HOLOMORPHIC CURVES WITH SHIFT-INVARIANT HYPERPLANE PREIMAGES

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ABSTRACT. If $f : \mathbb{C} \to \mathbb{P}^n$ is a holomorphic curve of hyper-order less than one for which $2n + 1$ hyperplanes in general position have forward invariant preimages with respect to the translation $\tau(z) = z + c$, then f is periodic with period $c \in \mathbb{C}$. This result, which can be described as a difference analogue of M. Green's Picard-type theorem for holomorphic curves, follows from a more general result presented in this paper. The proof relies on a new version of Cartan's second main theorem for the Casorati determinant and an extended version of the difference analogue of the lemma on the logarithmic derivatives, both of which are proved here. Finally, an application to the uniqueness theory of meromorphic functions is given, and the sharpness of the obtained results is demonstrated by examples.

1. INTRODUCTION

According to Picard's theorem all holomorphic mappings $f : \mathbb{C} \to \mathbb{P}^1 \setminus \{a, b, c\}$ are constants. For holomorphic curves in $\mathbb{P}^n$ where $n \geq 2$ Bloch [5] and Cartan [6] showed that if a non-constant holomorphic mapping $f : \mathbb{C} \to \mathbb{P}^n$ misses $n + 2$ hyperplanes in general position, then the image of f lies in a proper linear subspace of $\mathbb{P}^n$. Here a *hyperplane* H is the set of all points $x \in \mathbb{P}^n$, $x = [x_0 : \cdots : x_n]$, such that

$$\alpha_0 x_0 + \cdots + \alpha_n x_n = 0, \tag{1.1}$$

where $\alpha_j \in \mathbb{C}$ for $j = 0, \ldots, n$. The hyperplanes H_k, $k = 0, \ldots, m$, defined by $\alpha_{0,k} x_0 + \cdots + \alpha_{n,k} x_n = 0$ are said to be in *general position* if $m \geq n$ and any $n+1$ of the vectors $\alpha_k = (\alpha_{0,k}, \ldots, \alpha_{n,k}) \in \mathbb{C}^{n+1}$ are linearly independent.

Another natural generalization of Picard's theorem was given by Fujimoto [14] and Green [17], who showed that if $f : \mathbb{C} \to \mathbb{P}^n$ omits $n + p$ hyperplanes in general position where $p \in \{1, \ldots, n + 1\}$, then the image of f is contained in a linear subspace at most of dimension $[n/p]$. In particular, by taking $p = n + 1$ it follows that if the image of a holomorphic function $f : \mathbb{C} \to \mathbb{P}^n$ lies in the complement of $2n + 1$ hyperplanes in general position, then f must be a constant. Further extensions of Picard's theorem for holomorphic curves missing hyperplanes can be found, for instance, in [15, 18, 19].

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We say that the preimage of the hyperplane $H \subset \mathbb{P}^n$ under f is *forward invariant* with respect to the translation $\tau(z) = z + c$ if $\tau(f^{-1}(\{H\})) \subset f^{-1}(\{H\})$ where $f^{-1}(\{H\})$ and $\tau(f^{-1}(\{H\}))$ are multisets in which each point is repeated according to its multiplicity. Let, for instance, $\varphi(z)$ be an entire function given by the pullback divisor of the hyperplane H. If

$$\varphi(z) = \frac{\varphi^{(m)}(z_0)}{m!}(z - z_0)^m + O((z - z_0)^{m+1}), \quad \varphi^{(m)}(z_0) \neq 0,$$

and

$$\varphi(z + c) = \frac{\varphi^{(n)}(z_0)}{n!}(z - z_0)^n + O((z - z_0)^{n+1}), \quad \varphi^{(n)}(z_0) \neq 0,$$

for all z in a neighborhood of z_0 and $n \geq m > 0$, the point z_0 is a forward invariant element in a preimage of H with respect to $\tau(z)$, while if $m > n$ then z_0 is not a forward invariant element. By this definition the preimages of the usual Picard exceptional hyperplanes of f are special cases of forward invariant preimages since in this case $f^{-1}(\{H\}) = \emptyset$. One of the purposes of this paper is to show that analogous results to Picard's theorem for holomorphic curves $f : \mathbb{C} \to \mathbb{P}^n$ can be obtained even if the image of f intersects with the target hyperplanes in general position, provided that at the same time the preimages of these hyperplanes under f are forward invariant with respect to a translation, and the considered holomorphic curve does not grow too fast.

The growth is classified by the means of Nevanlinna theory in the following way. The *order of growth* of a holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n$ with homogeneous coordinate $f = [f_0 : \cdots : f_n]$ is defined by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log^+ T_f(r)}{\log r}, \tag{1.2}$$

where $\log^+ x = \max\{0, \log x\}$ for all $x \geq 0$, and

$$T_f(r) := \int_0^{2\pi} u(re^{i\theta})\frac{d\theta}{2\pi} - u(0), \quad u(z) = \sup_{k\in\{0,\ldots,n\}} \log|f_k(z)|, \tag{1.3}$$

is the *Cartan characteristic function* of f. Note here that this representation of f is to be reduced in the sense that the $n+1$ functions f_j are entire functions without common zeros. The *hyper-order* of a holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n$ is defined by

$$\varsigma(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ T_f(r)}{\log r} \tag{1.4}$$

and the usual Nevanlinna hyper-order is

$$\rho_2(w) = \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r, w)}{\log r},$$

where w is meromorphic in the complex plane and $T(r, w)$ is the Nevanlinna characteristic function. Since, by writing $w = [w_0 : w_1]$ where w_0 and w_1 are entire functions without common zeros, it follows that $\rho_2(w) = \varsigma(w)$, we will use the notation $\varsigma(w)$ from now on to denote the hyper-order of the meromorphic function w.

Let $c \in \mathbb{C}$, and let $\mathcal{P}_c^1$ be the field of period c meromorphic functions defined in $\mathbb{C}$ of hyper-order strictly less than one. The following theorem is a difference analogue of Picard's theorem for holomorphic curves.

Theorem 1.1. *Let $f : \mathbb{C} \to \mathbb{P}^n$ be a holomorphic curve such that $\varsigma(f) < 1$, let $c \in \mathbb{C}$ and let $p \in \{1, \dots, n+1\}$. If $n+p$ hyperplanes in general position have forward invariant preimages under f with respect to the translation $\tau(z) = z + c$, then the image of f is contained in a projective linear subspace over $\mathcal{P}^1_c$ of dimension $\leq [n/p]$.*

The following example shows that the growth condition $\varsigma(f) < 1$ in Theorem 1.1 cannot be replaced by $\varsigma(f) \leq 1$.

Example 1.2. Put $\omega = 2\pi i/(2\log 6)$ and consider a linearly non-degenerate holomorphic curve

$$f := [-\sin^2 \omega z : -\cos^2 \omega z : (\sin^2 \omega z) \exp e^z : (\cos^2 \omega z) \exp e^z] : \mathbb{C} \to \mathbb{P}^3,$$

which is not $(2\log 6)$-periodic and has the hyper-order $\varsigma(f) = 1$. Take the following seven hyperplanes located in general position in $\mathbb{P}^3$:

$$\begin{aligned}
H_1 &= \{w \,|\, h_1(w) := w_0 = 0\}\,,\\
H_2 &= \{w \,|\, h_2(w) := w_1 = 0\}\,,\\
H_3 &= \{w \,|\, h_3(w) := w_2 = 0\}\,,\\
H_4 &= \{w \,|\, h_4(w) := w_3 = 0\}\,,\\
H_5 &= \{w \,|\, h_5(w) := w_0 + w_1 + w_2 + w_3 = 0\}\,,\\
H_6 &= \{w \,|\, h_6(w) := w_0 + \eta_5 w_1 + \eta_5^2 w_2 + \eta_5^3 w_3 = 0\}\,,\\
H_7 &= \{w \,|\, h_7(w) := w_0 + \eta_7 w_1 + \eta_7^2 w_2 + \eta_7^3 w_3 = 0\}\,,
\end{aligned}$$

where $w = [w_0 : w_1 : w_2 : w_3]$ and η_5 and η_7 are the primitive fifth and seventh root of unity, respectively. Then we have

$$\begin{aligned}
h_1(f) &= -\sin^2 \omega z\,, \quad h_2(f) = -\cos^2 \omega z\,,\\
h_3(f) &= (\sin^2 \omega z) \exp e^z\,, \quad h_4(f) = (\cos^2 \omega z) \exp e^z,
\end{aligned}$$

all of whose zero preimages are forward invariant with respect to $\tau(z) = z + 2\log 6$, while

$$\begin{aligned}
h_5(f) &= (\sin^2 \omega z + \cos^2 \omega z)(\exp e^z - 1)\,,\\
h_6(f) &= \eta_5^2(\sin^2 \omega z + \eta_5 \cos^2 \omega z)(\exp e^z - \eta_5^3)\,,\\
h_7(f) &= \eta_7^2(\sin^2 \omega z + \eta_7 \cos^2 \omega z)(\exp e^z - \eta_7^5)\,,
\end{aligned}$$

each of whose zeros have forward invariant preimages with respect to $\tau(z) = z + 2\log 6$, or are points such that $\exp e^z = \alpha$ for some $\alpha \in \{1, \eta_5^3, \eta_7^5\}$. Then α is a 35^{th} root of unity and thus all preimages of these hyperplanes are $(2\log 6)$-forward invariant. On the other hand, the image of f is contained in a projective linear subspace over $\mathcal{P}^1_{2\log 6}$ of dimension 1 (even though f is linearly non-degenerate in the usual sense) but '$[n/p]$' in Theorem 1.1 satisfies $[n/p] = [3/(7-3)] = 0$. In fact the image is on the projective line described by the two hyperplanes $(\cos^2 \omega z)w_0 - (\sin^2 \omega z)w_1 = 0$ and $(\cos^2 \omega z)w_2 - (\sin^2 \omega z)w_3 = 0$ over $\mathcal{P}^1_{2\log 6}$, and also it does not degenerate into a singleton in the space, since $\exp e^z \notin \mathcal{P}^1_{2\log 6}$.

An example demonstrating the sharpness of the upper bound $[n/p]$ in Theorem 1.1 is given in section 7 below. The following corollary is immediately obtained by applying Theorem 1.1 with $p = n + 1$.

Corollary 1.3. *Let $f : \mathbb{C} \to \mathbb{P}^n$ be a holomorphic curve such that $\varsigma(f) < 1$, and let $c \in \mathbb{C}$. If $2n+1$ hyperplanes in general position have forward invariant preimages under f with respect to the translation $\tau(z) = z+c$, then f is periodic with period c.*

If the preimage of a hyperplane under a holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n$ is empty, then it is clearly forward invariant with respect to all translations of the complex plane. Therefore, if f omits $2n+1$ hyperplanes in general position, then it follows by Corollary 1.3 that f is, in fact, a periodic holomorphic curve with all periods $c \in \mathbb{C}$. This is, of course, only possible when f is a constant function. We have just shown that Corollary 1.3 implies M. Green's Picard-type theorem for holomorphic curves [17] in the special case $\varsigma(f) < 1$.

A simple example shows that the growth condition $\varsigma(f) < 1$ in Corollary 1.3 cannot be significantly weakened. For $f(z) = [\exp(\exp(z)) : 1] : \mathbb{C} \to \mathbb{P}^1$ each of the n^{th} roots of unity $[1 : -\exp(2m\pi i/n)]$, $m \in \{1, \dots, n\}$, has a forward invariant preimage with respect to $\tau(z) = z + \log(n+1)$, but nevertheless $f(z) \not\equiv f(z + \log(n+1))$. Therefore f is an example of a holomorphic curve which has arbitrarily many target values with forward invariant preimages, even though it just barely fails to satisfy the condition $\varsigma(f) < 1$.

Finite-order meromorphic solutions of difference equations have been under careful study recently. Ruijsenaars has been studying minimal solutions of certain classes of linear difference equations as part of a programme of developing Hilbert space theory for analytic difference operators [37, 38]. In the nonlinear case, Ablowitz, Halburd and Herbst [1] suggested that the existence of sufficiently many finite-order meromorphic solutions can be used to detect difference equations of Painlevé type. Difference quotient estimates [24, 25, 10, 9] have proved to be useful tools in much of the recent analysis involving finite-order meromorphic solutions of difference equations (see, e.g. [11, 26, 27, 31]) but so far there is limited amount of information available on the behavior of fast growing solutions. Another main purpose of this paper is to show that if f is a meromorphic function such that $\varsigma(f) = \varsigma < 1$ and $\varepsilon > 0$, then

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\varsigma-\varepsilon}}\right) \tag{1.5}$$

for all r outside of a set of finite logarithmic measure (see Theorem 5.1 below). The type of difference analogue of the lemma on the logarithmic derivatives represented by (1.5) cannot be in general extended to meromorphic functions of hyper-order at least one, since $g(z) = \exp(2^z)$ satisfies $g(z+1)/g(z) = g(z)$, and so $m(r, g(z+1)/g(z)) = T(r,g)$.

The remainder of the paper is organized in the following way. Section 2 contains a difference analogue of Cartan's generalization of the second main theorem of Nevanlinna theory, which will be applied in section 3 to obtain a difference analogue of Borel's theorem on linear combinations of entire functions without zeros. These results are some of the main components in the proof of Theorem 1.1 in section 10. Applications of these results to uniqueness theory of meromorphic functions are discussed in section 4. The proof of the difference Cartan in section 9 relies on a logarithmic difference estimate given in section 5, and proved in section 8. A discussion on q-difference analogues of the above results is given in section 6, and, finally, the sharpness of some of the main results is considered in section 7.

2. Difference analogue of Cartan's second main theorem

The second main theorem of Nevanlinna theory [34] is a deep generalization of Picard's theorem for meromorphic functions in the complex plane, and a cornerstone on which the whole value distribution theory lies. Cartan's version of the second main theorem [7] is a generalization of this result to holomorphic curves [32], and it has also turned out to be a useful tool for certain problems in the complex plane, for instance, in considering Waring's problem for analytic functions [29] and unique range sets for entire functions [22, 23].

We now recall some of the known properties of the Cartan characteristic function from [23, 32]. For instance, if $g = [g_0 : \cdots : g_n]$ with $n \geq 1$ is a reduced representation of g, then $T_g(r) \to \infty$ as $r \to \infty$, and if at least one quotient g_j/g_m is a transcendental function, then $T_g(r)/\log r \to \infty$ as $r \to \infty$. Moreover, if $f_0, \ldots, f_q$ are $q+1$ linear combinations of the functions $g_0, \ldots, g_n$ over $\mathbb{C}$, where $q > n$, such that any $n+1$ of the $q+1$ functions $f_0, \ldots, f_q$ are linearly independent, then

$$T\left(r, \frac{f_\mu}{f_\nu}\right) \leq T_g(r) + O(1) \tag{2.1}$$

where $r \to \infty$, and μ and ν are distinct integers in the set $\{0, \ldots, q\}$. Moreover, if $n = 1$, then (2.1) becomes an asymptotic identity.

The order of a holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n$ is independent of the reduced representation of f. For if $[f_0 : \cdots : f_n]$ and $[F_0 : \cdots : F_n]$ are two reduced representations of the curve f, then, since the f_j's and F_j's are entire and

$$\max_{j=0,\ldots,n} |f_j(z)| \neq 0 \text{ and } \max_{j=0,\ldots,n} |F_j(z)| \neq 0,$$

it follows that there exists a nowhere vanishing entire function h such that

$$F_j(z) = h(z) f_j(z)$$

for all $z \in \mathbb{C}$ and $j \in \{0, \ldots, n\}$. By writing $F = [F_0 : \cdots : F_n]$ and defining

$$T_F(r) = \int_0^{2\pi} U(re^{i\theta}) \frac{d\theta}{2\pi} - U(0), \quad U(z) = \sup_{k \in \{0,\ldots,n\}} \log |F_k(z)|,$$

it follows that

$$T_F(r) = T_f(r) + \frac{1}{2\pi} \int_0^{2\pi} \log |h(re^{i\theta})| d\theta - \log |h(0)|.$$

However, since $h(z)$ is entire and nowhere zero, it follows that $\log |h(z)|$ is harmonic, and therefore

$$\log |h(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |h(re^{i\theta})| d\theta.$$

Hence $T_F(r) = T_f(r)$ is independent of the representation of f in terms of projective coordinates, and so the order of f is well defined by (1.2). We refer to [32] for the full description of Cartan's value distribution theory, and [8, 28, 40] for the standard notation of Nevanlinna theory.

Let $g(z)$ be a meromorphic function, and let $c \in \mathbb{C}$. We will use the short notation

$$g(z) \equiv g, \quad g(z+c) \equiv \overline{g}, \quad g(z+2c) \equiv \overline{\overline{g}} \quad \text{and} \quad g(z+nc) \equiv \overline{g}^{[n]}$$

to suppress the z-dependence of $g(z)$. The *Casorati determinant* of $g_0, \dots, g_n$ is then defined by

$$C(g_0, \dots, g_n) = \begin{vmatrix} g_0 & g_1 & \cdots & g_n \\ \overline{g}_0 & \overline{g}_1 & \cdots & \overline{g}_n \\ \vdots & \vdots & \ddots & \vdots \\ \overline{g}_0^{[n]} & \overline{g}_1^{[n]} & \cdots & \overline{g}_n^{[n]} \end{vmatrix}.$$

In Cartan's generalization of the second main theorem the ramification term is expressed in terms of the Wronskian determinant of a set of linearly independent entire functions. The following theorem is a difference analogue of Cartan's result where the ramification term has been replaced by a quantity expressed in terms of the Casorati determinant of functions which are linearly independent over a field of periodic functions.

Theorem 2.1. *Let $n \geq 1$, and let $g_0, \dots, g_n$ be entire functions, linearly independent over $\mathcal{P}_c^1$, such that $\max\{|g_0(z)|, \dots, |g_n(z)|\} > 0$ for each $z \in \mathbb{C}$, and*

$$\varsigma := \varsigma(g) < 1, \quad g = [g_0 : \cdots : g_n]. \tag{2.2}$$

Let $\varepsilon > 0$. If $f_0, \dots, f_q$ are $q+1$ linear combinations of the $n+1$ functions $g_0, \dots, g_n$, where $q > n$, such that any $n+1$ of the $q+1$ functions $f_0, \dots, f_q$ are linearly independent, and

$$L = \frac{f_0 \overline{f}_1 \overline{\overline{f}}_2 \cdots \overline{f}_n^{[n]} f_{n+1} \cdots f_q}{C(g_0, g_1, \dots, g_n)}, \tag{2.3}$$

then

$$(q-n)T_g(r) \leq N\left(r, \frac{1}{L}\right) - N(r, L) + o\left(\frac{T_g(r)}{r^{1-\varsigma-\varepsilon}}\right) + O(1),$$

where r approaches infinity outside of an exceptional set E of finite logarithmic measure (i.e. $\int_{E \cap [1,\infty)} dt/t < \infty$).

In [25] an analogue of the second main theorem for the difference operator $\Delta_c f = f(z+c) - f(z)$ was introduced. We will now show that, for constant targets, Theorem 2.1 is a generalization of this result in a similar way as Nevanlinna's second main theorem follows by Cartan's result.

Let w be a meromorphic function such that the usual Nevanlinna hyper-order satisfies $\varsigma(w) < 1$. Then there exist linearly independent entire functions g_0 and g_1 with no common zeros such that $w = g_0/g_1$, and, according to (2.1), it follows that $\varsigma(g) < 1$ for $g = [g_0 : g_1]$. Note that in general the entire functions g_0 and g_1 themselves may be of hyper-order greater or equal to one (see [4]).

Let $a_j \in \mathbb{C}$ for $j = 0, \dots, q-1$, and denote $f_j = g_0 - a_j g_1$ and $f_q = g_1$. Then, by Theorem 2.1, it follows that

$$(q-1)T_g(r) \leq N\left(r, \frac{1}{L}\right) - N(r, L) + o(T_g(r)) \tag{2.4}$$

where

$$L = \frac{f_0 f_1 \cdots f_{q-1} \overline{g}_1}{g_0 \overline{g}_1 - \overline{g}_0 g_1}.$$

We define the counting function $\widetilde{N}$ for $a \in \mathbb{C}$ as in [25] by

$$\widetilde{N}\left(r, \frac{1}{w-a}\right) = \int_0^r \frac{\widetilde{n}(r,a) - \widetilde{n}(0,a)}{t}\, dt + \widetilde{n}(0,a) \log r \tag{2.5}$$

where $\widetilde{n}(r,a)$ counts the number of a-points of w with multiplicity of $w(z_0)=a$ counted according to multiplicity of a at z_0 *minus* the order of the (possible) zero of $\Delta_c w$ at z_0. The pole counting function is then

$$\widetilde{N}(r,w)=\widetilde{N}\left(r,\frac{1}{1/w}\right). \tag{2.6}$$

Since $\varsigma(w)<1$ it follows by Lemma 8.3 below that $N(r,\overline{w})\leq N(r+|c|,w)=N(r,w)+o(N(r,w))$ outside of a possible exceptional set of finite logarithmic measure. Therefore, by interpreting (2.4) in terms of the counting functions (2.5) and (2.6), and using (2.1) we have

$$(q-1)T(r,w)\leq \widetilde{N}(r,w)+\sum_{j=0}^{q-1}\widetilde{N}\left(r,\frac{1}{w-a_j}\right)-N_0\left(r,\frac{1}{\Delta_c w}\right)+o(T(r,w))$$

where $N_0(r,1/\Delta_c w)$ counts the number of those zeros of $\Delta_c w$ which do not coincide with any of the a_j-points or poles of w, and r runs to infinity outside of a set of finite logarithmic measure. This is an extension of [25, Theorem 2.5] as desired.

3. Difference analogue of Borel's theorem

According to Borel's theorem, if $h_0,\dots,h_n$ are entire functions without zeros, then the only possible solutions of the equation

$$h_0+\cdots+h_n=0 \tag{3.1}$$

are trivial solutions of the form

$$h_0+\cdots+h_n=\sum_{k=1}^{l}\sum_{i\in S_k}c_{i,j_k}h_{j_k},$$

where S_k, $k=1,\dots,l$, is the partition of $\{0,\dots,n\}$ formed so that i and j are in S_k if and only if $h_i/h_j\in\mathbb{C}$, and $\sum_{i\in S_k}c_{i,j_k}=0$ for all $k=1,\dots,l$ (see, e.g. [32, p. 186] or [36, p. 124]). The following difference analogue of Borel's theorem will be one of the key results needed in the proof of Theorem 1.1.

Theorem 3.1. *Let $c\in\mathbb{C}$, and let $g=[g_0:\cdots:g_n]$ be a holomorphic curve such that $\varsigma(g)<1$ and such that preimages of all zeros of $g_0,\dots,g_n$ are forward invariant with respect to the translation $\tau(z)=z+c$. Let*

$$S_1\cup\cdots\cup S_l$$

be the partition of $\{0,\dots,n\}$ formed in such a way that i and j are in the same class S_k if and only if $g_i/g_j\in\mathcal{P}_c^1$. If

$$g_0+\ldots+g_n=0, \tag{3.2}$$

then

$$\sum_{i\in S_k}g_i=0$$

for all $k\in\{1,\dots,l\}$.

For the proof of Theorem 3.1 we need two lemmas. The first one classifies linear dependence of the coordinate functions of g over the field $\mathcal{P}_c^1$.

Lemma 3.2. *If the holomorphic curve $g = [g_0 : \cdots : g_n]$ satisfies $\varsigma(g) < 1$ and if $c \in \mathbb{C}$, then $C(g_0, \ldots, g_n) \equiv 0$ if and only if the entire functions $g_0, \ldots, g_n$ are linearly dependent over the field $\mathcal{P}_c^1$.*

Since the periodic functions $\mathcal{P}_c^1$ are constants with respect to the difference operator $\Delta_c f = f(z+c) - f(z)$, Lemma 3.2 is a natural difference analogue of the fact that entire functions $f_0, \ldots, f_n$ are linearly dependent over $\mathbb{C}$ if and only if the Wronskian $W(f_0, \ldots, f_n)$ vanishes identically.

Proof of Lemma 3.2: Suppose first that $g_0, \ldots, g_n$ are linearly dependent over $\mathcal{P}_c^1$. Then there exist $A_0, \ldots, A_n \in \mathcal{P}_c^1$ such that $A_0 g_0 + \cdots + A_n g_n = 0$, and so

$$\left\{ \begin{array}{rcl} A_0 g_0 + \cdots + A_n g_n & = & 0 \\ A_0 \overline{g}_0 + \cdots + A_n \overline{g}_n & = & 0 \\ & \vdots & \\ A_0 \overline{g}_0^{[n]} + \cdots + A_n \overline{g}_n^{[n]} & = & 0. \end{array} \right. \tag{3.3}$$

The determinant of the coefficient matrix corresponding to the system (3.3) is the Casoratian $C(g_0, \ldots, g_n)$. Since (3.3) has a nontrivial solution, it follows that $C(g_0, \ldots, g_n) \equiv 0$.

We apply induction on n to prove the converse assertion. In the case $n = 1$ suppose that $C(g_0, g_1) \equiv 0$, and consider the system of equations

$$\left\{ \begin{array}{rcl} A_0 g_0 + A_1 g_1 & = & 0 \\ A_0 \overline{g}_0 + A_1 \overline{g}_1 & = & 0 \end{array} \right. \tag{3.4}$$

which is equivalent to

$$\left\{ \begin{array}{rcl} A_0 g_0 + A_1 g_1 & = & 0 \\ A_1 C(g_0, g_1) & = & 0. \end{array} \right.$$

Since $C(g_0, g_1) \equiv 0$, it follows that $A_0 = g_1/g_0$ and $A_1 = -1$ is a solution of (3.4). Moreover, since $\varsigma(g) < 1$ by assumption, also $\varsigma(\tilde{g}) < 1$ where $\tilde{g} = [g_0 : g_1]$. Therefore, by (2.1), the usual Nevanlinna hyper-order of A_0 satisfies $\varsigma(A_0) = \varsigma(g_1/g_0) \leq \varsigma(\tilde{g}) \leq \varsigma(g) < 1$. Since clearly $A_1 \in \mathcal{P}_c^1$, all we need to do to complete the proof in the case $n = 1$ is to show that A_0 is periodic with period c. By applying the difference operator $\Delta_c f = f(z+c) - f(z)$ to the first equation in (3.4), we have

$$A_0 \Delta_c g_0 + \overline{g}_0 \Delta_c A_0 - \Delta_c g_1 = 0. \tag{3.5}$$

On the other hand, (3.4) yields

$$A_0 \Delta_c g_0 - \Delta_c g_1 = 0,$$

which, combined with (3.5), implies that $\Delta_c A_0 \equiv 0$. We conclude that $A_0 \in \mathcal{P}_c^1$.

Suppose now that $C(g_0, \ldots, g_j) \equiv 0$ implies that $g_0, \ldots, g_j$ are linearly dependent over $\mathcal{P}_c^1$ for all $j \in \{1, \ldots, k-1\}$ where $k \leq n$, and assume that $C(g_0, \ldots, g_k) \equiv 0$. Then the linear system

$$\left\{ \begin{array}{rcl} A_0 g_0 + \cdots + A_k g_k & = & 0 \\ A_0 \overline{g}_0 + \cdots + A_k \overline{g}_k & = & 0 \\ & \vdots & \\ A_0 \overline{g}_0^{[k]} + \cdots + A_k \overline{g}_k^{[k]} & = & 0 \end{array} \right. \tag{3.6}$$

has at least one redundant equation and can be written as

$$\left\{\begin{array}{lcl} A_0 g_0 + \cdots + A_{k-1} g_{k-1} & = & g_k \\ A_0 \overline{g}_0 + \cdots + A_{k-1} \overline{g}_{k-1} & = & \overline{g}_k \\ & \vdots & \\ A_0 \overline{g}_0^{[k-1]} + \cdots + A_{k-1} \overline{g}_{k-1}^{[k-1]} & = & \overline{g}_k^{[k-1]} \end{array}\right. \tag{3.7}$$

where we have made the choice $A_k = -1$. If $C(g_0, \ldots, g_{k-1}) \equiv 0$, then $g_0, \ldots, g_{k-1}$ (and thus also $g_0, \ldots, g_k$) are linearly dependent over $\mathcal{P}_c^1$ by the induction assumption. If $C(g_0, \ldots, g_{k-1}) \not\equiv 0$, then by Cramer's rule for each $i = 0, \ldots, k-1$ we have

$$A_i = \frac{C(g_0, \ldots, g_k, \ldots, g_{k-1})}{C(g_0, \ldots, g_{k-1})},$$

where g_k occurs in the i^{th} entry of the Casorati determinant in the numerator instead of g_i. By writing A_i in the form

$$A_i = \frac{g_i \overline{g}_i \cdots \overline{g}_i^{[k-1]} C(g_0/g_i, \ldots, g_k/g_i, \ldots, g_{k-1}/g_i)}{g_k \overline{g}_k \cdots \overline{g}_k^{[k-1]} C(g_0/g_k, \ldots, g_{k-1}/g_k)}$$

it can be seen that

$$T(r, A_i) = O\left(\sum_{j=0}^{k} \sum_{l=0}^{k-1} \left(T\left(r, \frac{\overline{g}_j^{[l]}}{\overline{g}_i^{[l]}}\right) + T\left(r, \frac{\overline{g}_j^{[l]}}{\overline{g}_k^{[l]}}\right)\right)\right)$$

for all $i = 0, \ldots, k-1$. Now, since $T(r, f(z+c)) = O(T(r+|c|, f))$ for all functions $f(z)$ meromorphic in the complex plane (see, e.g., [16, pp. 66–67]) it follows by the assumption $\varsigma(g) < 1$ and (2.1) that $\varsigma(A_i) < 1$ for all $i = 0, \ldots, k-1$.

We still need to prove that A_i is periodic with period c for all $i = 0, \ldots, k-1$ ($A_k \equiv -1$ and so it is trivially periodic). By applying the operator Δ_c to all equations in the system (3.7), it follows that

$$\left\{\begin{array}{lcl} A_0 \Delta_c g_0 + \cdots + A_{k-1} \Delta_c g_{k-1} + \overline{g}_0 \Delta_c A_0 + \cdots + \overline{g}_{k-1} \Delta_c A_{k-1} & = & \Delta_c g_k \\ A_0 \Delta_c \overline{g}_0 + \cdots + A_{k-1} \Delta_c \overline{g}_{k-1} + \overline{\overline{g}}_0 \Delta_c A_0 + \cdots + \overline{\overline{g}}_{k-1} \Delta_c A_{k-1} & = & \Delta_c \overline{g}_k \\ & \vdots & \\ A_0 \Delta_c \overline{g}_0^{[k-1]} + \cdots + A_{k-1} \Delta_c \overline{g}_{k-1}^{[k-1]} + \overline{g}_0^{[k]} \Delta_c A_0 + \cdots + \overline{g}_{k-1}^{[k]} \Delta_c A_{k-1} & = & \Delta_c \overline{g}_k^{[k-1]}. \end{array}\right. \tag{3.8}$$

On the other hand, from (3.6), we obtain

$$\left\{\begin{array}{lcl} A_0 \Delta_c g_0 + \cdots + A_{k-1} \Delta_c g_{k-1} & = & \Delta_c g_k \\ A_0 \Delta_c \overline{g}_0 + \cdots + A_{k-1} \Delta_c \overline{g}_{k-1} & = & \Delta_c \overline{g}_k \\ & \vdots & \\ A_0 \Delta_c \overline{g}_0^{[k-1]} + \cdots + A_{k-1} \Delta_c \overline{g}_{k-1}^{[k-1]} & = & \Delta_c \overline{g}_k^{[k-1]}. \end{array}\right. \tag{3.9}$$

By combining (3.8) and (3.9) we finally obtain

$$\left\{\begin{array}{lcl} \overline{g}_0 \Delta_c A_0 + \cdots + \overline{g}_{k-1} \Delta_c A_{k-1} & = & 0 \\ \overline{\overline{g}}_0 \Delta_c A_0 + \cdots + \overline{\overline{g}}_{k-1} \Delta_c A_{k-1} & = & 0 \\ & \vdots & \\ \overline{g}_0^{[k]} \Delta_c A_0 + \cdots + \overline{g}_{k-1}^{[k]} \Delta_c A_{k-1} & = & 0 \end{array}\right. \tag{3.10}$$

which has only the trivial solution if $C(g_0,\dots,g_{k-1})\not\equiv 0$. Therefore $\Delta_c A_0\equiv\cdots\equiv\Delta_c A_{k-1}\equiv 0$, and so $A_i\in\mathcal{P}_c^1$ for all $i=0,\dots,k-1$. In the case $C(g_0,\dots,g_{k-1})\equiv 0$ the functions $g_0,\dots,g_{k-1}$ are linearly dependent over $\mathcal{P}_c^1$ by the induction assumption. □

Lemma 3.3. *Let $c\in\mathbb{C}$, and let $g=[g_0:\cdots:g_n]$ be a holomorphic curve such that $\varsigma(g)<1$ and such that preimages of all zeros of $g_0,\dots,g_n$ are forward invariant with respect to the translation $\tau(z)=z+c$. If $g_i/g_j\notin\mathcal{P}_c^1$ for all $i,j\in\{0,\dots,n\}$ such that $i\neq j$, then $g_0,\dots,g_n$ are linearly independent over $\mathcal{P}_c^1$.*

Proof. We will show that if $g_0,\dots,g_n$ are linearly dependent over $\mathcal{P}_c^1$, then it follows that there exist $i,j\in\{0,\dots,n\}$, $i\neq j$, such that $g_i/g_j\in\mathcal{P}_c^1$. Towards that end, suppose that $A_0,\dots,A_n\in\mathcal{P}_c^1$ such that

$$A_0g_0+\cdots+A_{n-1}g_{n-1}=A_ng_n \tag{3.11}$$

and not all A_j are identically zero. Without loss of generality we may assume that none of the functions A_j are identically zero. From the assumptions of the lemma and from the fact that $A_0,\dots,A_n$ are periodic, it follows that there exists a meromorphic function $F(z)$ such that $FA_0g_0,\dots,FA_ng_n$ are entire functions without common zeros and such that the preimages of all zeros of $FA_0g_0,\dots,FA_ng_n$ are forward invariant with respect to the translation $\tau(z)$. Moreover, since $A_0,\dots,A_n\in\mathcal{P}_c^1$, the function $F(z)$ satisfies

$$\limsup_{r\to\infty}\frac{\log^+\log^+\left(N(r,F)+N(r,1/F)\right)}{\log r}<1 \tag{3.12}$$

(but in general the hyper-order of F might not be less than 1). We define $G=[FA_0g_0:\cdots:FA_{n-1}g_{n-1}]$. Since $FA_0g_0,\dots,FA_ng_n$ do not have any common zeros, it follows from (3.11) that $FA_0g_0,\dots,FA_{n-1}g_{n-1}$ cannot have any common zeros either. Therefore the $T_G(r)$ is well defined. Furthermore

$$\begin{aligned}
T_G(r)&=\int_0^{2\pi}\sup_{k\in\{0,\dots,n-1\}}\log|FA_kg_k(re^{i\theta})|\frac{d\theta}{2\pi}+O(1)\\
&=\int_0^{2\pi}\sup_{k\in\{0,\dots,n-1\}}(\log|g_k(re^{i\theta})|+\log|A_k(re^{i\theta})|)\frac{d\theta}{2\pi}\\
&\quad+\int_0^{2\pi}\log|F(re^{i\theta})|\frac{d\theta}{2\pi}+O(1)\\
&\le T_g(r)+\sum_{j=0}^{n-1}m(r,A_j)+\int_0^{2\pi}\log|F(re^{i\theta})|\frac{d\theta}{2\pi}+O(1).
\end{aligned} \tag{3.13}$$

Since Poisson-Jensen formula implies that

$$\int_0^{2\pi}\log|F(re^{i\theta})|=N\left(r,\frac{1}{F}\right)-N(r,F)+O(1), \tag{3.14}$$

and since $A_0,\dots,A_{n-1}\in\mathcal{P}_c^1$, it follows by combining (3.12) and (3.13) that $\varsigma(G)<1$. Suppose that $FA_0g_0,\dots,FA_{n-1}g_{n-1}$ are linearly independent over $\mathcal{P}_c^1$. Then, $C(FA_0g_0,\dots,FA_{n-1}g_{n-1})\not\equiv 0$ by Lemma 3.2, and so Theorem 2.1 applied with

G and $FA_0g_0, \ldots, FA_ng_n$ yields

$$T_G(r) \leq \sum_{j=0}^{n-1} N\left(r, 1/FA_j\overline{g}_j^{[j]}\right) + N\left(r, 1/FA_ng_n\right) - N\left(r, 1/C(FA_0g_0, \ldots, FA_{n-1}g_{n-1})\right) + o(T_G(r)) + O(1) \tag{3.15}$$

for all r outside of an exceptional set of finite logarithmic measure.

Since the preimages of all zeros of $FA_0g_0, \ldots, FA_ng_n$ are forward invariant with respect to $\tau(z)$, all zeros of FA_jg_j, $j = 0, \ldots, n-1$, are zeros of the Casorati determinant $C(FA_0g_0, \ldots, FA_{n-1}g_{n-1})$ with the same or higher multiplicity. Moreover, since $FA_0g_0, \ldots, FA_ng_n$ do not have any common zeros, it follows in particular that for each $z_0 \in \mathbb{C}$ such that $FA_ng_n(z_0) = 0$ with multiplicity m_0 there exist $k_0 \in \{0, \ldots, n-1\}$ such that $FA_{k_0}g_{k_0}(z_0) \neq 0$. Using (3.11) we may write

$$\begin{aligned} &C(FA_0g_0, \ldots, FA_{n-1}g_{n-1}) \\ &\quad = C(FA_0g_0, \ldots, FA_{k_0-1}g_{k_0-1}, FA_ng_n, FA_{k_0+1}g_{k_0+1}, \ldots, FA_{n-1}g_{n-1}) \end{aligned}$$

which implies that $C(FA_0g_0, \ldots, FA_{n-1}g_{n-1})$ has a zero at z_0 with multiplicity m_0 at least. Also, at any common zero the functions $FA_{j_k}g_{j_k}$ with multiplicities m_{j_k}, $k = 1, \ldots, \ell$, where $\{j_1, \ldots, j_\ell\} \subset \{1, \ldots, n\}$ and $\ell \leq n-2$, the Casorati determinant $C(FA_0g_0, \ldots, FA_{n-1}g_{n-1})$ has a zero of multiplicity $\geq \sum_{k=1}^{\ell} m_{j_k}$. Therefore,

$$\sum_{j=0}^{n-1} N\left(r, 1/FA_j\overline{g}_j^{[j]}\right) + N\left(r, 1/FA_ng_n\right) \leq N\left(r, 1/C(FA_0g_0, \ldots, FA_{n-1}g_{n-1})\right),$$

and so inequality (3.15) yields $T_G(r) = O(1)$. But this is only possible when G is a constant curve, which implies that $g_0, \ldots, g_{n-1}$ (and so also $FA_0g_0, \ldots, FA_{n-1}g_{n-1}$) are linearly dependent over $\mathcal{P}_c^1$. Therefore there exist $B_0, \ldots, B_{n-1} \in \mathcal{P}_c^1$ such that

$$B_0g_0 + \cdots + B_{n-2}g_{n-2} = B_{n-1}g_{n-1},$$

where not all B_j are identically zero. By continuing in this fashion it follows after at most $n-2$ iterations of the above reasoning that $g_i/g_j \in \mathcal{P}_c^1$ for some $i \neq j$. □

Proof of Theorem 3.1: Using the fact that $g_i = A_{i,j_k}g_{j_k}$ for some $A_{i,j_k} \in \mathcal{P}_c^1$ whenever the indexes i and j_k are in the same class S_k, equation (3.2) may be written as

$$\sum_{k=0}^{n} g_k = \sum_{k=1}^{l} \sum_{i \in S_k} A_{i,j_k}g_{j_k} = \sum_{k=1}^{l} B_kg_{j_k} = 0,$$

where $B_k = \sum_{i \in S_k} A_{i,j_k}$. By Lemma 3.3 $B_k \equiv 0$ for all $k = 1, \ldots, l$, and so

$$\sum_{i \in S_k} g_i = \sum_{i \in S_k} A_{i,j_k}g_{j_k} = B_kg_{j_k} = 0$$

for all $k = 1, \ldots, l$. □

4. Applications to the uniqueness of meromorphic functions

Nevanlinna has shown that if two non-constant meromorphic functions f and g share five distinct values ignoring multiplicities (IM), then $f \equiv g$. Similarly, if f and g share four values counting multiplicities (CM), then there exists a Möbius transformation T such that $f \equiv T \circ g$. These results are known as Nevanlinna's five

and four values theorems, respectively [35]. Gundersen has proved that the assertion of the four values theorem remains valid when it is assumed that two values are shared IM and two are shared CM [21]. He has also given a counterexample which demonstrates that in general this assumption cannot be further weakened to 4 IM [20]. The case where one value is shared CM and three values IM is still open.

A difference analogue of the five value theorem states that if instead of multiplicities we ignore those values which have forward invariant preimages, then either $f \equiv g$ or both f and g are periodic functions [25]. In this section we apply Lemma 3.3 to show that the assumption 4 CM can be weakened to a difference analogue of the 4 IM assumption for meromorphic functions of hyper-order strictly less that one.

We denote by $\mathcal{S}(f)$ the set of all meromorphic functions a such that $T(r,a) = o(T(r,f))$ where r approaches infinity outside of a set of finite logarithmic measure. Functions in the set $\mathcal{S}(f)$ are called *small compared to* f, or *slowly moving* with respect to f. Moreover, we say that two meromorphic functions f and g share a periodic function $a \in \mathcal{P}^1_c \setminus \{\infty\}$, *ignoring c-separated pairs* (IcP), when for all $z \in \mathbb{C}$ exactly one of the following assertions is valid for the ratio $r(z) := \{f(z) - a(z)\}/\{g(z) - a(z)\}$.

(i) $r(z)$ is regular and does not vanish,
(ii) $r(z)$ vanishes but $r(z+c)/r(z)$ is regular,
(iii) $r(z)$ has a pole but $r(z)/r(z+c)$ is regular.

We say also that two non-constant meromorphic functions f and g share the constant function $a(z) \equiv \infty$ IcP, if their reciprocals $1/f$ and $1/g$ share the constant 0 IcP, that is, if for all $z \in \mathbb{C}$ we have exactly one of the followings.

(i) both $f(z)$ and $g(z)$ are regular,
(ii) $f(z)$ is not regular but $f(z)g(z+c)/f(z+c)g(z)$ is regular,
(iii) $g(z)$ is not regular but $f(z+c)g(z)/f(z)g(z+c)$ is regular.

The following theorem is a difference analogue of the four value theorem where 4 CM has been replaced by 4 IcP.

Theorem 4.1. *Let $c \in \mathbb{C} \setminus \{0\}$, and let f and g be meromorphic functions such that $\max\{\varsigma(f), \varsigma(g)\} < 1$. If f and g share the distinct functions $a_1, a_2, a_3, a_4 \in \mathcal{P}^1_c$ IcP, then*

$$f = \frac{Ag + B}{Cg + D}, \tag{4.1}$$

where $A, B, C, D \in \mathcal{P}^1_c$.

Note that the functions a_1, a_2, a_3, a_4 need not be small compared to f or g. The following example shows that the transformation (4.1) cannot be replaced by the identity $f = g$.

Example 4.2. Denote by $\operatorname{sn}(z,k) \equiv \operatorname{sn}(z)$ the elliptic function with the elliptic modulus $k \in (0,1)$ and the complete elliptic integral K. The function $\operatorname{sn}(z)$ is periodic with the periods $4K$ and $2iK'$, and it attains the value zero at points $2nK + 2miK'$ and has its poles at $2nK + (2m+1)iK'$, where $n, m \in \mathbb{Z}$. Therefore the meromorphic functions

$$f(z) = \frac{\cos^2(\pi z/K) + \sin^2(\pi z/K)\operatorname{sn}(z)}{\cos(\pi z/K) + \sin(\pi z/K)\operatorname{sn}(z)}$$

and

$$g(z) = \frac{\cos^2(\pi z/K)\,\mathrm{sn}(z) + \sin^2(\pi z/K)}{\cos(\pi z/K)\,\mathrm{sn}(z) + \sin(\pi z/K)}$$

share four small periodic functions $\sin(\pi z/K)$, $\cos(\pi z/K)$,

$$\cos(\pi z/K) + \sin(\pi z/K) \quad \text{and} \quad \frac{1}{\cos(\pi z/K) + \sin(\pi z/K)}$$

ignoring $2K$-separated pairs.

Proof of Theorem 4.1: By using a Möbius transformation over the field $\mathcal{P}_c^1$ (i.e. a so called quasi-Möbius transformation), if necessary, we may assume that f and g share $0, 1, a, \infty$ IcP, where $a \in \mathcal{P}_c^1 \setminus \{0, 1, \infty\}$. Therefore, by [33, Theorem 1]

$$\frac{f}{g} = \frac{\pi_1}{\kappa_1}, \quad \frac{f-1}{g-1} = \frac{\pi_2}{\kappa_2}, \quad \frac{f-a}{g-a} = \frac{\pi_3}{\kappa_3}, \tag{4.2}$$

where the preimages of the zeros of the entire functions π_j and κ_j are forward invariant and $\varsigma(\pi_j) < 1$ and $\varsigma(\kappa_j) < 1$ for $j = 1, 2, 3$. Note that in general the functions π_j and κ_j may have common zeros for $j = 1, 2, 3$. From (4.2) it follows that

$$(1-a)\pi_1\kappa_2\kappa_3 + a\pi_2\kappa_1\kappa_3 - \pi_1\pi_2\kappa_3 = \pi_3\kappa_1\kappa_2 - a\pi_1\pi_3\kappa_2 + (a-1)\pi_2\pi_3\kappa_1, \tag{4.3}$$

and so, by denoting

$$\begin{aligned} &g_0 := (1-a)\pi_1\kappa_2\kappa_3, \quad g_1 := a\pi_2\kappa_1\kappa_3, \quad g_2 := -\pi_1\pi_2\kappa_3, \\ &g_3 := -\pi_3\kappa_1\kappa_2, \quad g_4 := a\pi_1\pi_3\kappa_2, \quad g_5 := (a-1)\pi_2\pi_3\kappa_1, \end{aligned} \tag{4.4}$$

it follows that

$$g_5 = g_0 + g_1 + g_2 + g_3 + g_4. \tag{4.5}$$

Let $F(z)$ be a meromorphic function such that $Fg_0, \ldots, Fg_5$ are entire functions without common zeros. Then, similarly as in (3.12), (3.13) and (3.14), it follows that the holomorphic curve $G = [Fg_0 : \cdots : Fg_5]$ satisfies $\varsigma(G) < 1$. Therefore, since $Fg_0, \ldots, Fg_5$ are linearly dependent by (4.5), it follows by Lemma 3.3 that there exists a $\beta \in \mathcal{P}_c^1$ such that $g_k = \beta g_\ell$ for some $\ell, k \in \{0, \ldots, 5\}$ where $\ell \neq k$. We may assume without a loss of generality that $\ell > k$, and therefore by recalling the definition (4.4) it follows that β is one of the functions

$$\begin{aligned} &\frac{g_0}{g_1} = \frac{(1-a)\pi_1\kappa_2}{a\pi_2\kappa_1}, \quad \frac{g_0}{g_2} = \frac{(a-1)\kappa_2}{\pi_2}, \quad \frac{g_0}{g_3} = \frac{(a-1)\pi_1\kappa_3}{\pi_3\kappa_1}, \\ &\frac{g_0}{g_4} = \frac{(1-a)\kappa_3}{a\pi_3}, \quad \frac{g_0}{g_5} = -\frac{\pi_1\kappa_2\kappa_3}{\pi_2\pi_3\kappa_1}, \quad \frac{g_1}{g_2} = -\frac{a\kappa_1}{\pi_1}, \quad \frac{g_1}{g_3} = -\frac{a\pi_2\kappa_3}{\pi_3\kappa_2}, \\ &\frac{g_1}{g_4} = \frac{\pi_2\kappa_1\kappa_3}{\pi_1\pi_3\kappa_2}, \quad \frac{g_1}{g_5} = \frac{a\kappa_3}{(a-1)\pi_3}, \quad \frac{g_2}{g_3} = \frac{\pi_1\pi_2\kappa_3}{\pi_3\kappa_1\kappa_2}, \quad \frac{g_2}{g_4} = -\frac{\pi_2\kappa_3}{a\pi_3\kappa_2}, \\ &\frac{g_2}{g_5} = \frac{\pi_1\kappa_3}{(1-a)\pi_3\kappa_1}, \quad \frac{g_3}{g_4} = -\frac{\kappa_1}{a\pi_1}, \quad \frac{g_3}{g_5} = \frac{\kappa_2}{(1-a)\pi_2}, \quad \frac{g_4}{g_5} = \frac{a\pi_1\kappa_2}{(a-1)\pi_2\kappa_1}. \end{aligned} \tag{4.6}$$

Substituting (4.2) into (4.6) yields the desired quasi-Möbius transformation (4.1) in all cases $\beta = g_k/g_\ell$ where (k, ℓ) is not one of the pairs $(0, 5)$, $(1, 4)$ and $(2, 3)$. In order to deal with the remaining cases, suppose first that $\beta = g_0/g_5$. Then, by (4.5) it follows that

$$(1-\beta)g_5 = g_1 + g_2 + g_3 + g_4. \tag{4.7}$$

By applying the early part of the proof to equation (4.7) instead of (4.5), it follows that $g_{\tilde{k}} = \zeta g_{\tilde{\ell}}$ for some meromorphic function $\zeta \in \mathcal{P}_c^1$ and distinct indexes $\tilde{k}, \tilde{\ell} \in \{1, \ldots, 5\}$. If $(\tilde{k}, \tilde{\ell})$ is neither $(1, 4)$, nor $(2, 3)$ then we are lead to one of the quasi-Möbius cases of (4.6). Assume therefore that $(\tilde{k}, \tilde{\ell}) = (1, 4)$, which takes equation (4.7) in the form

$$(1 - \beta)g_5 = g_2 + g_3 + (1 + \zeta)g_4 \tag{4.8}$$

Now there are two possibilities. If at least one of the factors $(1-\beta)$ and $(1+\zeta)$ is non-zero, then there are at least three identically non-zero terms in the equation (4.8), and the early part of the proof can be again applied to deduce that $g_{\hat{k}} = \lambda g_{\hat{\ell}}$ for some meromorphic function $\lambda \in \mathcal{P}_c^1$ and distinct indexes $\hat{k}, \hat{\ell} \in \{2, \ldots, 5\}$. The only possible non-quasi-Möbius case left in (4.6) is now $(\hat{k}, \hat{\ell}) = (2, 3)$, which, combined with equation (4.8), yields

$$(1 - \beta)g_5 = (1 + \lambda)g_3 + (1 + \zeta)g_4. \tag{4.9}$$

By performing the reduction operation one more time to equation (4.9) yields one of the quasi-Möbius cases of (4.6).

We still need to consider the case where $(1 - \beta)$ and $(1 + \zeta)$ both vanish. But then $g_0 = g_5$ and $g_1 = -g_4$, which, together with (4.4) and (4.6) imply that

$$\frac{\pi_3}{\kappa_3} = \frac{\kappa_3}{\pi_3}. \tag{4.10}$$

By combining (4.2) and (4.10) it finally follows that either $f = g$, or $f = -g + 2a$. □

5. Logarithmic difference estimate with applications to difference equations

A difference analogue of the lemma on the logarithmic derivative for finite-order meromorphic functions was proved independently by Halburd and Korhonen [24, Lemma 2.3], [25, Theorem 2.1] and Chiang and Feng [10, Theorem 2.4]. The following theorem is an extension of these results to the case of hyper-order less than one.

Theorem 5.1. *Let f be a non-constant meromorphic function and $c \in \mathbb{C}$. If f is of finite order, then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = O\left(\frac{\log r}{r}\, T(r, f)\right) \tag{5.1}$$

for all r outside of a set E satisfying

$$\limsup_{r\to\infty} \frac{\int_{E\cap[1,r)} dt/t}{\log r} = 0, \tag{5.2}$$

i.e., outside of a set E of zero logarithmic density. If $\varsigma(f) = \varsigma < 1$ and $\varepsilon > 0$, then

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r, f)}{r^{1-\varsigma-\varepsilon}}\right) \tag{5.3}$$

for all r outside of a set of finite logarithmic measure.

In the finite-order case we have aimed for the cleanest possible statement with the expense of allowing a slightly larger exceptional set. By following the proof of (5.3) in Theorem 5.1, it follows that for all finite-order meromorphic functions f the estimate

$$m\left(r, \frac{f(z+c)}{f(z)}\right) \leq \frac{(\log r)^{3+\varepsilon}}{r}\, T(r,f)$$

holds outside of an exceptional set of finite logarithmic measure. Concerning the sharpness of these estimates, the gamma function $\Gamma(z)$, for instance, satisfies

$$m\left(r, \frac{\Gamma(z+1)}{\Gamma(z)}\right) = \log r \cong \frac{T(r,\Gamma)}{r}$$

as r approaches infinity. It is not clear, however, whether or not the factor $\log r$ can be removed in (5.1).

Concerning the estimate (5.3), by defining $g(z) := \exp(2^z)$, it follows that

$$m\left(r, \frac{g(z+1)}{g(z)}\right) = T(r,g)$$

which shows that the growth condition $\varsigma(f) < 1$ cannot be essentially weakened. Also if $\sigma(f) < \infty$, then $\varsigma(f) = 0$, and (5.3) reduces precisely into [25, Theorem 2.1].

We will now briefly discuss some applications of Theorem 5.1 in the theory of difference equations. Yanagihara [39] has shown that if w is a non-rational meromorphic solution of the difference equation

$$w(z+1) = R(z,w), \tag{5.4}$$

where $R(z,w)$ is rational in w having rational coefficients, then, for any $\varepsilon > 0$ and r sufficiently large, $T(r,w) \geq [(1-\varepsilon)\deg_w(R)]^r$. Therefore, if (5.4) has at least one non-rational meromorphic solution w of hyper-order strictly less than one, it follows that $\deg_w(R) = 1$, that is, equation (5.4) reduces to the difference Riccati equation. This result is sharp in the sense that the difference equation $w(z+1) = w(z)^2$ is satisfied by $w(z) = \exp(2^z)$, which has hyper-order exactly one. In the second-order case, it follows by the proof of [1, Theorem 3] that if

$$w(z+1) + w(z-1) = R(z,w) \tag{5.5}$$

has a meromorphic solution of hyper-order less than one, then $\deg_w(R) \leq 2$. In [26] it was shown that if (5.5) has at least one admissible finite-order meromorphic solution w, then either w satisfies a difference Riccati equation, or equation (5.5) can be transformed into one in a list of equations consisting of difference Painlevé equations and linear equations. Recall that a meromorphic solution w of a difference (or differential) equation is called *admissible* if all coefficients of the equation are in $\mathcal{S}(w)$. Following [26], and by applying Theorem 5.1 instead of [24, Lemma 2.3], the following version of [26, Theorem 1.1] is obtained.

Theorem 5.2. *If the equation* (5.5) *where $R(z,w)$ is rational in w and meromorphic in z, has an admissible meromorphic solution w such that $\varsigma(w) < 1$, then either w satisfies a difference Riccati equation, or equation* (5.5) *can be transformed by a*

linear change in w to one of the following equations:

$$\begin{aligned}
\overline{w}+w+\underline{w} &= \frac{\pi_1 z+\pi_2}{w}+\kappa_1\\
\overline{w}-w+\underline{w} &= \frac{\pi_1 z+\pi_2}{w}+(-1)^z\kappa_1\\
\overline{w}+\underline{w} &= \frac{\pi_1 z+\pi_3}{w}+\pi_2\\
\overline{w}+\underline{w} &= \frac{\pi_1 z+\kappa_1}{w}+\frac{\pi_2}{w^2}\\
\overline{w}+\underline{w} &= \frac{(\pi_1 z+\kappa_1)w+\pi_2}{(-1)^{-z}-w^2}\\
\overline{w}+\underline{w} &= \frac{(\pi_1 z+\kappa_1)w+\pi_2}{1-w^2}\\
\overline{w}w+w\underline{w} &= p\\
\overline{w}+\underline{w} &= p\,w+q
\end{aligned}$$

where $\overline{w}\equiv w(z+1)$, $\underline{w}\equiv w(z-1)$ and π_k,κ_k are periodic functions with period k.

Remark 5.3. In order to prove Theorem 5.2 one needs to extend the difference analogues of Clunie's and Mohon'ko's theorems used in [26] to meromorphic solutions of difference equations of hyper-order strictly less than one. This can be done by combining Theorem 5.1 with the proofs of [24, Theorem 3.1–3.2]. Similarly one can extend the generalization of [24, Theorem 3.1] obtained in [31] to meromorphic solutions f such that $\varsigma(f)<1$.

6. Counterparts in q-shifts

In this section we state a q-difference analogue of Theorem 1.1. Similarly to the Casorati determinant, we define the *q-Casorati determinant* of entire functions $g_0,\dots,g_n$ by

$$\hat{C}(g_0(z),\dots,g_n(z))=\begin{vmatrix} g_0(z) & g_1(z) & \cdots & g_n(z)\\ g_0(qz) & q_1(qz) & \cdots & g_n(qz)\\ \vdots & \vdots & \ddots & \vdots\\ g_0(q^nz) & g_1(q^nz) & \cdots & g_n(q^nz)\end{vmatrix}.$$

If $q\in\mathbb{C}\setminus\{0,1\}$, then the q-Casorati determinant vanishes identically on $\mathbb{C}$ if and only if the functions $g_0,\dots,g_n$ are linearly dependent over the field of functions $\phi(z)$ satisfying $\phi(qz)\equiv\phi(z)$. However, if $|q|\neq 1$, then the intersection of this field with the field of meromorphic functions consists only of constant functions, and we are therefore restricted to study hyperplanes over $\mathbb{C}$ in this context.

Theorem 6.1. *Let $f:\mathbb{C}\to\mathbb{P}^n$ be a holomorphic curve such that $\sigma(f)=0$, let $q\in\mathbb{C}\setminus\{0,1\}$ and let $p\in\{1,\dots,n+1\}$. If $n+p$ hyperplanes in general position have forward invariant preimages under f with respect to the rescaling $\tau(z)=qz$, then the image of f is contained in a projective linear subspace of dimension $\le[n/p]$.*

Theorem 6.1 can be proved by finding q-analogues of Theorems 2.1 and 3.1 and adapting of the proof of Theorem 1.1 suitably, where the q-difference analogue of the lemma on the logarithmic derivatives from [3] is used in the place of Theorem 5.1.

We omit further details of the proof. The following corollary is an immediate consequence of Theorem 6.1.

Corollary 6.2. *Let $f : \mathbb{C} \to \mathbb{P}^n$ be a holomorphic curve such that $\sigma(f) = 0$, and let $q \in \mathbb{C} \setminus \{0, 1\}$. If $2n + 1$ hyperplanes in general position have forward invariant preimages under f with respect to the rescaling $\tau(z) = qz$, then f is a constant.*

The order condition $\sigma(f) = 0$ in Theorem 6.1 and Corollary 6.2 cannot be simply dropped in the following sense.

Example 6.3. Five hyperplanes in $\mathbb{P}^2$ given by equations $h_1(w) := w_0 = 0, h_2(w) := w_1 = 0, h_3(w) := w_2 = 0, h_4(w) := w_0+w_1+w_2 = 0, h_5(w) := w_0+\omega w_1+\omega^2 w_2 = 0$ in general position have forward invariant preimages under the non-constant curve $f = [1 : \omega : e^z] : \mathbb{C} \to \mathbb{P}^2$ with $\omega = e^{2\pi i/3}$ with respect to the rescaling $\tau(z) = 4z$. In fact, $h_1(f) \equiv 1, h_2(f) \equiv \omega, h_3(f) = e^z$ are zero-free and the zeros of $h_4(f) = e^z - \omega^2, h_5(f) = \omega^2(e^z - \omega^2)$ are forward invariant, while $\sigma(f) = 1$.

Let f be a holomorphic curve such that $\sigma(f) = 0$ given by $f(z) := [1 : \omega : \Pi(z) + \omega^2]$ with the infinite product $\Pi(z) = \prod_{j=0}^{\infty}(1 - z/q^j)$ when $|q| > 1$. This $\Pi(z)$ satisfies $\Pi(qz) = (1 - qz)\Pi(z)$ and therefore the zeros are forward invariant with respect to $\tau(z) = qz$, while we have $h_1(f) \equiv 1, h_2(f) \equiv \omega$ again and also $h_4(f) = \Pi(z), h_5(f) = \omega^2\Pi(z)$ now. Of course, the zeros of $h_3(f) = \Pi(z) - \omega$ cannot be kept forward invariant anymore with the rescaling. This difference to Example 6.3 appears to stem from the fact that any non-constant entire function does not permit finite Picard exceptional values when the order of growth is less than one. This seems to indicate that Theorem 6.1 and Corollary 6.2 should remain true when $0 < \sigma(f) < 1$, but at the moment we have no proof of this. Confirming this conjecture would require a different approach to the one used here, since it has been shown by examples that the q-difference analogue of the lemma on the logarithmic derivative obtained in [3] cannot be extended to meromorphic functions of non-zero order. Moreover, if g is a transcendental entire function whose zeros are forward invariant with respect to a rescaling $r(z) = qz$, $q \notin \{0, 1\}$, then the function $\phi(z) := g(e^z)$ is entire and has the zeros that are forward invariant with respect to the shift $s(z) = z + c$ with $c = \log q \neq 0$. By an estimate due to J. Clunie in [12], it follows that

$$\log \max_{|w|=e^r} |g(w)| + O(1) \geq \log \max_{|z|=r} |\phi(z)| \geq \log \max_{|w|=ae^{br}} |g(w)| + O(1)$$

holds for some positive constants a, b with $b < 1$. Therefore, we see that $\varsigma(\phi) \leq 1$ if the order $\sigma(g)$ of g is finite, while $\sigma(g) = 0$ if $\varsigma(\phi) < 1$. This delicate growth balance between the functions g and ϕ must be taken into account in any attempt trying to demonstrate the conjecture.

Note: Any automorphism of $\mathbb{C}$ has the form $\tau(z) = qz + r$, $q \neq 0$, which is a composition of the shift $z + r/q$ and rescaling qz.

7. Sharpness of Theorems 1.1 and 6.1

Using similar methods to Green [17], we see that the dimension $[n/p]$ in Theorem 1.1 is the sharpest possible bound and is always attained for any given $n + p$ hyperplanes in general position. In fact, we only need to replace the choice of the exponential functions $\exp(g_m)$ with holomorphic mappings $g_m : \mathbb{C}^{[n/p]} \to \mathbb{P}^n$

$(m = 1, \dots, [n/p] + 1)$ by the entire functions $1/\Gamma(z/c + \omega^m)$ on $\mathbb{C}$ with the ordinary gamma function $\Gamma(z)$ and the primitive $([n/p]+1)$th root of unity ω, for instance. Following Green's argument, we obtain the holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n$ of order of growth one, whose image is nondegenerately included in an $[n/p]$ dimensional linear subspace of $\mathbb{P}^n$ and under which the $n + p$ hyperplanes over $\mathcal{P}_f = \{\pi \in \mathcal{P}_c : T(r, \pi) = o(r \log r)\}$ have forward invariant preimages with respect to the transformation $\tau(z) = z + c$. A similar argument will be used to demonstrate the sharpness of Theorem 6.1.

Let $f : \mathbb{C} \to \mathbb{P}^n$ be a linearly non-degenerate holomorphic curve and $\{H_j\}_{j=1}^q$ be a family of hyperplanes $H_j \subset \mathbb{P}^n$ in general position. Instead of considering each hyperplane H_j itself which is defined by

$$\hat{H}_j(w) := \sum_{k=0}^{n} h_{jk} w_k = 0 \quad (1 \le j \le q)\,,$$

we will mainly observe its representing vector $\boldsymbol{h}_j = (h_{j0}, \cdots, h_{jn}) \in \mathbb{C}^{n+1}$. Here we recall that $w = [w_0 : \cdots : w_n]$ is a homogeneous coordinate system of $\mathbb{P}^n$. Then it is convenient to use a symbol $(\bullet, \bullet)$ to denote a kind of 'inner product' in $\mathbb{C}^{n+1}$ given by $\big(\boldsymbol{h}_j, w\big) := \sum_{k=0}^n h_{jk} w_k = \hat{H}_j(w)$. Let $f := [f_0 : \cdots : f_n]$ be a reduced representation of the curve $f : \mathbb{C} \to \mathbb{P}^n$. Then $\big(\boldsymbol{h}_j, f(z)\big) := \sum_{k=0}^n h_{jk} f_k(z) = \hat{H}_j\big(f(z)\big)$ is an entire function on $\mathbb{C}$ for every j. By $\{\boldsymbol{e}_k\}_{k=0}^n$ we denote the standard basis of $\mathbb{C}^{n+1}$ throughout in this note, so that we have $(\boldsymbol{e}_k, f) = f_k$ $(0 \le k \le n)$.

Let m be a prime number and ε be a primitive mth root of unity. We take the set of $2m$ vectors $\mathcal{H} := \{\boldsymbol{h}_j : 1 \le j \le 2m\} \subset \mathbb{C}^m$ with

$$\boldsymbol{h}_j = \begin{cases} \boldsymbol{e}_{j-1} & (1 \le j \le m)\,, \\ \boldsymbol{v}_{j-m} & (m+1 \le j \le 2m)\,, \end{cases}$$

where

$$\begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_j \\ \vdots \\ \boldsymbol{v}_{m-1} \\ \boldsymbol{v}_m \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ 1 & \varepsilon & \cdots & \varepsilon^k & \cdots & \varepsilon^{m-1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \varepsilon^{j-1} & \cdots & \varepsilon^{(j-1)k} & \cdots & \varepsilon^{(j-1)(m-1)} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \varepsilon^{m-2} & \cdots & \varepsilon^{(m-2)k} & \cdots & \varepsilon^{(m-2)(m-1)} \\ 1 & \varepsilon^{m-1} & \cdots & \varepsilon^{(m-1)k} & \cdots & \varepsilon^{(m-1)(m-1)} \end{pmatrix},$$

which is a regular m-matrix $V_m = (\varepsilon^{\ell k})$ $(0 \le \ell, k \le m-1)$, in fact, a Vandermonde matrix.

Then we see that any m of the $2m$ vectors $\boldsymbol{h}_j$ in $\mathcal{H}$ are linearly independent over $\mathbb{C}$ in $\mathbb{C}^m$, so that those $2m$ vectors give the family of $2m$ hyperplanes in $\mathbb{P}^{m-1}(\mathbb{C})$ which are actually located in general position. In order to confirm this matter, we only need to know that every minor determinant of our Vandermonde matrix V_m does not vanish. As a matter of fact, it is only the reason why we have chosen m as a prime number that we can apply the following lemma for the purpose:

Lemma 7.1 ([13])**.** *Let m be a prime and let ε be a primitive mth root of unity in some field of characteristic zero. Suppose $a_1, \dots, a_\mu \in \mathbb{Z}$ are pairwise incongruent $(\mathrm{mod}\ m)$ and suppose the same for $b_1, \dots, b_\mu \in \mathbb{Z}$. Then the determinant of the matrix $\left(\varepsilon^{a_i b_j}\right)$ does not vanish.*

One sees that this is not the case unless m is prime: for example, when $m = 4$, $\varepsilon = \sqrt{-1}$, $a_1 = b_1 = 1$ and $a_2 = b_2 = 3$, then the determinant does vanish.

Now we assume concretely $m = 11$ so that $n = 10$. When $p = 3$, we consider four entire functions $\phi_j(z)$ to be determined concretely later and define the linearly non-degenerate holomorphic curve $f : \mathbb{C} \to \mathbb{P}^{10}(\mathbb{C})$ by

$$f(z) := \left[c_1\phi_1 : c_2\phi_1 : c_3\phi_1 : c_1\phi_2 : c_2\phi_2 : c_3\phi_2 : c_1\phi_3 : c_2\phi_3 : c_3\phi_3 : c_1\phi_4 : c_2\phi_4\right]$$

with some non-zero constants c_i $(i = 1, 2, 3)$ and take the set of 22 vectors $\mathcal{H} := \{\boldsymbol{h}_j : 1 \le j \le 22\} \subset \mathbb{C}^{11}$ as above:

$$\boldsymbol{h}_j = \begin{cases} \boldsymbol{e}_{j-1} & (1 \le j \le 11)\,, \\ \boldsymbol{v}_{j-11} & (12 \le j \le 22)\,. \end{cases}$$

Choosing $c_1 = \varepsilon$, $c_2 = -(\varepsilon + 1)$ and $c_3 = 1$, we have

$$c_1 + c_2 + c_3 = 0 \quad \text{and} \quad c_1 + \varepsilon c_2 + \varepsilon^2 c_3 = 0$$

as well as $c_1 + c_2 = -1$ and $c_1 + \varepsilon c_2 = -\varepsilon^2$. Then $n + p = 13$ vectors $\boldsymbol{h}_j$ $(1 \le j \le 13)$ give

$$\big(\boldsymbol{h}_j, f(z)\big) = d_j \phi_{k_j}(z) \quad (1 \le j \le 11)$$

with $d_j = c_i$ $(j \equiv i \mod 3)$ and $k_j = [(j-1)/3] + 1$ $(1 \le j \le 11)$ and also

$$\big(\boldsymbol{h}_{12}, f(z)\big) = -\phi_4(z)\,, \quad \big(\boldsymbol{h}_{13}, f(z)\big) = -\varepsilon^2 \phi_4(z)\,,$$

both of which follow from the choice of the three constants c_i $(i = 1, 2, 3)$. On the other hand, it follows from the definition that $f(\mathbb{C})$ is in the linear subspace of $\mathbb{P}^{10}(\mathbb{C})$ with dimension $3 = [10/3]$.

In the same way, given any p $(1 \le p \le 10)$, we can obtain a desired curve $f(z) : \mathbb{C} \to \mathbb{P}^{10}(\mathbb{C})$

$$f(z) := \Big[\overbrace{\underbrace{c_1\phi_1 : \cdots : c_p\phi_1}_{p} : \cdots : \underbrace{c_1\phi_{s-1} : \cdots : c_p\phi_{s-1}}_{p}}^{p\times[10/p]} : \underbrace{c_1\phi_s : \cdots : c_{11-p[10/p]}\phi_s}_{11-p[10/p]}\Big]$$

for $s = [10/p] + 1$ entire functions ϕ_k $(1 \le k \le s)$ and non-zero constants c_i $(1 \le i \le p)$ satisfying the simultaneous linear equations

$$\sum_{i=1}^{p} \varepsilon^{\ell(i-1)} c_i = 0 \quad (0 \le \ell \le p-1),$$

together with $p + 10$ hyperplanes defined by $\boldsymbol{h}_j$ $(1 \le j \le p + 10)$ which satisfy

$$\big(\boldsymbol{h}_j, f(z)\big) = d_j \phi_{k_j}(z) \quad (1 \le j \le 11)$$

with $d_j = c_i$ $(j \equiv i \mod p)$ and $k_j = [(j-1)/p] + 1$ $(1 \le j \le 11)$ and also

$$\big(\boldsymbol{h}_j, f(z)\big) = d_j \phi_s(z) \quad (12 \le j \le p + 10)$$

with $d_j = -\sum_{i=1}^{11-p[10/p]} \varepsilon^{(j-12)(i-1)} c_i (\neq 0)$ $(12 \le j \le p + 10)$, since $11 - p[10/p]$ does not coincide with p.

For each prime number m, we can similarly construct a corresponding holomorphic curve $f(z)$ following the idea of Green. When $n+1$ is not a prime, the choice of suitable hyperplanes would be a little complicated.

Remark 7.2. According to the value of p, the number $11-p[10/p]$ varies as follows:

p	1	2	3	4	5	6	7	8	9	10
$11-p[10/p]$	1	1	2	3	1	5	4	3	2	1

Therefore, if p is not any divisor of $n=10$ so that $[10/p] \neq 10/p$, then we can obtain a curve $g : \mathbb{C} \to \mathbb{P}^9(\mathbb{C})$ with our desired properties by projecting the curve f into $\mathbb{P}^{10}(\mathbb{C})$ and $p+10$ vectors $\boldsymbol{h}_j \in \mathbb{C}^{11}$ given above into the subspace

$$\left\{[w_0 : w_1 : \cdots : w_9 : 1] \,|\, [w_0 : w_1 : \cdots : w_9] \in \mathbb{P}^9(\mathbb{C})\right\}$$

and $\mathbb{C}^{10}_{11} := \mathbb{C}^{10} \times \{0\}$, respectively. Concretely consider the case when $p=3$. Then we give $g : \mathbb{C} \to \mathbb{P}^9(\mathbb{C})$ by

$$g(z) := \left[c_1\phi_1 : c_2\phi_1 : c_3\phi_1 : c_1\phi_2 : c_2\phi_2 : c_3\phi_2 : c_1\phi_3 : c_2\phi_3 : c_3\phi_3 : c_1\phi_4\right]$$

with constants $c_1 = \varepsilon$, $c_2 = -(\varepsilon+1)$ and $c_3 = 1$ as well as the following 12 vectors in $\mathbb{C}^{10}$:

$$\hat{\boldsymbol{h}}_j = \begin{cases} \hat{\boldsymbol{e}}_{j-1} & := \boldsymbol{e}_{j-1} \cap \mathbb{C}^{10} \ (1 \le j \le 10)\,, \\ \hat{\boldsymbol{v}}_{j-11} & := (1, 1, \cdots, 1) \ (j = 12)\,, \\ \hat{\boldsymbol{v}}_{j-11} & := (1, \varepsilon, \cdots, \varepsilon^9) \ (j = 13)\,. \end{cases}$$

Then we have

$$\left(\hat{\boldsymbol{h}}_j, g(z)\right) = d_j\phi_{k_j}(z) \quad (1 \le j \le 10)$$

with $d_j = c_i$ $(j \equiv i \mod 3)$ and $k_j = [(j-1)/3]+1$ $(1 \le j \le 10)$ and also

$$\left(\hat{\boldsymbol{h}}_{12}, g(z)\right) = \varepsilon\phi_4(z)\,, \quad \left(\hat{\boldsymbol{h}}_{13}, g(z)\right) = \varepsilon^{-1}\phi_4(z)\,.$$

On the other hand, it follows from the definition that $g(\mathbb{C})$ is in the linear subspace of $\mathbb{P}^9(\mathbb{C})$ with dimension $3 = [9/3]$.

In general we give

$$g(z) := \Big[\overbrace{\underbrace{c_1\phi_1 : \cdots : c_p\phi_1}_{p} : \cdots : \underbrace{c_1\phi_{s-1} : \cdots : c_p\phi_{s-1}}_{p}}^{p\times[10/p]} : \underbrace{c_1\phi_s : \cdots : c_{10-p[10/p]}\phi_s}_{10-p[10/p]}\Big]$$

for $s = [10/p]+1$ entire functions ϕ_k $(1 \le k \le s)$ and non-zero constants c_i $(1 \le i \le p)$ satisfying the simultaneous linear equations

$$\sum_{i=1}^{p} \varepsilon^{\ell(i-1)} c_i = 0 \quad (0 \le \ell \le p-1),$$

together with $p+9$ hyperplanes defined respectively by the vector $\hat{\boldsymbol{h}}_j$ which is the projection of $\boldsymbol{h}_j$ on $\mathbb{C}^{10}_{11}$ for each $j(\neq 11)$ with $1 \le j \le p+10$. They still satisfy

$$\left(\hat{\boldsymbol{h}}_j, g(z)\right) = d_j\phi_{k_j}(z) \quad (1 \le j \le 10)$$

with $d_j = c_i$ $(j \equiv i \mod p)$ and $k_j = [(j-1)/p]+1$ $(1 \le j \le 10)$ and also

$$\left(\hat{\boldsymbol{h}}_j, g(z)\right) = d_j\phi_s(z) \quad (12 \le j \le p+10)$$

with $d_j = -\sum_{i=1}^{10-p[10/p]} \varepsilon^{(j-12)(i-1)} c_i (\neq 0)$ $(12 \le j \le p+10)$, since $p > 10 - p[10/p] > 0$.

For any $q \in \mathbb{C}$ such that $|q| \in (0,1)$ the q-Gamma function $\Gamma_q(x)$ is defined by

$$\Gamma_q(x) := \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},$$

where $(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k)$ [2]. By defining

$$\gamma_q(z) := (1-q)^{x-1}\Gamma_q(x), \qquad z = q^x,$$

and $\gamma_q(0) := (q;q)_\infty$, it follows that $\gamma_q(z)$ is a zero-order meromorphic function with no zeros and having its poles exactly at the points $\{q^{-k}\}_{k=0}^{\infty}$. Therefore the preimages of the poles of $\gamma_q(z)$ are forward invariant with respect to the rescaling $\tau(z) = qz$. To show the sharpness of Theorems 1.1 and 6.1, we may take the functions $\phi_j(z)$ by

$$\phi_j(z) = \frac{1}{\Gamma\big(z+(j-1)/2\big)} \quad \text{and} \quad \phi_j(z) = \frac{1}{\gamma_q(q^{(j-1)/2}z)},$$

respectively.

8. The proof of Theorem 5.1

Lemma 8.1. *Let $a \in \mathbb{C}$, $c \in \mathbb{C}$ and $\delta \in (0,1)$. Then*

$$\int_0^{2\pi} \log^+ \left|1 + \frac{c}{re^{i\theta}-a}\right| \frac{d\theta}{2\pi} \le \frac{1}{\delta}\log^+\left(1 + \frac{|c|^\delta}{(1-\delta)}\frac{1}{r^\delta}\right) \tag{8.1}$$

for all $r > 0$.

Proof. By Jensen's inequality [8, p. 48], it follows that

$$\begin{aligned}\int_0^{2\pi} \log^+ \left|1 + \frac{c}{re^{i\theta}-a}\right| \frac{d\theta}{2\pi} &\le \frac{1}{\delta}\int_0^{2\pi} \log^+\left(1 + \left|\frac{c}{re^{i\theta}-a}\right|^\delta\right)\frac{d\theta}{2\pi} \\ &\le \frac{1}{\delta}\log^+\int_0^{2\pi}\left(1 + \left|\frac{c}{re^{i\theta}-a}\right|^\delta\right)\frac{d\theta}{2\pi}\end{aligned} \tag{8.2}$$

for all $r > 0$. Since $|re^{i\theta} - |a|| \ge r\theta\frac{2}{\pi}$ for all $0 \le \theta \le \frac{\pi}{2}$ and any $a \in \mathbb{C}$ (see, e.g., [16, p. 118]), we have

$$\int_0^{2\pi} \frac{d\theta}{|re^{i\theta}-a|^\delta} \le 4\int_0^{\frac{\pi}{2}} \frac{d\theta}{|re^{i\theta}-|a||^\delta} \le \frac{2\pi}{1-\delta}\frac{1}{r^\delta} \tag{8.3}$$

whenever $\delta \in (0,1)$. Inequality (8.1) follows by combining (8.2) and (8.3). □

The following lemma is an improved version of the inequality obtained in [24, Lemma 2.3] (see also [10, Theorem 2.4]). Its method of proof is based on a combination of the techniques used in the proofs of [30, Lemma 3] and [24, Lemma 2.3].

Lemma 8.2. *Let f be a meromorphic function such that $f(0) \neq 0, \infty$ and let $c \in \mathbb{C}$. Then for all $\alpha > 1$, $\delta \in (0,1)$ and $r > 0$,*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) \le \frac{K(\alpha,\delta,c)}{r^\delta}\left(T\big(\alpha(r+|c|), f\big) + \log^+\frac{1}{|f(0)|}\right),$$

where

$$K(\alpha,\delta,c) = \frac{4|c|^\delta(4\alpha+\alpha\delta+\delta)}{\delta(1-\delta)(\alpha-1)}.$$

Proof. By the Poisson-Jensen formula [28, Theorem 1.1],

$$\begin{aligned}\log\left|\frac{f(z+c)}{f(z)}\right| &= \int_0^{2\pi}\log|f(se^{i\theta})|\mathrm{Re}\left(\frac{se^{i\theta}+z+c}{se^{i\theta}-z-c}-\frac{se^{i\theta}+z}{se^{i\theta}-z}\right)\frac{d\theta}{2\pi}\\ &\quad+\sum_{|a_n|<s}\log\left|\frac{s(z+c-a_n)}{s^2-\bar{a}_n(z+c)}\cdot\frac{s^2-\bar{a}_nz}{s(z-a_n)}\right|\\ &\quad-\sum_{|b_m|<s}\log\left|\frac{s(z+c-b_m)}{s^2-\bar{b}_m(z+c)}\cdot\frac{s^2-\bar{b}_mz}{s(z-b_m)}\right|,\end{aligned}\tag{8.4}$$

where $|z|=r$, $s=\frac{\alpha+1}{2}(r+|c|)$, and $\{a_j\}$ and $\{b_m\}$ are the sequences of zeros and poles of f, respectively. Hence, by denoting $\{q_k\}:=\{a_j\}\cup\{b_m\}$ and integrating (8.4) over the set $\{\xi\in[0,2\pi):\left|\frac{f(re^{i\xi}+c)}{f(re^{i\xi})}\right|\geq 1\}$, it follows that

$$m\left(r,\frac{f(z+c)}{f(z)}\right)\leq S_1(r)+S_2(r),\tag{8.5}$$

where

$$S_1(r)=\int_0^{2\pi}\int_0^{2\pi}\left|\log|f(se^{i\theta})|\mathrm{Re}\left(\frac{2cse^{i\theta}}{(se^{i\theta}-re^{i\varphi}-c)(se^{i\theta}-re^{i\varphi})}\right)\right|\frac{d\theta}{2\pi}\frac{d\varphi}{2\pi}\tag{8.6}$$

and

$$\begin{aligned}S_2(r)&=\sum_{|q_k|<s}\int_0^{2\pi}\log^+\left|1+\frac{c}{re^{i\varphi}-q_k}\right|\frac{d\varphi}{2\pi}\\ &\quad+\sum_{|q_k|<s}\int_0^{2\pi}\log^+\left|1-\frac{c}{re^{i\varphi}+c-q_k}\right|\frac{d\varphi}{2\pi}\\ &\quad+\sum_{|q_k|<s}\int_0^{2\pi}\log^+\left|1+\frac{c}{re^{i\varphi}-\frac{s^2}{\bar{q}_k}}\right|\frac{d\varphi}{2\pi}\\ &\quad+\sum_{|q_k|<s}\int_0^{2\pi}\log^+\left|1-\frac{c}{re^{i\varphi}+c-\frac{s^2}{\bar{q}_k}}\right|\frac{d\varphi}{2\pi}.\end{aligned}$$

By interchanging the order of integration in (8.6) using Fubini's theorem, it follows that

$$S_1(r)=\int_0^{2\pi}\big|\log|f(se^{i\theta})|\big|\int_0^{2\pi}\left|\mathrm{Re}\left(\frac{2cse^{i\theta}}{(se^{i\theta}-re^{i\varphi}-c)(se^{i\theta}-re^{i\varphi})}\right)\right|\frac{d\varphi}{2\pi}\frac{d\theta}{2\pi}.$$

Therefore, by applying the inequality (8.3) and using the facts $s=(\alpha+1)(r+|c|)/2$, $s-r-|c|=(\alpha-1)(r+|c|)/2$ and $s-r\geq|c|$, we have

$$\begin{aligned}S_1(r)&\leq\frac{2|c|s}{(s-r-|c|)(s-r)^{1-\delta}}\int_0^{2\pi}\big|\log|f(se^{i\theta})|\big|\int_0^{2\pi}\frac{1}{|se^{i\theta}-re^{i\varphi}|^\delta}\frac{d\varphi}{2\pi}\frac{d\theta}{2\pi}\\ &\leq\frac{2|c|^\delta}{(1-\delta)}\cdot\frac{\alpha+1}{\alpha-1}\cdot\frac{1}{r^\delta}\left(m(s,f)+m\left(s,\frac{1}{f}\right)\right)\\ &\leq\frac{4|c|^\delta}{(1-\delta)}\cdot\frac{\alpha+1}{\alpha-1}\cdot\frac{1}{r^\delta}\left(T(s,f)+\log^+\frac{1}{|f(0)|}\right).\end{aligned}\tag{8.7}$$

Moreover, by using Lemma 8.1 and substituting $s = \frac{\alpha+1}{2}(r+|c|)$, it follows that

$$\begin{aligned} S_2(r) &\le \frac{4}{\delta}\left(n(s,f)+n\left(s,\frac{1}{f}\right)\right)\log^+\left(1+\frac{|c|^\delta}{(1-\delta)}\frac{1}{r^\delta}\right) \\ &\le \frac{16\alpha}{(\alpha-1)\delta}\left(T(\alpha(r+|c|),f)+\log^+\frac{1}{|f(0)|}\right)\log^+\left(1+\frac{|c|^\delta}{(1-\delta)}\frac{1}{r^\delta}\right) \\ &\le \frac{16|c|^\delta}{\delta(1-\delta)}\cdot\frac{\alpha}{\alpha-1}\cdot\frac{1}{r^\delta}\left(T(\alpha(r+|c|),f)+\log^+\frac{1}{|f(0)|}\right) \end{aligned} \tag{8.8}$$

The assertion follows by combining the inequalities (8.5), (8.7) and (8.8). □

Proof of Theorem 5.1: If f has either a zero or a pole at the origin, then the assertion can be proved by considering the function $w(z) = z^k f(z)$, where $k \in \mathbb{Z}$ is chosen such that $w(0) \neq 0, \infty$. Therefore it is sufficient to consider only the case where $f(0) \neq 0, \infty$. Let $\xi(x)$ and $\phi(s)$ be positive, nondecreasing and continuous functions defined for all sufficiently large x and s, respectively, and let $C > 1$. Then, by [8, Lemma 3.3.1], we have

$$T\left(s+\frac{\phi(s)}{\xi(T(s,f))},f\right) \le C\,T(s,f) \tag{8.9}$$

for all s outside of a set E satisfying

$$\int_{E\cap[s_0,R]}\frac{ds}{\phi(s)} \le \frac{1}{\log C}\int_e^{T(R,f)}\frac{dx}{x\xi(x)}+O(1) \tag{8.10}$$

where $R < \infty$.

Assume first that f is of finite order. By choosing $\phi(r) = r$, $\xi(x) = 1$ and $\alpha = 2$ in Lemma 8.2 and (8.9), it follows that

$$T(\alpha(r+|c|),f) = T\left(r+|c|+\frac{\phi(r+|c|)}{\xi(T(r+|c|,f))},f\right) \le C\,T(r+|c|,f) \tag{8.11}$$

for all r outside of a set E which according to (8.10) satisfies

$$\int_{E\cap[1,R]}\frac{ds}{s} \le \frac{1}{\log C}\int_e^{R^\rho}\frac{dx}{x}+O(1) \tag{8.12}$$

for some $\rho > 0$. Suppose that there exist a set F of strictly positive logarithmic density $d(F)$, say $d(F) = \ell > 0$, such that

$$\limsup_{\substack{r\to\infty,\\ r\in F}}\frac{T(2r,f)}{T(r,f)} = \infty. \tag{8.13}$$

Then, choosing $C = \exp(2\rho/\ell)$, it follows by (8.12) that

$$d(F) = \limsup_{R\to\infty}\frac{\int_{F\cap[1,R]}\frac{ds}{s}}{\log R} \le \frac{\rho}{\log C} = \ell/2$$

which contradicts the assumption $d(F) = \ell > 0$. Therefore, $T(2r,f) = O(T(r,f))$ for all r outside of an exceptional set E of zero logarithmic density, and so by choosing $\delta = 1 - 1/\log r$ in Lemma 8.2 it follows that

$$m\left(r,\frac{f(z+c)}{f(z)}\right) = O\left(\frac{\log r}{r}\,T(r+|c|,f)\right) \tag{8.14}$$

as $r \to \infty$ such that $r \notin E$.

If f is of infinite order and $\varsigma(f) < 1$, then by choosing $\phi(r) = r$, $\xi(x) = (\log x)^{1+\varepsilon/3}$ and

$$\alpha = 1 + \frac{\phi(r+|c|)}{(r+|c|)\xi(T(r+|c|,f))},$$

in Lemma 8.2 and (8.9), it follows that

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r+|c|,f)}{r^{1-\varsigma-\varepsilon}}\right) \tag{8.15}$$

as r approaches infinity outside of an r-set of finite logarithmic measure. Asymptotic relations (8.14) and (8.15) together with the following lemma, which is a generalization of [26, Lemma 2.1], yield the assertion of Theorem 5.1. □

Lemma 8.3. *Let $T : [0, +\infty) \to [0, +\infty)$ be a non-decreasing continuous function and let $s \in (0, \infty)$. If the hyper-order of T is strictly less than one, i.e.,*

$$\limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} = \varsigma < 1 \tag{8.16}$$

and $\delta \in (0, 1-\varsigma)$ then

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^{\delta}}\right) \tag{8.17}$$

where r runs to infinity outside of a set of finite logarithmic measure.

Proof. Let $\tilde{\delta} \in (\delta, 1-\varsigma)$, $\eta \in \mathbb{R}^+$ and assume that the set $F_\eta \subset [1, \infty)$ defined by

$$F_\eta = \left\{ r \in \mathbb{R}^+ : \frac{T(r+s) - T(r)}{T(r)} \cdot r^{\tilde{\delta}} \geq \eta \right\} \tag{8.18}$$

is of infinite logarithmic measure. Note that F_η is a closed set and therefore it has a smallest element, say r_0. Set $r_n = \min\{F_\eta \cap [r_{n-1} + s, \infty)\}$ for all $n \in \mathbb{N}$. Then the sequence $\{r_n\}_{n\in\mathbb{Z}^+}$ satisfies $r_{n+1} - r_n \geq s$ for all $n \in \mathbb{Z}^+$, $F_\eta \subset \bigcup_{n=0}^{\infty}[r_n, r_n + s]$ and

$$\left(1 + \frac{\eta}{{r_n}^{\tilde{\delta}}}\right) T(r_n) \leq T(r_{n+1}) \tag{8.19}$$

for all $n \in \mathbb{Z}^+$.

Let $\varepsilon > 0$, and suppose that there exists an $m \in \mathbb{Z}^+$ such that $r_n \geq n^{1+\varepsilon}$ for all $r_n \geq m$. But then,

$$\begin{aligned} \int_{F_\eta\cap[1,\infty)} \frac{dt}{t} &\leq \sum_{n=0}^{\infty} \int_{r_n}^{r_n+s} \frac{dt}{t} \leq \int_1^m \frac{dt}{t} + \sum_{n=1}^{\infty} \log\left(1 + \frac{s}{r_n}\right) \\ &\leq \sum_{n=1}^{\infty} \log\left(1 + sn^{-(1+\varepsilon)}\right) + O(1) < \infty \end{aligned}$$

which contradicts the assumption $\int_{F_\eta\cap[1,\infty)} \frac{dt}{t} = \infty$. Therefore the sequence $\{r_n\}_{n\in\mathbb{Z}^+}$ has a subsequence $\{r_{n_j}\}_{j\in\mathbb{Z}^+}$ such that $r_{n_j} \leq n_j^{1+\varepsilon}$ for all $j \in \mathbb{Z}^+$. By iterating (8.19) along the sequence $\{r_{n_j}\}_{j\in\mathbb{Z}^+}$, we have

$$T(r_{n_j}) \geq \prod_{\nu=0}^{n_j-1} \left(1 + \frac{\eta}{{r_\nu}^{\tilde{\delta}}}\right) T(r_0)$$

for all $j \in \mathbb{Z}^+$, and so

$$\begin{aligned}
\limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} &\geq \limsup_{j\to\infty} \frac{\log\big(\log T(r_0) + \sum_{\nu=0}^{n_j-1} \log(1+\eta/{r_\nu}^{\tilde\delta})\big)}{\log r_{n_j}} \\
&\geq \limsup_{j\to\infty} \frac{\log\big(\log T(r_0) + n_j \log(1+\eta/{r_{n_j}}^{\tilde\delta})\big)}{(1+\varepsilon)\log n_j} \\
&\geq \limsup_{j\to\infty} \frac{\log\left(\log T(r_0) + n_j \frac{\eta}{{n_j}^{(1+\varepsilon)\tilde\delta}} \log(1+\eta/{n_j}^{(1+\varepsilon)\tilde\delta})^{{n_j}^{(1+\varepsilon)\tilde\delta}/\eta}\right)}{(1+\varepsilon)\log n_j} \\
&\geq \limsup_{j\to\infty} \frac{[1-(1+\varepsilon)\tilde\delta]\log n_j}{(1+\varepsilon)\log n_j} \\
&\geq \frac{1}{1+\varepsilon} - \tilde\delta.
\end{aligned}$$

By letting $\varepsilon \to 0$, we obtain

$$\limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} \geq 1 - \tilde\delta$$

which contradicts (8.16) since $1-\tilde\delta > \varsigma$. Hence the logarithmic measure of F_η defined by (8.18) must be finite, and so

$$T(r+s) = T(r) + O\left(\frac{T(r)}{r^{\tilde\delta}}\right)$$

for all r outside of a set of finite logarithmic measure. Therefore the assertion (8.17) follows. □

9. The proof of Theorem 2.1

The following lemma is due to Cartan [7] (see also [23]).

Lemma 9.1 ([7])**.** *Let $n \geq 1$, let $z \in \mathbb{C}$ and let $g_0, \ldots, g_n$ be linearly independent entire functions such that $\max\{|g_0(z)|, \ldots, |g_n(z)|\} > 0$ for each $z \in \mathbb{C}$. If $f_0, \ldots, f_q$ are $q+1$ linear combinations of the $n+1$ functions $g_0, \ldots, g_n$, where $q > n$, such that any $n+1$ of the $q+1$ functions $f_0, \ldots, f_q$ are linearly independent, then there exists a positive constant A that does not depend on z, such that*

$$|g_j(z)| \leq A|f_{m_\nu}(z)|,$$

where $0 \leq j \leq n$, $0 \leq \nu \leq q-n$ and the integers $m_0, \ldots, m_q$ are chosen so that

$$|f_{m_0}(z)| \geq |f_{m_2}(z)| \geq \cdots \geq |f_{m_q}(z)|.$$

In particular, there exist at least $q-n+1$ functions f_j that do not vanish at z.

Proof of Theorem 2.1: The proof follows the original proof of Cartan's second main theorem, see, e.g., [7, 29, 23], taking into account the special properties of the Casorati determinant. Since the functions g_j, where $j = 0, \ldots, n$, are linearly independent over $\mathcal{P}_c^1$, it follows by Lemma 3.2 that $C(g_0, \ldots, g_n) \not\equiv 0$ and so the function L is well defined. The functions g_j, $j = 0, \ldots, n$, are also linearly independent over $\mathbb{C}$ (since $\mathbb{C} \subset \mathcal{P}_c^1$), and so by Lemma 9.1 the auxiliary function

$$v(z) = \max_{\{k_j\}_{j=0}^{q-n-1} \subset \{0,\ldots,q\}} \log|f_{k_0}(z)\cdots f_{k_{q-n-1}}(z)| \tag{9.1}$$

gives a finite real number for all $z \in \mathbb{C}$. Let $\{a_0, \dots, a_{q-n-1}\} \subset \{0, \dots, q\}$, and $\{b_0, \dots, b_n\} = \{0, \dots, q\} \setminus \{a_0, \dots, a_{q-n-1}\}$. Since $f_{b_0}, \dots, f_{b_n}$ are linearly independent linear combinations of $g_0, \dots, g_n$, it follows that $C(f_{b_0}, \dots, f_{b_n}) \not\equiv 0$, and

$$\begin{pmatrix} f_{b_0} & \cdots & f_{b_n} \\ \overline{f}_{b_0} & \cdots & \overline{f}_{b_n} \\ \vdots & \ddots & \vdots \\ \overline{f}^{[n]}_{b_0} & \cdots & \overline{f}^{[n]}_{b_n} \end{pmatrix} = \begin{pmatrix} g_0 & \cdots & g_n \\ \overline{g}_0 & \cdots & \overline{g}_n \\ \vdots & \ddots & \vdots \\ \overline{g}^{[n]}_0 & \cdots & \overline{g}^{[n]}_n \end{pmatrix} \begin{pmatrix} \pi_{00} & \cdots & \pi_{0n} \\ \pi_{10} & \cdots & \pi_{1n} \\ \vdots & \ddots & \vdots \\ \pi_{n0} & \cdots & \pi_{nn} \end{pmatrix}$$

where $\pi_{jm} \in \mathbb{C}$ for all $j = 0, \dots, n$ and $m = 0, \dots, n$. Therefore,

$$C(g_0, \dots, g_n) = A(b_0, \dots, b_n) C(f_{b_0}, \dots, f_{b_n}) \tag{9.2}$$

where $A(b_0, \dots, b_n) =: A_{\mathbf{b}} \in \mathbb{C} \setminus \{0\}$. By substituting (9.2) into (2.3), we have

$$\begin{aligned}
L &= \frac{f_0 \overline{f}_1 \cdots \overline{f}^{[n]}_n f_{n+1} \cdots f_q}{A_{\mathbf{b}} C(f_{b_0}, f_{b_1}, \dots, f_{b_n})} \\
&= \frac{f_0 \cdots f_q \cdot (\overline{f}_1/f_1) \cdots (\overline{f}^{[n]}_n/f_n)}{A_{\mathbf{b}} C(f_{b_0}, f_{b_1}, \dots, f_{b_n})} \\
&= \frac{f_{b_0} \overline{f}_{b_1} \cdots \overline{f}^{[n]}_{b_n} \cdot f_{a_0} \cdots f_{a_{q-n-1}} (\overline{f}_1/f_1) \cdots (\overline{f}^{[n]}_n/f_n) \cdot (f_{b_1}/\overline{f}_{b_1}) \cdots (f_{b_n}/\overline{f}^{[n]}_{b_n})}{A_{\mathbf{b}} C(f_{b_0}, f_{b_1}, \dots, f_{b_n})} \\
&= \frac{f_{a_0} \cdots f_{a_{q-n-1}} (\overline{f}_1/f_1) \cdot (f_{b_1}/\overline{f}_{b_1}) \cdots (\overline{f}^{[n]}_n/f_n) \cdot (f_{b_n}/\overline{f}^{[n]}_{b_n})}{\left(\dfrac{A_{\mathbf{b}} f_0 \overline{f}_0 \cdots \overline{f}^{[n]}_0 C(f_{b_0}/f_0, f_{b_1}/f_0, \dots, f_{b_n}/f_0)}{f_{b_0} \overline{f}_{b_1} \cdots \overline{f}^{[n]}_{b_n}} \right)} \\
&= \frac{f_{a_0} \cdots f_{a_{q-n-1}} (\overline{f}_1/\overline{f}_{b_1})/(f_1/f_{b_1}) \cdots (\overline{f}^{[n]}_n/\overline{f}^{[n]}_{b_n})/(f_n/f_{b_n})}{\left(\dfrac{A_{\mathbf{b}} f_0 \overline{f}_0 \cdots \overline{f}^{[n]}_0 C(f_{b_0}/f_0, f_{b_1}/f_0, \dots, f_{b_n}/f_0)}{f_{b_0} \overline{f}_{b_1} \cdots \overline{f}^{[n]}_{b_n}} \right)} \\
&= \frac{f_{a_0} \cdots f_{a_{q-n-1}} (\overline{f}_1/\overline{f}_{b_1})/(f_1/f_{b_1}) \cdots (\overline{f}^{[n]}_n/\overline{f}^{[n]}_{b_n})/(f_n/f_{b_n})}{\left(\dfrac{A_{\mathbf{b}} C(f_{b_0}/f_0, f_{b_1}/f_0, \dots, f_{b_n}/f_0)}{(f_{b_0}/f_0) \cdot (\overline{f}_{b_1}/\overline{f}_0) \cdots (\overline{f}^{[n]}_{b_n}/\overline{f}^{[n]}_0)} \right)}.
\end{aligned}$$

Therefore,

$$L = \frac{f_{a_0} \cdots f_{a_{q-n-1}}}{A_{\mathbf{b}} G}$$

where

$$G = \frac{\left(\dfrac{C(f_{b_0}/f_0, f_{b_1}/f_0, \dots, f_{b_n}/f_0)}{(f_{b_0}/f_0) \cdot (\overline{f}_{b_1}/\overline{f}_0) \cdots (\overline{f}^{[n]}_{b_n}/\overline{f}^{[n]}_0)} \right)}{(\overline{f}_1/\overline{f}_{b_1})/(f_1/f_{b_1}) \cdots (\overline{f}^{[n]}_n/\overline{f}^{[n]}_{b_n})/(f_n/f_{b_n})}. \tag{9.3}$$

By defining

$$w(z) = \max_{\{b_j\}_{j=0}^n \subset \{0, \dots, q\}} \log |A_{\mathbf{b}} G(z)|$$

it follows that $v(z) = \log|L(z)| + w(z)$ whenever $L(z)$ non-zero and finite, and so

$$\int_0^{2\pi} v(re^{i\theta})d\theta = \int_0^{2\pi} \log|L(re^{i\theta})|d\theta + \int_0^{2\pi} w(re^{i\theta})d\theta. \tag{9.4}$$

(If L has zeros or poles on the circle $\{z : |z| = r\}$, then the path of integration may be slightly amended in (9.4) so that any poles or zeros of L are avoided by the new path. The validity of (9.4) then follows by a limiting argument where the modified path is allowed to approach the circle $\{z : |z| = r\}$. See, e.g., [23] for more details.)

Let $\{c_0, \dots, c_{q-n-1}\}$ be the set of indexes for which the maximum in (9.1) is attained for a particular choice of $z \in \mathbb{C}$. Then by Lemma 9.1 it follows that $\log|g_j(z)| \le \log|f_{c_\nu}(z)| + \log A$ for all $0 \le j \le n$ and $0 \le \nu \le q-n-1$, and so

$$(q-n)T_g(r) \le \frac{1}{2\pi}\int_0^{2\pi} v(re^{i\theta})d\theta + O(1) \tag{9.5}$$

as $r \to \infty$.

Since the function G in (9.3) consists purely of sums, products and quotients of fractions of the form $(\overline{f}_j^{[l]}/\overline{f}_k^{[l]})/(f_j/f_k)$ where $l \in \{1, \dots, n\}$ and $j, k \in \{0, \dots, q\}$, it follows by Theorem 5.1 that

$$\int_0^{2\pi} w(re^{i\theta}) \le \sum_{j=0}^{q}\sum_{k=0}^{q} o\left(\frac{T(r, f_j/f_k)}{r^{1-\varsigma-\varepsilon}}\right) + O(1) \tag{9.6}$$

as r approaches infinity outside of an exceptional set of finite logarithmic measure. By combining (9.6) and (2.1) it follows that

$$\int_0^{2\pi} w(re^{i\theta}) = o\left(\frac{T_g(r)}{r^{1-\varsigma-\varepsilon}}\right) + O(1) \tag{9.7}$$

where r tends to infinity outside of an exceptional set of finite logarithmic measure.

Finally, by Jensen's formula,

$$\frac{1}{2\pi}\int_0^{2\pi} \log|L(re^{i\theta})|d\theta = N\left(r, \frac{1}{L}\right) - N(r, L) + O(1) \tag{9.8}$$

as $r \to \infty$, and therefore the assertion follows by combining (9.4), (9.5), (9.7) and (9.8). □

10. The proof of Theorem 1.1

Let $x = [x_0 : \cdots : x_n]$, and let $H_j(x)$ be the linear form defining the hyperplane $H_j(x) = 0$ for all $j = 1, \dots, n+p$. Since by assumption any $n+1$ of the hyperplanes H_k, $k = 1, \dots, n+p$, are linearly independent, it follows that any $n+2$ of the forms $H_j(x)$ satisfy a linear relation with non-zero coefficients in $\mathbb{C}$. By writing $\tau(z) = z + c$ and $f = [f_0 : \dots : f_n]$, where f_j's are entire functions without common zeros, it follows by assumption that the functions $h_k = H_k(f)$ satisfy

$$\left\{\tau(h_k^{-1}(\{0\}))\right\} \subset \left\{h_k^{-1}(\{0\})\right\} \tag{10.1}$$

for all $k = 1, \dots, n+p$, where $\{\cdot\}$ denotes a multiset which takes into account the multiplicities of its elements. The set of indexes $\{1, \dots, n+p\}$ may be split into disjoint equivalence classes S_k by saying that $i \sim j$ if $h_i = \alpha h_j$ for some $\alpha \in \mathcal{P}_c^1 \setminus \{0\}$. Therefore

$$\{1, \dots, n+p\} = \bigcup_{j=1}^{N} S_j$$

for some $N \in \{1, \dots, n+p\}$.

Suppose that the complement of S_k has at least $n+1$ elements for some $k \in \{1, \dots, N\}$. Choose an element $s_0 \in S_k$, and denote $U = \{1, \dots, n+p\} \setminus S_k \cup \{s_0\}$. Since the set U contains at least $n+2$ elements, there exists a subset $U_0 \subset U$ such that $U_0 \cap S_k = \{s_0\}$ and $\operatorname{card}(U_0) = n+2$. Therefore, there exists $\alpha_j \in \mathbb{C} \setminus \{0\}$ such that

$$\sum_{j \in U_0} \alpha_j H_j = 0,$$

and so

$$\sum_{j \in U_0} \alpha_j h_j = 0.$$

This contradicts Theorem 3.1, and so the set $\{1, \dots, n+p\} \setminus S_k$ has at most n elements. Hence S_k has at least p elements for all $k = 1, \dots, N$, and it follows that $N \leq (n+p)/p$.

Let V be any subset of $\{1, \dots, n+p\}$ with exactly $n+1$ elements. Then the forms H_j, $j \in V$, are linearly independent. By denoting $V_k = V \cap S_k$ it follows that

$$V = \bigcup_{k=1}^{N} V_k.$$

Since each set V_k gives raise to $\operatorname{card}(V_k) - 1$ equations over the field $\mathcal{P}_c^1$, it follows that we have at least

$$\sum_{j=1}^{N} \operatorname{card}(V_k) - 1 = n + 1 - N \geq n + 1 - \frac{n+p}{p} = n - \frac{n}{p}$$

linearly independent relations over the field $\mathcal{P}_c^1$. Therefore the image of f is contained in a linear subspace over $\mathcal{P}_c^1$ of dimension $\leq [n/p]$, as desired. □

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