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A R^4 non-renormalisation theorem in $\mathcal{N}=4$ supergravity

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Abstract

We consider the four-graviton amplitudes in CHL constructions providing four-dimensional $\mathcal{N}=4$ models with various vector multiplet. We show that in these models the two-loop amplitude factorizes a $\partial^2 R^4$. This implies a non-renormalisation theorem for the R^4 term, which forbids the appearance of a three-loop ultraviolet divergence in four dimensions in the four-graviton amplitude. We connect the special nature of the R^4 term to the $U(1)$ anomaly of pure $\mathcal{N}=4$ supergravity.

I. INTRODUCTION

$\mathcal{N}=4$ supergravity in four dimensions has sixteen real supercharges and $SU(4)$ for R-symmetry group. The gravity supermutiplet is composed of a spin 2 graviton and two spin 0 real scalars in the singlet representation of $SU(4)$, four spin 3/2 gravitini and four spin 1/2 fermions in the fundamental representation **4** of $SU(4)$, and six spin 1 gravi-photons in the **6** of $SU(4)$.

The only matter multiplet is the vector multiplet composed by one spin 1 vector which is $SU(4)$ singlet, four spin 1/2 fermions transforming in the fundamental of $SU(4)$, and six spin 0 real scalars transforming in the **6** of $SU(4)$. The vector multiplets may be carrying non-Abelian gauge group from a $\mathcal{N}=4$ super-Yang-Mills theory.

Pure $\mathcal{N}=4$ supergravity contains only the gravity supermultiplet and the two real scalar can be assembled into a complex axion-dilaton scalar S parametrizing the coset space $SU(1,1)/U(1)$. This multiplet can be coupled to n_v vector multiplets, whose scalar fields parametrize the coset space $SO(6,n_v)/SO(6)\times SO(n_v)$ [1].

$\mathcal{N}=4$ supergravity theories can be obtained by consistent dimensional reduction of $\mathcal{N}=1$ supergravity in $D=10$, or from various string theory models. For instance the reduction of the $\mathcal{N}=8$ gravity super-multiplet, leads to $\mathcal{N}=4$ gravity super-multiplet four spin 3/2 $\mathcal{N}=4$ super-multiplet, and six vector-multiplet

$$\begin{aligned}(2_{\mathbf{1}},3/2_{\mathbf{8}},1_{\mathbf{28}},1/2_{\mathbf{56}},0_{\mathbf{70}})_{\mathcal{N}=8} &= (2_{\mathbf{1}},3/2_{\mathbf{4}},1_{\mathbf{6}},1/2_{\mathbf{4}},0_{\mathbf{1+1}})_{\mathcal{N}=4} \\ &\oplus\ \mathbf{4}\,(3/2_{\mathbf{1}},1_{\mathbf{4}},1/2_{\mathbf{6+1}},0_{\mathbf{4+\bar{4}}})_{\mathcal{N}=4} \\ &\oplus\ \mathbf{6}\,(1_{\mathbf{1}},1/2_{\mathbf{4}},0_{\mathbf{6}})_{\mathcal{N}=4}\,.\end{aligned} \tag{I.1}$$

Removing the four spin 3/2 $\mathcal{N}=4$ supermultiplet leads to $\mathcal{N}=4$ supergravity coupled to $n_v=6$ vector multiplet.

In order to disentangle the contributions from the vector multiplets and the gravity supermultiplets, we will use CHL models [2–4] that allow to construct $\mathcal{N}=4$ four dimensional heterotic string with gauge groups of reduced rank. In this paper we will work at a generic point of the moduli space in the presence of (diagonal) Wilson lines where the gauge group is Abelian.

Various CHL compactifications in four dimensions can obtained by considering $\mathbb{Z}_N$ orbifold [3, 5, 6] of the heterotic string on $T^5\times S^1$. The orbifold acts on the current algebra and the right-moving compactified modes of the string (world-sheet supersymmetry is on the left moving sector) together with an order N shift along the S^1 direction. This leads to four-dimensional models $\mathcal{N} = 4$ with $n_v = 48/(N+1) - 2$ vector multiplets at a generic point of the moduli space. Models with $(n_v, N) \in \{(22,1), (14,2), (10,3), (6,5), (4,7)\}$ have been constructed. No no-go theorem are known ruling out the $n_v = 0$ case although it will probably not arise from an asymmetric orbifold construction.[1]

One important point about the CHL model is that orbifold action does not alter the left moving supersymmetric sector of the theory and the fermionic zero mode saturation in these models will be identical as in the toroidally compactified heterotic string. It was shown in [7–9] that the $t_8\mathrm{tr}(R^4)$ and $t_8\mathrm{tr}(R^2)^2$ are half-BPS statured couplings, receiving contributions only from the short multiplet of the $\mathcal{N} = 4$ super-algebra, and without perturbative contributions beyond one-loop. These non-renormalisation theorems were confirmed in [10] using the explicit evaluation of the genus-two four-graviton heterotic amplitude derived in [11–13].

We show in this paper that the genus-two four-graviton amplitude in CHL model satisfy the same non-renormalisation theorems, since the genus-two four-graviton amplitude factorizes the mass dimension ten $\partial^2 R^4$ operator in each kinematic channel. By taking the field theory limit of these amplitudes in four dimensions no reduction of derivative is found for generic numbers of vector multiplets. Since this result is independent on the number of vector multiplets in the model, we conclude that this rules out the appearance of a R^4 ultraviolet counter-term at three-loop order in four dimensional pure $\mathcal{N} = 4$ supergravity. Thus the four-graviton scattering amplitude is ultraviolet finite at three loops in four dimensions.

The paper is organized as follows. In section II we will give the form of the one- and two-loop four-graviton amplitude in orbifold CHL models. In section III we evaluate their field theory limit in four dimensions. In section IV we discuss the implication of these results for the ultraviolet properties of pure $\mathcal{N} = 4$ supergravity.

Note: As this paper was being finalized, the preprint [14] appeared on the arXiv. In this work the absence of three-loop divergence in the four-graviton amplitude in four dimensions is obtained by a direct field theory computation.

[1] We would like to thank A. Sen for a discussion on this point.

II. ONE- AND TWO-LOOP AMPLITUDES IN CHL MODELS

Our convention are that the left-moving sector of the heterotic string is the supersymmetric sector, while the right-moving contains the current algebra.

We evaluate the four-graviton amplitude in four dimensional CHL heterotic string models. We show that the fermionic zero mode saturation is model independent and similar to the torus compactification.

A. The one-loop amplitude in string theory

The expression of the one-loop four-graviton amplitude in CHL models in $D = 10 - d$ dimensions is an immediate extension of the amplitude derived in [15]

$$\mathcal{M}^{(n_v)}_{4,1-loop} = \mathcal{N}_1 \, t_8 F^4 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{2-\frac{d}{2}}} \mathcal{Z}^{(n_v)}_1 \int_{\mathcal{T}} \prod_{1\leq i<j\leq 4} \frac{d^2\nu_i}{\tau_2} \, \mathcal{W}^{(1)} \, e^{-\sum_{1\leq i<j\leq 4} 2\alpha' k_i\cdot k_j P(\nu_{ij})} \,. \tag{II.1}$$

Where $\mathcal{N}_1$ is a constant of normalisation, $\mathcal{F} := \{\tau = \tau_1 + i\tau_2, |\tau| \geq 1, |\tau_1| \leq \frac{1}{2}, \tau_2 > 0\}$ is a fundamental domain for $SL(2,\mathbb{Z})$ and the domain of integration $\mathcal{T}$ is defined as $\mathcal{T} := \{\nu = \nu^1 + i\nu^2; |\nu^1| \leq \frac{1}{2}, 0 \leq \nu^2 \leq \tau_2\}$. $\mathcal{Z}^{(n_v)}_1$ is the genus-one partition function of the CHL model.

The polarisation of the rth graviton is factorized as $h^{(r)}_{\mu\nu} = \epsilon^{(r)}_\mu \, \tilde\epsilon^{(r)}_\nu$. We introduce the notation $t_8 F^4 := t_8^{\mu_1\cdots\mu_8} \prod_{r=1}^4 k^{(r)}_{\mu_{2r-1}} \, \epsilon^{(r)}_{\mu_{2r}}$. The quantity $\mathcal{W}^{(1)}$ arises from the contractions of the right-moving part of the graviton vertex operator

$$\mathcal{W}^{(1)} := \frac{\langle \prod_{j=1}^4 \epsilon^j \cdot \bar\partial X(z_j) e^{ik_j\cdot x(z_j)} \rangle}{\langle \prod_{j=1}^4 e^{ik_j\cdot x(z_j)} \rangle} = \prod_{r=1}^4 \tilde\epsilon^{(r)}_{\nu_r} \, t^{\nu_1\cdots\nu_4}_{4;1} \,, \tag{II.2}$$

with $\hat t^{\nu_1\cdots\nu_4}_{4;1}$ the quantity evaluated in [15]

$$\hat t^{\nu_1\cdots\nu_4}_{4;1} := Q^{\nu_1}_1 \cdots Q^{\nu_4}_4 + \frac{2}{\alpha'} \left(Q^{\nu_1}_1 Q^{\nu_2}_2 \delta^{\nu_3\nu_4} T(\nu_{34}) + perms\right) + \frac{4}{\alpha'^2} \left(\delta^{\nu_1\nu_2}\delta^{\nu_3\nu_4} T(\nu_{12}) T(\nu_{34}) + perms\right) , \tag{II.3}$$

where

$$Q^\mu_I := \sum_{r=1}^4 k^{(r)\mu} \, \bar\partial P(\nu_{Ir}|\tau); \qquad T(\nu) := \bar\partial^2_\nu P(\nu|\tau) \,.$$

We follow the notations and conventions of [16, 17]. The genus one propagator is given by

$$P(\nu|\tau) := -\frac{1}{4}\,\log\left|\frac{\theta_1(\nu|\tau)}{\theta_1'(0|\tau)}\right|^2 + \frac{\pi\nu_2^2}{2\tau_2}\,. \tag{II.4}$$

In the $\alpha' \to 0$ limit relevant for the field theory analysis in section III, with all the radii of compactification $R_i^2 \sim \alpha'$, the mass of the Kaluza-Klein excitations and winding modes go to infinity and the genus-one partition function $\mathcal{Z}_1^{(n_v)}$ has the following expression in $\bar q = \exp(-2i\pi\bar\tau)$

$$\mathcal{Z}_1^{(n_v)} = \frac{1}{\bar q} + c^1_{n_v} + O(\bar q)\,. \tag{II.5}$$

The $1/\bar q$ contribution is the "tachyonic" pole, $c^1_{n_v}$ depends on the number of vector multiplet and higher powers of $\bar q$ are massive string contribution that will not contribute in the field theory limit.

B. The two-loop amplitude in string theory

By applying the technics of [10–13], for evaluating the heterotic string two-loop amplitude we obtain that the four gravitons amplitude in the CHL models is given by

$$\mathcal{M}^{(n_v)}_{4,2-loop} = \mathcal{N}_2\,\frac{t_8F^4}{64\pi^{14}}\int\frac{|d^3\Omega|^2}{(\det\Im\mathrm{m}\Omega)^{5-\frac{d}{2}}}\,\mathcal{Z}_2^{(n_v)}\int\prod_{i=1}^4 d^2\nu_i\,\mathcal{W}^{(2)}\,\mathcal{Y}_s\,e^{-\sum_{1\le i<j\le 4}2\alpha' k^i\cdot k^j P(\nu_{ij})} \tag{II.6}$$

where $\mathcal{N}_2$ is a normalization constant, $\mathcal{Z}_2^{(n_v)}(\Omega,\bar\Omega)$ is the genus-two partition function and

$$\mathcal{W}^{(2)} := \frac{\langle\prod_{j=1}^4\epsilon^j\cdot\bar\partial X(z_j)e^{ik_j\cdot x(z_j)}\rangle}{\langle\prod_{j=1}^4 e^{ik_j\cdot x(z_j)}\rangle} = \prod_{i=1}^4\tilde\epsilon_i^{\nu_i}\,t_{4;2}^{\nu_1\cdot\nu_4}\,. \tag{II.7}$$

The tensor $t_{4;2}^{\nu_1\cdot\nu_4}$ is the genus-two equivalent of the genus-one tensor given in (II.3)

$$t_{4;2}^{\nu_1\cdots\nu_4} = Q_1^{\nu_1}\cdots Q_4^{\nu_4} + \frac{2}{\alpha'}Q_1^{\nu_1}Q_2^{\nu_2}T(\nu_{34})\delta^{\nu_3\nu_4} + \frac{4}{(\alpha')^2}\,\delta^{\nu_1\nu_2}\delta^{\nu_3\nu_4}T(\nu_{12})T(\nu_{34}) + perms\,, \tag{II.8}$$

this time expressed in terms of the genus-two bosonic propagator

$$P(\nu_1-\nu_2|\Omega) := -\log|E(\nu_1,\nu_2|\Omega)|^2 + 2\pi(\Im\mathrm{m}\Omega)^{-1}_{IJ}\,(\Im\mathrm{m}\int_{\nu_1}^{\nu_2}\omega_I)(\Im\mathrm{m}\int_{\nu_1}^{\nu_2}\omega_J)\,. \tag{II.9}$$

where $E(\nu)$ is the genus-two prime form, Ω is the period matrix and the ω_I with $I=1,2$ are the holomorphic abelian differentials. We refer to [13, Appendix A] for the main properties of these objects.

The $\mathcal{Y}_S$ quantity, arising from several contributions in the RNS formalism and from the fermionic zero modes in the pure spinor formalism [18, 19], is given by

$$3\mathcal{Y}_S = (k_1 - k_2) \cdot (k_3 - k_4)\, \Delta_{12}\Delta_{34} + (13)(24) + (14)(23)\,, \tag{II.10}$$

with

$$\Delta(z, w) = \omega_1(z)\omega_2(w) - \omega_1(w)\omega_2(z)\,. \tag{II.11}$$

Using the identity $\Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{42} + \Delta_{14}\Delta_{23} = 0$ we have the equivalent form $\mathcal{Y}_S = -3\,(s\Delta_{14}\Delta_{23} - t\Delta_{12}\Delta_{34})$.

We use a parametrisation of the period matrix reflecting the symmetries of the field theory vacuum two-loop diagram considered in the next section

$$\Omega := \begin{pmatrix} \tau_1 + \tau_3 & \tau_3 \\ \tau_3 & \tau_2 + \tau_3 \end{pmatrix}\,. \tag{II.12}$$

With this parametrisation the expression for $\mathcal{Z}_2^{(n_v)}(\Omega, \bar\Omega)$ is completely symmetric in the variables $q_I = \exp(2i\pi\tau_I)$ with $I = 1, 2, 3$.

In the limit relevant for the field theory analysis in section III, where $r, R_i \to 0$ with $r^2, R_i^2 \gg \alpha'$, the partition function of the CHL model has the following $\bar q_i$-expansion [20]

$$\mathcal{Z}_2^{(n_v)} = \frac{1}{\bar q_1 \bar q_2 \bar q_3} + a_{n_v} \sum_{1\leq i<j\leq 3} \frac{1}{\bar q_i \bar q_j} + b_{n_v} \sum_{1\leq i\leq 3} \frac{1}{\bar q_i} + c_{n_v} + O(q_i)\,. \tag{II.13}$$

III. THE FIELD THEORY LIMIT

In this section we extract the field theory limit of the string theory amplitudes compactified to four dimensions. We consider the low-energy limit $\alpha' \to 0$ with the radii of the torus proportional to $\sqrt{\alpha'}$ so that all the massive Kaluza-Klein states, winding states and excited string states decouple.

In order to simplify the analyzing we make the following choice of polarisations $(1^{++}, 2^{++}, 3^{--}, 4^{--})$ and of reference momenta[2] $q_1 = q_2 = k_3$ and $q_3 = q_4 = k_1$, such that $2t_8F^4 = \langle k_1\, k_2\rangle^2\, [k_3\, k_4]^2$, and $4t_8t_8R^4 = \langle k_1\, k_2\rangle^4\, [k_3\, k_4]^4$. With these choices the expression for $\mathcal{W}^{(g)}$ reduces to

[2] Our conventions are that null vector $k^2 = 0$ is parametrized as $k_{\alpha\dot\alpha} = k_\alpha \bar k_{\dot\alpha}$. The spin 1 polarisations of positive and negative helicities are given by $\epsilon^+(k, q)_{\alpha\dot\alpha} := \frac{q_\alpha \bar k_{\dot\alpha}}{\sqrt{2}\,\langle q\, k\rangle}$, $\epsilon^-(k, q)_{\alpha\dot\alpha} := -\frac{k_\alpha \bar q_{\dot\alpha}}{\sqrt{2}\,[q\, k]}$, where q is a reference momentum. One finds that $t_8F^{(1)+}\cdots F^{(4)+} = t_8F^{(1)+}\cdots F^{(4)+} = 0$ and $t_8F^{(1)-}F^{(2)-}F^{(3)+}F^{(4)+} = \frac{1}{16}\langle k_1\, k_2\rangle^2\, [k_3\, k_4]^2$

$$
\begin{aligned}
\mathcal{W}^{(g)} &= t_8 t_8 R^4 \, (\bar\partial P(\nu_{12}) - \bar\partial P(\nu_{14}))(\bar\partial P(\nu_{21}) - \bar\partial P(\nu_{24}))(\bar\partial P(\nu_{32}) - \bar\partial P(\nu_{34}))(\bar\partial P(\nu_{42}) - \bar\partial P(\nu_{43})) \\
&+ \frac{t_8 t_8 R^4}{u} \, \bar\partial^2 P(\nu_{24})(\bar\partial P(\nu_{12}) - \bar\partial P(\nu_{14}))(\bar\partial P(\nu_{32}) - \bar\partial P(\nu_{34})) \, . \qquad \text{(III.1)}
\end{aligned}
$$

where $s = (k_1 + k_2)^2$, $t = (k_1 + k_4)^2$ and $u = (k_1 + k_3)^2$. We introduce the notation $\mathcal{W}^{(g)} = t_8 t_8 R^4 \, (\mathcal{W}_1^{(g)} + u^{-1} \mathcal{W}_2^{(g)})$.

The main result of this section is that the one-loop amplitudes factorizes a $t_8 t_8 R^4$ and that the two-loop amplitudes factorizes a $\partial^2 t_8 t_8 R^4$ term. A more detailed analysis will be given in the work [20].

A. The one-loop amplitude in field theory

In the field theory limit $\alpha' \to 0$ with $\tau_2 \to \infty$ and $t = \alpha' \tau_2$ fixed, we define $\nu^2 = \tau_2 \, \omega$ for $\nu = \nu^1 + i \nu^2$.

Because of the $1/\bar q$ pole in the partition function (II.5) the integration over τ_1 yields two contributions

$$
\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \, \mathcal{Z}_1^{(n_v)} F(\tau, \bar\tau) = F_1 + c^1_{n_v} \, F_0 \, , \qquad \text{(III.2)}
$$

where $F(\tau, \bar\tau) = F_0 + \bar q F_1 + c.c. + O(\bar q^2)$ represents the integrand of the one-loop amplitude.

The bosonic propagator can be split in an asymptotic value for $\tau_2 \to \infty$ (the field theory limit) and a correction [16]

$$
P(\nu|\tau) = P^\infty(\nu|\tau) + \hat P(\nu|\tau) \qquad \text{(III.3)}
$$

where

$$
\begin{aligned}
P^\infty(\nu|\tau) &= \frac{\pi \nu_2^2}{2\tau_2} - \frac{1}{4} \ln \left| \frac{\sin(\pi\nu)}{\pi} \right|^2 \\
\hat P(\nu|\tau) &= - \sum_{m \geq 1} \left(\frac{q^m}{1 - q^m} \frac{\sin^2(m\pi\nu)}{m} + c.c. \right) + C(\tau), \qquad \text{(III.4)}
\end{aligned}
$$

where $q = \exp(2 i \pi \tau)$ and $C(\tau)$ is a zero mode contribution which drops out of the amplitude due to the momentum conversation [16].

We decompose the asymptotic propagator $P^\infty(\nu|\tau) = \frac{\pi}{2} \tau_2 \, P^{FT}(\omega) + \delta(\nu)$ into a piece that will dominate in the field theory limit

$$P^{FT}(\omega) = \omega^2 - |\omega|\,, \tag{III.5}$$

and a contribution $\delta(\nu)$ from the massive string modes [16, appendix A]

$$\delta(\nu) = \sum_{m\neq 0} \frac{1}{4|m|}\, e^{2i\pi m\nu^1 - 2\pi|m\nu^2|}\,. \tag{III.6}$$

The expression for Q_I^μ and T in (II.4) become

$$\begin{aligned} Q_I^\mu &= Q_I^{FT\,\mu} + \delta Q_I^\mu - \pi \sum_{r=1}^{4} k^{(r)\mu}\, \sin(2\pi\bar\nu_{Ir})\,\bar q + o(\bar q^2) \\ T(\bar\nu) &= T^{FT}(\omega) + \delta T(\bar\nu) + 2\pi\cos(2\pi\bar\nu)\,\bar q + o(\bar q^2)\,, \end{aligned} \tag{III.7}$$

where

$$Q_I^{FT\,\mu} := -\frac{\pi}{2}\,(2K^\mu + q_I^\mu) \tag{III.8}$$

$$K^\mu := \sum_{r=1}^{4} k^{(r)\mu}\,\omega_r \tag{III.9}$$

$$q_I^\mu := \sum_{r=1}^{4} k^{(r)\mu}\,\mathrm{sign}(\omega_I - \omega_r) \tag{III.10}$$

$$T^{FT}(\omega) \;=\; \frac{\pi\alpha'}{t}(1 - \delta(\omega))\,, \tag{III.11}$$

and

$$\begin{aligned} \delta Q_I^\mu(\bar\nu) &= \sum_{r=1}^{4} k^{(r)\mu}\,\bar\partial\delta(\bar\nu_{Ir}) = -\frac{i\pi}{2}\sum_{r=1}^{4}\mathrm{sign}(\nu_{Ir}^2)k^{(r)\mu}\sum_{m\geq 1} e^{-\mathrm{sign}(\nu_{Ir}^2)\,2i\pi m\bar\nu_{Ir}} \\ \delta T(\nu) &= \bar\partial^2\delta(\bar\nu) = -\pi^2\sum_{m\geq 1} m\, e^{-\mathrm{sign}(\nu_{Ir}^2)\,2i\pi m\bar\nu_{Ir}}\,. \end{aligned} \tag{III.12}$$

We introduce the notation

$$Q^{(1)}(\omega) := \sum_{1\leq i<j\leq 4} k_i\cdot k_j\, P^{FT}(\omega_{ij})\,, \tag{III.13}$$

such that $\partial_{\omega_i} Q^{(1)} = k_i \cdot Q_i^{FT}$.

In the field theory limit $\alpha' \to 0$ with all the radii of compactification $R_i^2 \sim \alpha'$ the integrand of the string amplitude in (II.1) becomes

$$M_{4;1}^{(n_v)} = N_1\, t_8 t_8 R^4 \int_0^\infty \frac{d\tau_2}{\tau_2^{2-\frac{d}{2}}} \int_{\Delta_\omega} \prod_{i=1}^{3} d\omega_i\; e^{t\, Q^{(1)}(\omega)} \times \tag{III.14}$$
$$\times \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1\, \frac{1 + c^1_{n_v} \bar q + o(\bar q^2)}{\bar q}\, (\mathcal{W}_1^{(1)} + \frac{1}{s_{13}} \mathcal{W}_2^{(1)}) \times$$
$$\times \exp\left(\sum_{1\leq i<j\leq 4} 2\alpha'\, k_i \cdot k_j \left(\delta(\nu_{ij}) - \sum_{m\geq 1} \bar q\, \sin^2(\pi \bar\nu_{ij}) + O(\bar q) \right) \right),$$

here N_1 is a constant of normalisation. The domain of integration $\Delta_\omega = [0,1]^3$ is decomposed into three regions $\Delta_w = \Delta_{(s,t)} \cup \Delta_{(s,u)} \cup \Delta_{(t,u)}$ given by the union of the (s,t), (s,u) and (t,u) domains. In the $\Delta_{(s,t)}$ domain the integration is performed over $0 \leq \omega_1 \leq \omega_2 \leq \omega_3 \leq 1$ where $Q^{(1)}(\omega) = -s\omega_1(\omega_3 - \omega_2) - t(\omega_2 - \omega_1)(1 - \omega_3)$ with equivalent formulas obtained by permuting the external legs labels in the (t,u) and (s,u) regions (see [16] for details).

The leading contribution to the amplitude is given by

$$M_{4;1}^{(n_v)} = N_1\, t_8 t_8 R^4 \int_0^\infty \frac{d\tau_2}{\tau_2^{2-\frac{d}{2}}} \int_{\Delta_\omega} \prod_{i=1}^{3} d\omega_i\; e^{t\, Q^{(1)}(\omega)} \times \tag{III.15}$$
$$\times \int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1 \left(\left(\mathcal{W}_1^{(1)} + \frac{1}{u} \mathcal{W}_2^{(1)} \right)\Big|_0 (c^1_{n_v} - \sum_{1\leq i<j\leq 4} 2\alpha'\, k_i \cdot k_j\, \sin^2(\pi \bar\nu_{ij})) + \left(\mathcal{W}_1^{(1)} + \frac{1}{u} \mathcal{W}_2^{(1)} \right)\Big|_1 \right).$$

where $(\mathcal{W}_1^{(1)} + \frac{1}{u}\mathcal{W}_2^{(1)})|_0$ and $(\mathcal{W}_1^{(1)} + \frac{1}{u}\mathcal{W}_2^{(1)})|_1$ are respectively the zeroth and first order in the $\bar q$ expansion of $\mathcal{W}_i^{(1)}$.

Performing the integrations over the ν_i^1 variables leads to the following structure for the amplitude reflecting the decomposition in (I.1)

$$M_{4;1}^{(n_v)} = N_1 \frac{\pi^4}{4} \left(c^1_{n_v}\, M_{4;1}^{\mathcal{N}=4 \text{ matter}} + M_{4;1}^{\mathcal{N}=8} - 4 M_{4;1}^{\mathcal{N}=4\ spin\ \frac{3}{2}} \right). \tag{III.16}$$

The contribution from the $\mathcal{N}=8$ supergravity multiplet is given by the quantity evaluated in [21]

$$M_{4;1}^{\mathcal{N}=8} = t_8 t_8 R^4 \int_{\Delta_\omega} d^3\omega\, \Gamma\left(2+\epsilon\right)\, (Q^{(1)})^{-2-\epsilon}, \tag{III.17}$$

where we have specified the dimension $D = 4 - 2\epsilon$ and $Q^{(1)}$ is defined in (III.13). The contribution from the $\mathcal{N}=4$ matter fields vector super-multiplets

$$M_{4;1}^{\mathcal{N}=4\ matter} = t_8 t_8 R^4\, \frac{\pi^4}{16} \int_{\Delta_\omega} d^3\omega \Big[\Gamma\,(1+\epsilon)\ (Q^{(1)})^{-1-\epsilon}\, W_2^{(1)} + \Gamma\,(2+\epsilon)\ (Q^{(1)})^{-2-\epsilon}\, W_1^{(1)}\Big] \tag{III.18}$$

where $W_i^{(1)}$ with $i = 1, 2$ are the field theory limits of the $\mathcal{W}_i^{(1)}$'s

$$\begin{aligned} W_2^{(1)} &= \frac{1}{u}\,(2\omega_2 - 1 + \mathrm{sign}(\omega_3 - \omega_2))(2\omega_2 - 1 + \mathrm{sign}(\omega_1 - \omega_2))\,(1 - \delta(\omega_{24})) \\ W_1^{(1)} &= 2(\omega_2 - \omega_3)(\mathrm{sign}(\omega_1 - \omega_2) + 2\omega_2 - 1) \times \\ &\quad \times (\mathrm{sign}(\omega_2 - \omega_1) + 2\omega_1 - 1)(\mathrm{sign}(\omega_3 - \omega_2) + 2\omega_2 - 1)\,. \end{aligned} \tag{III.19}$$

Finally, the $\mathcal{N} = 4$ spin 3/2 gravitino multiplet running in the loop

$$M_{4;1}^{\mathcal{N}=4\ \mathrm{spin}\ \frac{3}{2}} = t_8 t_8 R^4 \int_{\Delta_\omega} d^3\omega\, \Gamma\,(2+\epsilon)\, \tilde{W}_2^{(1)}\, (Q^{(1)})^{-2-\epsilon}, \tag{III.20}$$

where

$$\tilde{W}_2^{(1)} = (2\omega_2 - 1 + \mathrm{sign}(\omega_3 - \omega_2))(2\omega_2 - 1 + \mathrm{sign}(\omega_1 - \omega_2))\,. \tag{III.21}$$

Using the dictionary given in [22, 23], we recognize that the amplitudes in (III.18) and (III.20) are combinations of scalar box integral functions $I_4^{(D=4-2\epsilon)}[\ell^n]$ evaluated in $D = 4 - 2\epsilon$ with $n = 4, 2, 0$ powers of loop momentum and $I_4^{(D=6-2\epsilon)}[\ell^n]$ with $n = 2, 0$ powers of loop momentum evaluated in $D = 6 - 2\epsilon$ dimensions. The $\mathcal{N} = 8$ supergravity part in (III.17) is only given by a scalar box amplitude function $I_4^{(D=4-2\epsilon)}[1]$ evaluated in $D = 4 - 2\epsilon$ dimensions.

In [20] we show the presence of rational terms in these $\mathcal{N} = 4$ amplitudes in agreement with the analysis of [24–27].

B. The two-loop amplitude in field theory

We will follow the notations of [28, section 2.1] where the two-loop four-graviton amplitude in $\mathcal{N} = 8$ supergravity was presented in the world-line formalism. In the field theory limit $\alpha' \to 0$ the imaginary part of the genus-two period matrix Ω becomes the period matrix $K := \alpha' \Im\mathrm{m}\Omega$ of the two-loop graph in figure 1

$$K := \begin{pmatrix} L_1 + L_3 & L_3 \\ L_3 & L_2 + L_3 \end{pmatrix}\,. \tag{III.22}$$

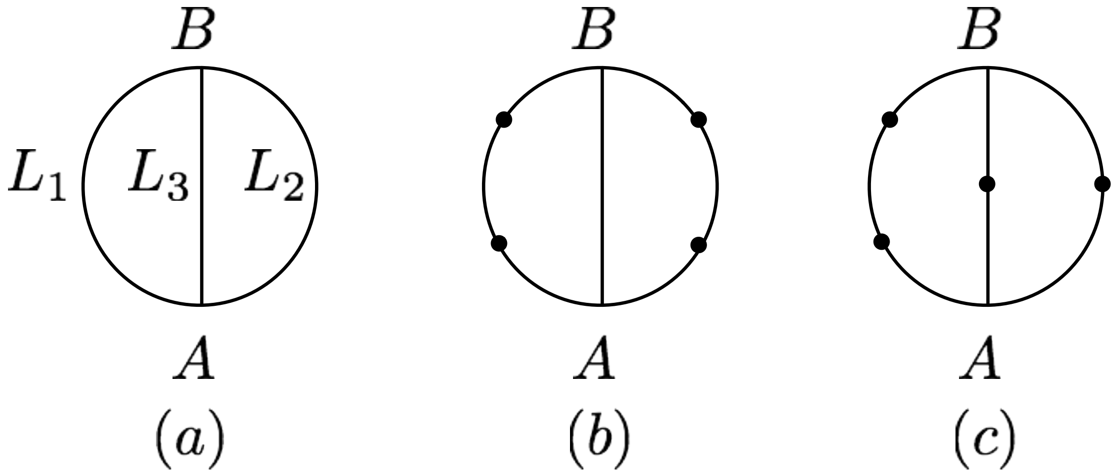

FIG. 1. *Parametrisation of the two-loop diagram in field theory. Figure (a) is the vacuum diagram and the definition of the proper times, and figures (b) and (c) the two configurations contributing to the four-point amplitude.*

We set $L_i = \alpha' \tau_i$ and $\Delta = \det K = L_1L_2 + L_1L_3 + L_2L_3$. The position of a point on the line $l = 1, 2, 3$ of length L_l will be denoted by $t^{(l)}$. We choose the point A to be the origin of the coordinate system, i.e. $t^{(l)} = 0$ means the point is located at position A, and $t^{(l)} = L_l$ on the lth line means the point is located at position B.

It is convenient to introduce the rank two vectors $v_i = t_i^{(l_i)}\, u^{(l_i)}$ where

$$u^{(1)} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u^{(2)} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u^{(3)} := \begin{pmatrix} -1 \\ -1 \end{pmatrix}. \qquad \text{(III.23)}$$

The v_i are the field theory degenerate form of the Abel map of a point on the Riemann surface to its divisor. The vectors $u^{(i)}$ are the degenerate form of the integrals of the holomorphic one-forms ω_I. If the integrations on each line is oriented from A to B, the integration element on line i is $du^{l_i} = dt_i\, u^{(l_i)}$. The canonical homology basis (A_i, B_i) of the genus two Riemann surface degenerate to $(0, b_i)$, with $b_i = L_i \cup \bar{L}_3$. $\bar{L}_3$ means that we circulate on the middle line from B to A. With these definitions we can reconstruct the period matrix (III.22) from

$$\begin{aligned}
\oint_{b_1} du \cdot u^{(1)} &= \int_0^{L_1} dt_1 + \int_0^{L_3} dt_3 = L_1 + L_3 \\
\oint_{b_2} du \cdot u^{(2)} &= \int_0^{L_2} dt_1 + \int_0^{L_3} dt_3 = L_2 + L_3 \\
\oint_{b_1} du \cdot u^{(2)} &= \int_0^{L_3} dt_3 = L_3 \\
\oint_{b_2} du \cdot u^{(1)} &= \int_0^{L_3} dt_3 = L_3 \,,
\end{aligned} \tag{III.24}$$

in agreement with the corresponding relations on the Riemann surface $\oint_{B_I} \omega_J = \Omega_{IJ}$. In the field theory limit of $\mathcal{Y}_S$ (II.10) becomes

$$3Y_S = (k_1 - k_2) \cdot (k_3 - k_4)\, \Delta_{12}^{FT} \Delta_{34}^{FT} + (13)(24) + (14)(23) \tag{III.25}$$

where

$$\Delta_{ij}^{FT} = \epsilon^{IJ} u_I^{(l_i)} u_J^{(l_j)} \,. \tag{III.26}$$

Notice that $\Delta_{ij}^{FT} = 0$ when the point i and j are on the same line (i.e. $l_i = l_j$). Therefore Y_S vanishes if three points are on the same line, and the only non-vanishing configurations are the one depicted in figure 1(b)-(c).

In the field theory limit the leading contribution to Y_S is given by

$$Y_S = \begin{cases} s & \text{for } l_1 = l_2 \text{ or } l_3 = l_4 \\ t & \text{for } l_1 = l_4 \text{ or } l_3 = l_2 \\ u & \text{for } l_1 = l_3 \text{ or } l_2 = l_4 \end{cases} . \tag{III.27}$$

The bosonic propagator in (II.9) becomes

$$P_2^{FT}(v_i - v_j) := -\frac{1}{2}\, d(v_i - v_j) + \frac{1}{2}\, (v_i - v_j)^T K^{-1} (v_i - v_j) \,, \tag{III.28}$$

where $d(v_i - v_j)$ is given by $|t_i^{(l_i)} - t_j^{(l_j)}|$ if the two points are on the same line $l_i = l_j$ or $t_i^{(l_i)} + t_j^{(l_j)}$ is the two point are on different lines $l_i \neq l_j$.

We find that

$$\partial_{ij} P_2^{FT}(v_i - v_j) = (u_i - u_j)^T K^{-1} (v_i - v_j) + \begin{cases} \mathrm{sign}(t_i^{(l_i)} - t_j^{(l_j)}) & \text{if } l_i = l_j \\ 0 & \text{otherwise} \end{cases} , \tag{III.29}$$

and

$$\partial^2_{ij} P_2^{FT}(v_i - v_j) = (u_i - u_j)^T K^{-1} (u_i - u_j) + \begin{cases} 2\delta(t_i^{(l_i)} - t_j^{(l_j)}) & \text{if } l_i = l_j \\ 0 & \text{otherwise} \end{cases} , \qquad \text{(III.30)}$$

We define the quantity

$$Q^{(2)} = \sum_{1\leq i<j\leq 4} k_i \cdot k_j \, P_2^{FT}(v_i - v_j) \, . \qquad \text{(III.31)}$$

In this limit the expansion of CHL model partition function $\mathcal{Z}_2^{(n_v)}$ is given by in (II.13) where $O(q_i)$ do not contribute to the field theory limit. The integration over the real part of the component of the period matrix projects the integrand in the following way

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} d^3\Re\mathrm{e}\Omega \, \mathcal{Z}_2^{(n_v)} F(\Omega, \bar\Omega) \; = c_{n_v} F_0 + F_{123} + a_{n_v} \left(F_{12} + F_{13} + F_{23}\right) + b_{n_v} \left(F_1 + F_2 + F_3\right) \quad \text{(III.32)}$$

where $F(\Omega, \bar\Omega) = F_0 + \sum_{i=1}^3 \bar q_i F_i + \sum_{1\leq i<j\leq 3} \bar q_i \bar q_j F_{ij} + \bar q_1 \bar q_2 \bar q_3 \, F_{123} + c.c. + O(q_i \bar q_i)$ represents the integrand of the two-loop amplitude.

When performing the field theory limit the integral takes the form[3]

$$M_{4;2}^{(n_v)} = N_2 \, t_8 t_8 R^4 \int_0^\infty \frac{d^3 L_i}{\Delta^{2+\epsilon}} \oint d^4 t_i \, Y_S \, [W_1^{(2)} + W_2^{(2)}] \, e^{Q^{(2)}} \, . \qquad \text{(III.33)}$$

The contribution $W_1^{(2)}$ leads two-loop double-box integrals $I^{(D=4-2\epsilon)}_{double-box}[\ell^n]$ with $n = 4, 2, 0$ up to four powers of loop momentum and $s/u \, I^{(D=4-2\epsilon)}_{double-box}[\ell^m]$ with $m = 2, 0$ with up to two powers of loop momentum evaluated in $D = 4 - 2\epsilon$. Everything multiplied by $s \times t_8 t_8 R^4$ or $t \times t_8 t_8 R^4$ or $u \times t_8 t_8 R^4$ depending on the channel according to the decomposition of Y_S in (III.27).

The contribution $W_2^{(2)}$ leads to two-loop double-box integrals $I^{(D=6-2\epsilon)}_{double-box}[\ell^n]$ with $n = 2, 0$ up to two powers of loop momentum evaluated in $D = 6 - 2\epsilon$ multiplied by $\frac{s}{u} \times t_8 t_8 R^4$ or $\frac{t}{u} \times t_8 t_8 R^4$ or $t_8 t_8 R^4$ depending on the channel according to the decomposition of Y_S in (III.27).

We therefore conclude that the field theory limit of the four-graviton two-loop amplitude of the CHL models with various number of vector multiplet has a $\partial^2 R^4$ term factorizing the $D = 4 - 2\epsilon$ amplitudes.

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[3] A detailed analysis of these integrals will be given in [20].

IV. NON RENORMALISATION THEOREMS

The analysis performed in this paper shows that the two-loop four-graviton amplitude in $\mathcal{N}=4$ pure supergravity factorizes a $\partial^2 R^4$ operator in each kinematical sectors. This implies a non-renormalisation theorem for the R^4 term, which forbids the appearance of a three-loop ultraviolet divergence in four dimensions in the four-graviton amplitude.[4]

Since a fully supersymmetric R^4 three-loop ultraviolet counter-terms in four dimensions has been constructed in [29] one can wonder why no divergence occur. We provide a few arguments that could explain why the R^4 term is a protected operator in $\mathcal{N}=4$ pure supergravity.

It was argued in [7–9] that the R^4 is a half-BPS protected operator and does not receive perturbative corrections beyond one-loop in heterotic string compactification. These non-renormalisation theorems were confirmed in [10] using the explicit evaluation of the genus-two four-graviton heterotic amplitude derived in [11–13]. In $D=4$ dimensions the CHL model with $4 \leq n_v \leq 22$ vector multiplets obtained by the asymmetric orbifold construction satisfy the same non-renormalisation theorems. For these models the moduli space is $SU(1,1)/U(1) \times SO(6,n_v)/SO(6) \times SO(n_v)$. Since the axion-dilaton parametrizes the $SU(1,1)/U(1)$ factor it is natural to conjecture that this moduli space will stay factorized and that one can decouple the contributions from the vector multiplets. If one can set to zero all the vector multiplets, this analysis shows the existence of the R^4 non renormalisation theorem in the pure $\mathcal{N}=4$ supergravity case.

It was shown in [29] that the $SU(1,1)$-invariant superspace volume vanishes and the R^4 super-invariant was constructed as an harmonic superspace integral over 3/4 of the full superspace. The structure of the amplitudes analyzed in this paper and the absence of three-loop divergence points to the fact that this partial superspace integral is an F-term.

The existence of an off-shell formulation for $\mathcal{N}=4$ conformal supergravity and linearized $\mathcal{N}=4$ supergravity with six vector multiplets [30–32] makes this F-term nature plausible in the Poincaré pure supergravity.

What makes the $\mathcal{N}=4$ supergravity case special compared to the other $5 \leq \mathcal{N} \leq 8$ cases is the anomalous $U(1)$ symmetry [33]. Therefore even without the existence of an off-shell formalism, this anomaly could make the R^4 term special and be the reason why

[4] This has been confirmed by the recent field theory evaluation in [14].

it turns out to be ruled out as a possible counter-term in four-graviton amplitude in four dimensions. Because of the $U(1)$-anomaly full superspace integrals of functions of the axion-dilaton superfield $\mathbb{S} = S + \cdots$ are allowed [29]

$$I = \kappa_{(4)}^4 \int d^4x d^{16}\theta \, E(x,\theta) \, F(\mathbb{S}) = \kappa_{(4)}^4 \int d^4x \sqrt{-g} \, f(S) \, R^4 + \text{susy completion} \,, \qquad \text{(IV.1)}$$

suggesting a three-loop divergence in the higher-point amplitudes with four gravitons and scalar fields. Since one can write full superspace for $\partial^2 R^4$ in terms of the gravitino $\int d^{16}\theta \, E(x,\theta)(\chi\bar{\chi})^2$, one should expect a four-loop divergence in the four-graviton amplitude in four dimensions.

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