelator of the $1D$ Glauber-Ising model indicates that the correspondence (7) should be more than just a patching-up of data for the autoresponse function.

What does this mean for the existence of local scale-invariance in non-equilibrium dynamics ? In a few exactly solved systems (where the dynamical exponent $z=2$) we have found exact agreement and in several models as generic as kinetic Ising models or the contact process eqs. (8,9) describe the data very well for t/s not too small. On the other hand, field-theoretical studies of the critical $\mathrm{O}(n)$ model in both $4-\varepsilon$ dimensions [5] and in $2+\varepsilon$ dimensions [23], although they agree with LSI at the lowest orders in ε, continue to find discrepancies with either (8) or (9) at some higher order. However, the available field-theoretical results are still far from the numerical data.‖ But since we have shown that LSI reproduces the known exact results of both $R(t,s)$ and $C(t,s)$ of the $1D$ Ising model it might be too simplistic to argue that LSI could at best describe gaussian fluctuations. A better understanding of the dynamical symmetries of non-equilibrium critical dynamics remains a challenging problem.

‖ The second-order calculation in $4-\varepsilon$ dimensions for $n=1$ is a little closer to the numerical data than LSI with $a'=a$ [17].

Appendix. Two-time autocorrelations for $z = 2$

If the dynamical exponent $z = 2$, local scale-invariance reduces to Schrödinger-invariance. We have already described in the past [10] how two-time autocorrelation functions can be calculated in the case $\xi = 0$ and we now wish to extend that treatment to the more general correspondence (7). We consider a Langevin equation of the form $\partial_t \phi = -D\frac{\delta \mathcal{H}}{\delta \phi} - Dv(t)\phi + \eta$ where $\mathcal{H}$ is the hamiltonian, D the diffusion constant, the gaussian noise η has zero mean and variance $\langle \eta(t, \boldsymbol{r})\eta(s, \boldsymbol{r}')\rangle = 2DT\delta(t-s)\delta(\boldsymbol{r}-\boldsymbol{r}')$ and T is the bath temperature. The potential $v(t)$ acts as a Langrange multiplier which can be used to describe explicitly the breaking of time-translational invariance. Here we restrict to situations where

$$k(t) := \exp\left[-D\int_0^t \mathrm{d}u\, v(u)\right] \sim t^F \tag{A1}$$

Then is has been shown [10] that for systems at criticality

$$\begin{aligned} C(t,s) = \langle\phi(t)\phi(s)\rangle &= DT\int \mathrm{d}u\,\mathrm{d}\boldsymbol{R}\, \left\langle \phi(t,\boldsymbol{y})\phi(s,\boldsymbol{y})\widetilde{\phi}^2(u,\boldsymbol{r}+\boldsymbol{y})\right\rangle_0 \\ &= DT\int \mathrm{d}u\,\mathrm{d}\boldsymbol{R}\, \frac{k(t)k(s)}{k(u)^2}\mathcal{R}_0^{(3)}(t,s,u;\boldsymbol{R}) \end{aligned} \tag{A2}$$

where $\mathcal{R}_0^{(3)}$ is the well-known three-point response function for $v(t) = 0$ which can be found from its Schrödinger-covariance and reads [24]

$$\begin{aligned} \mathcal{R}_0^{(3)}(t,s,u;\boldsymbol{r}) &= \mathcal{R}_0^{(3)}(t,s,u)\exp\left[-\frac{\mathcal{M}}{2}\frac{t+s-2u}{(s-u)(t-u)}\boldsymbol{r}^2\right]\Psi\left(\frac{t-s}{(t-u)(s-u)}\boldsymbol{r}^2\right) \\ \mathcal{R}_0^{(3)}(t,s,u) &= \Theta(t-u)\Theta(s-u)(t-u)^{-\widetilde{x}_2}(s-u)^{-\widetilde{x}_2}(t-s)^{-x+\widetilde{x}_2} \end{aligned}$$

where Ψ is an undetermined scaling function and the causality conditions $t > u, s > u$ are noted explicitly. In writing this, we have dropped a term coming from the correlations in the initial state which merely produces finite-time corrections to the leading scaling behaviour, see [4, 5, 10].

We now generalize this to the primary scaling operators according to (7). The operator Φ has the scaling dimension $x + 2\xi$, the composite scaling operator $\widetilde{\Phi}^2$ has the scaling dimension $2\widetilde{x}_2 + 4\widetilde{\xi}_2$.¶ We then obtain for the physical autocorrelation function, up to normalization and with $t > s$

$$\begin{aligned} C(t,s) &= (ts)^{\xi}\int \mathrm{d}u\,\mathrm{d}\boldsymbol{R}\,\left\langle\Phi(t,\boldsymbol{y})\Phi(s,\boldsymbol{y})\widetilde{\Phi}^2(u,\boldsymbol{R}+\boldsymbol{y})\right\rangle_0 u^{2\widetilde{\xi}_2} \\ &= (ts)^{\xi}(t-s)^{-x-2\xi+\widetilde{x}_2+2\widetilde{\xi}_2-d/2}\int_0^s \mathrm{d}u\,\frac{k(t)k(s)}{k(u)^2}u^{2\widetilde{\xi}_2}\left[(t-u)(s-u)\right]^{-\widetilde{x}_2-2\widetilde{\xi}_2+d/2} \\ &\quad\times\int \mathrm{d}\boldsymbol{R}\,\exp\left[-\frac{\mathcal{M}}{2}\frac{t+s-2u}{t-s}\boldsymbol{R}^2\right]\Psi\left(\boldsymbol{R}^2\right) \end{aligned}$$

¶ For bosonic free fields, one would have $\widetilde{x}_2 = \widetilde{x}$ and $\widetilde{\xi}_2 = \widetilde{\xi}$.

$$
= s^{1+d/2-x-\widetilde{x}_2} \left(\frac{t}{s}\right)^{\xi+\digamma} \left(\frac{t}{s}-1\right)^{\widetilde{x}_2+2\widetilde{\xi}_2-x-2\xi-d/2}
$$

$$
\times \int_0^1 \mathrm{d}v\, v^{2\widetilde{\xi}_2-\digamma} \left[\left(\frac{t}{s}-v\right)\left(1-v\right)\right]^{d/2-\widetilde{x}_2-2\widetilde{\xi}_2} \mathbf{\Psi}\left(\frac{t/s+1-2v}{t/s-1}\right) \tag{A3}
$$

and where the function $\mathbf{\Psi}$ is defined by the integral over $\boldsymbol{R}$. By comparison with the standard scaling from for $C(t,s)$, we read off $b = x+\widetilde{x}_2-1-d/2$ and $\lambda_C = 2(x-\digamma)+2\xi$.+ Furthermore, since $1+a' = x+2\xi$, it turns out that the form of the scaling function $f_C(y)$ is described by just one more parameter $\mu := \xi + \widetilde{\xi}_2$ and we finally have

$$
C(t,s) = C_0 s^b \left(\frac{t}{s}\right)^{1+a'-\lambda_C/2} \left(\frac{t}{s}-1\right)^{b-2a'-1+2\mu}
$$

$$
\times \int_0^1 \mathrm{d}v\, v^{\lambda_C+2\mu-2-2a'} \left[\left(\frac{t}{s}-v\right)\left(1-v\right)\right]^{a'-b-2\mu} \mathbf{\Psi}\left(\frac{t/s+1-2v}{t/s-1}\right) \tag{A4}
$$

and we have also reintroduced a normalization constant C_0. This should hold for simple (non-glassy) magnets with $z=2$ and in situations where the initial correlations have no effect on the leading scaling behaviour; of course the scaling limit $s\to\infty$ and $t/s = y > 1$ fixed is understood.

As an illustration, we consider the $1D$ Glauber-Ising model. At $T=0$, the exact two-time autocorrelation function is [14]

$$
C(t,s) = \frac{2}{\pi}\arctan\left(\sqrt{\frac{2}{t/s-1}}\right) \tag{A5}
$$

This holds true not only for the usually considered short-ranged initial conditions but also for long-ranged initial spin-spin correlations $\langle\sigma_r(0)\sigma_0(0)\rangle \sim r^{-\nu}$ with $\nu > 0$ (for $\nu = 0$ an analogous result holds for the connected autocorrelator) [14]. In addition, the exponents a, a' and λ_R are independent of ν.

In previous work [10], we have already explained the form of the exact autoresponse function $R(t,s)$ in terms of the correspondence eq. (7) (see table 1) but we had to leave open the analogous question for $C(t,s)$. In order to account for (A5), we remark that for $t=s$, the autocorrelator should not be singular. This requires

$$
\mathbf{\Psi}(w) = w^{b-2a'-1+2\mu} \quad \text{for } w \gg 1 \tag{A6}
$$

The most simple way to realize this is to require that (A6) holds for all values of w. This kind of assumption was already seen to become exact in models described by an

+ A similar calculation for the autoresponse function gives, up to normalization,

$$
R(t,s) = s^{-(x+\widetilde{x})/2} (t/s)^{\xi+\digamma} (t/s-1)^{-x-2\xi}\, \delta_{x+2\xi,\widetilde{x}+2\widetilde{\xi}}
$$

which reproduces again eq. (8), hence $\lambda_R = 2(x-\digamma)+2\xi = \lambda_C$ as expected [4] for non-disordered systems without long-range initial correlations.

underlying bosonic free field-theory [10]. Recalling from table 1 that $b = a = 0$ and $\lambda_C = 1$ and assuming (A6) to hold for all w, we obtain

$$C(t,s) \approx C_0 \int_0^1 \mathrm{d}v\, v^{2\mu} \left[\left(\frac{t}{s} - v \right) \left(1 - v \right) \right]^{-2\mu - 1/2} \left(\frac{t}{s} + 1 - 2v \right)^{2\mu} \qquad \text{(A7)}$$

Because the exact result (A5) is independent of the initial correlations, the comparison with the expression (A7) derived from the thermal noise is justified. The exact Glauber-Ising result (A5) is recovered from (A7) for $\mu = -1/4$ and $C_0 = \sqrt{2}/\pi$.

This is the first example of a model with $a' \neq a$ where the scaling of *both* the autoresponse and of the autocorrelation functions can be explained in terms of LSI.

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[1] J.L. Cardy in E. Brézin and J. Zinn-Justin (eds), *Fields, strings and critical phenomena*, Les Houches XLIX, North Holland (Amsterdam 1990).
[2] J.M Drouffe and C. Itzykson, *Théorie statistique des champs*, Éditions CNRS (Paris 1988), vol. 1.
[3] I. Montvay and G. Münster, *Quantum fields on a lattice*, Cambridge University Press (1994).
[4] A.J. Bray, Adv. Phys. **43**, 357 (1994).
[5] P. Calabrese and A. Gambassi, J. Phys. A: Math. Gen. **38**, R181 (2005).
[6] M. Henkel, M. Pleimling and R. Sanctuary (eds), *Ageing and the glass transition*, Springer Lecture Notes in Physics (Springer, Heidelberg 2006).
[7] C. de Dominicis and L. Peliti, Phys. Rev. **B18**, 353 (1978).
[8] H.K. Janssen, in G. Györgyi et *al.* (eds) *From phase transitions to chaos*, World Scientific (Singapour 1992), p. 68
[9] M. Henkel, Nucl. Phys. **B641**, 405 (2002).
[10] A. Picone and M. Henkel, Nucl. Phys. **B688**, 217 (2004).
[11] M. Henkel and M. Pleimling, J. Phys. Cond. Matt. **17**, S1899 (2005).
[12] L. Berthier, J.-L. Barrat and J. Kurchan, Eur. Phys. J. **B11**, 635 (1999).
[13] G.F. Mazenko, Phys. Rev. **E69**, 016114 (2004).
[14] C. Godrèche and J.-M. Luck, J. Phys. A: Math. Gen. **33**, 9141 (2000); E. Lippiello and M. Zannetti, Phys. Rev. **E61**, 3369 (2000); M. Henkel and G.M. Schütz, J. Phys. A: Math. Gen. **37**, 591 (2004).
[15] P. Mayer, S. Léonard, L. Berthier, J.P. Garrahan and P. Sollich, Phys. Rev. Lett. **96**, 030602 (2006).
[16] P. Mayer, PhD thesis, King's college London (2004).
[17] M. Pleimling and A. Gambassi, Phys. Rev. **B71**, 180401(R) (2005).
[18] M. Henkel, M. Pleimling, C. Godrèche and J.-M. Luck, Phys. Rev. Lett. **87**, 265701 (2001).
[19] K. Oerding and F. van Wijland, J. Phys A: Math. Gen. **31**, 7011 (1998).
[20] J.J. Ramasco, M. Henkel, M.A. Santos and C.A. de Silva Santos, J. Phys. A: Math. Gen. **37**, 10497 (2004).
[21] T. Enss, M. Henkel, A. Picone and U. Schollwöck, J. Phys. A: Math Gen. **37**, 10479 (2004).

[22] H. Hinrichsen, `cond-mat/0603369` and `private communication`
[23] A.A. Fedorenko and S. Trimper, Europhys. Lett. **74**, 89 (2006).
[24] M. Henkel, J. Stat. Phys. **75**, 1023 (1994).