

The asymptotics of stationary electro-vacuum metrics in odd space-time dimensions

Robert Beig*
Institut für Theoretische Physik
Universität Wien

Piotr T. Chruściel†
LMPT, Fédération Denis Poisson
Tours

December 1, 2006

Abstract

We show that stationary, asymptotically flat solutions of the electro-vacuum Einstein equations are analytic at i^0 , for a large family of gauges, in odd space-time dimensions higher than seven. The same is true in space-time dimension five for static vacuum solutions with non-vanishing mass.

1 Introduction

There is currently interest in asymptotically flat solutions of the vacuum Einstein equations in higher dimensions [5, 9]. It is thus natural to enquire which part of our body of knowledge of $(3 + 1)$ -dimensional solutions carries over to higher dimensions. In this note we study that question for asymptotic expansions at spatial infinity of stationary or static electro-vacuum metrics. We prove analyticity at i^0 , up to a conformal factor, for a family of natural geometric gauges, in even dimensions $n \geq 6$. The same result is established in space-dimension $n = 4$ for static vacuum metrics with non-vanishing ADM mass.

2 Static vacuum metrics

We write the space-time metric in the form

$$ds^2 = -e^{2u} dt^2 + e^{-\frac{2u}{n-2}} \tilde{g}_{ij} dx^i dx^j ,$$

*E-mail robert.beig@univie.ac.at

†E-mail Piotr.Chrusciel@lmpt.univ-tours.fr, URL www.phys.univ-tours.fr/~piotr

where \tilde{g} is an asymptotically flat Riemannian metric, with $\partial_t u = \partial_t \tilde{g}_{ij} = 0$. The vacuum Einstein equations show that u is \tilde{g} -harmonic, with \tilde{g} satisfying the equation

$$\tilde{R}_{ij} = \frac{n-1}{n-2} \partial_i u \partial_j u, \quad (2.1)$$

where n is the space-dimension, and \tilde{R}_{ij} is the Ricci tensor of \tilde{g} . We assume $n \geq 3$ throughout.

It is a standard consequence of those equations that, in harmonic coordinates in the asymptotically flat region, and whatever $n \geq 3$, both u and \tilde{g}_{ij} have a full asymptotic expansion in terms of powers of $\ln r$ and inverse powers of r . Solutions without $\ln r$ terms are of special interest, because they can be used to construct smoothly compactifiable hyperboloidal initial surfaces. In even space-time dimension initial data sets containing such asymptotic regions, when close enough to Minkowskian data, lead to asymptotically simple spacetimes [1, 11]. It has been shown by Beig and Simon that logarithmic terms can always be gotten rid of by a change of coordinates when space-dimension equals three [4, 14].

From what has been said one can infer that the leading order corrections in the metric can be written in a Schwarzschild form, which in ‘‘isotropic’’ coordinates reads

$$\begin{aligned} g_m &= - \left(\frac{1 - m/2|x|^{n-2}}{1 + m/2|x|^{n-2}} \right)^2 dt^2 + \left(1 + \frac{m}{2|x|^{n-2}} \right)^{\frac{4}{n-2}} \left(\sum_{i=1}^n dx_i^2 \right) \\ &\approx - \left(1 - \frac{m}{\tilde{r}^{n-2}} \right)^2 dt^2 + \left(1 + \frac{m}{\tilde{r}^{n-2}} \right)^{\frac{2}{n-2}} \left(\sum_{i=1}^n dx_i^2 \right), \end{aligned} \quad (2.2)$$

where m is of course a constant, and $\tilde{r} = |x|$ is a radial coordinate in the asymptotically flat region. This gives the asymptotic expansion

$$u = -\frac{m}{\tilde{r}^{n-2}} + O(\tilde{r}^{-n+1}), \quad (2.3)$$

Further we have

$$\tilde{g}_{ij} = \delta_{ij} + O(\tilde{r}^{1-n}). \quad (2.4)$$

Equation (2.3) shows that for $m \neq 0$ the function

$$\omega := (u^2)^{\frac{1}{n-2}} \quad (2.5)$$

behaves asymptotically as \tilde{r}^{-2} , and can therefore be used as a conformal factor in the usual one-point compactification of the asymptotic region. Indeed, assuming that $m \neq 0$ and setting

$$g_{ij} := \omega^2 \tilde{g}_{ij}. \quad (2.6)$$

one obtains a $C^{n-2,1}$ metric¹ on the manifold obtained by adding a point (which we denote by i^0) to the asymptotically Euclidean region.

¹The differentiability class near i^0 can be established by examining Taylor expansions there.

From the fact that u is \tilde{g} -harmonic one finds

$$\Delta\omega = \mu , \quad (2.7)$$

where the auxiliary function μ is defined as

$$\mu := \frac{n}{2}\omega^{-1}g^{ij}\partial_i\omega\partial_j\omega . \quad (2.8)$$

Note that, in spite of the negative power of ω , this function can be extended by continuity to i^0 , the extended function, still denoted by μ , being of $C^{n-2,1}$ differentiability class.

Let L_{ij} be the Schouten tensor of g_{ij} ,

$$L_{ij} := \frac{1}{n-2}\left(R_{ij} - \frac{R}{2(n-1)}g_{ij}\right) . \quad (2.9)$$

Using tildes to denote the corresponding objects for the metric \tilde{g} , from (2.1) one obtains

$$\tilde{L}_{ij} = \frac{1}{4}\omega^{n-4}\left((n-1)\partial_i\omega\partial_j\omega - \frac{1}{2}g_{ij}g^{k\ell}\partial_k\omega\partial_\ell\omega\right) . \quad (2.10)$$

We see that for $n \geq 3$ the tensor \tilde{L}_{ij} is bounded on the one-point compactification at infinity, and for $n \geq 4$ it is as differentiable as $d\omega$ and the metric allow. This last property is not true anymore for $n = 3$, however the following objects are well behaved:

$$\tilde{L}_{ij}D^j\omega = \frac{2n-3}{4n}\omega^{n-3}\mu D_i\omega , \quad \tilde{L}_{i[j}D_{k]}\omega = -\frac{1}{4n}\omega^{n-3}\mu g_{i[j}D_{k]}\omega . \quad (2.11)$$

3 Conformal rescalings

We recall the well-known transformation law of the Schouten tensor under the conformal rescaling (2.6)

$$L_{ij} = \tilde{L}_{ij} + \omega^{-1}D_iD_j\omega - \frac{1}{2}\omega^{-2}g_{ij}g^{k\ell}\partial_k\omega\partial_\ell\omega ; \quad (3.1)$$

we emphasize that this holds whether or not ω is related to \tilde{L} as in (2.10). Taking a trace of (3.1) and using (2.7) one finds

$$R = \frac{(n-1)(n-2)}{2n}\omega^{n-3}\mu . \quad (3.2)$$

In our subsequent manipulations it is convenient to rewrite (3.1) as an equation for $D_iD_j\omega$,

$$D_jD_i\omega = \omega(L_{ij} - \tilde{L}_{ij}) + \frac{1}{n}\mu g_{ij} . \quad (3.3)$$

We note that the right-hand-side is well-behaved at $\omega = 0$ for all $n \geq 3$.

Let C_{ijk} denote the Cotton tensor,

$$C_{ijk} := D_kL_{ij} - D_jL_{ik} , \quad (3.4)$$

and let C_{ijkl} be the Weyl tensor of g . Note the identity

$$D^i C_{ijkl} = (3 - n)C_{jkl} . \quad (3.5)$$

Applying D_k to (3.3) and anti-symmetrising over j and k one obtains

$$\omega C_{ijk} + C_{kji\ell} D^\ell \omega = \tilde{L}_{ijk} , \quad (3.6)$$

where

$$\tilde{L}_{ijk} := 2D_{[k}(\omega \tilde{L}_{j]i}) + 2g_{i[j} \tilde{L}_{k]\ell} D^\ell \omega . \quad (3.7)$$

Writing down the second term in (3.7) and using (2.11) and again (3.3), we find that the terms with ω^{n-4} drop out and there results

$$\tilde{L}_{ijk} = \frac{1}{2} \omega^{n-2} \left(\underbrace{-(n-1)D_{[j} \omega (L_{k]i} - \tilde{L}_{k]i}) + g_{i[j} (L_{k]\ell} - \tilde{L}_{k]\ell}) D^\ell \omega}_{\text{}} \right) . \quad (3.8)$$

Here \tilde{L}_{ij} should be expressed in terms of ω , $d\omega$ and μ using (2.11). It should be emphasized that the underbraced expression is regular at $\omega = 0$.

Let B_{ij} denote the Bach tensor,

$$B_{ij} := D^k C_{ijk} - L^{k\ell} C_{kjil} . \quad (3.9)$$

Applying D^k to (3.6) and using (3.5) and (3.3) one obtains

$$B_{ij} - (n-4)\omega^{-1} C_{ijk} D^k \omega = \omega^{-1} D^k \tilde{L}_{ijk} - C_{kji\ell} \tilde{L}^{k\ell} . \quad (3.10)$$

Note that the factor ω^{-1} in front of the divergence $D^k \tilde{L}_{ijk}$ is compensated by ω^{n-2} in (3.7), so that for $n \geq 4$ the right-hand-side is a well-behaved function of the metric, of ω , and of their derivatives at zeros of ω . Alternatively we can, using (3.6), rewrite (3.10) as

$$B_{ij} + (n-4)\omega^{-2} C_{kijl} D^k \omega D^l \omega = \omega^{3-n} D^k (\omega^{n-4} \tilde{L}_{ijk}) - C_{kijl} \tilde{L}^{kl} . \quad (3.11)$$

Note that the right-hand-side of (3.11) is regular also for $n = 3$. Recall, now, the identity

$$\begin{aligned} B_{ij} &= \Delta L_{ij} - D_i D_j (\text{tr} L) + \mathcal{F}_{ij} \\ &= \frac{1}{n-2} \Delta R_{ij} - \frac{1}{2(n-1)} \left(\frac{1}{n-2} \Delta R g_{ij} + D_i D_j R \right) + \mathcal{F}_{ij} , \end{aligned} \quad (3.12)$$

where \mathcal{F}_{ij} depends upon the metric and its derivatives up to order two. We eliminate the Ricci scalar terms using (3.2). The terms involving derivatives of R will introduce derivatives of μ , which can be handled as follows. Differentiating (2.8) and using (3.3) one obtains

$$\begin{aligned} D_i \mu &= -n(L_{ij} - \tilde{L}_{ij}) D^j \omega \\ &= -n L_{ij} D^j \omega + \frac{2n-3}{4} \omega^{n-3} \mu D_i \omega , \end{aligned} \quad (3.13)$$

which allows us to eliminate each derivative of μ in terms of μ , ω and $d\omega$.

3.1 Space-dimensions three and four

In dimension three the term involving $\omega^{-2}C_{kjil}L^{k\ell}$ on the left-hand-side of (3.11) goes away because the Weyl tensor vanishes. In dimension four its coefficient vanishes. In those dimensions one therefore ends up with an equation of the form

$$\Delta R_{ij} = F_{ij}(n, \omega, d\omega, \partial^2\omega, g, \partial g, \partial^2g) . \quad (3.14)$$

with a tensor field F_{ij} which is well behaved at $\omega = 0$. Here we have used the expression of μ as a function of the metric, ∂g , $\partial\omega$ and $\partial^2\omega$ which follows from (2.7).

We can calculate the laplacian of μ by taking a divergence of (3.13) and eliminating again the second derivatives of ω in terms of μ , and the first derivatives of μ , as before. This leads to a fourth-order equation for ω of the form

$$\Delta^2\omega = F(n, \omega, d\omega, \partial^2\omega, g, \partial g, \text{Ric}) , \quad (3.15)$$

with F — well behaved at $\omega = 0$, where Ric stands for the Ricci tensor. Note that one should use the Bianchi identities to eliminate the term involving the divergence of L_{ij} which arises in the process:

$$D^j L_{ij} = \frac{1}{2(n-1)} D_i R .$$

In harmonic coordinates, Equations (3.14)-(3.15) can be viewed as a system of equations of fourth order for the metric g and the function ω , with diagonal principal part Δ^2 . The system is elliptic so that usual bootstrap arguments show smoothness of all fields. In fact the solutions are real-analytic by [13], as we wished to show.

3.2 Higher even dimensions

A natural generalisation of the Bach tensor in even dimensions $n \geq 6$ is the obstruction tensor \mathcal{O}_{ij} of Fefferman and Graham [10, 12]. It is of the form

$$\mathcal{O}_{ij} = \Delta^{\frac{n-4}{2}} [\Delta L_{ij} - D_i D_j(\text{tr}L)] + \mathcal{F}_{ij}^n , \quad (3.16)$$

where \mathcal{F}_{ij}^n is a tensor constructed out of the metric and its derivatives up to order $n - 2$. This leads us to expect that further differentiations of the equations above leads to a regular expression for $\Delta^{\frac{n-2}{2}} B_{ij}$ in terms of ω and its derivatives up to order $n - 3$. However, we have not been able to conclude using this approach. Instead, we proceed as in [6]:

In coordinates x^i which are harmonic with respect to the metric \tilde{g} , (2.1) and the harmonicity condition for u lead to a set of equations for u and

$$f := (\tilde{g}_{ij} - \delta_{ij})$$

of the form

$$\tilde{g}^{ij} \partial_i \partial_j f = F(f)(\partial f)^2 + (\partial u)^2 , \quad \tilde{g}^{ij} \partial_i \partial_j u = 0 .$$

Setting

$$\Omega = \frac{1}{r^2}, \quad \tilde{f} = \Omega^{-\frac{n-2}{2}} f, \quad \tilde{u} = \Omega^{-\frac{n-2}{2}} u, \quad y^i = \frac{x^i}{r^2},$$

one obtains a set of regular elliptic equations in the coordinates y^i after a conformal rescaling $\delta_{ij} \rightarrow \Omega^2 \delta_{ij}$ of the flat metric, provided that $n \geq 6$. The reader is referred to [6] for a detailed calculation in a Lorentzian setting, which carries over with minor modifications (due to the quadratic rather than linear zero of Ω) to the current situation; note that n in the calculations there should be replaced by $n-1$ for the calculations at hand. We further note that the leading order behavior of \tilde{f} is governed by the mass, which can be made arbitrarily small by a constant rescaling of the metric and of the original harmonic coordinates x^i ; this freedom can be made use of to ensure ellipticity of the resulting equations. Finally we emphasise that this result, contrary to the one for n equal three or four, does not require the non-vanishing of mass.

4 Stationary vacuum solutions

We consider Lorentzian metrics ${}^{n+1}g$ in odd space-time-dimension $n+1 \geq 7$, with Killing vector $X = \partial/\partial t$. In adapted coordinates those metrics can be written as

$${}^{n+1}g = -V^2(dt + \underbrace{\theta_i dx^i}_{=\theta})^2 + \underbrace{g_{ij} dx^i dx^j}_{=g}, \quad (4.1)$$

$$\partial_t V = \partial_t \theta = \partial_t g = 0. \quad (4.2)$$

The vacuum Einstein equations (with vanishing cosmological constant) read (see, e.g., [8])

$$\begin{cases} V \nabla^* \nabla V = \frac{1}{4} |\lambda|_g^2, \\ \text{Ric}(g) - V^{-1} \text{Hess}_g V = \frac{1}{2V^2} \lambda \circ \lambda, \\ \text{div}(V\lambda) = 0, \end{cases} \quad (4.3)$$

where

$$\lambda_{ij} = -V^2(\partial_i \theta_j - \partial_j \theta_i), \quad (\lambda \circ \lambda)_{ij} = \lambda_i^k \lambda_{kj}.$$

We consider metrics satisfying, for some $\alpha > 0$,

$$g_{ij} - \delta_{ij} = O(r^{-\alpha}), \quad \partial_k g_{ij} = O(r^{-\alpha-1}), \quad V = O(r^{-\alpha}), \quad \partial_k V = O(r^{-\alpha-1}). \quad (4.4)$$

As is well known [2], one can then introduce new coordinates, compatible with the above fall-off requirements, which are harmonic for g .

Next, a redefinition $t \rightarrow t + \psi$, introduces a gauge transformation

$$\theta \rightarrow \theta + d\psi,$$

and one can exploit this freedom to impose restrictions on θ . We will assume a condition of the form

$$g^{ij} \partial_i \theta_j = \underbrace{Q(g, V)}_p; \underbrace{\partial g, \partial V, \theta}_q, \quad (4.5)$$

where Q is a smooth function of the variables listed near $(\delta, 1; 0, 0, 0)$, with a zero of order two or higher *with respect to* q :

$$Q(p; 0) = \partial_q Q(p; 0) = 0 .$$

Examples include the *harmonic gauge*, $\square_{n+1} g t = 0$, which reads

$$\partial_i (\sqrt{\det g} V g^{ij} \theta_j) = 0 , \quad (4.6)$$

as well as the maximal gauge,

$$\partial_i \left(\frac{V^3 \sqrt{\det g} g^{ij}}{\sqrt{1 - V^2 g^{k\ell} \theta_k \theta_\ell}} \theta_j \right) = 0 . \quad (4.7)$$

Equation (4.6) can always be achieved by solving a linear equation for ψ , *cf.*, *e.g.*, [2, 7] for the relevant isomorphism theorems. On the other hand, (4.7) can always be solved outside of some large ball [3]. More generally, when non-linear in θ , equation (4.5) can typically be solved outside of some large ball using the implicit function theorem in weighted Hölder or weighted Sobolev spaces.

In harmonic coordinates, and in a gauge (4.5), the system (4.3) is elliptic and, similarly to the static case, standard asymptotic considerations show that g_{ij} is Schwarzschild in the leading order, and that there exist constants α_{ij} such that

$$\theta_i = \frac{\alpha_{ij} x^j}{r^n} + O(r^{-n}) .$$

To prove analyticity at i^0 one proceeds as in Section 3.2: thus, one first rewrites the second of equations (4.3) as an equation for

$$\tilde{g}_{ij} := e^{\frac{2u}{n-2}} g_{ij} \equiv V^{\frac{2}{n-2}} g_{ij} ,$$

which gets rid of the Hessian of V there. It should then be clear that, in coordinates which are harmonic for \tilde{g} , the first two equations in (4.3) have the right structure for the argument of Section 3.2. It remains to check the third one. For this we note that, in \tilde{g} -harmonic coordinates so that $\partial_i (\sqrt{\det \tilde{g}} \tilde{g}^{ij}) = 0$,

$$\begin{aligned} \operatorname{div}(V\lambda)_k &= \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g} V^3 g^{ij} (\partial_j \theta_k - \partial_k \theta_j) \right) \\ &= \frac{V^{\frac{n}{n-2}}}{\sqrt{\det \tilde{g}}} \partial_i \left(\sqrt{\det \tilde{g}} V^2 \tilde{g}^{ij} (\partial_j \theta_k - \partial_k \theta_j) \right) \\ &= V^{\frac{n}{n-2}} \tilde{g}^{ij} \partial_i \left(V^2 (\partial_j \theta_k - \partial_k \theta_j) \right) \\ &= V^{\frac{n}{n-2}} \left(\tilde{g}^{ij} \partial_i \partial_j \theta_k + 2V \tilde{g}^{ij} \partial_i V (\partial_j \theta_k - \partial_k \theta_j) \right. \\ &\quad \left. - \underbrace{\tilde{g}^{ij} \partial_i \partial_k \theta_j}_{=V^{-\frac{2}{n-2}} (\partial_k (g^{ij} \partial_i \theta_j) + \partial_k g^{ij} \partial_i \theta_j)} \right) . \end{aligned}$$

If Q in (4.5) is zero, then the vanishing of $\operatorname{div}(V\lambda)$ immediately gives an equation of the right form for θ . Otherwise, ∂Q leads to nonlinear terms of the form $\partial_x^2 g \theta$, *etc.*, which are again of the right form, see the calculations in [6]. Note that such terms do not affect the ellipticity of the equations because of their off-diagonal character.

5 Einstein-Maxwell equations

The above considerations immediately generalise to the stationary Einstein-Maxwell equations, with a Killing vector which approaches a time-translation in the asymptotically flat region. Indeed, the calculations of Section 4 carry over to this setting, as follows:

Stationary Maxwell fields can be described by a time-independent scalar field $\varphi = A_0$ and a vector potential $A = A_i dx^i$, again time-independent. Here one needs to assume that, in addition to (4.4), one has

$$A_\mu = O(r^{-\alpha}), \quad \partial_k A_\mu = O(r^{-\alpha-1}).$$

Maxwell fields lead to supplementary source terms in the right-hand-sides of (4.3) which are quadratic in the first derivatives of φ and A , hence of the right form for the argument so far. Next, if we write the Maxwell equations as

$$\frac{1}{\sqrt{^{n+1}g}} \partial_\mu \left(\sqrt{^{n+1}g} \, ^{n+1}g^{\mu\rho} \, ^{n+1}g^{\nu\sigma} \partial_{[\nu} A_{\sigma]} \right) = 0,$$

and impose the Lorenz gauge,

$$\frac{1}{\sqrt{^{n+1}g}} \partial_\mu \left(\sqrt{^{n+1}g} \, ^{n+1}g^{\mu\nu} A_\nu \right) = 0,$$

the equations $\partial_t A_\mu = 0$ allow one to rewrite the above as

$$g^{ij} \partial_i \partial_j a = H(f, V, \theta; \partial f, \partial V, \partial \theta; \partial a),$$

where $a = (\varphi, A_i)$, with a function H which is bilinear in the second and third groups of arguments. This is again of the right form, which finishes the proof of analyticity of $\tilde{f}, \tilde{\varphi}, \tilde{A}$ and $\tilde{\theta}$ at i_0 for even $n \geq 6$, where the original fields are related to the tilde-ones via a rescaling by $\Omega^{\frac{n-2}{2}}$, e.g. $\varphi = \Omega^{\frac{n-2}{2}} \tilde{\varphi}$, and so on.

References

- [1] M.T. Anderson and P.T. Chruściel, *Asymptotically simple solutions of the vacuum Einstein equations in even dimensions*, Commun. Math. Phys. **260** (2005), 557–577, gr-qc/0412020. MR MR2183957
- [2] R. Bartnik, *The mass of an asymptotically flat manifold*, Commun. Pure and Appl. Math. **39** (1986), 661–693.
- [3] R. Bartnik, P.T. Chruściel, and N. Ó Murchadha, *On maximal surfaces in asymptotically flat space-times.*, Commun. Math. Phys. **130** (1990), 95–109.
- [4] R. Beig and W. Simon, *Proof of a multipole conjecture due to Geroch*, Commun. Math. Phys. **78** (1980), 75–82.
- [5] P. Bizoń, T. Chmaj, and B.G. Schmidt, *Critical behavior in vacuum gravitational collapse in 4+1 dimensions*, Phys. Rev. Lett. **95** (2005), 071102, gr-qc/0506074.

- [6] Y. Choquet-Bruhat, P.T. Chruściel, and J. Loizelet, *Global solutions of the Einstein-Maxwell equations in higher dimension*, *Class. Quantum Grav.* (2006), 7383–7394.
- [7] D. Christodoulou and Y. Choquet-Bruhat, *Elliptic systems in $H_{s,\delta}$ spaces on manifolds which are Euclidean at infinity*, *Acta. Math.* **146** (1981), 129–150.
- [8] R. Coquereaux and A. Jadczyk, *Riemannian geometry, fiber bundles, Kaluza-Klein theories and all that*, *World Sci. Lect. Notes Phys.* **16** (1988), 1–345.
- [9] R. Emparan and H.S. Reall, *Black rings*, (2006), hep-th/0608012.
- [10] C. Fefferman and C.R. Graham, *Conformal invariants*, *Élie Cartan et les mathématiques d’aujourd’hui*, *The mathematical heritage of Élie Cartan*, *Sémin. Lyon 1984*, *Astérisque*, No.Hors Sér. 1985, 95–116.
- [11] H. Friedrich, *On the existence of n -geodesically complete or future complete solutions of Einstein’s field equations with smooth asymptotic structure*, *Commun. Math. Phys.* **107** (1986), 587–609.
- [12] C.R. Graham and K. Hirachi, *The ambient obstruction tensor and Q -curvature*, *AdS/CFT correspondence: Einstein metrics and their conformal boundaries*, *IRMA Lect. Math. Theor. Phys.*, vol. 8, *Eur. Math. Soc.*, Zürich, 2005, pp. 59–71.
- [13] C.B. Morrey, *Multiple integrals in the calculus of variation*, *Springer Verlag*, Berlin, Heidelberg, New York, 1966.
- [14] W. Simon and R. Beig, *The multipole structure of stationary space-times*, *Jour. Math. Phys.* **24** (1983), 1163–1171.