The exact evaluation of the corner-to-corner resistance of an $M \times N$ resistor network: Asymptotic expansion

J. W. Essam
Department of Mathematics
Royal Holloway College, University of London
Egham, Surrey TW20 0EX, England

F. Y. Wu
Department of Physics
Northeastern University
Boston, Massachusetts 02115, U.S.A.

PACS numbers: 01.55+b, 02.10.Yn
Key words: resistance, electrical networks, asymptotics, square lattice.

Abstract

We study the corner-to-corner resistance of an $M \times N$ resistor network with resistors r and s in the two spatial directions, and obtain an asymptotic expansion of its exact expression for large M and N. The resulting asymptotic expansion reproduces numerical results obtained from a Neville table determination in the case of $M = N$ and $r = s$.

1 Introduction

A classic problem in the theory of electric circuits is the computation of the resistance between two nodes in a resistor network. Formulated by Kirchhoff [1] more than 160 years ago, the problem has been studied by numerous authors over many years (see, for example, [2, 3]). Kirchhoff explored the graph-theoretical aspect of the algebraic formulation and obtained the two-point resistance in terms of 2-rooted spanning forests and spanning trees. But the formulation, while elegant, does not provide sufficient physical insights. Past studies have instead focused on infinite networks for which analysis can be carried to fruition [4].

The computation of the asymptotic expansion of the corner-to-corner resistance of a rectangular resistor network has been of interest for some time, as its value provides a lower bound to the resistance of compact percolation clusters in the Domany-Kinzel model of a directed percolation [5]. The corner-to-corner resistance has been studied by one of us (JWE) numerically using the method of a differential approximants [6] together with a Neville table analysis [7].

Recently, one of us (FYW) has re-visited the two-point resistance problem [8], and deduced a closed-form expression for the resistance between arbitrary two nodes for finite networks. However, the exact expression obtained in [8] is in the form of a double summation whose mathematical and physical contents are not immediately apparent. In this paper, we take a closer look at this summation formula and obtain its asymptotic expansion for large lattices.

The organization of this paper is as follows: In Sec. 2 we recall the expression of the corner-to-corner resistance in an $M \times N$ resistor network obtained in [8], and reduce it to a form more manageable for our purposes. The dominant term in the exact expression of the resistance is next deduced in Sec. 3 by using the Euler-Maclaurin summation formula. The asymptotic expansion of the exact expression for large M, N is obtained in Sec. 4 and we summarize the results in Sec. 5. We also show that the exact expression of the asymptotic expansion yields results in agreement with those determined numerically [7].

2 Formulation of the summation formula

Consider a rectangular $M \times N$ network of resistors with resistances r and s on edges of the network in the respective horizontal and vertical directions. For definiteness, we consider both M, N even, and expect the asymptotic expansion to be independent of this choice. The example of an $M = 6, N = 4$ network is shown in Fig. 1.

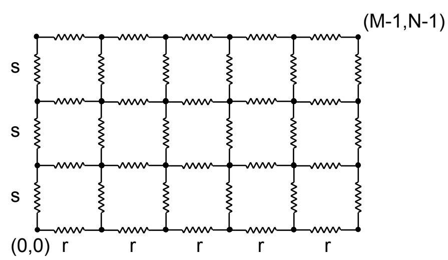

Figure 1: *An $M \times N$ resistor network.*

Using Eq. (37) of [8], the resistance between opposite corner nodes $(0, 0)$ and $(M - 1, N - 1)$ of the network is

$$R_{\{M\times N\}}(r,s) = \frac{r(M-1)}{N} + \frac{s(N-1)}{M} + \frac{2}{MN}\sum_{m=1}^{M-1}\sum_{n=1}^{N-1} \frac{\Big[\cos\left(\frac{1}{2}\theta_m\right)\cos\left(\frac{1}{2}\phi_n\right) - \cos\left(M-\frac{1}{2}\right)\theta_m \cos\left(N-\frac{1}{2}\right)\phi_n\Big]^2}{r^{-1}(1-\cos\theta_m) + s^{-1}(1-\cos\phi_n)} \qquad (1)$$

where $\theta_m = m\pi/M, \phi_n = n\pi/N$. Re-arranging the numerator in the summand, (1) becomes

$$R_{M\times N}(r,s) = \frac{r(M-1)}{N} + \frac{s(N-1)}{M}$$
$$+ \frac{8}{MN}\sum_{m=1}^{M-1}\sum_{n=1\,(m+n\,\text{odd})}^{N-1} \frac{\cos^2(\theta_m/2)\cos^2(\phi_n/2)}{r^{-1}(1-\cos\theta_m)+s^{-1}(1-\cos\phi_n)} \qquad (2)$$

There are two possibilities for the restriction $m+n=$ odd to hold, namely,

$$m = 2p-1, n = 2q, \quad p = 1,2,..,M/2,\ q = 1,2,...,N/2,$$
$$n = 2p-1, m = 2q, \quad p = 1,2,..,N/2,\ q = 1,2,...,M/2.$$

Splitting the sum into two parts accordingly and introducing the notation

$$A_q = \frac{q\pi}{N}, \quad B_p = \left(p-\frac{1}{2}\right)\frac{\pi}{M},$$

we obtain

$$R_{M\times N}(r,s) = (rs)^{\frac{1}{2}}[R_{M\times N}(r/s) + R_{N\times M}(s/r)] \qquad (3)$$

where

$$R_{M\times N}(\rho) = \frac{\sqrt{\rho}(M-1)}{N}$$
$$+ \frac{4\sqrt{\rho}}{MN}\sum_{p=1}^{M/2}\sum_{q=1}^{N/2}\left[\frac{\cos^2 A_q(1+\rho\sin^2 A_q)}{\rho\sin^2 A_q + \sin^2 B_p} - \cos^2 A_q\right]. \qquad (4)$$

Sums of the term $\cos^2 A_q$ can be carried out using the identity

$$\sum_{q=1}^{N/2}\cos^2\left(\frac{q\pi}{N}\right) = \frac{N}{4} - \frac{1}{2}. \qquad (5)$$

This yields

$$R_{M\times N}(\rho) = \sqrt{\rho}\left(\frac{M}{N} - \frac{1}{2}\right) + S_{M\times N}(\rho)$$

and

$$R_{M\times N}(r,s) = \sqrt{rs}\Big[\sqrt{\rho}\left(\frac{M}{N}-\frac{1}{2}\right) + \frac{1}{\sqrt{\rho}}\left(\frac{N}{M}-\frac{1}{2}\right)$$
$$+ S_{M\times N}(\rho) + S_{N\times M}(1/\rho)\Big] \qquad (6)$$

where

$$S_{M\times N}(\rho) = \frac{4\sqrt{\rho}}{N}\sum_{q=1}^{N/2}(\cos^2 A_q)(1+\rho\sin^2 A_q)\, S_{q,M,N}(\rho) \qquad (7)$$

with

$$S_{q,M,N}(\rho) = \frac{1}{M}\sum_{p=1}^{M/2}\left[\rho\sin^2 A_q + \sin^2 B_p\right]^{-1}$$
$$= \frac{1}{M}\sum_{k=0}^{(M/2)-1}\left[\rho\sin^2 A_q + \sin^2\left(\frac{(k+\frac{1}{2})\pi}{M}\right)\right]^{-1}. \qquad (8)$$

The summation (8) can be evaluated in a closed form (see Lemma (29) in Sec. 4). However, for a better understanding it is useful to sort out the dominant contribution in $S_{q,M,N}(\rho)$ by using the Euler-Maclaurin summation formula.

3 Evaluation of the summation by an integral

The dominant contribution of (8) can be obtained using the Euler-Maclaurin sum formula ([9] equation 5.8.18)

$$\sum_{k=0}^{r-1} g_{k+\frac{1}{2}} = \frac{1}{h}\int_{x_0}^{x_r} g(x)dx - \sum_{i=1}^{m}\frac{(1-2^{1-2i})B_{2i}h^{2i-1}}{(2i)!}\left[g^{(2i-1)}(x_r) - g^{(2i-1)}(x_0)\right] + E_m(\xi_m) \qquad (9)$$

where $g(x)$ is such that $g_i = g(x_0 + ih)$, the integer r is finite, and

$$E_m(\xi_m) = -r\frac{(1-2^{-1-2m})B_{2m+2}h^{2m+2}}{(2m+2)!}g^{(2m+2)}(\xi_m), \quad x_0 < \xi_m < x_r,$$

where B_{2m} are Bernoulli numbers.

Using (9) with $x_0 = 0$, $x_r = \pi/2$, $h = \pi/M$, $r = M/2$,

$$g(x) = \frac{1}{\rho\sin^2 A_q + \sin^2 x}, \qquad (10)$$

and noting that the odd derivatives vanish at the endpoints, we obtain

$$S_{q,M,N}(\rho) = \frac{1}{\pi}\int_0^{\pi/2} g(x)dx + E_m(q,N,\xi_m)$$
$$= \frac{1}{2\sqrt{\rho}\sin A_q\sqrt{1+\rho\sin^2 A_q}} + E_m(q,N,\xi_m), \qquad (11)$$

indicating that the dominant term in $S_{q,M,N}(\rho)$ is the first term in (11). The error term $E_m(q,N,\xi_m)$ can be written as

$$E_m(q,N,\xi_m) = (-1)^{m+1}\left(1 - \frac{1}{2^{2m+1}}\right)\frac{\zeta(2m+2)}{(2M)^{2m+1}}g^{(2m+2)}(q,\xi_m),$$
$$0 < \xi_m < \pi/2,$$

where $\zeta(2m+2)$ is the Riemann zeta function bounded in the above by $\zeta(2)=\pi^2/6$, and we have used $B_{2n}=(-1)^{n-1}2(2n)!\zeta(2n)/(2\pi)^{2n}$ ([10] equation 9.616). Since the denominator of (10) can be very small for small q, $E_m(q,N,\xi_m)$ does not necessarily vanish in the limit of $m\to\infty$.

Write the correction to the dominant contribution in $S_{M\times N}(\rho)$ as

$$\Delta_{M,N}(\rho)=S_{M\times N}(\rho)-S_N^{(1)}(\rho), \tag{12}$$

where

$$S_N^{(1)}(\rho)=\frac{2}{N}\sum_{q=1}^{N/2}\frac{\cos^2 A_q}{\sin A_q}\sqrt{1+\rho\sin^2 A_q} \tag{13}$$

is the dominant term depending only on N. Further write

$$\Delta_{M,N}(\rho)=\sum_{q=1}^{N/2}\Delta_{q,M,N}(\rho)\ , \tag{14}$$

with

$$\Delta_{q,M,N}(\rho)=D_{q,M,N}^{(1)}(\rho)-D_{q,N}^{(2)}(\rho), \tag{15}$$

$$D_{q,M,N}^{(1)}(\rho)=\frac{4\sqrt{\rho}}{MN}\sum_{k=0}^{(M/2)-1}\frac{\cos^2 A_q(1+\rho\sin^2 A_q)}{\rho\sin^2 A_q+\sin^2[(k+\frac{1}{2})\frac{\pi}{M}]} \tag{16}$$

$$D_{q,N}^{(2)}(\rho)=\frac{2}{N}\cdot\frac{\cos^2 A_q\sqrt{1+\rho\sin^2 A_q}}{\sin A_q}. \tag{17}$$

Numerical evaluation of the difference $\Delta_{q,M,N}(1)$ using (15) for $M=N$ and small values of q shows that it initially decreases with N but ultimately shows a rapid increase. For $q=1$ the turning point is $N=6$ and for $q=2$ it is $N=12$. However $\Delta_{q,M,N}(\rho)$ for fixed N decreases exponentially with increasing q, a fact which will be seen to hold for general M and N later (see Eq. (42) below). The sum in (14) therefore converges rapidly.

The asymptotic form of $S_N^{(1)}(\rho)$ is now deduced using the alternate Euler-Maclaurin sum formula ([9] equation 5.8.13)

$$\sum_{p=1}^{r}f_p=\frac{1}{h}\int_{x_0}^{x_r}f(x)dx+\frac{1}{2}\big[f(x_r)-f(x_0)\big]$$
$$+\sum_{i=1}^{m}\frac{B_{2i}h^{2i-1}}{(2i)!}\Big[f^{(2i-1)}(x_r)-f^{(2i-1)}(x_0)\Big]+E_m(\eta_m) \tag{18}$$

where r is finite and the error term is given by

$$E_m(\eta_m)=r\frac{B_{2m+2}h^{2m+2}}{(2m+2)!}f^{(2m+2)}(\eta_m),\qquad x_0<\eta_m<x_r. \tag{19}$$

But the direct application of (18) to effect the summation in (13) leads to a divergent integral so we add and subtract $1/A_q$ to the summand

and use (18) with $f(x)$ given by

$$f(x) \equiv f_\rho(x) = \frac{\cos^2 x}{\sin x}\sqrt{1+\rho\sin^2 x} - \frac{1}{x}. \tag{20}$$

Using $x_0 = 0, x_r = \pi/2, h = \pi/N, r = N/2$ and since $f_\rho(x)$ does not diverge at small x, the error term E_m is of the order of $O(N^{-(2m+1)})$ and can be neglected in $m \to \infty$. Denoting by $U_N(\rho)$ and $L_N(\rho)$ the respective correction to the integral at the upper an lower limits, we obtain

$$S_N^{(1)}(\rho) = I(\rho) + S_N + U_N(\rho) + L_N(\rho)\,, \tag{21}$$

where $I(\rho)$ is the integral

$$\begin{aligned} I(\rho) &= \frac{2}{\pi}\int_0^{\pi/2} f_\rho(x)dx \\ &= \frac{1}{\pi}\Big[-1 + 4\log 2 - 2\log\pi - \log(1+\rho) + \frac{\rho-1}{\sqrt{\rho}}\tan^{-1}\sqrt{\rho}\Big]. \end{aligned} \tag{22}$$

The second term in (21) is the added summation S_N, which can be evaluated using the result ([9] chapter 5, problem 26) as

$$S_N = \frac{2}{N}\sum_{q=1}^{N/2}\frac{1}{A_q} = \frac{2}{\pi}\sum_{q=1}^{N/2}\frac{1}{q} = \frac{2}{\pi}\Big(\log\frac{N}{2} + \gamma + \frac{1}{N} - \sum_{m=1}^{\infty}\frac{4^m B_{2m}}{2mN^{2m}}\Big), \tag{23}$$

where $\gamma = 0.577\,215\,664\,901\,53\ldots$ is Euler's constant.

The first part of $f_\rho(x)$ is antisymmetric about $\pi/2$ so the odd derivatives at the upper limit arise entirely from the $-1/x$ term and is independent of ρ. Hence for j odd $f_\rho^{(j)}(\pi/2) = (-1)^{j+1}j!(2/\pi)^{j+1}$ and the correction to the integral from the upper limit is

$$U_N(\rho) = \frac{1}{N}f\Big(\frac{\pi}{2}\Big) + \frac{2}{N}\sum_{i=1}^{m}\frac{B_{2i}h^{2i-1}}{(2i)!}f_\rho^{(2i-1)}\Big(\frac{\pi}{2}\Big) = \frac{-2}{\pi N} + \frac{2}{\pi}\sum_{i=1}^{m}\frac{4^i B_{2i}}{2iN^{2i}} \tag{24}$$

which, as $m \to \infty$, cancels terms of the inverse powers of N in S_N.

At the lower limit we have $f_\rho(0) = 0$ and

$$L_N(\rho) = -\frac{2}{\pi}\sum_{i=1}^{m}\frac{B_{2i}}{(2i)!}\Big(\frac{\pi}{N}\Big)^{2i} f_\rho^{(2i-1)}(0). \tag{25}$$

Using Bernoulli numbers $B_2 = 1/6, B_4 = -1/30, B_6 = 1/42$ ([9] equation 5.8.8), the leading terms in L_N are

$$L_N(\rho) = \frac{2}{\pi}\Big[-\frac{\pi^2}{12N^2}f_\rho^{(1)}(0) + \frac{\pi^4}{720N^4}f_\rho^{(3)}(0) - \frac{\pi^6}{30240N^6}f_\rho^{(5)}(0) + O\big(\frac{1}{N^8}\big)\Big] \tag{26}$$

with

$$\begin{aligned} f_\rho^{(1)}(0) &= \frac{1}{6}(-5+3\rho), \\ f_\rho^{(3)}(0) &= \frac{1}{60}(67 - 210\rho - 45\rho^2), \\ f_\rho^{(5)}(0) &= \frac{1}{126}(-95 + 3843\rho + 2835\rho^2 + 945\rho^3). \end{aligned} \tag{27}$$

Combining (21) - (24), we obtain from (12) the result

$$S_{M\times N}(\rho) = I(\rho) + \frac{2}{\pi}\Big[\log\frac{N}{2} + \gamma\Big] + L_N(\rho) + \sum_{q=1}^{N/2}\Delta_{q,M,N}(\rho), \quad (28)$$

where $\Delta_{q,M,N}(\rho)$ is given by (15).

4 Evaluation of $\Delta_{q,M,N}(\rho)$

4.1 Exact evaluation

We now evaluate the term $\Delta_{q,M,N}(\rho)$ in (28) exactly. The exact evaluation makes use of a summation identity which we state as a lemma.
Lemma:

$$\sum_{k=0}^{(M/2)-1}\frac{1}{\rho\sin^2 A_q + \sin^2[(k+\frac{1}{2})\frac{\pi}{M}]} = R(y^*) \equiv \frac{M\tanh(\pi y^*)}{2\sqrt{\rho}\sin A_q\sqrt{1+\rho\sin^2 A_q}}, \quad (29)$$

where $M =$ even and y^* is defined by

$$\sinh\frac{\pi y^*}{M} = \sqrt{\rho}\sin A_q. \quad (30)$$

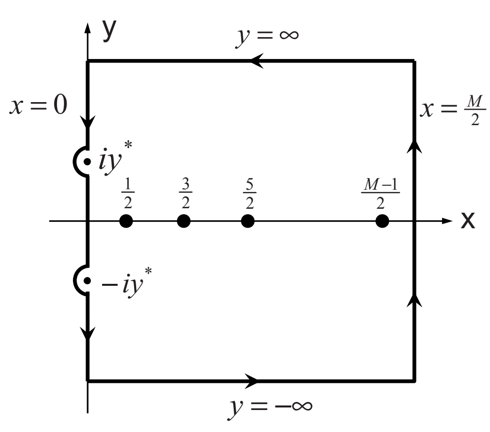

Figure 2: *Contour of integration C in* (31). *Solid circles denote simple poles enclosed by C.*

Proof. Consider the contour integral

$$J_{q,M,N}(\rho) = \frac{1}{2\pi i}\oint_C \frac{\pi\tan(\pi z)dz}{\sin^2(\frac{\pi z}{M}) + \rho\sin^2 A_q} \quad (31)$$

where the contour C consists of the lines

$$x = \frac{M}{2}, \quad y = -\infty, \quad y = \infty \tag{32}$$

and the imaginary axis $x = 0$ with two half circles of radii $\epsilon \to 0$ around the two points $z = \pm iy^*$ as shown in Fig. 2. The contour encloses $\frac{M}{2} + 2$ simple poles of the integrand at $z = \pm\, iy^*$ and $z = \frac{1}{2}, \frac{3}{2}, ..., \frac{M-1}{2}$. The residue is $R(y^*)$ at the simple poles on the y-axis and $-[\rho \sin^2 A_q + \sin^2(k+\frac{1}{2})\frac{\pi}{M}]^{-1}$ at $z = k + \frac{1}{2}$, $k = 0, 1, ...$

The integration along the contour C vanishes on the lines $y = \pm\infty$, and on the straight line portions of $x = 0, \frac{M}{2}$ since the integrand is odd in y. Hence the contour integral is nonzero only on the two half circles. The integrand is odd in z so that the integral along the lower half circle is equal to the integral in the anti-clockwise direction along the reflection of the upper half circle in the $y-$axis. The integral $J_{q,M,N}(\rho)$ along the contour C may therefore be obtained by integrating round a circle centered on iy^*. Thus, by the residue theorem, the residue at iy^* is equal to the sum of the residues of the $\frac{M}{2} + 2$ simple poles enclosed by C, hence

$$R(y^*) = 2R(y^*) - \sum_{k=0}^{(M/2)-1} \frac{1}{\sin^2[(k+\frac{1}{2})\frac{\pi}{M}] + \rho \sin^2 A_q}, \tag{33}$$

which yields (29). □

Substituting (29) into (16), one obtains

$$\begin{aligned} D^{(1)}_{q,M,N}(\rho) &= \frac{2}{N} \cdot \frac{\cos^2 A_q \sqrt{1 + \rho \sin^2 A_q}}{\sin A_q} \cdot \tanh \pi y^* \\ &= D^{(2)}_{q,N}(\rho) \cdot \tanh(\pi y^*), \end{aligned} \tag{34}$$

so that from (12) and (15) we obtain the result

$$S_{M\times N}(\rho) = S^{(1)}_N(\rho) + \sum_{q=1}^{N/2} \Delta_{q,M,N}(\rho)\ ,$$

with

$$\Delta_{q,M,N}(\rho) = D^{(2)}_{q,N}(\rho)\big[\tanh(\pi y^*) - 1\big], \tag{35}$$

where $S^{(1)}_N(\rho)$ has been evaluated in (21) and $D^{(2)}_{q,N}(\rho)$ is given by (17). Equation (17) can be further written as

$$\begin{aligned} D^{(2)}_{q,N}(\rho) &= \frac{2}{q\pi} + \frac{2}{N} f_\rho(\frac{\pi q}{N}) \\ &= \frac{1}{q\pi}\Big[2 + 2 f^{(1)}_\rho(0)\Big(\frac{q\pi}{N}\Big)^2 + \frac{1}{3} f^{(3)}_\rho(0)\Big(\frac{q\pi}{N}\Big)^4 + \cdots\Big]. \end{aligned} \tag{36}$$

where the derivatives are given in (27).

4.2 Asymptotic expansion

We now deduce the asymptotic expansion of $\Delta_{M,N}(\rho)$.

For large M, N with $M/N = \lambda$ fixed, we use

$$\sinh^{-1}(\sqrt{\rho}\sin x) = \sqrt{\rho}\, x\Big[1 - \frac{1+\rho}{6}x^2 + \frac{(1+\rho)(1+9\rho)}{120}x^4 + \cdots\Big] \tag{37}$$

and (30) to obtain

$$\pi y^* = (\pi\,\tilde{q})\Big[1 - \frac{1+\rho}{6}\Big(\frac{q\pi}{N}\Big)^2 + \frac{(1+\rho)(1+9\rho)}{120}\Big(\frac{q\pi}{N}\Big)^4 + \cdots\Big] \tag{38}$$

where $\tilde{q} = \lambda\sqrt{\rho}\, q$. This leads to

$$\begin{aligned}\tanh(\pi y^*) = \tanh(\pi\tilde{q}) &- \frac{1+\rho}{6}(\pi\,\tilde{q})\,\mathrm{sech}^2(\pi\,\tilde{q})\Big(\frac{\pi q}{N}\Big)^2 \\ &+ \Big[\frac{\pi\,\tilde{q}}{120}(1+\rho)(1+9\rho) \\ &\quad - \frac{(\pi\,\tilde{q})^2}{36}(1+\rho)^2\tanh(\pi\,\tilde{q})\Big]\mathrm{sech}^2(\pi\,\tilde{q})\Big(\frac{\pi q}{N}\Big)^4 + \cdots.\end{aligned} \tag{39}$$

Substituting (39) into (35), we obtain

$$\begin{aligned}\Delta_{q,M,N}(\rho) &= D^{(2)}_{q,N}(\rho)\big[\tanh(\pi y^*) - 1\big] \\ &= D^{(2)}_{q,N}(\rho)\big[\tanh(\pi\tilde{q}) - 1\big] \\ &\quad + D^{(2)}_{q,N}(\rho)\times(\pi\tilde{q})\Big[-\frac{1+\rho}{6}\,\mathrm{sech}^2(\pi\,\tilde{q})\Big(\frac{\pi q}{N}\Big)^2 \\ &\quad + (1+\rho)\mathrm{sech}^2(\pi\tilde{q})\Big[\frac{1+9\rho}{120} - \frac{1+\rho}{36}(\pi\tilde{q})\tanh(\pi\tilde{q})\Big]\Big(\frac{\pi q}{N}\Big)^4 \\ &\quad + \cdots\Big].\end{aligned} \tag{40}$$

This leads to, after introducing (36), the asymptotic expansion

$$\Delta_{q,M,N}(\rho) = \sum_{i=0}^{\infty}\frac{\Delta_{q,2i}(\lambda,\rho)}{N^{2i}} \tag{41}$$

with expansion coefficients

$$\begin{aligned}\Delta_{q,0}(\lambda,\rho) &= \frac{2}{\pi q}\big[\tanh(\pi\tilde{q}) - 1\big], \\ \Delta_{q,2}(\lambda,\rho) &= 2\pi q f^{(1)}_{\rho}(0)\big[\tanh(\pi\tilde{q}) - 1\big] - \frac{\lambda\sqrt{\rho}(\pi q)^2}{3}(1+\rho)\,\mathrm{sech}^2(\pi\tilde{q}), \\ \Delta_{q,4}(\lambda,\rho) &= \frac{(\pi q)^3 f^{(3)}_{\rho}(0)}{3}\big[\tanh(\pi\tilde{q}) - 1\big] \\ &\quad + \lambda\sqrt{\rho}(\pi q)^4(1+\rho)\Big[\frac{53-3\rho}{180} - \frac{(1+\rho)}{18}(\pi\tilde{q})\tanh(\pi\tilde{q})\Big]\,\mathrm{sech}^2(\pi\tilde{q}).\end{aligned} \tag{42}$$

As remarked earlier, values of these coefficients decrease exponentially as q increases.

q	$2\Delta_{q,0}$	$2\Delta_{q,2}$	$2\Delta_{q,4}$
1	-0.0047465399754997281	-0.082316647898659221	0.038515173969807909
2	$-4.4402067094342628\ 10^{-6}$	-0.00067582947581056974	-0.032940604383097552
3	$-5.5279070728467383\ 10^{-9}$	$-2.9215219029290850\ 10^{-6}$	-0.00060979439982744305
4	$-7.7422874638854272\ 10^{-12}$	$-9.8350012986547643\ 10^{-9}$	$-5.3264158004237130\ 10^{-6}$
5	$-1.1566622761121781\ 10^{-14}$	$-2.8935175541424704\ 10^{-11}$	$-3.2098297733739912\ 10^{-8}$
Σ_q	-0.0047509857178701073	-0.082995408760387631	0.004959416517708477

Table 1: The coefficients $\Delta_{q,2i}(1,1)$ in (47).

5 Results

5.1 Summary of asymptotic expansions

Results obtained so far may be summarised as follows: the resistance $R_{M\times N}(r,s)$ is given by (3), with $R_{M\times N}(\rho)$ expanded as

$$R_{M\times N}(\rho) = \frac{2}{\pi}\log N + \sqrt{\rho}\Big(\frac{M}{N} - \frac{1}{2}\Big) + \frac{1}{\pi}\Big[2\gamma - 1 + 2\log\Big(\frac{2}{\pi}\Big) - \log(1+\rho) + \frac{\rho-1}{\sqrt{\rho}}\tan^{-1}\sqrt{\rho}\Big] + L_N(\rho) + \sum_{q=1}^{N/2}\Delta_{q,M,N}(\rho), \quad (43)$$

where $\gamma = 0.577\,215\,664\,901\,53\ldots$ is Euler's constant, $L_N(\rho)$ is given by (25) and $\Delta_{q,M,N}(\rho)$ given by (41).

As $N\to\infty$ with $\lambda = M/N$ fixed, (43) can be written as

$$R_{M\times N}(\rho) = \frac{2}{\pi}\log N + C(\lambda,\rho) + \sum_{i=1}^{\infty}\frac{b_{2i}(\lambda,\rho)}{N^{2i}} \quad (44)$$

where

$$C(\lambda,\rho) = \sqrt{\rho}\Big(\lambda - \frac{1}{2}\Big) + \frac{1}{\pi}\Big[2\gamma - 1 + 2\log\Big(\frac{2}{\pi}\Big) - \log(1+\rho) + \frac{\rho-1}{\sqrt{\rho}}\tan^{-1}\sqrt{\rho}\Big] + \sum_{q=1}^{\infty}\Delta_{q,0}(\lambda,\rho),$$
$$b_{2i}(\lambda,\rho) = -\Big(\frac{2B_{2i}\pi^{2i-1}}{(2i)!}\Big)f_\rho^{(2i-1)}(0) + \sum_{q=1}^{\infty}\Delta_{q,2i}(\lambda,\rho). \quad (45)$$

Here, the Bernoulli numbers are $B_2 = 1/6, B_4 = -1/30, B_6 = 1/42$ ([9] equation 5.8.8). The function $f_\rho(x)$ is defined by (20) and its first few derivatives are given in (27). Equation (41) gives an expansion of $\Delta_{q,M,N}(\rho)$ in inverse powers of N^2 correct to $O(1/N^4)$ and the coefficients decay exponentially with q so that accurate results may be obtained using only the first few terms of the sum. This is illustrated in table 1 in the case $\lambda = \rho = 1$.

5.2 The case M=N, r=s=1

For an $N \times N$ network with $r = s = 1$ we have $\lambda = \rho = 1$. From (3) and (44) we obtain

$$\begin{aligned} R_{N\times N}(1,1) &= 2R_{N\times N}(1) \\ &= \frac{4}{\pi}\log N + c_0 + \frac{c_2}{N^2} + \frac{c_4}{N^4} + O(\frac{1}{N^6}), \end{aligned} \tag{46}$$

where

$$\begin{aligned} c_0 &= 2C(1,1) + 2\sum_{q=1}^{\infty}\Delta_{q,0}(1,1) \\ &= 1 + \frac{2}{\pi}\left[2\gamma - 1 + \log\Big(\frac{2}{\pi^2}\Big)\right] + \frac{4}{\pi}\sum_{q=1}^{\infty}\Big(\frac{\tanh(\pi q) - 1}{q}\Big) \\ &= (0.082\ 069\ 879\ 627\ 328\cdots) - (0.004\ 750\ 985\ 717\ 870\ 046\ 5\cdots) \\ &= 0.077\ 318\ 893\ 909\ 458\cdots, \\ c_2 &= -2\pi B_2 f_1^{(1)}(0) + 2\sum_{q=1}^{\infty}\Delta_{q,2}(1,1) \\ &= 0.266\ 070\ 441\ 638\ 478\ \cdots, \\ c_4 &= -\frac{\pi^3 B_4}{6}f_1^{(3)}0) + 2\sum_{q=1}^{\infty}\Delta_{q,4}(1,1) \\ &= -0.534\ 779\ 473\ 843\ 066\ \cdots. \end{aligned} \tag{47}$$

where we have used the data in table 1. This reproduces numerical values of the coefficient c_0 determined from a differential approximant analysis [6] of the first 29 values of $R_{N\times N}(1,1)$ together with a Neville table analysis [7]. We have further extended the Neville table analysis of [7] to the next two coefficients, and obtained results in agreement with the theoretical values of c_2 and c_4.

Finally, the asymptotic expansion (46) is to be compared to that of the resistance between nodes $(0,0)$ and $(N-1, N-1)$ in an infinite square lattice [4],

$$R_{N\times N,\infty}(1,1) = \frac{1}{\pi}\Big[\log N + \gamma + 2\log 2\Big] + \cdots. \tag{48}$$

Acknowledgments

FYW would like to thank David Wallace for the hospitality at the Issac Newton Institute for Mathematical Sciences where this research was initiated. We are grateful to Wentao Lu for help in the preparation of the manuscript.

References

1. Kirchhoff G., 1847, Ann. Phys. und Chemie, **72**, 497-508 (1847).
2. van der Pol, B., 1959, Lectures in Applied Mathematics, Vol. 1, Ed. M. Kac (Interscience Publ. London) pp. 237-257.
3. Doyle, P. G. and J. L. Snell, Random walks and electric networks, The Carus Mathematical Monograph, Series 22 (The Mathematical Association of America, USA, 1984), pp. 83-149.
4. Cserti, J., 2000, Am. J. Phys. **68**, 896-906.
5. Domany, E. and W. Kinzel, 1984 Phys. Rev. Lett. **53**, 311-4.
6. Guttmann, A. J. 1989, *Phase Transitions and Critical Phenomena*, Vol. 13, Ed. C. Domb and J. L. Lebowitz, Academic Press, 1-229.
7. Essam, J. W., D. TanlaKishani and F. M. Bhatti, unpublished report.
8. Wu, F. Y., 2004, J. Phys. A: Math. Gen. **37**, 6653-73.
9. Hildebrand, F. B., 1956, *Introduction to Numerical Analysis*, Tata McGraw-Hill publishing company, Bombay-Dehli.
10. Gradshteyn, I. S., and I. M. Ryzhik, 1994, *Table of Integrals, Series, and Products*, Academic Press, New York.