Multiple cover formula of generalized DT invariants II: Jacobian localizations

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Abstract

The generalized Donaldson-Thomas invariants counting one dimensional semistable sheaves on Calabi-Yau 3-folds are conjectured to satisfy a certain multiple cover formula. This conjecture is equivalent to Pandharipande-Thomas's strong rationality conjecture on the generating series of stable pair invariants, and its local version is enough to prove. In this paper, using Jacobian localizations and parabolic stable pair invariants introduced in the previous paper, we reduce the conjectural multiple cover formula for local curves with at worst nodal singularities to the case of local trees of smooth rational curves.

1 Introduction

This paper is a sequel of the author's previous paper [23], and we study the conjectural multiple cover formula of generalized Donaldson-Thomas (DT) invariants counting one dimensional semistable sheaves on Calabi-Yau 3-folds. Our main result is to reduce the multiple cover formula for local curves with at worst nodal singularities to that for local trees of $\mathbb{P}^1$. The latter case is easier to study, and we actually prove the multiple cover formula in some cases using our main result. The idea consists of twofold: using the notion of parabolic stable pairs introduced in [23], and the localizations with respect to the actions of Jacobian groups on the moduli spaces of parabolic stable pairs.

1.1 Conjectural multiple cover formula

Let X be a smooth projective Calabi-Yau 3-fold over $\mathbb{C}$, i.e.

$$\bigwedge^3 T_X^{\vee} \cong \mathcal{O}_X, \quad H^1(X, \mathcal{O}_X) = 0.$$

Given data,

$$n \in \mathbb{Z}, \quad \beta \in H_2(X, \mathbb{Z}),$$

the *generalized DT invariant* is introduced by Joyce-Song [15], Kontsevich-Soibelman [17],

$$N_{n,\beta} \in \mathbb{Q}. \tag{1}$$

The invariant (1) counts one dimensional semistable sheaves F on X satisfying

$$\chi(F) = n, \quad [F] = \beta.$$

(cf. Subsection 2.3.) The above invariant is expected to satisfy the following multiple cover conjecture:

Conjecture 1.1. [15, Conjecture 6.20], [24, Conjecture 6.3] *We have the following formula,*

$$N_{n,\beta} = \sum_{k \ge 1, k|(n,\beta)} \frac{1}{k^2} N_{1,\beta/k}.$$

The motivation of the above conjecture is that it is equivalent to Pandharipande-Thomas's (PT) strong rationality conjecture [21]. (See [24, Theorem 6.4].) The PT strong rationality conjecture claims the product expansion formula (called *Gopakumar-Vafa form*) of the generating series of rank one DT type invariants, which should be true if we believe GW/DT correspondence [19].

There is also a local version of the invariant (1) and its conjectural multiple cover formula. Namely for a one cycle γ on X, we can associate the invariant,

$$N_{n,\gamma} \in \mathbb{Q},$$

which counts one dimensional semistable sheaves F on X satisfying

$$\chi(F) = n, \quad [F] = \gamma,$$

where the second equality is an equality as a one cycle. The above local invariant is also expected to satisfy the multiple cover formula,

$$N_{n,\gamma} = \sum_{k \ge 1, k|(n,\gamma)} \frac{1}{k^2} N_{1,\gamma/k}. \tag{2}$$

The local version (2) is enough to prove Conjecture 1.1. (cf. [23, Proposition 4.17].) The purpose of this paper is to study the conjectural formula (2) via Jacobian localization technique.

1.2 Main result

Let X be as before, γ a one cycle on X and $C \subset X$ the support of γ. The invariant $N_{n,\gamma}$ can be shown to be zero if there is an irreducible component of C whose geometric genus is bigger than or equal to one. (cf. Lemma 2.11.) Therefore in discussing the formula (2), we may assume that C is a rational curve, i.e. the normalization of C is a disjoint union of $\mathbb{P}^1$. The simple cases are $C = \mathbb{P}^1$, or C is a tree of $\mathbb{P}^1$. The main result of this paper is to show that, when C has at worst nodal singularities, then the formula (2) follows from the same formula for local trees of $\mathbb{P}^1$. More precisely, suppose that C is a rational curve with at worst nodal singularities, and

$$C \subset U \subset X \tag{3}$$

a sufficiently small analytic neighborhood of C in X. We consider data,

$$(C' \subset U') \stackrel{\sigma'}{\to} (C \subset X),$$

where C' is a reduced curve, U' is a three dimensional complex manifold and σ' is a local immersion. The above data is called a *cyclic neighborhood* if it is given as a composition of cyclic coverings of U. (See Definition 3.3 for more precise definition.) For any one cycle γ' on U' supported on C', we can similarly construct the invariant

$$N_{n,\gamma'}(U') \in \mathbb{Q}.$$

(cf. Subsection 3.6.) Our main result is as follows:

Theorem 1.2. [Theorem 4.7] *Let X be a smooth projective Calabi-Yau 3-fold over $\mathbb{C}$, $C \subset X$ a reduced rational curve with at worst nodal singularities, and γ a one cycle on X supported on C. Suppose that for any cyclic neighborhood $(C' \subset U') \stackrel{\sigma'}{\to} (C \subset X)$ with C' a tree of $\mathbb{P}^1$, the following conditions hold:*

- *The moduli stack of one dimensional semistable sheaves on U' is locally written as a critical locus of some holomorphic function on a complex manifold up to some group action. (cf. Conjecture 3.6.)*
- *For any one cycle γ' on U' with $\sigma'_*\gamma' = \gamma$, the invariant $N_{n,\gamma'}(U')$ satisfies the formula*

$$N_{n,\gamma'}(U') = \sum_{k\ge 1, k|(n,\gamma')} \frac{1}{k^2} N_{1,\gamma'/k}(U').$$

Then the invariant $N_{n,\gamma}$ satisfies the formula (2).

There are several situations in which the cyclic neighborhood $C' \subset U'$ satisfies the assumptions in Theorem 1.2, e.g. C' is a chain of super rigid rational curves in U'. Roughly speaking, we will give the following applications in Section 5:

- If γ is supported on an irreducible rational curve with one node, or a circle of $\mathbb{P}^1$, we explicitly compute the invariant $N_{n,\gamma}$. (cf. Theorem 5.4.)
- We prove the local multiple cover formula of $N_{n,\gamma}$ if $\gamma = p[C]$ for an irreducible rational curve C with at worst nodal singularities, and p is a prime number. (cf. Theorem 5.5.)
- We give some evidence of the conjecture in [25, Conjecture 1.3] on the Euler characteristic invariants of local K3 surfaces. (cf. Theorem 5.11.)

The first and the second applications will be given under a certain assumption on an analytic neighborhood of a one cycle γ. (cf. Definition 5.1.)

1.3 Idea for a local curve with one node

Here we explain the idea of the proof of Theorem 1.2 in a simple example. Let

$$C \subset X$$

be an irreducible rational curve with one node $x \in C$. Suppose that a one cycle γ on X is supported on C. Then for any analytic neighborhood U as in (3), the Jacobian group $\mathrm{Pic}^0(U)$ acts on the moduli space which defines $N_{n,\gamma}$. If we take U to be homotopically equivalent to C, then $\mathrm{Pic}^0(C) \cong \mathbb{C}^*$ is considered to be a subgroup of $\mathrm{Pic}^0(U)$. So we would like to apply $\mathrm{Pic}^0(C)$-localization on the invariant $N_{n,\gamma}$. In order to see this, we need to find $\mathrm{Pic}^0(C)$-fixed semistable sheaves on U supported on C.

If we take U as above, then we have

$$\pi_1(C) \cong \pi_1(U) \cong \mathbb{Z}.$$

Hence if we take the universal covering space of U,

$$f_U \colon \widetilde{U} \to U, \tag{4}$$

then $\widetilde{U}$ admits a $\mathbb{Z}$-action, and it contains the universal cover of C denoted by $\widetilde{C}$. A key observation is that a stable sheaf on U supported on C is $\mathrm{Pic}^0(C)$-fixed if and only if it is a push-forward of some sheaf on $\widetilde{U}$ supported on $\widetilde{C}$, which is unique up to $\mathbb{Z}$-action on $\widetilde{U}$.

The universal cover $\widetilde{C} \to C$ is described in the following way. Let

$$\mathbb{P}^1 \cong C^{\dagger} \to C$$

be the normalization and $x_1, x_2 \in C^{\dagger}$ the preimage at the node $x \in C$. We take an infinite number of copies of $\{C^{\dagger}, x_1, x_2\}$, denoted by

$$\{C_i, x_{1,i}, x_{2,i}\}, \quad i \in \mathbb{Z}.$$

Then $\widetilde{C}$ is an infinite chain of smooth rational curves,

$$\widetilde{C} = \cdots \cup C_{-1} \cup C_0 \cup C_1 \cdots \cup C_i \cup C_{i+1} \cup \cdots,$$

where C_i and C_{i+1} are attached along $x_{2,i}$ and $x_{1,i+1}$. (See Figure 1.)

For instance, let us look at the invariant $N_{0,2C}$. By the above argument, we may expect the formula,

$$N_{n,2C} = \sum_{i \ge 0} N_{n,C_0+C_i}(\widetilde{U}). \tag{5}$$

Now by the assumptions in Theorem 1.2, we obtain

$$N_{0,2C_0}(\widetilde{U}) = N_{1,2C_0}(\widetilde{U}) + \frac{1}{4} N_{1,C_0}(\widetilde{U}),$$
$$N_{0,C_0+C_1}(\widetilde{U}) = N_{1,C_0+C_1}(\widetilde{U}).$$

The above localization argument also implies $N_{1,C_0}(\widetilde{U}) = N_{1,C}$, and it is also easy to see $N_{n,C_0+C_i}(\widetilde{U}) = 0$ for $i \ge 2$. Thus we obtain

$$N_{0,2C} = N_{1,2C} + \frac{1}{4}N_{1,C},$$

which is nothing but the desired formula (2) for $\gamma = 2C$. This picture is quite similar to the multiple cover formula for genus zero Gromov-Witten invariants of a local nodal curve with one node [4].

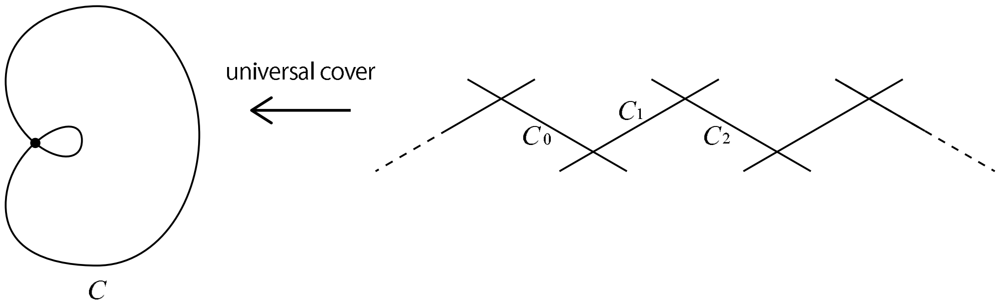

Figure 1: Universal cover $C \leftarrow \widetilde{C}$

1.4 Parabolic stable pairs

In the previous subsection, we explained the idea of the multiple cover formula in a simple example. However it is not obvious to realize the story there directly, especially the formula (5) seems to be hard to deduce. The issue is that, since the definition of $N_{n,\gamma}$ involves Joyce's log stack function [14], denoted by $\epsilon_{n,\gamma}$ in Subsection 2.3, the above localization argument seems to be very hard to apply. Namely, we have to compare the contribution of $\epsilon_{n,\gamma}$ on the $\mathbb{C}^*$-fixed points with that on the universal cover. But to do this, we also have to 'localize' the product structure on the Hall algebra, which seems to require a new technique. In order to overcome this technical difficulty, we use the idea of *parabolic stable pairs*, introduced in the previous paper [23]. By definition, a parabolic stable pair consists of a pair,

$$(F, s), \quad s \in F \otimes \mathcal{O}_H,$$

where F is a one dimensional semistable sheaf on X, H is a fixed divisor in X, satisfying a certain stability condition. (cf. Definition 2.7.) In [23], we constructed invariants counting parabolic stable pairs, and showed that Conjecture 1.1 is equivalent to a certain product expansion formula of the generating series of parabolic stable pair invariants. The moduli space of parabolic stable pairs is a scheme, (not a stack,) and $\mathrm{Pic}^0(C)$ also acts on the moduli space of (local) parabolic stable pairs. There is no technical difficulty in applying $\mathrm{Pic}^0(C)$-localizations to parabolic stable pair invariants, and the arguments similar to the previous subsection work for parabolic stable pairs.

When the one cycle γ on X is supported on a nodal curve which has more than one nodes, then its universal covering space is much more complicated. Instead of taking the universal cover, we take cyclic neighborhoods and proceed the induction argument. Combining the above ideas, (Jacobian localizations, parabolic stable pairs, induction via cyclic neighborhoods,) we are able to prove Theorem 1.2.

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2 Multiple cover formula of generalized DT invariants

In this section, we recall (generalized) DT invariants on Calabi-Yau 3-folds and the conjectural multiple cover formula. In what follows, X is a smooth projective Calabi-Yau 3-fold over $\mathbb{C}$, i.e.

$$\bigwedge^3 T_X^\vee \cong \mathcal{O}_X, \quad H^1(X, \mathcal{O}_X) = 0.$$

We fix an ample line bundle $\mathcal{O}_X(1)$ and set $\omega = c_1(\mathcal{O}_X(1))$. Below we say a coherent sheaf F on X *d-dimensional* if the support of F is d-dimensional.

2.1 Semistable sheaves

Let us recall the notion of one dimensional ω-semistable sheaves on X. They are defined by the notion of slope: for a one dimensional coherent sheaf F, its slope is defined by

$$\mu_\omega(F) := \frac{\chi(F)}{[F] \cdot \omega}.$$

Here $\chi(F)$ is the holomorphic Euler characteristic of F and $[F]$ is the fundamental one cycle associated to F, defined by

$$[F] = \sum_\eta (\mathrm{length}_{\mathcal{O}_{X,\eta}} F)\overline{\{\eta\}}. \tag{6}$$

In the above sum, η runs all the codimension two points in X.

Definition 2.1. *A one dimensional coherent sheaf F on X is ω-(semi)stable if for any subsheaf $0 \neq F' \subsetneq F$, we have the inequality,*

$$\mu_\omega(F') < (\leq) \mu_\omega(F).$$

Note that any one dimensional ω-semistable sheaf F is pure, i.e. there is no zero dimensional subsheaf in F. Also we say that F is *strictly ω-semistable* if F is ω-semistable but not ω-stable. For the detail of (semi)stable sheaves, see [10].

2.2 DT invariants

Let us take data,

$$n \in \mathbb{Z}, \quad \beta \in H_2(X, \mathbb{Z}). \tag{7}$$

The (generalized) DT invariant is the $\mathbb{Q}$-valued invariant,

$$N_{n,\beta} \in \mathbb{Q}, \tag{8}$$

counting one dimensional ω-semistable sheaves F on X satisfying

$$[F] = \beta, \quad \chi(F) = n. \tag{9}$$

Here by an abuse of notation, we denote by $[F]$ the homology class of the one cycle (6).

The invariant (8) is defined in the following way. Let

$$M_n(X, \beta) \tag{10}$$

be the coarse moduli space of one dimensional ω-semistable sheaves F on X satisfying (9). There are some criterions for the moduli space (10) to be fine. For instance suppose that the following condition holds:

$$\text{g.c.d.}(\omega \cdot \beta, n) = 1, \tag{11}$$

e.g. $n = 1$. Then there is no strictly ω-semistable sheaf on X satisfying (9), and (10) is a fine projective scheme over $\mathbb{C}$. In this case, the moduli space (10) carries a symmetric perfect obstruction theory, hence the zero dimensional virtual cycle [22].

Definition 2.2. *If the condition (11) holds, then we define $N_{n,\beta}$ to be*

$$N_{n,\beta} = \int_{[M_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

Another way to define $N_{n,\beta}$ is to use Behrend's constructible function [1]. Recall that for any $\mathbb{C}$-scheme M, Behrend constructs a canonical constructible function,

$$\nu \colon M \to \mathbb{Z},$$

such that if M carries a symmetric perfect obstruction theory, then we have

$$\int_{[M]^{\mathrm{vir}}} 1 = \int_M \nu d\chi, \\ = \sum_{m \in \mathbb{Z}} m \cdot \chi(\nu^{-1}(m)).$$

Hence by using the Behrend function ν on $M_n(X, \beta)$, the invariant (8) can also be also expressed as

$$N_{n,\beta} = \int_{M_n(X,\beta)} \nu d\chi. \tag{12}$$

2.3 Generalized DT invariants

In a general choice of (7), the condition (11) may not hold, and there may be strictly ω-semistable sheaves F satisfying (9). In this case, the invariant (8) is one of *generalized DT invariants* introduced by Joyce-Song [15] and Kontsevich-Soibelman [17]. It requires sophisticated techniques on Hall algebras of coherent sheaves to define them, and we need some more preparations for this. Since we will not need the detail of the definition of (8) in a general case, we just give a rough explanation.

A strictly ω-semistable sheaf has non-trivial automorphisms, and we need to involve the contributions of the automorphism groups with the invariant (8). For this purpose, we need to work with the moduli stack,

$$\mathcal{M}_n(X, \beta), \tag{13}$$

which parameterizes ω-semistable one dimensional sheaves F satisfying (9). The stack (13) is known to be an Artin stack of finite type over $\mathbb{C}$.

The Behrend functions on $\mathbb{C}$-schemes naturally extend to constructible functions on Artin stacks of finite type over $\mathbb{C}$. (cf. [15, Proposition 4.4].) However the stack (13) may have stabilizer groups whose Euler characteristic are zero, e.g. $\mathrm{GL}(2, \mathbb{C})$. Hence the integration of the Behrend function (12), replacing $M_n(X, \beta)$ by $\mathcal{M}_n(X, \beta)$, does not make sense. The idea of the definition of generalized DT invariant is that, instead of working with the stack (13), we should work with the 'logarithm' of (13) in the Hall algebra of coherent sheaves, denoted by $H(X)$.

The algebra $H(X)$ is, as a $\mathbb{Q}$-vector space, spanned by the isomorphism classes of symbols,

$$[\rho \colon \mathcal{X} \to \mathcal{C}oh(X)].$$

Here $\mathcal{X}$ is an Artin stack of finite type with affine geometric stabilizers, and $\mathcal{C}oh(X)$ is the stack of all the coherent sheaves on X. There is an associative $*$-product on $H(X)$ based on Ringel Hall algebras. For the detail, see [13, Theorem 5.2].

The stack (13) is considered to be an element of $H(X)$, by regarding it as an open substack of $\mathcal{C}oh(X)$,

$$\delta_{n,\beta} := [\mathcal{M}_n(X, \beta) \hookrightarrow \mathcal{C}oh(X)] \in H(X).$$

The 'logarithm' of $\delta_{n,\beta}$, denoted by $\epsilon_{n,\beta} \in H(X)$, is defined by the rule,

$$\sum_{n/\omega\cdot\beta=\mu} \epsilon_{n,\beta} = \log\left(1 + \sum_{n/\omega\cdot\beta=\mu} \delta_{n,\beta}\right),$$

for any $\mu \in \mathbb{Q}$ in a certain completion of the algebra $(H(X), *)$. In other words, $\epsilon_{n,\beta}$ is given by

$$\epsilon_{n,\beta} = \sum_{l\ge 1} \frac{(-1)^{l-1}}{l} \sum_{\substack{\beta_1,\cdots,\beta_l \in H_2(X,\mathbb{Z}), \\ n_1,\cdots,n_l\in\mathbb{Z}, \\ n_i/\omega\cdot\beta_i = n/\omega\cdot\beta}} \delta_{n_1,\beta_1} * \cdots * \delta_{n_l,\beta_l}.$$

The above sum is easily shown to be a finite sum.

The important fact is that $\epsilon_{n,\beta}$ is supported on 'virtual indecomposable sheaves'. Roughly speaking this implies that, modulo some relations in $H(X)$, the element $\epsilon_{n,\beta}$ is written as

$$\epsilon_{n,\beta} = \sum_i a_i[\rho_i \colon [M_i/\mathbb{C}^*] \to \mathcal{C}oh(X)],$$

where $a_i \in \mathbb{Q}$, M_i are quasi-projective varieties on which $\mathbb{C}^*$ act trivially. The invariant (8) is then defined by the weighted Euler characteristics of M_i, weighted by the Behrend function ν on $\mathcal{C}oh(X)$ pulled back by ρ_i. Namely, $N_{n,\beta}$ is defined by

$$N_{n,\beta} := -\sum_i a_i \int_{M_i} \rho_i^* \nu d\chi. \tag{14}$$

Here we need to change the sign due to the appearance of the trivial $\mathbb{C}^*$-action.

We have skipped lots of details in the above definition of (8). For more detail, we refer [15]. Also see [24, Section 4] for a more direct explanation.

Remark 2.3. *In priori, we need to choose an ample divisor ω to define $N_{n,\beta}$. However it can be shown that $N_{n,\beta}$ does not depend on a choice of ω. (cf. [15, Theorem 6.16].)*

2.4 Local generalized DT invariants

There is also a local version of (generalized) DT invariant, which we explain below. Let us fix a reduced curve C in X,

$$i \colon C \hookrightarrow X,$$

with irreducible components $C_1, \cdots, C_N$. Then a one cycle γ on X supported on C is identified with an element of $H_2(C,\mathbb{Z})$,

$$\gamma \in H_2(C,\mathbb{Z}) \cong \bigoplus_{i=1}^N \mathbb{Z}[C_i].$$

Suppose that $\beta = i_* \gamma$ and $n \in \mathbb{Z}$ satisfies the condition (11). Then we have the fine moduli space (10), and the closed subscheme,

$$M_n(C, \gamma) \subset M_n(X, \beta), \tag{15}$$

corresponding to ω-stable sheaves F satisfying

$$[F] = \gamma, \quad \chi(F) = n. \tag{16}$$

Here $[F] = \gamma$ is an equality as a one cycle on X. Then the local DT invariant is defined by

$$N_{n,\gamma} := \int_{M_n(C,\gamma)} \nu d\chi. \tag{17}$$

Here ν is the Behrend function on $M_n(X, \beta)$ restricted to $M_n(C, \gamma)$. We remark that ν may not coincide with the Behrend function on $M_n(C, \gamma)$.

Even if (n, β) does not satisfy the condition (11), we can similarly define the local generalized DT invariant,

$$N_{n,\gamma} \in \mathbb{Q}, \tag{18}$$

counting one dimensional ω-semistable sheaves F satisfying (16). Instead of using the stack (13), we use the substack,

$$\mathcal{M}_n(C, \gamma) \subset \mathcal{M}_n(X, \beta), \tag{19}$$

parameterizing one dimensional ω-semistable sheaves F on X satisfying (16). We can similarly take the logarithm of the substack (19) in the Hall algebra $H(X)$, and the invariant (18) is defined by integrating the Behrend function on $\mathcal{C}oh(X)$ over it. See [23, Subsection 4.4] for some more detail. Similarly to $N_{n,\beta}$, the local invariant $N_{n,\gamma}$ also does not depend on ω. (cf. Remark 2.3.)

2.5 Multiple cover formula

As we discussed in the previous subsections, the invariant (8) is an integer if the condition (11) is satisfied. In particular, for $\beta \in H_2(X, \mathbb{Z})$, we have the $\mathbb{Z}$-valued invariant,

$$N_{1,\beta} \in \mathbb{Z}.$$

The above invariant is introduced by Katz [16] as a sheaf theoretic definition of genus zero Gopakumar-Vafa invariant. On the other hand if (n, β) does not satisfy the condition (11), then $N_{n,\beta}$ may not be an integer and hence does not coincide with $N_{1,\beta}$. However the invariants $N_{n,\beta}$ for $n \neq 1$ are conjectured to be related to $N_{1,\beta}$ via the multiple cover formula:

Conjecture 2.4. [15, Conjecture 6.20], [24, Conjecture 6.3] *We have the following formula,*

$$N_{n,\beta} = \sum_{k \geq 1, k | (n,\beta)} \frac{1}{k^2} N_{1,\beta/k}. \tag{20}$$

In [24, Theorem 6.4], it is shown that the above conjecture is equivalent to Pandharipande-Thomas's strong rationality conjecture [21, Conjecture 3.14]. We refer [24, Section 6] for discussions on strong rationality conjecture and its relation to Conjecture 2.4.

For a reduced curve $C \subset X$, $n \in \mathbb{Z}$ and $\gamma \in H_2(C, \mathbb{Z})$, we have the local (generalized) DT invariants as in (18). The local version of the above conjecture is also similarly formulated:

Conjecture 2.5. [23, Conjecture 4.13] *For $n \in \mathbb{Z}$ and $\gamma \in H_2(C, \mathbb{Z})$, we have the formula,*

$$N_{n,\gamma} = \sum_{k \geq 1, k|(n,\gamma)} \frac{1}{k^2} N_{1,\gamma/k}. \tag{21}$$

As shown in [23, Corollary 4.18], the local multiple cover formula is enough to show the global multiple cover formula:

Lemma 2.6. [23, Corollary 4.18] *For $n \in \mathbb{Z}$ and $\beta \in H_2(X, \mathbb{Z})$, suppose that the formula (21) holds for any reduced curve $i \colon C \hookrightarrow X$ and $\gamma \in H_2(C, \mathbb{Z})$ with $\beta = i_* \gamma$. Then $N_{n,\beta}$ satisfies the formula (20).*

As we discussed in the Introduction, our purpose is to study Conjecture 2.5 in terms of Jacobian localizations and parabolic stable pair invariants, which we recall in the next subsection.

2.6 (Local) parabolic stable pair theory

The notion of parabolic stable pairs is introduced in [23]. It is determined by fixing a divisor,

$$H \in |\mathcal{O}_X(h)|,$$

for some $h > 0$. In what follows, we say a one cycle γ on X intersects with H transversally if it satisfies $\dim H \cap \gamma = 0$. Equivalently, any irreducible component in γ is not contained in H.

Definition 2.7. *For a fixed divisor H on X as above, a parabolic stable pair is defined to be a pair*

$$(F, s), \quad s \in F \otimes \mathcal{O}_H, \tag{22}$$

such that the following conditions are satisfied.

- *The sheaf F is a one dimensional ω-semistable sheaf on X.*
- *The one cycle $[F]$ intersects with H transversally.*
- *For any surjection $F \stackrel{\pi}{\twoheadrightarrow} F'$ with $\mu_\omega(F) = \mu_\omega(F')$, we have*

$$(\pi \otimes \mathcal{O}_H)(s) \neq 0.$$

The moduli space of parabolic stable pairs (F, s) satisfying $[F] = \beta$, $\chi(F) = n$ is denoted by

$$M_n^{\mathrm{par}}(X, \beta). \tag{23}$$

By [23, Theorem 2.10], if H satisfies an additional condition given in [23, Lemma 2.9], then the moduli space (23) is a projective scheme even if (n, β) does not satisfy the condition (11). In the case that H does not satisfy the condition in [23, Lemma 2.9], the moduli space (23) is at least a quasi-projective variety. (cf. [23, Remark 2.13].)

Suppose that a reduced one dimensional subscheme $i\colon C \hookrightarrow X$ satisfies $\dim H \cap C = 0$. Then for any $\gamma \in H_2(C, \mathbb{Z})$ with $\beta = i_* \gamma$, we have the subscheme,

$$M_n^{\mathrm{par}}(C, \gamma) \subset M_n^{\mathrm{par}}(X, \beta), \tag{24}$$

corresponding to parabolic stable pairs (F, s) with F supported on C, $[F] = \gamma$ as a one cycle on X and $\chi(F) = n$.

Let

$$\nu_M \colon M_n^{\mathrm{par}}(X, \beta) \to \mathbb{Z},$$

be the Behrend's constructible function [1] on $M_n^{\mathrm{par}}(X, \beta)$. The local parabolic stable pair invariant is defined in the following way.

Definition 2.8. *For $\gamma \in H_2(C, \mathbb{Z})$, we define $\mathrm{DT}_{n,\gamma}^{\mathrm{par}} \in \mathbb{Z}$ to be*

$$\mathrm{DT}_{n,\gamma}^{\mathrm{par}} := \int_{M_n^{\mathrm{par}}(C,\gamma)} \nu_M d\chi. \tag{25}$$

Here as in the local DT theory, we use the Behrend function on $M_n^{\mathrm{par}}(X, \beta)$, not on $M_n^{\mathrm{par}}(C, \gamma)$, to define the local invariant.

2.7 Multiple cover formula via parabolic stable pairs

In [23], we established a relationship between (local) parabolic stable pair invariants and (local) generalized DT invariants. As a result, conjectures in Subsection 2.5 can be translated into a formula relating (local) parabolic stable pair invariants and (local) DT invariants, which are both integer valued.

Let $C \subset X$ be a reduced curve, with irreducible components $C_1, \cdots, C_N$, which intersects with H transversally. As in Definition 2.8, we have the local parabolic stable pair invariants w.r.t. H. For each $\mu \in \mathbb{Q}$, we set the generating series $\mathrm{DT}^{\mathrm{par}}(\mu, C)$ to be

$$\mathrm{DT}^{\mathrm{par}}(\mu, C) := 1 + \sum_{\substack{n \in \mathbb{Z},\ \gamma \in H_2(C,\mathbb{Z})_{>0}, \\ n/\omega \cdot \gamma = \mu}} \mathrm{DT}_{n,\gamma}^{\mathrm{par}} q^n t^\gamma.$$

Here $H_2(C, \mathbb{Z})_{>0} \subset H_2(C, \mathbb{Z})$ is defined by

$$H_2(C, \mathbb{Z})_{>0} := \left\{ \sum_{i=1}^{N} a_i [C_i] : a_i \geq 0 \right\} \setminus \{0\} \subset H_2(C, \mathbb{Z}).$$

The statement of Conjecture 2.5 can be translated into a product expansion formula (26) of $\mathrm{DT}^{\mathrm{par}}(\mu, C)$ below:

Proposition 2.9. **[23, Proposition 4.5]** *We have the formula (21) for any* $(n,\gamma) \in \mathbb{Z} \oplus H_2(C,\mathbb{Z})_{>0}$ *with* $n/\omega \cdot \gamma = \mu$ *if and only if the following formula holds,*

$$\mathrm{DT}^{\mathrm{par}}(\mu, C) = \prod_{\substack{\gamma \in H_2(C,\mathbb{Z})_{>0}, \\ n/\omega\cdot\gamma=\mu}} \left(1-(-1)^{\gamma\cdot H} q^n t^{\gamma}\right)^{(\gamma\cdot H)N_{1,\gamma}}. \tag{26}$$

If we are interested in the formula (21) for a specified (n,γ), then it is enough to check the formula (28) below: let us take the logarithm of $\mathrm{DT}^{\mathrm{par}}(\mu, C)$ and write

$$\log \mathrm{DT}^{\mathrm{par}}(\mu, C) = \sum_{\substack{\gamma \in H_2(C,\mathbb{Z})_{>0}, \\ n/\omega\cdot\gamma=\mu}} \widehat{\mathrm{DT}}^{\mathrm{par}}_{n,\gamma} q^n t^{\gamma}.$$

Note that $\widehat{\mathrm{DT}}^{\mathrm{par}}_{n,\gamma}$ is written as

$$\widehat{\mathrm{DT}}^{\mathrm{par}}_{n,\gamma} = \sum_{l\ge 1} \frac{(-1)^{l-1}}{l} \sum_{\substack{\gamma_1+\cdots+\gamma_l=\gamma, \gamma_i\in H_2(C,\mathbb{Z})_{>0} \\ n_1+\cdots+n_l=n, n_i\in\mathbb{Z}, \\ n_i/\omega\cdot\gamma_i=n/\omega\cdot\gamma}} \prod_{i=1}^{l} \mathrm{DT}^{\mathrm{par}}_{n_i,\gamma_i}. \tag{27}$$

Then we should have the formula,

$$\widehat{\mathrm{DT}}^{\mathrm{par}}_{n,\gamma} = \sum_{k\ge 1, k|(n,\gamma)} \frac{(-1)^{\gamma\cdot H-1}}{k^2} (\gamma\cdot H) N_{1,\gamma/k}. \tag{28}$$

Here the RHS of (28) is $q^n t^{\gamma}$-coefficient of the RHS of (26). Note that (28) is a relationship between $\mathbb{Z}$-valued invariants. (cf. [23, Corollary 4.18].)

2.8 Jacobian actions on the moduli space of parabolic stable pairs

In this subsection, we discuss Jacobian actions on the moduil space of parabolic stable pairs.

Let $i\colon C \hookrightarrow X$ be a reduced curve, and $H \subset X$ a divisor which intersects with C transversally. Let us take

$$n\in\mathbb{Z},\ \gamma\in H_2(C,\mathbb{Z}),$$

and set $\beta = i_*\gamma \in H_2(X,\mathbb{Z})$. Let U be a complex analytic neighborhood of C in X,

$$C\subset U\subset X.$$

Then we have the analytic open subset of the moduli space (10),

$$M_n(U,\beta)\subset M_n(X,\beta),$$

corresponding to ω-semistable one dimensional sheaves F with $\mathrm{Supp}(F)\subset U$.

Let $\mathrm{Pic}^0(U)$ be the group of line bundles on U, whose restriction to any projective curve in U has degree zero. Then we have the action of $\mathrm{Pic}^0(U)$ on $M_n(U, \beta)$ via

$$L \cdot F = F \otimes L,$$

for $L \in \mathrm{Pic}^0(U)$ and $F \in M_n(U, \beta)$. The $\mathrm{Pic}^0(U)$-action preserves the closed subscheme,

$$M_n(C, \gamma) \subset M_n(U, \beta),$$

where the LHS is given in (15).

Let us consider parabolic stable pairs w.r.t. the divisor H as above. Similarly, we have the analytic open subspace,

$$M_n^{\mathrm{par}}(U, \beta) \subset M_n^{\mathrm{par}}(X, \beta),$$

corresponding to parabolic stable pairs (F, s) with $\mathrm{Supp}(F) \subset U$. Let $\widehat{\mathrm{Pic}^0}(U)$ be the group defined by

$$\widehat{\mathrm{Pic}^0}(U) := \{(L, \phi) : L \in \mathrm{Pic}^0(U), \lambda \colon \mathcal{O}_{H \cap U} \stackrel{\cong}{\to} \mathcal{O}_{H \cap U} \otimes L\}. \tag{29}$$

Note that the forgetting map $\widehat{\mathrm{Pic}^0}(U) \ni (L, \phi) \mapsto L \in \mathrm{Pic}^0(U)$ is surjective if U is a sufficiently small analytic neighborhood of C. The group $\widehat{\mathrm{Pic}^0}(U)$ acts on $M_n^{\mathrm{par}}(U, \beta)$ via

$$(L, \lambda) \cdot (F, s) = (F \otimes L, s'), \tag{30}$$

where s' is the image of s by the isomorphism,

$$\mathrm{id}_F \otimes \lambda \colon F \otimes \mathcal{O}_H \stackrel{\cong}{\to} F \otimes L \otimes \mathcal{O}_H.$$

The above isomorphism makes sense since F is supported on U. Obviously the action (30) preserves the closed subspace,

$$M_n^{\mathrm{par}}(C, \gamma) \subset M_n^{\mathrm{par}}(U, \beta), \tag{31}$$

where the LHS is given by the LHS of (24). Also the action (30) is compatible with the $\mathrm{Pic}^0(U)$-action on $M_n(U, \beta)$ and the forgetting morphisms,

$$M_n^{\mathrm{par}}(U, \beta) \ni (F, s) \mapsto F \in M_n(U, \beta),$$
$$\widehat{\mathrm{Pic}^0}(U) \ni (L, \lambda) \mapsto L \in \mathrm{Pic}^0(U).$$

Remark 2.10. *By Chow's theorem, the complex analytic spaces $M_n(U, \beta)$, $M_n^{\mathrm{par}}(U, \beta)$ are regarded as the moduli spaces of ω-semistable sheaves, parabolic stable pairs on U in an analytic sense respectively. Hence the above $\mathrm{Pic}^0(U)$, $\widehat{\mathrm{Pic}^0}(U)$-actions make sense.*

2.9 Local multiple cover formula in simple cases

Finally in this section, we discuss some situations in which the formula (21) is easily proved. Let $C \subset U \subset X$ be as in the previous subsection. In the following lemma, which is partially obtained in [15, Proposition 6.19], we reduce the problem to the case that C has only rational irreducible components.

Lemma 2.11. *Let $C_1, \cdots, C_N$ be the irreducible components of C, and take*

$$\gamma = \sum_{i=1}^{N} a_i[C_i] \in H_2(C, \mathbb{Z})_{>0}.$$

Suppose that there is $1 \le i \le N$ such that $a_i > 0$ and the geometric genus of C_i is is bigger than or equal to one. Then for any $n \in \mathbb{Z}$, we have $N_{n,\gamma} = 0$. In particular, the formula (21) holds.

Proof. Let us consider the $\mathrm{Pic}^0(U)$ action on $M_n(U, \gamma)$ as in the previous subsection. Note that any point $p \in M_n(U, \gamma)$ is represented by an ω-semistable sheaf F which is a direct sum of ω-stable sheaves. If p is fixed by the action of $\mathcal{L} \in \mathrm{Pic}^0(U)$, then we have $F \otimes \mathcal{L} \cong F$. For the normalization $f \colon C_i^{\dagger} \to C_i$, we have

$$f^*(F|_{C_i}) \cong f^*(F|_{C_i}) \otimes f^*(\mathcal{L}|_{C_i}).$$

Taking the determinant of the both sides, we have

$$f^*(\mathcal{L}|_{C_i})^{\otimes k} \cong \mathcal{O}_{C_i^{\dagger}},$$

for some $k \in \mathbb{Z}_{\ge 1}$. Given γ, there is only a finite number of possibilities for the above k, say $k_1, \cdots, k_l$. Since $\mathrm{Pic}^0(C_i^{\dagger})$ is a complex torus of positive dimension, we can find a subgroup

$$S^1 \subset \mathrm{Pic}^0(C_i^{\dagger}), \tag{32}$$

which does not pass through any k_i-torsion points for $1 \le i \le l$. On the other hand, we have the composition of the pull-backs

$$\mathrm{Pic}^0(U) \to \mathrm{Pic}^0(C_i) \to \mathrm{Pic}^0(C_i^{\dagger}). \tag{33}$$

Since U is a sufficiently small analytic neighborhood of C, an argument similar to Subsection 3.2 below shows that both of the arrows in (33) are surjective. Furthermore, the same argument also easily shows that there is a subgroup $S^1 \subset \mathrm{Pic}^0(U)$ which restricts to the subgroup (32) under the restriction (33). Then the action of $\mathrm{Pic}^0(U)$ on $M_n(C, \gamma)$ restricted to $S^1 \subset \mathrm{Pic}^0(U)$ is free, hence the same localization argument of [15, Proposition 6.19] shows the vanishing $N_{n,\gamma} = 0$. $\square$

Next we discuss the case that the class $\gamma \in H_2(C, \mathbb{Z})$ is primitive, i.e. γ is not a multiple of some other element of $H_2(C, \mathbb{Z})$.

Lemma 2.12. *Suppose that $\gamma \in H_2(C, \mathbb{Z})$ is primitive. Then $N_{n,\gamma}$ does not depend on n. In particular, the formula (21) holds.*

Proof. Let $\mathrm{Coh}_C(X)$ be the category of coherent sheaves on X supported on C. We first generalize μ_ω-stability to twisted stability on $\mathrm{Coh}_C(X)$. Let $C \subset U \subset X$ be a sufficiently small analytic neighborhood, and take an element

$$B + i\omega \in H^2(U, \mathbb{C}),$$

such that $\omega|_C$ is ample. For a one dimensional sheaf $F \in \mathrm{Coh}_C(X)$, we set $\mu_{B,\omega}(F) \in \mathbb{Q}$ to be

$$\mu_{B,\omega}(F) := \frac{\chi(F) - [F] \cdot B}{[F] \cdot \omega}.$$

Similarly to Definition 2.1, we have the notion of $\mu_{B,\omega}$-stability on $\mathrm{Coh}_C(X)$, called *twisted stability*. As in the case of μ_ω-stability, we can construct the moduli stack $\mathcal{M}_n(C, \gamma, B+i\omega)$ parameterizing $\mu_{B,\omega}$-semistable objects $F \in \mathrm{Coh}_C(X)$ with $[F] = \gamma$ and $\chi(F) = n$, and the generalized DT invariant defined by the above moduli stack. The same argument of [15, Theorem 6.16] shows that the resulting invariant does not depend on a choice of B and ω, thus coincides with $N_{n,\gamma}$.

Let $C_1, \cdots, C_N$ be the irreducible components of C, and set $\gamma = \sum_{i=1}^N a_i[C_i]$. Since γ is primitive, we have g.c.d.$(a_1, \cdots, a_N) = 1$. Hence we can find $m_1, \cdots, m_N \in \mathbb{Z}$ such that $\sum_{i=1}^N m_i a_i = 1$. Let us take divisors $D_1, \cdots, D_N$ on U such that $D_i \cdot C_j = \delta_{ij}$, and set $D = \sum_{i=1}^N m_i D_i$. (This is possible since U is taken to be a sufficiently small analytic neighborhood of C in X.) Then we have the isomorphism of stacks,

$$\mathcal{M}_n(C, \gamma, B + i\omega) \stackrel{\cong}{\to} \mathcal{M}_{n+1}(C, \gamma, B - D + i\omega),$$

given by $F \mapsto F \otimes \mathcal{O}_U(D)$. Since the generalized DT invariants do not depend on B and ω, the above isomorphism of stacks immediately implies $N_{n,\gamma} = N_{n+1,\gamma}$ for all $n \in \mathbb{Z}$. □

3 Cyclic covers of nodal rational curves

Let X be a smooth projective Calabi-Yau 3-fold over $\mathbb{C}$. In what follows, we fix a connected reduced curve C and an embedding,

$$i \colon C \hookrightarrow X,$$

satisfying the following conditions.

- Any irreducible component of C has geometric genus zero. (We call such a curve as a *rational curve*.)
- The curve C has at worst nodal singularities.

Note that for our purpose, we can always assume the first condition by Lemma 2.11. The geometric genus of C is defined by

$$g(C) := \dim H^1(C, \mathcal{O}_C).$$

When $g(C) = 0$, each irreducible component of C is $\mathbb{P}^1$, and the dual graph of C is simply connected. (See Subsection 3.1 below.) In this case, we say C is a *tree of* $\mathbb{P}^1$. Below we assume that $g(C) > 0$.

We also fix an ample divisor H in X which is smooth, connected, and intersects with C transversally at non-singular points of C.

3.1 Jacobian group of C

We first recall the description of the Jacobian group of a nodal curve C. Suppose that C has δ_n-nodes and has δ_c-irreducible components. Let us take the normalization of C,

$$f \colon C^{\dagger} \to C.$$

We have the exact sequence of sheaves,

$$0 \to \mathcal{O}_C \to f_* \mathcal{O}_{C^{\dagger}} \to \mathbb{C}^{\oplus \delta_n} \to 0.$$

By the long exact sequence of cohomologies, we obtain the isomorphism,

$$\mathbb{C}^{\delta_n - \delta_c + 1} \cong H^1(C, \mathcal{O}_C).$$

In particular the arithmetic genus $g(C)$ satisfies

$$g(C) = \delta_n - \delta_c + 1.$$

Combining the above argument with the standard exact sequence,

$$0 \to \mathbb{Z} \to \mathcal{O}_C \to \mathcal{O}_C^* \to 1,$$

we can easily see that $H^1(C, \mathbb{Z})$ generates $H^1(C, \mathcal{O}_C)$ as a $\mathbb{C}$-vector space,

$$H^1(C, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^1(C, \mathcal{O}_C). \tag{34}$$

Hence we have the isomorphisms,

$$\begin{aligned} (\mathbb{C}^*)^{g(C)} &\cong H^1(C, \mathcal{O}_C)/H^1(C, \mathbb{Z}) \\ &\cong \operatorname{Pic}^0(C). \end{aligned} \tag{35}$$

We can interpret the above isomorphism in terms of the dual graph Γ_C associated to C, determined in the following way:

- The vertices and edges of Γ_C correspond to irreducible components of C and nodal points of C respectively.
- For an edge e corresponding to a nodal point $x \in C$, it connects vertices v_1, v_2 if the corresponding irreducible components C_1, C_2 satisfies $p \in C_1 \cap C_2$. (Note that the case of $v_1 = v_2$ corresponds to a self node.)

Then Γ_C is a connected graph satisfying $b_1(\Gamma_C) = g(C)$. We can interpret (35) as the isomorphism,

$$H^1(\Gamma_C, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^* \cong \operatorname{Pic}^0(C). \tag{36}$$

The isomorphism (36) can be constructed in the following way. For an oriented loop α in Γ_C, we choose an edge $e \subset \alpha$ so that $\Gamma_C \setminus \{e\}$ is still connected. Let $x \in C$ be a nodal point corresponding to the edge e. Let v_1, v_2 be vertices in Γ_C connected by e so that

e starts from v_1 and ends at v_2. Let C_1, C_2 be the irreducible components of C which correspond to v_1, v_2 respectively. We partially normalize C at x, and obtain $C_x^{\dagger}$,

$$f_x \colon C_x^{\dagger} \to C. \tag{37}$$

For $i = 1, 2$, let $C_i^{\dagger}$ be the irreducible component of $C_x^{\dagger}$ which is mapped to C_i by f_x. The preimage of x by f_x is denoted by

$$x_i \in C_i^{\dagger}, \quad i = 1, 2.$$

(In case of $v_1 = v_2$, i.e. $x \in C$ is a self node, we need to fix a correspondence between an orientation of e and a numbering of two points $f_x^{-1}(x)$.) For $z \in \mathbb{C}^*$, we glue the trivial line bundle on $C_x^{\dagger}$ by the isomorphism at x_i,

$$\mathcal{O}_{C_x^{\dagger}} \otimes k(x_1) \ni a \mapsto za \in \mathcal{O}_{C_x^{\dagger}} \otimes k(x_2).$$

The above gluing procedure produces a line bundle on C. The resulting line bundle is independent of a choice of e, and denoted by $L_{\alpha,z}$. The isomorphism (36) is given by sending $\alpha \otimes z$ to $L_{\alpha,z}$.

3.2 Jacobian group of an analytic neighborhood of C

Let $C \subset X$ be as in the previous subsection. We take an analytic open neighborhood U of C in X,

$$C \subset U \subset X.$$

If we take U sufficiently small so that it is homotopically equivalent to C, we have the commutative diagram of exact sequences,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(U,\mathbb{Z}) & \longrightarrow & H^1(U,\mathcal{O}_U) & \longrightarrow & \mathrm{Pic}^0(U) & \longrightarrow & 1 \\
 & & \downarrow \cong & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(C,\mathbb{Z}) & \longrightarrow & H^1(C,\mathcal{O}_C) & \longrightarrow & \mathrm{Pic}^0(C) & \longrightarrow & 1.
\end{array}$$

Here all the vertical morphisms are pull-backs with respect to the inclusion $C \hookrightarrow U$. Let W be the sub $\mathbb{C}$-vector space of $H^1(U, \mathcal{O}_U)$ generated by $H^1(U, \mathbb{Z})$. Then the above commutative diagram and the isomorphism (34) implies that

$$\mathrm{Pic}^0(C) \cong W/H^1(U,\mathbb{Z}) \subset \mathrm{Pic}^0(U). \tag{38}$$

By composing the isomorphism (36) and the embedding (38), we obtain the embedding,

$$H^1(\Gamma_C, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^* \hookrightarrow \mathrm{Pic}^0(U). \tag{39}$$

The embedding (39) can be described in the following way. For each nodal point $x \in C^{\mathrm{sing}}$, let us fix a norm $\|*\|$ on an analytic neighborhood of x in X, and set

$$V_x(\epsilon) = \{x' \in X : \|x' - x\| < \epsilon\}, \tag{40}$$

for $\epsilon > 0$. We construct the following open subsets in U,

$$U_x(\epsilon) := U \cap V_x(\epsilon),$$
$$U'_x(\epsilon) := U \setminus \overline{V}_x(\epsilon).$$

Then for $0 < \epsilon' < \epsilon$, the collection $\{U_x(\epsilon), U'_x(\epsilon')\}$ is an open cover of U.

Suppose that $x \in C$ is contained in two irreducible components C_1, C_2. Then we have

$$U_x(\epsilon) \cap U'_x(\epsilon') \cap C = \coprod_{j=1}^{2} (U_x(\epsilon) \cap U'_x(\epsilon') \cap C_j),$$

and each $U_x(\epsilon) \cap U'_x(\epsilon) \cap C_j$ is homeomorphic to an open annulus in $\mathbb{C}$. Hence if U is chosen to be sufficiently small, we have the decomposition,

$$U_x(\epsilon) \cap U'_x(\epsilon') = W_1 \coprod W_2, \tag{41}$$

such that we have

$$U_x(\epsilon) \cap U'_x(\epsilon') \cap C_j \subset W_j.$$

Let α be an oriented loop in Γ_C and take $z \in \mathbb{C}^*$. As in the previous subsection, we take an edge $e \subset \alpha$ such that $\Gamma_C \setminus \{e\}$ is connected. If e corresponds to the node $x \in C$, we construct the line bundle on U by gluing the trivial line bundles on $U_x(\epsilon)$ and $U'_x(\epsilon')$ by the isomorphism,

$$\begin{aligned} \mathcal{O}_{W_1} \oplus \mathcal{O}_{W_2} \ni (a_1, a_2) & \\ & \mapsto (za_1, a_2) \in \mathcal{O}_{W_1} \oplus \mathcal{O}_{W_2}. \end{aligned}$$

Here we have used the identification by (41),

$$\mathcal{O}_{U_x(\epsilon)\cap U'_x(\epsilon')} = \mathcal{O}_{W_1} \oplus \mathcal{O}_{W_2}.$$

The resulting line bundle is independent of e, and denoted by $\mathcal{L}_{\alpha,z}$. Note that $\mathcal{L}_{\alpha,z}$ restricts to a line bundle $L_{\alpha,z}$ on C, constructed in the previous subsection. When x is a self node, $\mathcal{L}_{\alpha,z}$ can be similarly constructed by replacing C_1, C_2 by analytic branches of C near x. The embedding (39) is given by sending $\alpha \otimes z$ to $\mathcal{L}_{\alpha,z}$.

If we fix an oriented loop α in Γ_C, we have the complex subtorus,

$$\mathbb{C}^* \subset \mathrm{Pic}^0(U), \tag{42}$$

given by the embedding $z \mapsto \mathcal{L}_{\alpha,z}$. Here recall that, in the first part of this section, we took a divisor $H \subset X$ so that it intersects with C at non-singular points on C. Therefore if we furthermore fix an edge $e \subset \alpha$, the above construction of $\mathcal{L}_{\alpha,z}$ yields a canonical isomorphism,

$$\phi_{\alpha,e,z} \colon \mathcal{O}_{H\cap U} \stackrel{\cong}{\to} \mathcal{O}_{H\cap U} \otimes \mathcal{L}_{\alpha,z}.$$

This implies that the embedding (42) lifts to a group homomorphism,

$$\mathbb{C}^* \hookrightarrow \widehat{\mathrm{Pic}^0}(U), \tag{43}$$

given by

$$\alpha \otimes z \mapsto (\mathcal{L}_{\alpha,z}, \phi_{\alpha,e,z}),$$

which is an embedding. Here $\widehat{\mathrm{Pic}^0}(U)$ is defined by (29). Combined with the argument in Subsection 2.8, we have the action of the subtorus (43) on the moduli space of parabolic stable pairs $M_n^{\mathrm{par}}(U,\beta)$, which restricts to the action on the subspace (31).

3.3 Cyclic covers of U

Let us fix a loop $\alpha \subset \Gamma_C$ and consider the subtorus (42), $z \mapsto \mathcal{L}_{\alpha,z}$. The root of unity $z = e^{2\pi i/m}$ corresponds to the line bundle,

$$\mathcal{L}_{\alpha,m} := \mathcal{L}_{\alpha,e^{2\pi i/m}} \in \mathrm{Pic}^0(U),$$

which is an m-torsion element, i.e. there is an isomorphism of line bundles,

$$\psi_{\alpha,m} \colon \mathcal{O}_U \xrightarrow{\cong} \mathcal{L}_{\alpha,m}^{\otimes m}. \tag{44}$$

Given an isomorphism $\psi_{\alpha,m}$ as above, we can construct the complex manifold $\widetilde{U}_{\alpha,m}$ as follows:

$$\widetilde{U}_{\alpha,m} := \{y \in \mathcal{L}_{\alpha,m} : y^{\otimes m} = \psi_{\alpha,m}(1)\}.$$

Here we have regarded the line bundle $\mathcal{L}_{\alpha,m}$ as its total space on U. The projection $\mathcal{L}_{\alpha,m} \to U$ induces the morphism,

$$\sigma_{\alpha,m} \colon \widetilde{U}_{\alpha,m} \to U, \tag{45}$$

which is a covering map of covering degree m. By taking the pull-back of $C \subset U$ by $\sigma_{\alpha,m}$, we obtain the m-fold étale cover of C,

$$\sigma_{\alpha,m}|_{\widetilde{C}_{\alpha,m}} \colon \widetilde{C}_{\alpha,m} \to C. \tag{46}$$

Note that $\widetilde{U}_{\alpha,m}$ is a complex manifold containing $\widetilde{C}_{\alpha,m}$, and satisfies

$$\bigwedge^3 T^{\vee}_{\widetilde{U}_{\alpha,m}} \cong \mathcal{O}_{\widetilde{U}_{\alpha,m}}.$$

The m-fold cover (46) is determined by the isomorphism (44) restricted to C, which is described in the following way. Let us choose an edge $e \subset \alpha$ corresponding to the node $x \in C$, and take a partial normalization $C_x^{\dagger}$ as in (37). Let $x_1, x_2 \in C_x^{\dagger}$ be the preimages of x. We take the m-copies of $\{C_x^{\dagger}, x_1, x_2\}$,

$$\{C_{x,i}, x_{1,i}, x_{2,i}\}, \quad i \in \mathbb{Z}/m\mathbb{Z}.$$

Then $\widetilde{C}_{\alpha,m}$ is given by

$$\widetilde{C}_{\alpha,m} = \bigcup_{i \in \mathbb{Z}/m\mathbb{Z}} C_{x,i}, \tag{47}$$

where $C_{x,i}$ and $C_{x,i+1}$ are glued at $x_{1,i}$ and $x_{2,i+1}$. (See Figure 2.)

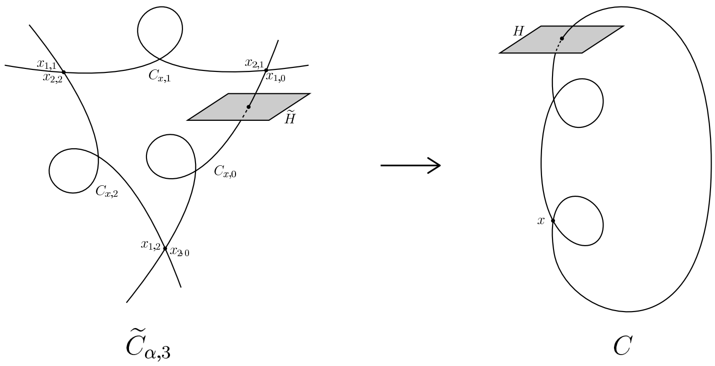

Figure 2: 3-fold cover of a rational curve with two nodes

3.4 Coherent sheaves on $\widetilde{U}$

Let us consider the m-fold cover (45) constructed in the previous subsection. In this subsection, we investigate coherent sheaves on the covering (45).

In what follows, we fix α and m, so we omit these symbols in the notation, e.g. we write $\widetilde{U}_{\alpha,m}$, $\widetilde{C}_{\alpha,m}$ as $\widetilde{U}$, $\widetilde{C}$, etc. We first discuss the category of coherent sheaves on the covering $\widetilde{U}$. By the construction, the $\mathcal{O}_U$-algebra $\sigma_*\mathcal{O}_{\widetilde{U}}$ is given by

$$\sigma_*\mathcal{O}_{\widetilde{U}} \cong \mathcal{O}_U \oplus \mathcal{L}^{-1} \oplus \cdots \oplus \mathcal{L}^{-m+1}.$$

The algebra structure on the RHS is given by, for $(a,b) \in \mathcal{L}^{-i} \times \mathcal{L}^{-j}$,

$$(a,b) \mapsto \begin{cases} a \otimes b, & i+j < m, \\ \psi(a\otimes b), & i+j \geq m. \end{cases}$$

Here $\psi = \psi_{\alpha,m}$ is the isomorphism given in (44). By the above isomorphism of $\mathcal{O}_U$-algebras, it is easy to show the following lemma:

Lemma 3.1. *The push-forward functor σ_* identifies the category $\mathrm{Coh}(\widetilde{U})$ with the category of pairs,*

$$(F, \phi_F), \quad \phi_F \colon F \to F \otimes \mathcal{L},$$

where $F \in \mathrm{Coh}(U)$ and ϕ_F is a morphism in $\mathrm{Coh}(U)$ satisfying

$$\overbrace{\phi_F \circ \cdots \circ \phi_F}^{m} = \mathrm{id}_F \otimes \psi, \tag{48}$$

as morphisms $F \to F \otimes \mathcal{L}^{\otimes m}$.

Note that a choice of a loop $\alpha \subset \Gamma_C$ and an edge $e \subset \alpha$ yields a lift of (42) by (43), which sends $z \in \mathbb{C}^*$ to $(\mathcal{L}_{\alpha,z}, \phi_{\alpha,e,z})$ in the notation of Subsection 3.2. In particular, a cyclic subgroup of order m in $\mathrm{Pic}^0(U)$ generated by the line bundle $\mathcal{L} = \mathcal{L}_{\alpha, e^{2\pi i/m}}$ lifts to an embedding into $\widehat{\mathrm{Pic}^0}(U)$,

$$\mathbb{Z}/m\mathbb{Z} \subset \mathbb{C}^* \subset \widehat{\mathrm{Pic}^0}(U). \tag{49}$$

In other words, the structure sheaf $\mathcal{O}_{H\cap U}$ of the divisor $H \cap U$ in U is equipped with an isomorphism,

$$\lambda_H \colon \mathcal{O}_H \stackrel{\cong}{\to} \mathcal{O}_H \otimes \mathcal{L}, \tag{50}$$

such that $\phi_{\mathcal{O}_H} := \lambda_H$ satisfies the condition (48). Here we have denoted $H \cap U$ just by H for simplicity. Hence λ_H determines a lift of $\mathcal{O}_H$ to a coherent sheaf on $\widetilde{U}$. This lift is a structure sheaf of some divisor in $\widetilde{U}$,

$$\widetilde{H} \subset \widetilde{U}, \tag{51}$$

which intersects with $\widetilde{C}$ transversally. (cf. Figure 2.) Later we will need the following lemma on the compatibility of ψ with λ_H.

Lemma 3.2. *(i) The isomorphism ψ in (44) induces the isomorphism,*

$$\widetilde{\psi} \colon \mathcal{O}_{\widetilde{U}} \stackrel{\cong}{\to} \sigma^* \mathcal{L}. \tag{52}$$

(ii) The isomorphism λ_H in (50) induces the isomorphism,

$$\widetilde{\lambda}_H \colon \bigoplus_{g \in \mathbb{Z}/m\mathbb{Z}} g_* \mathcal{O}_{\widetilde{H}} \stackrel{\cong}{\to} \sigma^* \mathcal{O}_H. \tag{53}$$

Here $\mathbb{Z}/m\mathbb{Z}$ acts on $\widetilde{U}$ as a deck transformation of the covering $\sigma \colon \widetilde{U} \to U$.

(iii) The compositions,

$$(\widetilde{\lambda}_H \otimes \mathrm{id}_{\sigma^*\mathcal{L}})^{-1} \circ (\widetilde{\psi} \otimes \mathrm{id}_{\sigma^*\mathcal{O}_H}) \circ \widetilde{\lambda}_H, \tag{54}$$

$$(\widetilde{\lambda}_H \otimes \mathrm{id}_{\sigma^*\mathcal{L}})^{-1} \circ \sigma^* \lambda_H \circ \widetilde{\lambda}_H, \tag{55}$$

determine two isomorphisms,

$$\bigoplus_{g \in \mathbb{Z}/m\mathbb{Z}} g_* \mathcal{O}_{\widetilde{H}} \to \bigoplus_{g \in \mathbb{Z}/m\mathbb{Z}} g_* \mathcal{O}_{\widetilde{H}} \otimes \sigma^* \mathcal{L}, \tag{56}$$

Both of (54) and (55) preserve direct summands of both sides of (56), and they are related by

$$(54)|_{g_*\mathcal{O}_{\widetilde{H}}} = e^{2\pi g i/m} \cdot (55)|_{g_*\mathcal{O}_{\widetilde{H}}}.$$

Proof. (i) By Lemma 3.1, it is enough to construct $\sigma_* \mathcal{O}_{\widetilde{U}}$-module isomorphism $\sigma_* \widetilde{\psi}$ between $\sigma_* \mathcal{O}_{\widetilde{U}}$ and $\sigma_* \sigma^* \mathcal{L} \cong \mathcal{L} \otimes \sigma_* \mathcal{O}_{\widetilde{U}}$. It is constructed as

$$\begin{aligned}&(x_0, x_1, \cdots, x_{m-1}) \in \mathcal{O}_U \oplus \mathcal{L} \oplus \cdots \oplus \mathcal{L}^{-m+1} \\ &\quad \mapsto (\psi(x_{m-1}), x_0, \cdots, x_{m-2}) \in \mathcal{L} \oplus \mathcal{O}_U \oplus \cdots \oplus \mathcal{L}^{-m+2}.\end{aligned} \tag{57}$$

(ii) As in (i), it is enough to construct $\sigma_* \widetilde{\lambda}_H$. Note that $\sigma_* g_* \mathcal{O}_{\widetilde{H}}$ is isomorphic to $\mathcal{O}_H$, whose $\sigma_* \mathcal{O}_{\widetilde{U}}$-module structure is given by $e^{2\pi g i/m} \cdot \lambda_H$. The morphism $\sigma_* \widetilde{\lambda}_H$ restricted to $\sigma_* g_* \mathcal{O}_{\widetilde{H}}$ is constructed to be

$$\begin{aligned}x \in \mathcal{O}_H \mapsto &(x, e^{-2\pi g i/m} \lambda_H^{-1}(x), \cdots, e^{-2\pi(m-1)g i/m} \lambda_H^{-m+1}(x)) \\ &\in \mathcal{O}_H \oplus \mathcal{L}|_H^{-1} \oplus \cdots \oplus \mathcal{L}|_H^{-m+1} \cong \sigma_* \sigma^* \mathcal{O}_H.\end{aligned} \tag{58}$$

It is easy to check that the $\sigma_* \widetilde{\lambda}_H$ constructed as above is an isomorphism of $\sigma_* \mathcal{O}_{\widetilde{U}}$-modules.

(iii) The statement of (iii) easily follows by comparing (57) and (58). □

3.5 Cyclic neighborhoods

In this subsection, we generalize the construction in the previous subsections and define the notion of cyclic neighborhoods. This will be used for the induction argument in the proof of Theorem 4.7 below.

Definition 3.3. *Let X, $C \subset U \subset X$ be as before. Let C' be a connected nodal curve which is embedded into a three dimensional complex manifold U'. We say $C' \subset U'$ is a cyclic neighborhood of $C \subset X$ if there is a sequence of local immersions,*

$$\sigma' \colon U' = U_{(R)} \stackrel{\sigma'_{(R)}}{\to} U_{(R-1)} \to \cdots \to U_{(1)} \stackrel{\sigma'_{(1)}}{\to} U_{(0)} = U, \tag{59}$$

and connected nodal curves $C_{(i)} \subset U_{(i)}$ such that the following conditions hold:

- *For each i, $U_{(i)}$ is a small analytic neighborhood of $C_{(i)}$, and $C_{(R)} = C'$, $C_{(0)} = C$.*
- *For each i, the map $\sigma'_{(i)} \colon U_{(i)} \to U_{(i-1)}$ factorizes as*

  $$\sigma'_{(i)} \colon U_{(i)} \subset \widetilde{U}_{(i-1)} \stackrel{\sigma_{(i)}}{\to} U_{(i-1)},$$

  where $\sigma_{(i)}$ is a cyclic covering with respect to some non-trivial loop in the graph $\Gamma_{C_{(i-1)}}$, and $U_{(i)} \subset \widetilde{U}_{(i-1)}$ is an open immersion.
- *The curves $C_{(i)}$ satisfy $\sigma_{(i)}(C_{(i)}) \subset C_{(i-1)}$.*

Below, a cyclic neighborhood $C' \subset U'$ of $C \subset X$ will be written as

$$(C' \subset U') \stackrel{\sigma'}{\to} (C \subset X),$$

where σ' is a composition of local immersions (59). In order to discuss parabolic stable pair invariants on cyclic neighborhoods, we define the notion of a lift of $H \subset X$ as follows:

Definition 3.4. *For a cyclic neighborhood $C' \subset U'$ of $C \subset X$, a divisor $H' \subset U'$ is called a lift of $H \subset X$ if the following conditions hold:*

- *There is a sequence (59) together with divisors $H_{(i)} \subset U_{(i)}$ such that $H_{(i)}$ is obtained by lifting $H_{(i-1)}$ to $\widetilde{U}_{(i-1)} \to U_{(i-1)}$ as in (51) and restricting to $U_{(i)} \subset \widetilde{U}_{(i-1)}$.*
- *We have $H \cap U = H_{(0)}$ and $H' = H_{(R)}$.*

Given a cyclic neighborhood $(C' \subset U') \stackrel{\sigma'}{\to} (C \subset X)$ with a lift $H' \subset U'$ of $H \subset X$, we can similarly define the notion of parabolic stable pairs,

$$(F', s') \quad s' \in F' \otimes \mathcal{O}_{H'}, \tag{60}$$

with F' one dimensional $\sigma'^{*}\omega$-semistable coherent sheaf on U', satisfying the same axiom as in Definition 2.7. Here we have to assume that the support of F' is compact in order to define $\sigma'^{*}\omega$-semistability. In what follows, we always assume that $\sigma'^{*}\omega$-semistable sheaves on U' have compact supports.

3.6 Moduli spaces and counting invariants on cyclic neighborhoods

Let $(C' \subset U') \stackrel{\sigma'}{\to} (C \subset X)$ be a cyclic neighborhood and $H' \subset U'$ a lift of $H \subset X$. Since U' is just a complex manifold, we need to work with an analytic category in discussing moduli spaces of semistable sheaves or parabolic stable pairs. A general moduli theory of sheaves on complex analytic spaces seems to be not yet established. However, since a cyclic neighborhood admits a sequence (59), we can inductively show the existence of analytic moduli spaces on cyclic neighborhoods. The result is formulated as follows:

Lemma 3.5. *In the above situation, take $n \in \mathbb{Z}$ and $\beta' \in H_2(U', \mathbb{Z})$.*

(i) There is an analytic stack of finite type $\mathcal{M}_n(U', \beta')$, which parameterizes $\sigma'^{*}\omega$-semistable one dimensional sheaves $F' \in \mathrm{Coh}(U')$ satisfying

$$[F'] = \beta', \quad \chi(F') = n'. \tag{61}$$

(ii) There is an analytic space of finite type $M_n^{\mathrm{par}}(U', \beta')$, which represents a functor of families of parabolic stable pairs (F', s') satisfying (61).

Proof. (i) If $(C', U') = (C, U)$, then $\mathcal{M}_n(U', \beta')$ is obtained as an analytic open substack of an Artin stack $\mathcal{M}_n(X, \beta')$. Suppose that U' is an open subset of an m-fold cover $\sigma \colon \widetilde{U} \to U$ given by $\mathcal{L} \in \mathrm{Pic}^0(U)$, as in Subsection 3.4 and Subsection 3.5. As an abstract stack, there is a 1-morphism,

$$\mathcal{M}_n(\widetilde{U}, \beta') \to \mathcal{M}_n(U, \sigma_* \beta'), \tag{62}$$

by sending F' to $\sigma_* F'$. In fact, since we have the decomposition,

$$\sigma^* \sigma_* F' \cong \bigoplus_{g \in \mathbb{Z}/m\mathbb{Z}} g_* F',$$

the sheaf $\sigma_* F'$ is ω-semistable by [10, Lemma 3.2.2]. By Lemma 3.1, the fiber of (62) at $[F] \in \mathcal{M}_n(U, \sigma_*\beta')$ is given by the closed subset of the finite dimensional vector space,

$$\phi_F \in \mathrm{Hom}(F, F \otimes \mathcal{L})$$

satisfying (48). Therefore (62) is representable, and $\mathcal{M}_n(U', \beta')$ is an analytic stack of finite type. A general case is obtained by applying the above argument to the sequence (59).

(ii) Let $\mathcal{M}_n^{\mathrm{par}}(U', \beta')$ be an abstract stack of families of parabolic stable pairs on U'. We have the forgetting 1-morphism,

$$\mathcal{M}_n^{\mathrm{par}}(U', \beta') \to \mathcal{M}_n(U', \beta'), \tag{63}$$

sending (F', s') to F'. The fiber of (63) at $[F'] \in \mathcal{M}_n(U', \beta')$ is given by an open subset of

$$s' \in F' \otimes \mathcal{O}_{H'} \cong \mathbb{C}^{\beta' \cdot H'},$$

giving parabolic stable pair structures on F'. Hence (63) is a representable smooth morphism, and in particular $\mathcal{M}_n^{\mathrm{par}}(U', \beta')$ is an analytic stack of finite type. However, as in [23, Lemma 2.7], there are no non-trivial stabilizer groups in $\mathcal{M}_n^{\mathrm{par}}(U', \beta')$. This implies that $\mathcal{M}_n^{\mathrm{par}}(U', \beta')$ is represented by an analytic space of finite type, $M_n^{\mathrm{par}}(U', \beta')$. □

In the above situation, let us take

$$\gamma' \in H_2(C', \mathbb{Z}), \quad i'_*\gamma' = \beta',$$

where $i' \colon C' \hookrightarrow U'$ is the embedding. Similarly to (19), there is the sub analytic stack,

$$\mathcal{M}_n(C', \gamma') \subset \mathcal{M}_n(U', \beta'), \tag{64}$$

parameterizing $\sigma'^*\omega$-semistable sheaves F' with $[F'] = \gamma'$ as a one cycle supported on C' and $\chi(F') = n$. Also similarly to (24), we have the sub analytic space,

$$M_n^{\mathrm{par}}(C', \gamma') \subset M_n^{\mathrm{par}}(U', \beta'),$$

parameterizing parabolic stable pairs (F', s') as above. It is straightforward to generalize the notion of Hall algebras, Behrend functions, to our analytic category on U'. Consequently, we have the invariants,

$$N_{n,\gamma'}(U') \in \mathbb{Q}, \quad \mathrm{DT}_{n,\gamma'}^{\mathrm{par}}(U') \in \mathbb{Z}, \tag{65}$$

as in (18), (25) respectively. By replacing $\mathrm{DT}_{n,\gamma}^{\mathrm{par}}$ by $\mathrm{DT}_{n,\gamma'}^{\mathrm{par}}(U')$ in the RHS of (27), we can also define the invariant,

$$\widehat{\mathrm{DT}}_{n,\gamma'}^{\mathrm{par}}(U') \in \mathbb{Q}. \tag{66}$$

In principle, as we discussed in Subsection 2.7, the same arguments in the proof of [23, Corollary 4.18] should show the equivalence between the multiple cover formula of $N_{n,\gamma'}(U')$

and the formula (28) for $\widehat{\mathrm{DT}}^{\mathrm{par}}_{n,\gamma'}(U')$. However there is one technical obstruction to do this, namely we need to show that the moduli stack $\mathcal{M}_n(U',\beta')$ is locally written as a critical locus of some holomorphic function. (This is used in the proof of [23, Theorem 3.16].) The moduli stack $\mathcal{M}_n(U,\beta)$ satisfies this condition, due to the fact that U is an open subset of a projective Calabi-Yau 3-fold X, and the result by Joyce-Song [15, Theorem 5.3]. Unfortunately we are not able to prove this critical locus condition for $\mathcal{M}_n(U',\beta')$. The required condition is formulated in the following conjecture:

Conjecture 3.6. *Let X be a smooth projective Calabi-Yau 3-fold over $\mathbb{C}$, and C a connected nodal curve with $C \subset X$. Let $(C' \subset U') \stackrel{\sigma'}{\to} (C \subset X)$ be a cyclic neighborhood, and for a point $[F'] \in \mathcal{M}_n(U',\beta')$, let G be a maximal reductive subgroup in $\mathrm{Aut}(F')$. Then there exists a G-invariant analytic open subset V of 0 in $\mathrm{Ext}^1(F',F')$, a G-invariant holomorphic function $f\colon V \to \mathbb{C}$ with $f(0) = df|_0 = 0$, and a smooth morphism of complex analytic stacks,*

$$[\{df = 0\}/G] \to \mathcal{M}_n(U',\beta'),$$

of relative dimension $\dim \mathrm{Aut}(F') - \dim G$.

The above conjecture is a complex analytic version of [15, Theorem 5.3], and true if U' is an open subset of a projective Calabi-Yau 3-fold by [15, Theorem 5.3]. As an analogy of Proposition 2.9, under the assumption of Conjecture 3.6, we have the following:

Proposition 3.7. *Let $C' \subset U'$ be a cyclic neighborhood of $C \subset X$ with a lift $H' \subset U'$ of $H \subset X$. Suppose that $C' \subset U'$ satisfies the condition of Conjecture 3.6. Then we have the formula*

$$N_{n,\gamma'}(U') = \sum_{k\ge 1, k|(n,\gamma')} \frac{1}{k^2} N_{1,\gamma'/k}(U'), \tag{67}$$

if and only if we have the formula,

$$\widehat{\mathrm{DT}}^{\mathrm{par}}_{n,\gamma'}(U') = \sum_{k\ge 1, k|(n,\gamma')} \frac{(-1)^{\gamma'\cdot H'-1}}{k^2} (\gamma'\cdot H') N_{1,\gamma'/k}(U'). \tag{68}$$

Proof. Since we assume Conjecture 3.6, the same argument of [23, Corollary 4.18] works. □

4 Counting invariants under cyclic coverings

This section is a core of this paper. We will compare the invariants (65), (66) constructed in the previous section under cyclic coverings. Using this, we will show our main result which reduces the multiple cover formula of $N_{n,\gamma}$ to that of $N_{n,\gamma'}(U')$ for all cyclic neighborhood $C' \subset U'$ with C' a tree of $\mathbb{P}^1$.

4.1 Comparison of moduli spaces of parabolic stable pairs

Let $C \subset U \subset X$ be as in the previous section. As in Subsection 3.4 and Subsection 3.5, let $\sigma \colon \widetilde{U} \to U$ be a cyclic covering of order m, and $\widetilde{H} \subset \widetilde{U}$ is a lift of H. Note that $\widetilde{C} \subset \widetilde{U}$ is a cyclic neighborhood of $C \subset X$, so we have the moduli space of parabolic stable pairs on $\widetilde{U}$ by Lemma 3.5. In this subsection, we compare moduli spaces of parabolic stable pairs under the above covering.

Note that by (49) and the argument in Subsection 2.8, we have the $\mathbb{C}^*$-action on the moduli spaces in (31), which restricts to the $\mathbb{Z}/m\mathbb{Z}$-action on these moduli spaces. We have the following lemma:

Lemma 4.1. *For $\widetilde{\beta} \in H_2(\widetilde{U}, \mathbb{Z})$ with $\beta = \sigma_* \widetilde{\beta}$, there is a natural morphism of complex analytic spaces,*

$$\sigma_* \colon M_n^{\mathrm{par}}(\widetilde{U}, \widetilde{\beta}) \to M_n^{\mathrm{par}}(U, \beta)^{\mathbb{Z}/m\mathbb{Z}}. \tag{69}$$

Proof. For a point $(\widetilde{F}, \widetilde{s}) \in M_n^{\mathrm{par}}(\widetilde{U}, \widetilde{\beta})$, we construct the pair

$$(F, s) := \sigma_*(\widetilde{F}, \widetilde{s}),$$

in the following way: first we set $F := \sigma_* \widetilde{F}$, which is ω-semistable as in the proof of Lemma 3.5 (i). Then we have

$$\begin{aligned} F \otimes \mathcal{O}_H &\cong \sigma_*(\widetilde{F} \otimes \sigma^* \mathcal{O}_H) \\ &\cong \bigoplus_{g \in \mathbb{Z}/m\mathbb{Z}} \sigma_*(\widetilde{F} \otimes g_* \mathcal{O}_{\widetilde{H}}), \end{aligned}$$

where the second isomorphism is induced by (53). Then we have the embedding into the direct summand,

$$\sigma_* \colon \widetilde{F} \otimes \mathcal{O}_{\widetilde{H}} \hookrightarrow F \otimes \mathcal{O}_H, \tag{70}$$

and we set $s := \sigma_* \widetilde{s}$.

We would like to see that (F, s) is a parabolic stable pair on U. Suppose by contradiction that there is a surjection $\pi \colon F \twoheadrightarrow F'$, where F' is an ω-semistable sheaf with $\mu_\omega(F) = \mu_\omega(F')$, satisfying

$$(\pi \otimes \mathcal{O}_H)(s) = 0. \tag{71}$$

By taking the adjunction, we have the non-zero map,

$$\widetilde{F} \to \sigma^! F' = \sigma^* F'. \tag{72}$$

By [10, Lemma 3.2.2], $\sigma^* F'$ is $\sigma^*\omega$-semistable with $\mu_{\sigma^*\omega}(\widetilde{F}) = \mu_{\sigma^*\omega}(\sigma^* F')$, hence the image of the morphism (72), denoted by A, is also $\sigma^*\omega$- semistable with $\mu_{\sigma^*\omega}(A) = \mu_{\sigma^*\omega}(\widetilde{F})$. We have the sequence,

$$\widetilde{F} \otimes \mathcal{O}_{\widetilde{H}} \twoheadrightarrow A \otimes \mathcal{O}_{\widetilde{H}} \hookrightarrow \sigma^* F' \otimes \mathcal{O}_{\widetilde{H}},$$

which takes $\widetilde{s}$ to zero by the construction of s and (71). Since the right arrow of the above sequence is injective, the surjection $\widetilde{F} \twoheadrightarrow A$ violates the condition of parabolic stability of $(\widetilde{F}, \widetilde{s})$. This is a contradiction, hence (F, s) is a parabolic stable pair.

We check that (F, s) is $\mathbb{Z}/m\mathbb{Z}$-invariant. This is equivalent to that the parabolic stable pair

$$(F \otimes \mathcal{L}, (\mathrm{id}_F \otimes \lambda_H)(s)),$$

is isomorphic to (F, s), where λ_H is given in (50). Since $F = \sigma_* \widetilde{F}$, the isomorphism (52) and the projection formula induce the isomorphism $\psi_F := \sigma_*(\mathrm{id}_{\sigma_* \widetilde{F}} \otimes \widetilde{\psi})$,

$$\psi_F \colon \sigma_* \widetilde{F} \to \sigma_*(\widetilde{F} \otimes \sigma^* \mathcal{L}) \cong \sigma_* \widetilde{F} \otimes \mathcal{L}. \tag{73}$$

We need to check that

$$(\psi_F \otimes \mathcal{O}_H)(s) = (\mathrm{id}_F \otimes \lambda_H)(s).$$

The above equality follows from that s comes from the LHS of (70) and Lemma 3.2 (iii).

The above argument shows that we have a set theoretic map σ_*. It is straightforward to generalize the above arguments to families of parabolic stable pairs. Namely for a complex analytic space S, let

$$\mathrm{Hom}(S, M_n^{\mathrm{par}}(U, \beta)),$$

be the set of morphisms from S to $M_n^{\mathrm{par}}(U, \beta)$ as complex analytic spaces. Then, since $M_n^{\mathrm{par}}(U, \beta)$ is a fine moduli space, giving an element in $\mathrm{Hom}(S, M_n^{\mathrm{par}}(U, \beta))$ is equivalent to giving a flat family of parabolic stable pairs over S. We can easily generalize the construction of σ_* to a functorial map,

$$\mathrm{Hom}(S, M_n^{\mathrm{par}}(\widetilde{U}, \widetilde{\beta})) \to \mathrm{Hom}(S, M_n^{\mathrm{par}}(U, \beta))^{\mathbb{Z}/m\mathbb{Z}},$$

which gives a morphism (69) as complex analytic spaces. □

We have the following proposition.

Proposition 4.2. *The morphism (69) induces the isomorphism of complex analytic spaces,*

$$\sigma_* \colon \coprod_{\substack{\widetilde{\beta} \in H_2(\widetilde{U}, \mathbb{Z}), \\ \sigma_* \widetilde{\beta} = \beta}} M_n^{\mathrm{par}}(\widetilde{U}, \widetilde{\beta}) \xrightarrow{\cong} M_n^{\mathrm{par}}(U, \beta)^{\mathbb{Z}/m\mathbb{Z}}, \tag{74}$$

which restricts to the isomorphism,

$$\sigma_* \colon \coprod_{\substack{\widetilde{\gamma} \in H_2(\widetilde{C}, \mathbb{Z}), \\ \sigma_* \widetilde{\gamma} = \gamma}} M_n^{\mathrm{par}}(\widetilde{C}, \widetilde{\gamma}) \xrightarrow{\cong} M_n^{\mathrm{par}}(C, \gamma)^{\mathbb{Z}/m\mathbb{Z}}. \tag{75}$$

Proof. It is enough to show the isomorphism (74). We first show that the morphism σ_* is injective. Suppose that there are two parabolic stable pairs,

$$(\widetilde{F}_i, \widetilde{s}_i) \in M_n^{\mathrm{par}}(\widetilde{U}, \widetilde{\beta}_i), \ i = 1, 2,$$

which are sent to the same point by σ_*. This implies that there is an isomorphism of sheaves,

$$\phi \colon \sigma_* \widetilde{F}_1 \xrightarrow{\cong} \sigma_* \widetilde{F}_2,$$

such that $\phi \otimes \mathrm{id}_{\mathcal{O}_H}$ sends $\sigma_* \widetilde{s}_1$ to $\sigma_* \widetilde{s}_2$. Let us check that ϕ is $\sigma_* \mathcal{O}_{\widetilde{U}}$-module homomorphism. This is equivalent to that the following diagram commutes:

$$\begin{array}{ccc} \sigma_* \widetilde{F}_1 & \xrightarrow{\phi} & \sigma_* \widetilde{F}_2 \\ \psi_1 \downarrow & & \downarrow \psi_2 \\ \sigma_* \widetilde{F}_1 \otimes \mathcal{L} & \xrightarrow{\phi \otimes \mathrm{id}_{\mathcal{L}}} & \sigma_* \widetilde{F}_2 \otimes \mathcal{L}. \end{array} \tag{76}$$

Here ψ_i are induced by the projection formula and the isomorphism (52) as in (73). We check that

$$\{(\psi_2 \circ \phi) \otimes \mathrm{id}_{\mathcal{O}_H}\} (\sigma_* \widetilde{s}_1) = \{\{(\phi \otimes \mathrm{id}_{\mathcal{L}}) \circ \psi_1\} \otimes \mathrm{id}_{\mathcal{O}_H}\} (\sigma_* \widetilde{s}_1), \tag{77}$$

as elements in $\sigma_* \widetilde{F}_2 \otimes \mathcal{L} \otimes \mathcal{O}_H$. In fact if (77) holds, then $\phi^{-1} \circ \psi_2^{-1} \circ (\phi \otimes \mathrm{id}_{\mathcal{L}}) \circ \psi_1$ is an automorphism of $(\sigma_* F_1, \sigma_* s_1)$, hence identity by [23, Lemma 2.7], i.e. the diagram (76) commutes.

The equality (77) can be checked as follows. The LHS of (77) is

$$\begin{aligned} &(\psi_2 \otimes \mathrm{id}_{\mathcal{O}_H})(\phi \otimes \mathrm{id}_{\mathcal{O}_H})(\sigma_* \widetilde{s}_1) \\ &= (\psi_2 \otimes \mathrm{id}_{\mathcal{O}_H})(\sigma_* s_2) \\ &= (\mathrm{id}_{\sigma_* \widetilde{F}_2} \otimes \lambda_H)(\sigma_* \widetilde{s}_2), \end{aligned}$$

where we have used Lemma 3.2 (iii) for the second equality. Similarly the RHS of (77) is

$$\begin{aligned} &(\phi \otimes \mathrm{id}_{\mathcal{L} \otimes \mathcal{O}_H})(\psi_1 \otimes \mathrm{id}_{\mathcal{O}_H})(\sigma_* \widetilde{s}_1) \\ &= (\phi \otimes \mathrm{id}_{\mathcal{L} \otimes \mathcal{O}_H})(\mathrm{id}_{\sigma_* \widetilde{F}_1} \otimes \lambda_H)(\sigma_* \widetilde{s}_1) \\ &= (\mathrm{id}_{\sigma_* \widetilde{F}_2} \otimes \lambda_H)(\phi \otimes \mathrm{id}_{\mathcal{O}_H})(\sigma_* \widetilde{s}_1) \\ &= (\mathrm{id}_{\sigma_* \widetilde{F}_2} \otimes \lambda_H)(\sigma_* \widetilde{s}_2). \end{aligned}$$

Therefore the equality (77) holds.

Now since the diagram (76) commutes, the isomorphism ϕ lifts to an isomorphism $\widetilde{\phi}$ between $\widetilde{F}_1$ and $\widetilde{F}_2$, such that $\widetilde{\phi} \otimes \mathrm{id}_{\mathcal{O}_{\widetilde{H}}}$ takes $\widetilde{s}_1$ to $\widetilde{s}_2$. Hence $(\widetilde{F}_1, \widetilde{s}_1)$ and $(\widetilde{F}_2, \widetilde{s}_2)$ are isomorphic, and σ_* is injective.

Next we prove that σ_* is surjective. Let us take a point $(F, s) \in M_n^{\mathrm{par}}(U, \beta)$, which is $\mathbb{Z}/m\mathbb{Z}$-invariant. This means that there is an isomorphism of sheaves,

$$\phi_F \colon F \to F \otimes \mathcal{L},$$

which satisfies that

$$(\phi_F \otimes \mathrm{id}_{\mathcal{O}_H})(s) = (\mathrm{id}_F \otimes \lambda_H)(s). \tag{78}$$

We show that ϕ_F satisfies the condition (48). We consider the morphism,

$$(\mathrm{id}_F \otimes \psi)^{-1} \circ \overbrace{\phi_F \circ \cdots \circ \phi_F}^{m} \colon F \to F. \tag{79}$$

After applying $\otimes \mathcal{O}_H$, the above morphism takes s to s since (78) holds and $\phi_{\mathcal{O}_H} = \lambda_H$ satisfies the condition (48). Therefore the morphism (78) is an identity by [23, Lemma 2.7], which implies that ϕ_F satisfies (48).

Since ϕ_F satisfies (48), the sheaf F is written as $\sigma_* \widetilde{F}$ for some $\widetilde{F} \in \mathrm{Coh}(\widetilde{U})$, such that the morphism ϕ_F is identified with

$$\psi_F \colon \sigma_* \widetilde{F} \to \sigma_*(\widetilde{F} \otimes \sigma^* \mathcal{L}) \cong \sigma_* \widetilde{F} \otimes \mathcal{L}, \tag{80}$$

which is a composition of (52) and the projection formula as in (73). To show that σ_* is surjective, it is enough to check that $s = \sigma_* \widetilde{s}$ for some $\widetilde{s} \in \widetilde{F} \otimes \mathcal{O}_{\widetilde{H}}$. This follows from (78), the fact that ϕ_F is identified with (80) and Lemma 3.2 (iii).

Now we have proved that σ_* is a set theoretic bijection. Similarly to Lemma 4.1, the above arguments can be easily generalized to families of parabolic stable pairs. Namely in the notation of the proof of Lemma 4.1, we have the bijection,

$$\sigma_* \colon \coprod_{\sigma_* \widetilde{\beta} = \beta} \mathrm{Hom}(S, M_n^{\mathrm{par}}(\widetilde{U}, \widetilde{\beta})) \stackrel{\cong}{\to} \mathrm{Hom}(S, M_n^{\mathrm{par}}(U, \beta))^{\mathbb{Z}/m\mathbb{Z}}.$$

Therefore the morphism σ_* is an isomorphism as complex analytic spaces. □

4.2 Comparison of moduli spaces of stable sheaves

Similarly to the previous subsection, we can also compare moduli spaces of one dimensional stable sheaves under the covering $\sigma \colon \widetilde{U} \to U$. Note that the moduli space $M_1(U, \beta)$ consists of one dimensional ω-stable sheaves on U, and it admits $\mathrm{Pic}^0(U)$-action. In particular it restricts to $\mathbb{C}^*$-action w.r.t. the embedding (42), and we have the $\mathbb{Z}/m\mathbb{Z}$-action by the embedding (49).

Let

$$M_1(\widetilde{C}, \widetilde{\gamma}) \subset M_1(\widetilde{U}, \widetilde{\beta}), \tag{81}$$

be the coarse moduli spaces of analytic stacks (64) for $n=1$, $U' = \widetilde{U}$, $\gamma' = \widetilde{\gamma}$ and $\beta' = \widetilde{\beta}$. Similarly to $M_1(U, \beta)$, points in the analytic spaces (81) correspond to $\sigma^* \omega$-stable sheaves. We have the following proposition.

Proposition 4.3. *We have the morphisms of complex analytic spaces,*

$$\sigma_* \colon \coprod_{\substack{\widetilde{\beta} \in H_2(\widetilde{U}, \mathbb{Z}), \\ \sigma_* \widetilde{\beta} = \beta}} M_1(\widetilde{U}, \widetilde{\beta}) \longrightarrow M_1(U, \beta)^{\mathbb{Z}/m\mathbb{Z}}, \tag{82}$$

$$\sigma_* \colon \coprod_{\substack{\widetilde{\gamma} \in H_2(\widetilde{C}, \mathbb{Z}), \\ \sigma_* \widetilde{\gamma} = \gamma}} M_1(\widetilde{C}, \widetilde{\gamma}) \longrightarrow M_1(C, \gamma)^{\mathbb{Z}/m\mathbb{Z}}. \tag{83}$$

If $m \gg 0$, then the above morphisms are covering maps of covering degree m.

Proof. It is enough to show the claim for (82). A proof similar to Lemma 4.1 shows the existence of the morphism (82). In order to show that (82) is a covering map, it is enough to show that there is a free $\mathbb{Z}/m\mathbb{Z}$-action on the LHS of (82) whose quotient space is isomorphic to the RHS of (82).

Note that $\sigma\colon \widetilde{U} \to U$ is a covering map whose covering transformation group is $\mathbb{Z}/m\mathbb{Z}$. Hence $\mathbb{Z}/m\mathbb{Z}$ acts on the LHS of (82) by $F \mapsto g_\ast F$ for $g \in \mathbb{Z}/m\mathbb{Z}$. Note that the support of F are connected, and if $m \gg 0$ and $g \neq 0$, then the support of F and that of $g_\ast F$ are different. Hence F and $g_\ast F$ are not isomorphic for $g \neq 0$, which implies that $\mathbb{Z}/m\mathbb{Z}$-action on the LHS of (82) is free.

For two $\sigma^\ast\omega$-stable sheaves $\widetilde{F}_i$, $i = 1, 2$, corresponding to points in the LHS of (82), suppose that $\sigma_\ast \widetilde{F}_1$ and $\sigma_\ast \widetilde{F}_2$ are isomorphic. By adjunction, there is a non-trivial morphism,

$$\sigma^\ast \sigma_\ast \widetilde{F}_2 \cong \bigoplus_{g \in \mathbb{Z}/m\mathbb{Z}} g_\ast \widetilde{F}_2 \to \widetilde{F}_1.$$

Hence there are $g \in \mathbb{Z}/m\mathbb{Z}$ and a non-trivial morphism $g_\ast \widetilde{F}_2 \to \widetilde{F}_1$. Since both of $\widetilde{F}_1$ and $g_\ast \widetilde{F}_2$ are $\sigma^\ast\omega$-stable with $\mu_{\sigma^\ast\omega}(\widetilde{F}_1) = \mu_{\sigma^\ast\omega}(g_\ast \widetilde{F}_2)$, we have $g_\ast \widetilde{F}_2 \cong \widetilde{F}_1$.

Next we check that (82) is surjective. For an ω-stable sheaf $F \in M_1(U, \beta)$, suppose that F is $\mathbb{Z}/m\mathbb{Z}$-invariant. This implies that there is an isomorphism of sheaves,

$$\phi_F \colon F \to F \otimes \mathcal{L}.$$

The morphism ϕ_F may not satisfy the condition (48). However since F is ω-stable, we have $\mathrm{Aut}(F) = \mathbb{C}^\ast$, so by replacing ϕ_F by a non-zero multiple, we can assume that ϕ_F satisfies (48). Hence F is isomorphic to $\sigma_\ast \widetilde{F}$ for some sheaf $\widetilde{F} \in \mathrm{Coh}(\widetilde{U})$. The sheaf $\widetilde{F}$ must be $\sigma^\ast\omega$-stable since $\sigma_\ast \colon \mathrm{Coh}(\widetilde{U}) \to \mathrm{Coh}(U)$ is an exact functor. This shows that (82) is surjective.

The above argument shows that $\sigma_\ast$ induces a bijection between the quotient space of the LHS of (82) by the $\mathbb{Z}/m\mathbb{Z}$-action and the RHS of (82). Similarly to Lemma 4.1, Proposition 4.2, the above bijection is an isomorphism between complex analytic spaces. Namely for a complex analytic space S, it is straightforward to generalize the above argument to the bijection,

$$\sigma_\ast \colon \left(\coprod_{\sigma_\ast \widetilde{\beta} = \beta} \mathrm{Hom}(S, M_1(\widetilde{U}, \widetilde{\beta})) \right) / (\mathbb{Z}/m\mathbb{Z}) \xrightarrow{\cong} \mathrm{Hom}(S, M_1(U, \beta))^{\mathbb{Z}/m\mathbb{Z}}.$$

Therefore we obtain the desired assertion. □

4.3 The formula for $\widehat{\mathrm{DT}}^{\mathrm{par}}_{n,\gamma}$ under the cyclic covering

In this subsection, we investigate the formula (28) under the cyclic covering $\sigma \colon \widetilde{U} \to U$. In the notation of previous subsections, we denote by

$$\nu_{\widetilde{M}^{\mathrm{par}}}, \ \nu_{M^{\mathrm{par}}}, \ \nu_{\widetilde{M}}, \ \nu_M,$$

the Behrend functions on the spaces,

$$M_n^{\mathrm{par}}(\widetilde{U}, \widetilde{\beta}), \ M_n^{\mathrm{par}}(U, \beta), \ M_1(\widetilde{U}, \widetilde{\beta}), \ M_1(U, \beta),$$

respectively. We need the following compatibility of the above Behrend functions on the morphisms discussed in Proposition 4.2 and Proposition 4.3. The proof is postponed until Section 6.

Lemma 4.4. *If m is a sufficiently big odd number, we have the following:*
(i) Under the morphism (74), we have the identity,

$$(\sigma_*)^* \nu_{M^{\mathrm{par}}}|_{M_n^{\mathrm{par}}(\widetilde{U}, \widetilde{\beta})} = (-1)^{\beta \cdot H - \widetilde{\beta} \cdot \widetilde{H}} \nu_{\widetilde{M}^{\mathrm{par}}}.$$

(ii) Under the morphism (82), we have the identity,

$$(\sigma_*)^* \nu_M = \nu_{\widetilde{M}}.$$

Proof. The proof will be given in Subsection 6.4. ☐

As a corollary of Proposition 4.3, Proposition 4.2 and Lemma 4.4, we have the following:

Corollary 4.5. *For $\gamma \in H_2(C, \mathbb{Z})$ and $n \in \mathbb{Z}$, we take a sufficiently big odd number m and an m-fold cover $\sigma \colon \widetilde{U} \to U$ as in (46). We have the formulas,*

$$\mathrm{DT}_{n,\gamma}^{\mathrm{par}} = \sum_{\sigma_* \widetilde{\gamma} = \gamma} (-1)^{\gamma \cdot H - \widetilde{\gamma} \cdot \widetilde{H}} \, \mathrm{DT}_{n, \widetilde{\gamma}}^{\mathrm{par}}(\widetilde{U}), \tag{84}$$

$$N_{1,\gamma} = \frac{1}{m} \sum_{\sigma_* \widetilde{\gamma} = \gamma} N_{1, \widetilde{\gamma}}(\widetilde{U}). \tag{85}$$

Proof. Let us consider $\mathbb{C}^*$-actions on $M_n^{\mathrm{par}}(C, \gamma)$, $M_1(C, \gamma)$, determined by the embeddings (43), (42) respectively. Then for $m \gg 0$, we have

$$M_n^{\mathrm{par}}(C, \gamma)^{\mathbb{C}^*} = M_n^{\mathrm{par}}(C, \gamma)^{\mathbb{Z}/m\mathbb{Z}}, \tag{86}$$

$$M_1(C, \gamma)^{\mathbb{C}^*} = M_1(C, \gamma)^{\mathbb{Z}/m\mathbb{Z}}. \tag{87}$$

Therefore the formulas (84), (85) follow from Proposition 4.2, Proposition 4.3, Lemma 4.4, (86), (87) and the $\mathbb{C}^*$-localizations. ☐

Furthermore we have the following proposition.

Proposition 4.6. *In the situation of Corollary 4.5, suppose that the following formula holds on $\widetilde{U}$,*

$$\widehat{\mathrm{DT}}_{n, \widetilde{\gamma}}^{\mathrm{par}}(\widetilde{U}) = \sum_{k \ge 1, k | (n, \widetilde{\gamma})} \frac{(-1)^{\widetilde{\gamma} \cdot \widetilde{H} - 1}}{k^2} (\widetilde{\gamma} \cdot \widetilde{H}) N_{1, \widetilde{\gamma}/k}(\widetilde{U}), \tag{88}$$

for any $\widetilde{\gamma} \in H_2(\widetilde{C}, \mathbb{Z})$ with $\sigma_* \widetilde{\gamma} = \gamma$. Then the formula (28) holds.

Proof. By Corollary 4.5, the LHS of (28) is

$$\sum_{l\geq 1}\frac{(-1)^{l-1}}{l}\sum_{\substack{\gamma_1+\cdots+\gamma_l=\gamma,\\ n_1+\cdots+n_l=n,\\ n_i/\omega\cdot\gamma_i=n/\omega\cdot\gamma}}\prod_{i=1}^{l}\left(\sum_{\sigma_*\widetilde{\gamma}_i=\gamma_i}(-1)^{\gamma_i\cdot H-\widetilde{\gamma}_i\cdot\widetilde{H}}\,\mathrm{DT}^{\mathrm{par}}_{n_i,\widetilde{\gamma}_i}(\widetilde{U})\right) \tag{89}$$

$$= \sum_{\sigma_*\widetilde{\gamma}=\gamma}(-1)^{\gamma\cdot H-\widetilde{\gamma}\cdot\widetilde{H}}\sum_{l\geq 1}\frac{(-1)^{l-1}}{l}\sum_{\substack{\widetilde{\gamma}_1+\cdots+\widetilde{\gamma}_l=\widetilde{\gamma},\\ n_1+\cdots+n_l=n,\\ n_i/\sigma^*\omega\cdot\widetilde{\gamma}_i=n/\sigma^*\omega\cdot\widetilde{\gamma}}}\prod_{i=1}^{l}\mathrm{DT}^{\mathrm{par}}_{n_i,\widetilde{\gamma}_i}(\widetilde{U})$$

$$= \sum_{\sigma_*\widetilde{\gamma}=\gamma}\sum_{k\geq 1,k|(n,\widetilde{\gamma})}\frac{(-1)^{\gamma\cdot H-1}}{k^2}(\widetilde{\gamma}\cdot\widetilde{H})N_{1,\widetilde{\gamma}/k}(\widetilde{U}) \tag{90}$$

$$= \sum_{k\geq 1,k|(n,\gamma)}\frac{(-1)^{\gamma\cdot H-1}}{k^2}\sum_{\sigma_*\widetilde{\gamma}=\gamma,\ k|\widetilde{\gamma}}(\widetilde{\gamma}\cdot\widetilde{H})N_{1,\widetilde{\gamma}/k}(\widetilde{U})$$

$$= \sum_{k\geq 1,k|(n,\gamma)}\frac{(-1)^{\gamma\cdot H-1}}{k^2}\cdot\frac{1}{m}\sum_{\substack{\sigma_*\widetilde{\gamma}=\gamma,\ k|\widetilde{\gamma}\\ g\in\mathbb{Z}/m\mathbb{Z}}}(g_*\widetilde{\gamma}\cdot\widetilde{H})N_{1,g_*\widetilde{\gamma}/k}(\widetilde{U})$$

$$= \sum_{k\geq 1,k|(n,\gamma)}\frac{(-1)^{\gamma\cdot H-1}}{k^2}\cdot\frac{1}{m}\sum_{\sigma_*\widetilde{\gamma}=\gamma,\ k|\widetilde{\gamma}}(\gamma\cdot H)N_{1,\widetilde{\gamma}/k}(\widetilde{U})$$

$$= \sum_{k\geq 1,k|(n,\gamma)}\frac{(-1)^{\gamma\cdot H-1}}{k^2}(\gamma\cdot H)N_{1,\gamma/k}. \tag{91}$$

Here we have used (84), (88), (85) in (89), (90), (91) respectively. Therefore the formula (28) holds. ∎

4.4 Reduction to trees of $\mathbb{P}^1$

Now we show our main result.

Theorem 4.7. *Let X be a smooth projective Calabi-Yau 3-fold over $\mathbb{C}$, $C\subset X$ a reduced rational curve with at worst nodal singularities, and take $\gamma\in H_2(C,\mathbb{Z})$. Suppose that for any cyclic neighborhood $(C'\subset U')\stackrel{\sigma'}{\to}(C\subset X)$ with C' a tree of $\mathbb{P}^1$, the following conditions hold:*

- *The cyclic neighborhood $C'\subset U'$ satisfies the condition of Conjecture 3.6.*
- *For any $\gamma'\in H_2(C',\mathbb{Z})$ with $\sigma'_*\gamma'=\gamma$, the invariant $N_{n,\gamma'}(U')$ satisfies the formula (67).*

Then $N_{n,\gamma}$ satisfies the formula (21).

Proof. An element $\gamma\in H_2(C,\mathbb{Z})_{>0}$ can be written as

$$\gamma=\sum_{i=1}^{N}a_i[C_i],$$

for $a_i \in \mathbb{Z}_{\geq 0}$ where $C_1, \cdots, C_N$ are irreducible components of C. The support of γ, denoted by C_γ, is defined to be the reduced curve,

$$C_\gamma := \bigcup_{a_i > 0} C_i \subset C.$$

We also set $d(\gamma)$ and $l(\gamma)$ to be

$$d(\gamma) := \sum_{i=1}^{N} a_i, \quad l(\gamma) := \sharp\{1 \leq i \leq N : a_i > 0\}.$$

Note that we have $l(\gamma) \leq d(\gamma)$.

We note that, if $\widehat{\mathrm{DT}}^{\mathrm{par}}_{n,\gamma}$ or $N_{1,\gamma}$ is non-zero, then C_γ is a connected curve. In fact, by [23, Equation (95)], the invariant $\widehat{\mathrm{DT}}^{\mathrm{par}}_{n,\gamma}$ is a multiple of $N_{n,\gamma}$. And if $N_{n,\gamma}$ is non-zero, then the same argument of [25, Lemma 11.6] shows that C_γ is connected. Hence we may assume that C is connected and $C_\gamma = C$. Note that $l(\gamma) = N$ in this case.

If $g(C) = 0$, then the result follows from the assumption. Suppose that $g(C) > 0$, and take a sufficiently small analytic neighborhood $C \subset U$ in X. Then we can take an m-fold cover

$$\sigma \colon \widetilde{U} \to U, \quad \widetilde{C} = \sigma^{-1}(C),$$

as in Subsection 3.3, for a sufficiently big odd number m. We take $\widetilde{\gamma} \in H_2(\widetilde{C}, \mathbb{Z})$ satisfying that $\sigma_* \widetilde{\gamma} = \gamma$, $C_{\widetilde{\gamma}}$ is connected and intersects with $\widetilde{H}$. Then either one of the following conditions hold:

$$l(\widetilde{\gamma}) > l(\gamma) = N, \text{ or} \tag{92}$$

$$l(\widetilde{\gamma}) = l(\gamma) = N, \ g(C_{\widetilde{\gamma}}) < g(C_\gamma) = g(C). \tag{93}$$

In fact, since $C_{\widetilde{\gamma}} \to C$ is surjective, we have $l(\widetilde{\gamma}) \geq l(\gamma)$. Suppose that $l(\widetilde{\gamma}) = l(\gamma)$. Then for each irreducible component $C_j \subset C$, the preimage $\sigma^{-1}(C_j) \cap C_{\widetilde{\gamma}}$ is also irreducible. Because $C_{\widetilde{\gamma}}$ is connected, this easily implies that $C_{\widetilde{\gamma}}$ is written as

$$C_{\widetilde{\gamma}} = A \cup \tau\overline{(C_{x,i} \setminus A)},$$

where $A \subset C_{x,i}$ is a connected subcurve for some $i \in \mathbb{Z}/m\mathbb{Z}$ in the notation of (47) and $\tau = 1 \in \mathbb{Z}/m\mathbb{Z}$. The curves $A \subset C_{x,i}$ and $\tau\overline{(C_{x,i} \setminus A)} \subset C_{x,i+1}$ are connected at the node $x_{1,i} = x_{2,i+1}$. Since $g(C_{x,i}) = g(C) - 1$, it follows that

$$\begin{aligned} g(C_{\widetilde{\gamma}}) &= g(C_{x,i}) - \sharp(A \cap \overline{(C_{x,i} \setminus A)}) + 1 \\ &= g(C) - \sharp(A \cap \overline{(C_{x,i} \setminus A)}) \\ &< g(C). \end{aligned}$$

Therefore one of (92) or (93) holds.

Now we replace $\widetilde{U}$ by a small analytic neighborhood of $C_{\widetilde{\gamma}}$, say $U_{(1)}$, and set

$$\gamma_{(1)} = \widetilde{\gamma}, \ C_{(1)} = C_{\widetilde{\gamma}}, \ H_{(1)} = \widetilde{H} \cap U_{(1)}.$$

Repeating the same procedures, we obtain the sequence of local immersions,

$$\cdots \to U_{(i)} \stackrel{\sigma_{(i)}}{\to} U_{(i-1)} \to \cdots \stackrel{\sigma_{(2)}}{\to} U_{(1)} \stackrel{\sigma}{\to} U_{(0)} = U, \tag{94}$$

and data,

$$C_{(i)} \subset U_{(i)}, \ \gamma_{(i)} \in H_2(C_{(i)}, \mathbb{Z}), \ H_{(i)} \subset U_{(i)},$$

where $C_{(i)}$ is a connected nodal curve, $\gamma_{(i)}$ satisfies $C_{\gamma_{(i)}} = C_{(i)}$ and $H_{(i)}$ is a lift of H. Note that for each i, $C_{(i)} \subset U_{(i)}$ is a cyclic neighborhood of $C \subset X$. Similarly as above, either one of the following conditions hold:

$$\begin{aligned} &l(\gamma_{(i)}) < l(\gamma_{(i+1)}), \quad \text{or} \\ &l(\gamma_{(i)}) = l(\gamma_{(i+1)}), \ g(C_{(i+1)}) < g(C_{(i)}). \end{aligned}$$

Because we have

$$l(\gamma_{(i)}) \le d(\gamma_{(i)}) = d(\gamma),$$

the sequence (94) terminates at some i, say $i = R$. Then we have $g(C_{(R)}) = 0$, i.e. $C_{(R)}$ is a tree of $\mathbb{P}^1$.

Below we say that the invariant $\widehat{\mathrm{DT}}^{\mathrm{par}}_{n,\gamma_{(i)}}(U_{(i)})$ satisfies (68) if the formula (68) holds for $U' = U_{(i)}$, $C' = C_{(i)}$ and $\gamma' = \gamma_{(i)}$. By the assumption and Proposition 3.7, the invariant $\widehat{\mathrm{DT}}^{\mathrm{par}}_{n,\gamma_{(i)}}(U_{(i)})$ satisfies (68) when $i = R$. Also it is straightforward to see that the argument of Proposition 4.6 for $C \subset U$ can be applied to $C_{(i)} \subset U_{(i)}$. Therefore $\widehat{\mathrm{DT}}^{\mathrm{par}}_{n,\gamma_{(i)}}(U_{(i)})$ satisfies (68) if the same formula holds on any cyclic covering $\widetilde{U}_{(i)} \to U_{(i)}$. By the induction argument, it follows that the invariant $\widehat{\mathrm{DT}}^{\mathrm{par}}_{n,\gamma_{(i)}}(U_{(i)})$ satisfies (68) for all i. Hence (28) holds, and the formula (21) holds as well. ∎

4.5 Euler characteristic version

For $n \in \mathbb{Z}$ and $\beta \in H_2(X, \mathbb{Z})$, there is also the Euler characteristic version of the invariant $N_{n,\beta}$, as discussed in [26], [27], [25]. Namely in the definition of $N_{n,\beta}$, we replace the Behrend function ν by the identity function. The resulting invariant is denoted by

$$N^{\chi}_{n,\beta} \in \mathbb{Q}. \tag{95}$$

If γ is a one cycle on X, the Euler characteristic version of the local invariant $N_{n,\gamma}$ can be similarly defined,

$$N^{\chi}_{n,\gamma} \in \mathbb{Q}.$$

The argument of Theorem 4.7 can be also applied to the invariant $N_{n,\gamma}$, which is easier since we do not have to take care of the Behrend functions at all. Also in this case, one may expect the formula,

$$N^{\chi}_{n,\gamma} = \sum_{k \ge 1, k|(n,\gamma)} \frac{1}{k^2} N^{\chi}_{1,\gamma/k}. \tag{96}$$

However unfortunately, the above formula is known to be false as the following example indicates:

Example 4.8. *Let $C \subset X$ be a smooth rational curve whose normal bundle is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Then the same computation of $N_{0,m[C]}$ in [23, Example 4.14] shows that*

$$N_{0,m[C]}^{\chi} = \frac{(-1)^{m-1}}{m^2}.$$

On the other hand, $N_{1,m[C]}^{\chi} = 1$ if $m = 1$ and 0 if $m \geq 2$. Hence (96) does not hold.

Although the formula (96) is not true in general, there are some situations in which the formula (96) should hold, as discussed in [25]. Similarly to Theorem 4.7, such a case can be reduced to the cases of trees of $\mathbb{P}^1$ on cyclic neighborhoods of $C \subset X$. Note that, for a cyclic neighborhood $C' \subset U'$ of $C \subset X$ and $\gamma' \in H_2(C', \mathbb{Z})$, we can also define the Euler characteristic invariant,

$$N_{n,\gamma'}^{\chi}(U') \in \mathbb{Q},$$

by replacing the Behrend function by the identity function in the definition of $N_{n,\gamma'}(U')$. We have the following theorem:

Theorem 4.9. *Let X be a smooth projective Calabi-Yau 3-fold over $\mathbb{C}$, $C \subset X$ a reduced rational curve with at worst nodal singularities, and take $\gamma \in H_2(C, \mathbb{Z})$. Suppose that for any cyclic neighborhood $(C' \subset U') \stackrel{\sigma'}{\to} (C \subset X)$ with C' a tree of $\mathbb{P}^1$, and an element $\gamma' \in H_2(C', \mathbb{Z})$ with $\sigma'_* \gamma' = \gamma$, the following equality holds:*

$$N_{n,\gamma'}^{\chi}(U') = \sum_{k \geq 1, k|(n,\gamma')} \frac{1}{k^2} N_{1,\gamma'/k}^{\chi}(U'). \tag{97}$$

Then $N_{n,\gamma}^{\chi}$ satisfies the formula (96).

Proof. The same proof of Theorem 4.7 works. The only difference is that we do not have to take care of the Behrend functions in deducing the equivalence between the formula (97) and the formula for parabolic stable pair invariants. Namely for any cyclic neighborhood $C' \subset U'$ of $C \subset X$ with a lift $H' \subset U'$ of $H \subset X$, and $\gamma' \in H_2(C', \mathbb{Z})$, we can define the Euler characteristic versions of parabolic stable pair invariants,

$$\mathrm{DT}_{n,\gamma'}^{\mathrm{par},\chi}(U') \in \mathbb{Z}, \quad \widehat{\mathrm{DT}}_{n,\gamma'}^{\mathrm{par},\chi}(U') \in \mathbb{Q},$$

by replacing the Behrend function by the identity function in the definitions of (65), (66) respectively. The same proof of [23, Corollary 4.18] shows that the formula (97) is equivalent to the formula, (without assuming Conjecture 3.6,)

$$\widehat{\mathrm{DT}}_{n,\gamma'}^{\mathrm{par},\chi}(U') = (\gamma' \cdot H') \sum_{k \geq 1, k|(n,\gamma')} \frac{1}{k^2} N_{1,\gamma'/k}^{\chi}(U').$$

Then we can apply the same induction argument as in the proof of Theorem 4.7, and conclude the assertion. ∎

5 Applications

In this section, we apply Theorem 4.7 to prove the multiple cover formula under some situations.

5.1 0-super rigid and surface type neighborhoods

Let $C \subset U \subset X$ be as in the previous sections. We study the local multiple cover formula in the following situations: $C \subset U$ is 0-super rigid or $C \subset U$ is of surface type. These concepts are defined as follows:

Definition 5.1. *(i) We say $C \subset U$ is 0-super rigid if for any projective curve C' of arithmetic genus 0 and a local immersion $f \colon C' \to C$, we have*

$$H^0(C', f^* N_{C/U}) = 0.$$

(ii) We say $C \subset U$ is of surface type if there is a complex surface U_0 and an analytic neighborhood Δ of $0 \in \mathbb{C}$ such that

$$U \cong U_0 \times \Delta,$$

and C is contained in $U_0 \times \{0\}$

The 0-super rigidity is a genericity condition for the pair $(C \subset U)$, and a concept adapted in [5]. The following proposition shows that there are some situations in which the assumptions in Theorem 4.7 are satisfied.

Proposition 5.2. *Let $C' \subset U'$ be a cyclic neighborhood of $C \subset U$, with C' a tree of $\mathbb{P}^1$.*

(i) Suppose that $C \subset U$ is 0-super rigid and C' is a chain of $\mathbb{P}^1$, say $C'_1, \cdots, C'_N$. Then $C' \subset U'$ satisfies the condition of Conjecture 3.6. Moreover, for any $n \in \mathbb{Z}$, and $a'_1, \cdots, a'_N \in \mathbb{Z}_{\geq 1}$, we have

$$N_{n, a'_1[C'_1] + \cdots + a'_N[C'_N]}(U') = \begin{cases} 1/k^2, & a'_1 = \cdots = a'_N = k, \ k|n, \\ 0, & \text{otherwise.} \end{cases} \tag{98}$$

In particular, the invariant $N_{n,\gamma'}(U')$ satisfies the formula (67).

(ii) Suppose that $C \subset U$ is of surface type and the dual graph of C' is of ADE type. Then $C' \subset U'$ satisfies the condition of Conjecture 3.6. If C' is a chain of $\mathbb{P}^1$, say $C'_1, \cdots, C'_N$, then for any $n \in \mathbb{Z}$, and $a'_1, \cdots, a'_N \in \mathbb{Z}_{\geq 1}$, we have

$$N_{n, a'_1[C'_1] + \cdots + a'_N[C'_N]}(U') = \begin{cases} -1/k^2, & a'_1 = \cdots = a'_N, \ k|n, \\ 0, & \text{otherwise.} \end{cases} \tag{99}$$

In particular, the invariant $N_{n,\gamma'}(U')$ satisfies the formula (67).

Proof. (i) The proof of [5, Lemma 3.1] shows that each irreducible component C'_i is a $(-1,-1)$-curve in U', and there is a bimeromorphic contraction,

$$f \colon U' \to U'',$$

which contracts C' to a cA_N-singularity $0 \in U''$. The argument of Van den Bergh [7] works in our situation, and we have the derived equivalence,

$$\Phi\colon D^b \operatorname{Coh}_{C'}(U') \cong D^b \mathrm{Mod}_{\mathrm{nil}}(A),$$

for some non-commutative $\mathcal{O}_{U''}$-algebra A, such that $\Phi^{-1}\mathrm{Mod}_{\mathrm{nil}}(A)$ corresponds to Bridgeland's perverse coherent sheaves with 0-perversity [3]. Here $\operatorname{Coh}_{C'}(U')$ is the category of coherent sheaves on U' supported on C', and $\mathrm{Mod}_{\mathrm{nil}}(A)$ is the category of finite dimensional nilpotent right A-modules. Let us take $\mathcal{L} \in \operatorname{Pic}(U')$ such that $\mathcal{L}|_{C'}$ is an ample line bundle. By the construction of perverse coherent sheaves in [3], an object $E \in \operatorname{Coh}_{C'}(U)$ satisfies $\Phi(E) \in \mathrm{Mod}_{\mathrm{nil}}(A)$ if and only if $R^1 f_* E = 0$. The latter condition is satisfied if we replace E by $E \otimes \mathcal{L}^{\otimes m}$ for $m \gg 0$. Since any object $[E] \in \mathcal{M}_n(U', \beta')$ is supported on C', the above argument implies that the moduli stack $\mathcal{M}_n(U', \beta')$ is regarded as an analytic open substack of objects in $\Phi^{-1}\mathrm{Mod}_{\mathrm{nil}}(A) \otimes \mathcal{L}^{\otimes -m}$ for $m \gg 0$. On the other hand, the algebra A is a Calabi-Yau 3-algebra. Hence the completion $\widehat{A}$ of A at $0 \in U''$ is written as a completion of a path algebra of a quiver Q with a super potential W by [6]. Since each component of the moduli stack of representations of (Q, W) is written as a quotient stack of a critical locus of some holomorphic function on a finite dimensional vector space, (cf. [15, Subsection 7.2],) we conclude that $\mathcal{M}_n(U', \beta')$ satisfies the condition of Conjecture 3.6.

Next let us take $n \in \mathbb{Z}$ and $a'_1, \cdots, a'_N \geq 1$. By the argument in [4, Proposition 2.10], there is a family of complex manifolds U'_t for $t \in \mathbb{C}$ such that $U'_0 = U'$ and curves in U'_ε for $0 < \varepsilon \ll 1$ consist of only $(-1,-1)$-curves. A subcuve $C'' \subset C'$ deforms to a curve in U'_ε if and only if C'' is a sub $\mathbb{P}^1$-chain of C'. Let $C'_\varepsilon \subset U'_\varepsilon$ be a $(-1,-1)$-curve, obtained by deforming C'. For $(n,k) \in \mathbb{Z}^{\oplus 2}$ with $k \geq 1$, we have $N_{n,k[C'_\varepsilon]} \neq 0$ only if $k|n$, and in this case we have (cf. [23, Example 4.14],)

$$N_{n,k[C'_\varepsilon]}(U'_\varepsilon) = \frac{1}{k^2}.$$

On the other hand, the invariant $N_{n,\beta'}(U')$ is invariant under deformation of U' by [15, Corollary 5.28]. (See Remark 5.3 below.) Hence the LHS of (98) is non-zero only if $a'_1 = \cdots = a'_N (= k)$, $k|n$, and equal to $1/k^2$ in this case.

(ii) If $C \subset U$ is of surface type, then the cyclic neighborhood $C' \subset U'$ is also of surface type: there is a complex surface U'_0 such that $U' \cong U'_0 \times \Delta$ and $C' \subset U'_0 \times \{0\}$. Then C' is a tree of $\mathbb{P}^1$ of ADE type in the surface U'_0, hence there is a bimeromorphic morphism to a singular complex surface U''_0,

$$U'_0 \to U''_0,$$

whose exceptional locus is C'. Since U''_0 is a small analytic neighborhood of C', U''_0 is an analytic neighborhood of a singular point in U''_0, which is isomorphic to an analytic neighborhood of the quotient singularity $\mathbb{C}^2/G$ for a finite subgroup in $G \subset \mathrm{SL}(2,\mathbb{C})$.

Let

$$f\colon V \to \mathbb{C}^2/G, \tag{100}$$

be the minimal resolution of singularities. Note that C' is regarded as an exceptional locus of (100). The above argument shows that U' is isomorphic to an analytic neighborhood of C' in $V \times \mathbb{C}$, where C' lies in $V \times \{0\}$. As explained in [9, Subsection 2.2], we have the derived equivalence,

$$\Phi \colon D^b \operatorname{Coh}(V \times \mathbb{C}) \cong D^b \operatorname{Rep}(Q, W),$$

for a certain quiver Q with a superpotential W, and $\Phi^{-1}\operatorname{Rep}(Q, W)$ is Bridgeland's perverse coherent sheaves. Then the same argument of (i) shows that $\mathcal{M}_n(U', \beta')$ satisfies the condition of Conjecture 3.6.

Let us compute the LHS of (99) when C' is a chain of $\mathbb{P}^1$. In the surface type case, the moduli space of stable pairs (101) in Remark 5.3 below is not compact, so the deformation argument is more subtle. Instead, we use the explicit result of the computation of DT type invariants on $V \times \mathbb{C}$ in [9]. In [9], it is proved that the generating series of PT invariants is written as a Gopakumar-Vafa form, that is a local version of the conjecture in [24, Conjecture 6.2]. By [24, Theorem 6.4], Conjecture 2.4 is equivalent to [24, Conjecture 6.2], hence $N_{n,\gamma'}(U')$ satisfies (67) for any $n \in \mathbb{Z}$ and $\gamma' \in H_2(C', \mathbb{Z})$. By [9, Corollary 1.6] and [24, Theorem 6.4], for $a'_1, \cdots, a'_N \in \mathbb{Z}_{\ge 1}$, we have

$$N_{1, a'_1[C'_1] + \cdots + a'_N[C'_N]}(U') = \begin{cases} -1, & a'_1 = \cdots = a'_N = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore the result of (ii) holds. □

Remark 5.3. *The argument of [15, Corollary 5.28] works when the ambient space U' is a projective Calabi-Yau 3-fold. In our case, U' is an analytic small neighborhood of C', so we need to modify the argument. Let $\mathcal{L}$ be a line bundle on U' such that $\mathcal{L}|_{C'}$ is ample. Then we consider moduli space of pairs,*

$$(F, u), \tag{101}$$

where F is a compactly supported one dimensional coherent sheaf on U', and $u \in H^0(F \otimes \mathcal{L}^{\otimes m})$ for $m \gg 0$, satisfying the stability condition as in [15, Definition 5.20]. If $C' \subset U'$ is a chain of $(-1,-1)$-curves, then any sheaf F as above is supported on C', hence the moduli space of pairs (101) is a projective scheme. Let us consider the object,

$$E = (\mathcal{L}^{\otimes -m} \stackrel{u}{\to} F) \in D^b \operatorname{Coh}(U').$$

Although U' is non-compact, the groups $\operatorname{Ext}^i_{U'}(E, E)$ for $i = 1, 2$ are finite dimensional, hence determine a symmetric perfect obstruction theory on the moduli space of pairs (101) as in [15, Theorem 5.23]. Then the deformation invariance of $N_{n,\beta'}(U')$ follows from the same argument of [15, Corollary 5.28].

5.2 Local generalized DT invariants on a nodal rational curve of type I_N

In this subsection, using the results in the previous subsection, we compute some generalized DT invariants which have not been computed so far. Recall that a nodal curve C is called type I_N if C is one of the following:

- C is of type I_1 if C is an irreducible rational curve with one node.
- C is of type I_N for $N \geq 2$ if C is a circle of irreducible components $C_1, \cdots, C_N$ such that $C_i \cong \mathbb{P}^1$ for all i.

Note that the above notation is used in the Kodaira's classification of singular fibers of elliptic fibrations. (See Figure 3.)

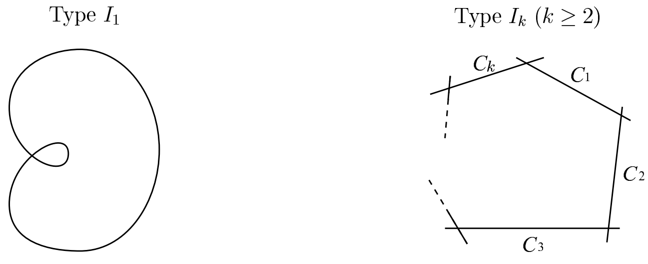

Figure 3:

We have the following theorem.

Theorem 5.4. *Let C be a nodal curve of type I_N with irreducible components $C_1, \cdots, C_N$, which is embedded into a Calabi-Yau 3-fold X. Suppose that an analytic neighborhood $C \subset U \subset X$ is either 0-super rigid or of surface type. Then for $n \in \mathbb{Z}$ and $a_1, \cdots, a_N \in \mathbb{Z}_{\geq 1}$, the invariant $N_{n,a_1[C_1]+\cdots+a_N[C_N]}$ is non-zero only if $a_1 = \cdots = a_N$, say m. In this case, we have*

$$N_{n,m[C_1]+\cdots+m[C_N]} = \begin{cases} \sum_{k\geq 1, k|(n,m)} N/k^2, & 0\text{-super rigid case}, \\ -\sum_{k\geq 1, k|(n,m)} N/k^2, & \text{surface type case}. \end{cases}$$

In particular, the invariant $N_{n,a_1[C_1]+\cdots+a_N[C_N]}$ satisfies the formula (21).

Proof. Let (C', U') be a cyclic neighborhood of $C \subset U$. Then the construction of cyclic coverings in Subsection 3.3 shows that, if C' is a tree of $\mathbb{P}^1$, then it must be a chain of $\mathbb{P}^1$. By Proposition 5.2, the assumptions in Theorem 4.7 are satisfied, hence the invariant $N_{n,a_1[C_1]+\cdots+a_N[C_N]}$ satisfies the formula (21). It is enough to compute $N_{1,a_1[C_1]+\cdots+a_N[C_N]}$, which follows from the computation on cyclic neighborhoods in Proposition 5.2 and the comparison formula under the covering (85). For instance if $C \subset U$ is 0-super rigid, then it easily follows that

$$N_{1,a_1[C_1]+\cdots+a_N[C_N]} = \begin{cases} N, & a_1 = \cdots = a_N = 1, \\ 0, & \text{otherwise}. \end{cases}$$

The case of surface type is similar. $\square$

5.3 Local generalized DT invariants on irreducible nodal rational curves

In this subsection, using Lemma 2.12, Proposition 5.2 and Theorem 4.7, we study the local multiple cover formula of generalized DT invariants on irreducible nodal curves. We have the following result:

Theorem 5.5. *Let C be an irreducible rational curve with at worst nodal singularities, embedded into a Calabi-Yau 3-fold X. For a sufficiently small analytic neighborhood $C \subset U \subset X$, suppose that it is 0-super rigid or of surface type. Moreover assume that any cyclic neighborhood $C' \subset U'$ with C' tree of $\mathbb{P}^1$ satisfies the condition of Conjecture 3.6. The for any prime number p, the invariant $N_{n,p[C]}$ satisfies the formula (21).*

Proof. Let $(C' \subset U') \stackrel{\sigma'}{\to} (C \subset X)$ be a cyclic neighborhood of $C \subset X$ with C' tree of $\mathbb{P}^1$. By Theorem 4.7, it is enough to show that the invariant $N_{n,\gamma'}(U')$ satisfies the formula (67) for any $n \in \mathbb{Z}$ and $\gamma' \in H_2(C', \mathbb{Z})_{>0}$ with $\sigma'_*\gamma' = p[C]$. Let $C'_1, \cdots, C'_N$ be the irreducible components of C', and we write $\gamma' \in H_2(C', \mathbb{Z})_{>0}$ as $\gamma' = \sum_{i=1}^N a'_i[C'_i]$ for $a'_i \in \mathbb{Z}_{\geq 0}$. Then $\sigma'_*\gamma' = p[C]$ is equivalent to that

$$a'_1 + \cdots + a'_N = p. \tag{102}$$

Suppose that there are at least two $1 \leq i \leq N$ with $a'_i > 0$. Then, since p is a prime number, the equality (102) implies that

$$\text{g.c.d.}(a'_1, \cdots, a'_N) = 1,$$

i.e. $\gamma' \in H_2(C', \mathbb{Z})$ is primitive. Then the invariant $N_{n,\gamma'}(U')$ satisfies (67) by Lemma 2.12. If there is only one $1 \leq i \leq N$ with $a'_i > 0$, then we can assume that C' is a single $\mathbb{P}^1$. In this case, the invariant $N_{n,\gamma'}(U')$ satisfies the formula (67) by Proposition 5.2. ☐

In the situation of the above theorem, we are not able to prove the condition of Conjecture 3.6 on cyclic neighborhoods. However we don't have to take care of this condition in the Euler characteristic version. We have the following:

Theorem 5.6. *Let C be an irreducible rational curve with at worst nodal singularities, embedded into a Calabi-Yau 3-fold X. For a sufficiently small analytic neighborhood $C \subset U \subset X$, suppose that it is of surface type. The for any prime number p, the invariant $N^{\chi}_{n,p[C]}$ satisfies the formula (96).*

Proof. The same proof of Theorem 5.5 works, using Theorem 4.9 instead of Theorem4.7. In the notation of the proof of Theorem 5.5, suppose that there are at least two $1 \leq i \leq N$ with $a'_i > 0$. Then the same proof of Lemma 2.12 shows the equality $N^{\chi}_{n,\gamma'}(U') = N^{\chi}_{1,\gamma'}(U')$. Otherwise, we may assume that C' is a single $\mathbb{P}^1$. In this case, the invariant $N^{\chi}_{n,p[C']}(U')$ can be checked to satisfy (97) by comparing the formula in [27, Theorem 1.3] and the Euler characteristic version of the formula in [9, Theorem 1.2]. ☐

Remark 5.7. *The result of Theorem 5.6 is not true for a 0-super rigid case, as we discussed in Example 4.8.*

In the situation of Theorem 5.6, we can say more for the invariants $N^{\chi}_{n,m[C]}$ when m is small:

Lemma 5.8. *In the situation of Theorem 5.6, the invariant $N^{\chi}_{n,m[C]}$ satisfies the formula (96) when $m \le 10$.*

Proof. First suppose that $m < 10$. In the notation of the proof of Theorem 5.5, suppose that $\gamma' = \sum_{i=1}^{N} a_i'[C_i']$ satisfies $\sigma_\ast \gamma' = m[C]$, i.e.

$$a_1' + \cdots + a_N' = m.$$

Then we have either g.c.d.$(a_1', \cdots, a_N') = 1$ or γ is supported on an ADE configuration of $\mathbb{P}^1$. As in the proof of Theorem 5.6, using the results in [9], [27], it is straightforward to check the Euler characteristic version of the results in Proposition 5.2 (ii). Therefore the result follows from the Euler characteristic version of Lemma 2.12 and Theorem 4.9. When $m = 10$, we have the following exceptional case: $N = 5$, $C_1', \cdots, C_5'$ satisfy

$$C_1' \cdot C_i' = 1, \ (i \ge 2), \quad C_i' \cdot C_j' = 0, \ (i, j \ge 2).$$

and $\gamma' = 2[C']$. In this case, $C_1', \cdots, C_5'$ is not an ADE configuration. However we can check that $N^{\chi}_{n,2[C']}(U')$ satisfies (97) by a direct calculation as in [25, Proposition 6.9]. In fact, by the Riemann-Roch theorem, one can show that there is no stable sheaf F' on U' with $[F'] = \gamma'$. Then a computation similar to [25, Proposition 6.9] works, whose detail is left to the readers. ☐

Remark 5.9. *The result of Lemma 5.8 can be generalized as follows. Let C be a reduced (not necessary irreducible) rational curve with at worst nodal singularities, which is embedded into a Calabi-Yau 3-fold X, Suppose that a sufficiently small analytic neighborhood $C \subset U \subset X$ is of surface type. Let $C_1, \cdots, C_N$ be the irreducible components of C. Then the invariant $N^{\chi}_{n,a_1[C_1]+\cdots+a_N[C_N]}$ satisfies the formula (96) if $a_1, \cdots, a_N$ satisfies*

$$a_1 + \cdots + a_N \le 10.$$

The proof is same as in Lemma 5.8.

5.4 Euler characteristic invariants on K3 surfaces

Let S be a smooth projective K3 surface over $\mathbb{C}$, i.e.

$$K_S = \mathcal{O}_S, \quad H^1(S, \mathcal{O}_S) = 0,$$

and X is the total space of the canonical line bundle on S, i.e.

$$X = S \times \mathbb{C}.$$

In [25], we established a formula relating Euler characteristic of the moduli space of PT stable pairs and Joyce type Euler characteristic invariants counting semistable sheaves on the fibers of the projection

$$X = S \times \mathbb{C} \to \mathbb{C}.$$

For a vector

$$v = (r, \beta, n) \in H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z}),$$

the latter invariant was denoted by

$$J(r, \beta, n) \in \mathbb{Q}.$$

The invariant $J(v)$ is the Euler characteristic version of DT type invariants on X, counting $p^*\omega$-semistable sheaves $F \in \mathrm{Coh}(X)$, with compact supports, and satisfies

$$\mathrm{ch}(p_* F)\sqrt{\mathrm{td}_S} = v.$$

Here $p \colon X = S \times \mathbb{C} \to S$ is the first projection, and ω is an ample divisor on S. In the notation of Subsection 4.5, after identifying $H^2(S, \mathbb{Z})$ with $H_2(X, \mathbb{Z})$, we have

$$J(0, \beta, n) = N^{\chi}_{n,\beta}. \tag{103}$$

If $v \in H^*(S, \mathbb{Z})$ is a primitive algebraic class, then $J(v)$ is written as

$$J(v) = \chi(\mathrm{Hilb}^{(v,v)/2+1}(S)). \tag{104}$$

(cf. [25, Equation (65)].) Here $\mathrm{Hilb}^m(S)$ is the Hilbert scheme of m-points in S and $(*, *)$ is the Mukai inner product,

$$((r_1, \beta_1, n_1), (r_2, \beta_2, n_2)) = \beta_1 \cdot \beta_2 - r_1 n_2 - r_2 n_1.$$

The RHS of (105) is determined by the Göttsche's formula [8],

$$\sum_{m \ge 0} \chi(\mathrm{Hilb}^m(S)) q^m = \prod_{m \ge 1} \frac{1}{(1 - q^m)^{24}}.$$

If v is not necessary primitive, we proposed the following multiple cover conjecture in [25, Conjecture 1.3]:

Conjecture 5.10. **([25, Conjecture 1.3])** *If $v \in H^*(S, \mathbb{Z})$ is an algebraic class, we have the equality,*

$$J(v) = \sum_{k \ge 1, k|v} \frac{1}{k^2} \chi(\mathrm{Hilb}^{(v/k, v/k)/2+1}(S)). \tag{105}$$

Using the results in this paper, we can give some evidence of the conjecture. Below we say a (not necessary reduced) curve $C \subset S$ is *rational*, or has *at worst nodal singularities* if the reduced curve C^{red} satisfies the corresponding property. We have the following theorem:

Theorem 5.11. *Let S be a smooth projective K3 surface over $\mathbb{C}$ and $X = S \times \mathbb{C}$. Suppose that $\mathrm{Pic}(S)$ is generated by an ample line bundle L on S, such that any rational member in $|L|$ has at worst nodal singularities. Then the invariant $J(0, mc_1(L), n)$ satisfies the formula (105) if $m \le 10$ or m is a prime number. In particular, if $L^2 = 2d - 2$ for $d \in \mathbb{Z}$ and p is a prime number, we have*

$$J(0, pc_1(L), 0) = \chi(\mathrm{Hilb}^{(d-1)p^2+1}(S)) + \frac{1}{p^2}\chi(\mathrm{Hilb}^d(S)). \tag{106}$$

Proof. Let γ be a one cycle on X whose support is compact. We show that $N^{\chi}_{n,\gamma}$ satisfies the formula (96) when the homology class of γ coincides with $mc_1(L)$ under the isomorphism $H_2(X,\mathbb{Z}) \cong H^2(S,\mathbb{Z})$.

As in the proof of Theorem 4.7, we may assume that γ is connected. Then γ is supported on $S \times \{t\}$ for some $t \in \mathbb{C}$, hence regarded as $\gamma \in |mL|$. We write γ as

$$\gamma = a_1[C_1] + \cdots + a_N[C_N],$$

where $C_1, \cdots, C_N$ are irreducible curves in $S \times \{t\}$. By Lemma 2.11, we may assume that all C_i have geometric genus zero. By the assumption, the reduced curve $C = \cup_{i=1}^{N} C_i$ has at worst nodal singularities, and there is an analytic neighborhood $C \subset U \subset X$ of surface type. Note that $C_i \in |m_i L|$ for some $m_i \in \mathbb{Z}_{\ge 1}$, and we have

$$a_1 m_1 + \cdots + a_N m_N = m.$$

Since $m \le 10$ or m is a prime number, either one of the following conditions hold:

(i) $a_1 + \cdots + a_N \le 10$.

(ii) g.c.d.$(a_1, \cdots, a_N) = 1$.

(iii) $N = 1$, $a_1 = m$, $m_1 = 1$.

In these cases, the formula (96) follows from Remark 5.9, Lemma 2.12 and Theorem 5.6.

The proof of Lemma 2.6 in [23, Corollary 4.11] is also applied to the Euler characteristic version in our situation. Consequently, we have the global multiple cover formula,

$$N^{\chi}_{n,mc_1(L)} = \sum_{k \ge 1, k|(n,m)} \frac{1}{k^2} N^{\chi}_{1,mc_1(L)/k}. \tag{107}$$

On the other hand, since $v = (0, ac_1(L), 1) \in H^*(X,\mathbb{Z})$ is primitive for any $a \in \mathbb{Z}$, we have

$$N^{\chi}_{1,ac_1(L)} = \chi(\mathrm{Hilb}^{(v,v)/2+1}(S)),$$

by (104) and (103). Also noting (103) in the LHS of (107), the invariant $J(0, mc_1(L), n)$ satisfies the formula (105). $\square$

Remark 5.12. *The assumption of the K3 surface S in Theorem 5.11 is a genericity condition for polarized K3 surfaces. Namely for $d \in \mathbb{Z}$, let us consider the moduli space of polarized K3 surfaces (S, L) with $L^2 = 2d - 2$. Then (S, L) satisfies the assumption in Theorem 5.11 when (S, L) is a general point of the above moduli space.*

Remark 5.13. *In [25, Proposition 6.9], we proved the formula (106) for $d = 2$ and $p = 2$. Even in this case, it is not obvious to compute the LHS of (106) directly. Indeed in [25, Proposition 6.9], we used the result by Mozgovoy [20]. The proof of Theorem 5.11 does not require the result of [20], and can be applied to more general cases.*

Remark 5.14. *Note that there is a weight two Hodge structure on $H^*(S,\mathbb{C})$ by*

$$H^{*2,0} = H^{2,0}, \ H^{*0,2} = H^{0,2},$$
$$H^{*1,1} = H^{0,0} \oplus H^{1,1} \oplus H^{2,2}.$$

Let G be the group of Hodge isometries of $H^*(S,\mathbb{Z})$. Then for any $g \in G$ and $v \in H^*(S,\mathbb{Z})$, it is proved in [25, Theorem 4.21] that

$$J(gv) = J(v). \tag{108}$$

Let $v = (0, mc_1(L), n) \in H^*(S,\mathbb{Z})$ be as in the statement of Theorem 5.11. The result of Theorem 5.11 and the formula (108) imply that $J(gv)$ also satisfies the formula (105) for any $g \in G$.

6 Identity of Behrend functions

In this section, we give a proof of Lemma 4.4. In principle, the result is a consequence of $\mathbb{C}^*$-localizations of the Behrend functions as in [2, Proposition 3.3], [18, Theorem C], together with the results in Proposition 4.2. However, as we discussed in [23, Remark 2.4], we are unable to find a symmetric perfect obstruction theory on the moduli space of parabolic stable pairs, which prevents us to use the $\mathbb{C}^*$-localizations on the Behrend functions directly. Instead, we consider some other moduli spaces which admit $\mathbb{C}^*$-equivariant symmetric perfect obstruction theories, and apply $\mathbb{C}^*$-localizations to the Behrend functions on them. Then the result follows by comparing their Behrend functions with those on the moduli spaces of parabolic stable pairs.

6.1 Deformations of sheaves

In this subsection, we recall a result on deformations of sheaves on algebraic varieties given by Huybrechts-Thomas [11], which will be used in the next subsection. Let X be a smooth projective variety and T an affine scheme with a closed point $0 \in T$. Suppose that we are given a T-flat coherent sheaf on $X \times T$,

$$A \in \mathrm{Coh}(X \times T).$$

We would like to extend A to a square zero extension $j \colon T \hookrightarrow \overline{T}$, i.e. there is an ideal $J \subset \mathcal{O}_{\overline{T}}$ such that

$$\mathcal{O}_T \cong \mathcal{O}_{\overline{T}}/J, \quad J^2 = 0. \tag{109}$$

Let us take the distinguished triangle in $D^b\,\mathrm{Coh}(X \times T)$,

$$Q_A \to \mathbf{L}j^* j_* A \to A. \tag{110}$$

Here the right arrow is the adjunction, and we have denoted $\mathrm{id}_X \times j$ just by j for simplicity. Following [11], we construct the morphism

$$\pi_A \colon Q_A \to A \otimes_{\mathcal{O}_T} J[1],$$

in the following way. Let h be the embedding

$$h \colon X \times T \times X \times T \hookrightarrow X \times T \times X \times \overline{T},$$

and H the object

$$H := \mathbf{L}h^* h_* \Delta_* \mathcal{O}_{X\times T},$$

where Δ is the diagonal embedding of $X \times T$. By [11, Subsection 3.1], there are distinguished triangles on $X \times T$,

$$\begin{aligned} &\tau^{\le -1}H \to H \to \Delta_* \mathcal{O}_{X\times T}, \\ &\tau^{\le -2}H \to \tau^{\le -1}H \stackrel{\pi_H}{\to} \Delta_* J[1]. \end{aligned} \tag{111}$$

Applying Fourier-Mukai transforms of A for the triangle (111), we obtain the triangle (110). Then the morphism π_A is obtained by taking the Fourier-Mukai transform of A with for the morphism π_H.

On the other hand, suppose that the following morphism on $X \times \overline{T}$ is given,

$$e_A \colon j_* A \to j_*(A \otimes J)[1].$$

Then we construct the morphism Ψ_{e_A} to be the composition,

$$\Psi_{e_A} \colon Q_A \to \mathbf{L}j^* j_* A \stackrel{\mathbf{L}j^* e_A}{\to} \mathbf{L}j^* j_*(A \otimes J)[1] \to A \otimes J[1].$$

Here the left arrow is given by the left arrow of (110), and the right arrow is given by the adjunction. Let us take the cone of e_A,

$$j_*(A \otimes J) \to \overline{A} \to j_* A \stackrel{e_A}{\to} j_*(A \otimes J)[1].$$

Note that $\overline{A}$ is a coherent sheaf on $X \times \overline{T}$. By [11, Theorem 3.3], we have the following criteion for an object $\overline{A}$ to be a deformation of A:

Theorem 6.1. **([11, Theorem 3.3])** *The sheaf $\overline{A} \in \mathrm{Coh}(X \times \overline{T})$ is flat over $\overline{T}$ with $\overline{A}|_{X\times T} \cong A$ if and only if we have the equality,*

$$\pi_A = \Psi_{e_A}. \tag{112}$$

6.2 Moduli spaces of simple sheaves and relative Quot schemes

Let X be a smooth projective Calabi-Yau 3-fold over $\mathbb{C}$, and $H \subset X$ a smooth and connected divisor. Recall that, giving a parabolic stable pair (F, s) on X is equivalent to giving a pair,

$$N_{H/X}[-1] \to F,$$

where $N_{H/X}$ is the normal bundle of H in X, satisfying a certain stability condition. (cf. [23, Proposition 3.9].) By taking the cone, we obtain the exact sequence of sheaves,

$$0 \to F \to E \to N_{H/X} \to 0. \tag{113}$$

Note that $N_{H/X} \cong \mathcal{O}_H(H)$ is a simple sheaf, i.e. $\mathrm{End}(N_{H/X}) = \mathbb{C}$. Together with the parabolic stability of (F, s), the sheaf E is also a simple sheaf. (cf. [23, Corollary 3.10].)

We consider the moduli space of simple sheaves on X, which we denote by $\mathcal{M}$. The space $\mathcal{M}$ is known to be an algebraic space locally of finite type over $\mathbb{C}$. (cf. [12].) The universal sheaf is denoted by

$$\mathcal{E} \in \mathrm{Coh}(X \times \mathcal{M}). \tag{114}$$

Let

$$\tau \colon \mathcal{Q} \to \mathcal{M}, \tag{115}$$

be the algebraic space representing the relative Quot-functor for the family of simple sheaves (114). Namely for each $[E] \in \mathcal{M}$ the fiber of $\mathcal{Q} \to \mathcal{M}$ is Grothendieck's Quot-scheme parameterizing quotient sheaves $E \twoheadrightarrow E'$. We have the morphism,

$$\iota \colon M_n^{\mathrm{par}}(U, \beta) \to \mathcal{Q}, \tag{116}$$

sending a parabolic stable pair (F, s) to the surjection $E \twoheadrightarrow N_{H/X}$ given by the sequence (113). We have the following lemma.

Lemma 6.2. *The morphism τ is étale at any point in the image of ι.*

Proof. Let (F, s) be a parabolic stable pair on X and $E \twoheadrightarrow N_{H/X}$ a point of $\mathcal{Q}$ determined by (113). Let T be an affine scheme, $0 \in T$ a closed point, $j \colon T \hookrightarrow \overline{T}$ a square zero extension with an ideal $J \subset \mathcal{O}_{\overline{T}}$ as in (109). Suppose that there is a commutative diagram,

$$\begin{array}{ccc} T & \xrightarrow{f} & \mathcal{Q} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \tau} \\ \overline{T} & \xrightarrow{h} & \mathcal{M}, \end{array} \tag{117}$$

such that f sends $0 \in T$ to $(E \twoheadrightarrow N_{H/X}) \in \mathcal{Q}$. It is enough to show that, after replacing T by an affine open neighborhood of $0 \in T$, the morphism f uniquely extends to $\overline{f} \colon \overline{T} \to \mathcal{Q}$ which commutes with all the arrows in (117).

By pulling back $\mathcal{E} \in \mathrm{Coh}(X \times \mathcal{M})$ to T by the composition $\tau \circ f \colon T \to \mathcal{M}$, we obtain the sheaf $\mathcal{E}_T \in \mathrm{Coh}(X \times T)$, which is flat over T, and restricts to the sheaf E on $X \times \{0\}$. The morphism f corresponds to the exact sequence of sheaves on $X \times T$,

$$0 \to \mathcal{F}_T \to \mathcal{E}_T \to \mathcal{N}_T \to 0, \tag{118}$$

which restrict to the exact sequence (113) on $X \times \{0\}$. Also the morphism h corresponds to a $\overline{T}$-flat sheaf on $X \times \overline{T}$,

$$\mathcal{E}_{\overline{T}} \in \mathrm{Coh}(X \times \overline{T}),$$

which restricts to $\mathcal{E}_T$ on $X \times T$. The existence of unique $\overline{f}$ is equivalent to the existence of unique (up to isomorphism) exact sequence of sheaves on $X \times \overline{T}$,

$$0 \to \mathcal{F}_{\overline{T}} \to \mathcal{E}_{\overline{T}} \to \mathcal{N}_{\overline{T}} \to 0, \tag{119}$$

which restricts to the exact sequence (118) on $X \times T$.

Let us consider the distinguished triangle on $X \times \overline{T}$,

$$j_*(\mathcal{E}_T \otimes J) \to \mathcal{E}_{\overline{T}} \to j_*\mathcal{E}_T \stackrel{e_{\mathcal{E}}}{\to} j_*(\mathcal{E}_T \otimes J)[1].$$

Since $\mathcal{E}_{\overline{T}}$ is a deformation of $\mathcal{E}_T$ to $X \times \overline{T}$, we have

$$\pi_{\mathcal{E}_T} = \Psi_{e_{\mathcal{E}}}, \tag{120}$$

in the notation of the previous subsection, by Theorem 6.1. Also we have the distinguished triangles on $X \times \overline{T}$,

$$\begin{array}{ccccc} j_*\mathcal{F}_T & \longrightarrow & j_*\mathcal{E}_T & \longrightarrow & j_*\mathcal{N}_T \\ & & \downarrow{\scriptstyle e_{\mathcal{E}}} & & \\ j_*(\mathcal{F}_T \otimes J)[1] & \longrightarrow & j_*(\mathcal{E}_T \otimes J)[1] & \longrightarrow & j_*(\mathcal{N}_T \otimes J)[1]. \end{array} \tag{121}$$

Since $\mathrm{Hom}(F, N_{H/X}[i]) = 0$ for $i = 0, 1$, we have (after shrinking T if necessary)

$$\mathrm{Hom}_{X\times T}(\mathcal{F}_T, \mathcal{N}_T \otimes J[i]) = 0, \tag{122}$$

for $i = 0, 1$. By the distinguished triangle,

$$\mathcal{F}_T \otimes J[1] \to \mathbf{L}j^*j_*\mathcal{F}_T \to \mathcal{F}_T,$$

and the vanishing (122), we see that

$$\begin{aligned} \mathrm{Hom}_{X\times\overline{T}}(j_*\mathcal{F}_T, j_*(\mathcal{N}_T \otimes J)[i]) &\cong \mathrm{Hom}_{X\times T}(\mathbf{L}j^*j_*\mathcal{F}_T, \mathcal{N}_T \otimes J[i]) \\ &= 0, \end{aligned}$$

for $i = 0, 1$. Therefore there are unique morphisms,

$$\begin{aligned} e_{\mathcal{F}} &\colon j_*\mathcal{F}_T \to j_*(\mathcal{F}_T \otimes J)[1], \\ e_{\mathcal{N}} &\colon j_*\mathcal{N}_T \to j_*(\mathcal{N}_T \otimes J)[1], \end{aligned}$$

which make the diagram (121) commutative. We need to show that $e_{\mathcal{F}}$ and $e_{\mathcal{N}}$ determine deformations of $\mathcal{F}_T$ and $\mathcal{N}_T$ to $X \times \overline{T}$. In order to see these, we consider the commutative diagram,

$$\begin{array}{ccccc} Q_{\mathcal{F}_T} & \longrightarrow & Q_{\mathcal{E}_T} & \longrightarrow & Q_{\mathcal{N}_T} \\ {\scriptstyle \pi_{\mathcal{F}}-\Psi_{e_{\mathcal{F}}}}\downarrow & & {\scriptstyle \pi_{\mathcal{E}}-\Psi_{e_{\mathcal{E}}}}\downarrow & & {\scriptstyle \pi_{\mathcal{N}}-\Psi_{e_{\mathcal{N}}}}\downarrow \\ \mathcal{F}_T \otimes J[1] & \longrightarrow & \mathcal{E}_T \otimes J[1] & \longrightarrow & \mathcal{N}_T \otimes J[1]. \end{array}$$

The middle arrow is zero by (120). Also we have $\mathrm{Hom}(Q_{\mathcal{F}_T}, \mathcal{N}_T \otimes J) = 0$ since $\mathcal{H}^i(Q_{\mathcal{F}_T}) = 0$ for $i \geq 0$. Therefore by the above commutative diagram, we have

$$\pi_{\mathcal{F}} = \Psi_{e_{\mathcal{F}}},$$

which implies that $e_{\mathcal{F}}$ determines a deformation of $\mathcal{F}_T$ to $\mathcal{F}_{\overline{T}}$ by Theorem 6.1. A similar argument shows that $e_{\mathcal{N}}$ determines a deformation of $\mathcal{N}_T$ to $\mathcal{N}_{\overline{T}}$.

By taking the cones e_* for $* = \mathcal{F}, \mathcal{E}, \mathcal{N}$ in the diagram (121), we obtain the exact sequence of sheaves,

$$0 \to \mathcal{F}_{\overline{T}} \to \mathcal{E}_{\overline{T}} \to \mathcal{N}_{\overline{T}} \to 0,$$

where $\mathcal{F}_{\overline{T}}$ and $\mathcal{N}_{\overline{T}}$ are deformations of $\mathcal{F}_T$, $\mathcal{N}_{\overline{T}}$ determined by $e_{\mathcal{F}}$, $e_{\mathcal{N}}$ respectively. It is straightforward to check that the above extension of (118) to $X \times \overline{T}$ is unique up to isomorphisms, and we leave the readers to check the detail. ☐

6.3 Some identities of Behrend functions

Let $C \subset U \subset X$ and $H \subset X$ be as in the previous sections. Let

$$\mathcal{Q}_U \subset \mathcal{Q}, \quad \mathcal{M}_U \subset \mathcal{M},$$

be sufficiently small analytic neighborhoods of the images of ι, $\tau \circ \iota$ respectively. Here τ, ι are defined in (115), (116). By Lemma 6.2, the morphism τ restricts to a local immersion,

$$\tau \colon \mathcal{Q}_U \to \mathcal{M}_U. \tag{123}$$

The arguments similar to Subsection 2.8 and Subsection 4.1 show that $\mathcal{M}_U$ and $\mathcal{Q}_U$ admit $\mathbb{C}^*$-actions, where $\mathbb{C}^*$ is the subtorus (42), so that the morphisms (116) and (123) are $\mathbb{C}^*$-equivariant. Let

$$\nu_{\mathcal{Q}},\ \nu_{\mathcal{Q}^{\mathbb{C}^*}},\ \nu_{\mathcal{M}},\ \nu_{\mathcal{M}^{\mathbb{C}^*}},$$

be the Behrend functions on $\mathcal{Q}$, $\mathcal{Q}_U^{\mathbb{C}^*}$, $\mathcal{M}$, $\mathcal{M}_U^{\mathbb{C}^*}$ respectively. We have the following lemma:

Lemma 6.3. *For $(F,s) \in M_n^{\mathrm{par}}(U,\beta)^{\mathbb{C}^*}$ and the associated element $p = \iota(F,s) \in \mathcal{Q}_U^{\mathbb{C}^*}$, we have*

$$\nu_{\mathcal{Q}}(p) = (-1)^{\dim T_p \mathcal{Q} - \dim T_p \mathcal{Q}_U^{\mathbb{C}^*}} \nu_{\mathcal{Q}^{\mathbb{C}^*}}(p). \tag{124}$$

Proof. Let us write $p = (E \twoheadrightarrow N_{H/X}) \in \mathcal{Q}_U^{\mathbb{C}^*}$. Then by Lemma 6.2, we have

$$\nu_{\mathcal{Q}}(p) = \nu_{\mathcal{M}}([E]), \quad \nu_{\mathcal{Q}^{\mathbb{C}^*}}(p) = \nu_{\mathcal{M}^{\mathbb{C}^*}}([E]). \tag{125}$$

Next note that, by [11], the algebraic space $\mathcal{M}$ admits a symmetric perfect obstruction theory determined by the universal sheaf (114). It is easy to check that the symmetric perfect obstruction theory on $\mathcal{M}$, restricted to $\mathcal{M}_U$, is $\mathbb{C}^*$-equivariant. Therefore the $\mathbb{C}^*$-localizations of the Behrend functions given in [2, Proposition 3.3], [18, Theorem C] are applied. The result is

$$\nu_{\mathcal{M}}(E) = (-1)^{\dim T_{[E]}\mathcal{M} - \dim T_{[E]}\mathcal{M}^{\mathbb{C}^*}} \cdot \nu_{\mathcal{M}^{\mathbb{C}^*}}(E). \tag{126}$$

Again by Lemma 6.2, we have

$$\dim T_p \mathcal{Q} = \dim T_{[E]}\mathcal{M}, \quad \dim T_p \mathcal{Q}_U^{\mathbb{C}^*} = \dim T_{[E]}\mathcal{M}^{\mathbb{C}^*}. \tag{127}$$

The equality (124) follows from (125), (126) and (127). ☐

Next we compare the Behrend functions on $\nu_{\mathcal{Q}}$ and $\nu_{M^{\mathrm{par}}}$ under the morphism (116). (Recall that $\nu_{M^{\mathrm{par}}}$ is the Behrend function on $M_n(U,\beta)$.) In what follows, for $E_1, E_2 \in \mathrm{Coh}(X)$, we write

$$\hom(E_1, E_2) := \dim \mathrm{Hom}(E_1, E_2),$$
$$\mathrm{ext}^1(E_1, E_2) := \dim \mathrm{Ext}^1_X(E_1, E_2).$$

We have the following lemma:

Lemma 6.4. *For $(F,s) \in M_n^{\mathrm{par}}(U,\beta)$ with $p = \iota(F,s) \in \mathcal{Q}$, we have the equality,*

$$\nu_{\mathcal{Q}}(p) = (-1)^{\mathrm{ext}^1(N_{H/X}, N_{H/X})} \cdot \nu_{M^{\mathrm{par}}}(F,s). \tag{128}$$

Proof. Let us write $p = (E \twoheadrightarrow N_{H/X}) \in \mathcal{Q}$. We first note that, a point of $\mathcal{Q}_U$ near $p \in \mathcal{Q}$ is represented by an exact sequence

$$0 \to F' \to E' \to N' \to 0, \tag{129}$$

where F', E', N' are small deformations of sheaves F, E, $N_{H/X}$ in (113). Hence near $p \in \mathcal{Q}_U$, we have the 1-morphism,

$$\mathcal{Q}_U \to \mathcal{M} \times \mathcal{C}oh(X), \tag{130}$$

which sends the sequence (129) to (N', F'). Here $\mathcal{C}oh(X)$ is the stack of all the objects in $\mathrm{Coh}(X)$, as in Subsection 2.3. The fiber of the above 1-morphism at (N', F') is an open subset of $\mathrm{Ext}^1_X(N', F')$. Since we have

$$\mathrm{Ext}^i_X(N_{H/X}, F) \cong \begin{cases} \mathbb{C}^{\beta \cdot H}, & i = 1, \\ 0, & i \neq 1, \end{cases}$$

it follows that

$$\mathrm{Ext}^1_X(N', F') \cong \mathbb{C}^{\beta \cdot H}.$$

Therefore the morphism (130) is a smooth morphism of relative dimension $\beta \cdot H$.

Let us consider the Behrend function on the RHS of (130). It is easy to see that any small deformation of $N_{H/X} \cong \mathcal{O}_H(H)$ is obtained as a line bundle on a divisor in X. Hence we see that the algebraic space $\mathcal{M}$ is smooth of dimension $\mathrm{ext}^1(N_{H/X}, N_{H/X})$ at $[N_{H/X}] \in \mathcal{M}$. It follows that we have

$$\nu_{\mathcal{M}}(N_{H/X}) = (-1)^{\mathrm{ext}^1(N_{H/X}, N_{H/X})}.$$

If we denote by $\nu_{\mathcal{C}}$ the Behrend function on $\mathcal{C}oh(X)$, the above arguments imply

$$\nu_{\mathcal{Q}}(p) = \nu_{\mathcal{M}}(N_{H/X}) \cdot \nu_{\mathcal{C}}(F) \cdot (-1)^{\beta \cdot H}, \tag{131}$$
$$= (-1)^{\beta \cdot H + \mathrm{ext}^1(N_{H/X}, N_{H/X})} \cdot \nu_{\mathcal{C}}(F). \tag{132}$$

In (131), we have used the property of the Behrend function under smooth morphisms and products [1, Proposition 1.5].

On the other hand, there is a forgetting 1-morphism,

$$M_n^{\mathrm{par}}(U,\beta) \to \mathcal{C}oh(X),$$

sending (F,s) to F. The fiber of the above morphism at $[F]$ is an open subset of $F\otimes\mathcal{O}_H \cong \mathbb{C}^{\beta\cdot H}$, hence it is a smooth morphism of relative dimension $\beta\cdot H$. Therefore we have

$$\nu_{M^{\mathrm{par}}}(F,s) = (-1)^{\beta\cdot H}\cdot \nu_{\mathcal{C}}(F). \tag{133}$$

Combined with (132) and (133), we obtain the desired equality (128). □

6.4 Proof of Lemma 4.4

Finally in this section, we give a proof of Lemma 4.4.

Proof. Let $\sigma\colon \widetilde{U}\to U$ be the m-fold cyclic cover considered in the statement of Lemma 4.4. Then for $m\gg 0$, we have

$$M_n^{\mathrm{par}}(U,\beta)^{\mathbb{Z}/m\mathbb{Z}} = M_n^{\mathrm{par}}(U,\beta)^{\mathbb{C}^*}.$$

Hence by Proposition 4.2, for $(F,s)\in M_n^{\mathrm{par}}(U,\beta)^{\mathbb{C}^*}$, there is unique $(\widetilde{F},\widetilde{s})\in M_n^{\mathrm{par}}(\widetilde{U},\widetilde{\beta})$ such that $(F,s)=\sigma_*(\widetilde{F},\widetilde{s})$. Similarly to (113), the pair $(\widetilde{F},\widetilde{s})$ associates the exact sequence of sheaves on $\widetilde{U}$,

$$0\to \widetilde{F}\to\widetilde{E}\to N_{\widetilde{H}/\widetilde{U}}\to 0.$$

Let N' be a coherent sheaf on X, which is a small deformation of $N_{H/X}$. Then we can uniquely lift $N'|_U$ to a sheaf $\widetilde{N}'$ on $\widetilde{U}$ so that $\widetilde{N}'$ is a small deformation of the sheaf $N_{\widetilde{H}/\widetilde{U}}$ on $\widetilde{U}$. Let $\widetilde{\mathcal{Q}}$ be the analytic local moduli space parameterizing small deformations of surjections $\widetilde{E}\twoheadrightarrow N_{\widetilde{H}/\widetilde{U}}$,

$$\widetilde{E}'\twoheadrightarrow\widetilde{N}', \tag{134}$$

where (134) is a surjection of coherent sheaves on $\widetilde{U}$, and $\widetilde{N}'$ is a lift of a small deformation of $N_{H/X}$ restricted to U as above. An argument similar to the proof of Lemma 4.1 shows that there is a natural morphism,

$$\sigma_*\colon \widetilde{\mathcal{Q}}\to\mathcal{Q}_U^{\mathbb{Z}/m\mathbb{Z}} \tag{135}$$

satisfying that

$$\sigma_*(\widetilde{E}\twoheadrightarrow N_{\widetilde{H}/\widetilde{U}}) = (E\twoheadrightarrow N_{H/X}).$$

Also a proof similar to Proposition 4.2 shows that the morphism (135) is an isomorphism onto connected components of $\mathcal{Q}_U^{\mathbb{Z}/m\mathbb{Z}}$. Since we have $\mathcal{Q}_U^{\mathbb{Z}/m\mathbb{Z}}=\mathcal{Q}_U^{\mathbb{C}^*}$ for $m\gg 0$, it follows that

$$\nu_{\widetilde{\mathcal{Q}}}(\widetilde{E}\twoheadrightarrow N_{\widetilde{H}/\widetilde{U}}) = \nu_{\mathcal{Q}^{\mathbb{C}^*}}(E\twoheadrightarrow N_{H/X}). \tag{136}$$

Also similarly to (128), we have the equality,

$$\nu_{\widetilde{\mathcal{Q}}}(\widetilde{E} \twoheadrightarrow N_{\widetilde{H}/\widetilde{X}}) = (-1)^{\mathrm{ext}^1(N_{H/X}, N_{H/X})} \cdot \nu_{\widetilde{M}^{\mathrm{par}}}(\widetilde{F}, \widetilde{s}). \tag{137}$$

By (128), (124), (136) and (137), we obtain

$$\nu_{\widetilde{M}^{\mathrm{par}}}(\widetilde{F}, \widetilde{s}) = (-1)^{\dim T_{\widetilde{p}}\widetilde{\mathcal{Q}} - \dim T_p \mathcal{Q}} \cdot \nu_{M^{\mathrm{par}}}(F, s), \tag{138}$$

where $\widetilde{p} = (\widetilde{E} \twoheadrightarrow N_{\widetilde{H}/\widetilde{U}}) \in \widetilde{\mathcal{Q}}$.

Let us evaluate $\dim T_{\widetilde{p}}\widetilde{\mathcal{Q}} - \dim T_p\mathcal{Q}$. Since the morphism (130) is a smooth morphism of relative dimension $\beta \cdot H$, we have

$$\dim T_p \mathcal{Q} = \mathrm{ext}^1(F, F) - \hom(F, F) + \mathrm{ext}^1(N_{H/X}, N_{H/X}) + \beta \cdot H. \tag{139}$$

Similarly we have

$$\dim T_{\widetilde{p}} \widetilde{\mathcal{Q}} = \mathrm{ext}^1(\widetilde{F}, \widetilde{F}) - \hom(\widetilde{F}, \widetilde{F}) + \mathrm{ext}^1(N_{H/X}, N_{H/X}) + \widetilde{\beta} \cdot \widetilde{H}. \tag{140}$$

We have

$$\begin{aligned} &\mathrm{ext}^1(F, F) - \hom(F, F) - \mathrm{ext}^1(\widetilde{F}, \widetilde{F}) + \hom(\widetilde{F}, \widetilde{F}) \\ &= \mathrm{ext}^1(\sigma_*\widetilde{F}, \sigma_*\widetilde{F}) - \hom(\sigma_*\widetilde{F}, \sigma_*\widetilde{F}) - \mathrm{ext}^1(\widetilde{F}, \widetilde{F}) + \hom(\widetilde{F}, \widetilde{F}) \\ &= \sum_{0 \neq g \in \mathbb{Z}/m\mathbb{Z}} \{\mathrm{ext}^1(g_*\widetilde{F}, \widetilde{F}) - \hom(g_*\widetilde{F}, \widetilde{F})\} \end{aligned} \tag{141}$$

By the Riemann-Roch theorem and the Serre duality, we have

$$\begin{aligned} \mathrm{ext}^1(g_*\widetilde{F}, \widetilde{F}) - \hom(g_*\widetilde{F}, \widetilde{F}) &= \mathrm{ext}^1(\widetilde{F}, g_*\widetilde{F}) - \hom(\widetilde{F}, g_*\widetilde{F}) \\ &= \mathrm{ext}^1((-g)_*\widetilde{F}, \widetilde{F}) - \hom((-g)_*\widetilde{F}, \widetilde{F}). \end{aligned}$$

Therefore (141) is an even integer if m is an odd integer. By (139), (140), we have

$$\dim T_{\widetilde{p}}\widetilde{\mathcal{Q}} - \dim T_p \mathcal{Q} \equiv \widetilde{\beta} \cdot \widetilde{H} - \beta \cdot H, \quad (\mathrm{mod}\ 2).$$

Combined with (138), we obtain (i) of Lemma 4.4. The result of (ii) follows from (i) and (133). □

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