MATHEMATICS OF COMPUTATION
Volume 00, Number 0, Pages 000–000
S 0025-5718(XX)0000-0

COMPUTABLE ERROR BOUNDS FOR FINITE ELEMENT APPROXIMATION ON NON-POLYGONAL DOMAINS

MARK AINSWORTH AND RICHARD RANKIN

Abstract. Fully computable, guaranteed bounds are obtained on the error in the finite element approximation which take the effect of the boundary approximation into account. We consider the case of piecewise affine approximation of the Poisson problem with pure Neumann boundary data, and obtain a fully computable quantity which is shown to provide a guaranteed upper bound on the energy norm of the error. The estimator provides, up to a constant and oscillation terms, local lower bounds on the energy norm of the error.

1. Introduction

Whilst the topic of a posteriori error estimation for finite element approximation dates back over 50 years, it is only relatively recently that techniques have been developed that enable the computation of accurate, guaranteed error bounds [2, 4, 9, 12, 13, 15]. All of these works assume that the computational domain is polygonal and can be meshed exactly using finite elements. Of course, many problems arising in practical applications are posed on curvilinear domains and a decision has to be made on how to deal with the meshing. Although approaches are available that enable the use of curvilinear elements that match the domain exactly, in practice iso-parametric elements are used to approximate the computational domain. The approximation of the domain incurs an additional source of error that should be taken into account in both the a priori convergence analysis, and in the a posteriori error bounds.

A priori error bounds have been studied by various authors: problems with pure Dirichlet boundary conditions were considered in [7, 16, 17]; problems with homogeneous Robin and Dirichlet boundary conditions were considered in [11]; mixed Dirichlet-Neumann boundary conditions are considered in [5] for the Poisson problem in which Neumann boundary conditions are imposed on curved parts of the boundary whilst Dirichlet boundary conditions are imposed on straight parts of the boundary. The case of pure Neumann data is problematic because the compatibility condition on the Neumann data and the volumetric data is generally lost once the domain is approximated. The case of pure, homogeneous natural boundary conditions was considered by Strang and Fix [16], who stop short of dealing with non-homogenous data and simply assert their confidence in the errors being under control. Barrett and Elliott [6] considered the case of pure Neumann data and enforced the compatibility issue through a global perturbation of data.

A posteriori error analysis for curvilinear domains is much less well-developed. In [10] a posteriori error bounds were obtained for finite element approximation of

Received by the editor September 5, 2012.

2010 *Mathematics Subject Classification.* 65N15,65N30.

Support of the authors by the Engineering and Physical Sciences Research Council of Great Britain under the Numerical Algorithms and Intelligent Software (NAIS) for the evolving HPC platform grant EP/G036136/1 is gratefully acknowledged.

©XXXX American Mathematical Society

the Poisson problem on non-polygonal domains with pure Dirichlet boundary conditions. In common with many a posteriori error estimators of that period, the bounds involved multiplicative constants that are unknown meaning that one does not obtain an actual numerical bound on the error. In [8] a posteriori error bounds were obtained for the finite element approximation of the Poisson problem on polygonal domains containing non-polygonal holes on which homogeneous Neumann boundary conditions were imposed, again valid up to multiplicative constants.

The fully computable a posteriori error estimators referred to earlier are derived under the assumption that the domain is meshed exactly. One may ask whether such estimators continue to provide an upper bound in the presence of approximation of the domain. Consider the problem

$$\begin{aligned} -\Delta u &= -2 \text{ in } \Omega = \left\{(x,y) : x^2+y^2<1\right\} \\ \boldsymbol{n}\cdot\mathbf{grad}\, u &= 1 \text{ on } \partial\Omega \end{aligned}$$

with true solution given by $u = \frac{1}{2}\left(x^2+y^2\right)$. The estimator from [2] has been proved to provide a guaranteed upper bound on the energy norm of the error $|||e|||_\Omega$ when the domain Ω is a polygon. We investigate whether this estimator continues to offer an upper bound when the domain is curvilinear. The results shown in Table 1 show that the estimator η_0, does not provide an upper bound in this example.

NDOF	$\|\|\|e\|\|\|_\Omega$	η_0	$\eta_0/\ \|\|\|e\|\|\|_\Omega$
5	0.7008	0.4714	0.6726
7	0.6172	0.5494	0.8901
9	0.4984	0.4507	0.9043
12	0.4286	0.4050	0.9449
15	0.3748	0.3845	1.0261

Table 1. The performance of the estimator from [2].

This behaviour is not limited to this particular choice of estimator. In fact, as far as we are aware, there are no computable a posteriori error bounds available for the case where the domain on which the problem is posed is not a polygon. The current work seeks to develop fully computable, guaranteed bound on the error which takes the effect of the boundary approximation into account. We consider the case of piecewise affine approximation of the Poisson problem with pure Neumann boundary data, and obtain a fully computable quantity which is shown to provides a guaranteed upper bound on the energy norm of the error. The estimator provides, up to a constant and oscillation terms, local lower bounds on the energy norm of the error.

2. Preliminaries

2.1. **Discretisation of the domain.** The fact that the domain Ω is allowed to be curvilinear means that some care must be exercised in constructing a triangulation on which to approximate the problem. This section is concerned with formulating a precise set of conditions on the triangulation and establishing some preliminary consequences that will be needed later. Let $\mathcal{P}$ denote a set of nonoverlapping, shape-regular triangular elements such that the nonempty intersection of a distinct pair of elements is a single common node or single common edge of both elements. Such a partition $\mathcal{P}$ is locally quasi-uniform in the sense that the ratio of the diameters of any pair of neighbouring elements is uniformly bounded above and below. Throughout we shall use C and c to denote positive constants which are independent of the size of the elements in the mesh. The shape regularity of the elements in the mesh means that, for all $K \in \mathcal{P}$, the area $|K|$ of the element K satisfies

$$ch_K^2 \leq |K| \leq Ch_K^2 \tag{2.1}$$

where h_K denotes the diameter of K. Likewise, if we denote the set containing the individual edges of K by $\mathcal{E}_K$ then, for each $\gamma \in \mathcal{E}_K$, the length $|\gamma|$ of the edge γ satisfies

$$ch_K \leq |\gamma| \leq Ch_K \tag{2.2}$$

We define a polygonal approximation to the domain Ω to be $\Omega_{\mathcal{P}} = \bigcup_{K \in \mathcal{P}} K$. Let $\mathcal{E}$ denote the set of edges of the elements in $\mathcal{P}$. We define the set of interior edges to be

$$\mathcal{E}_I = \{\gamma \in \mathcal{E} : \ \gamma \in \mathcal{E}_K \cap \mathcal{E}_{K'} \text{ for distinct } K, K' \in \mathcal{P}\}$$

and the set of boundary edges to be

$$\mathcal{E}_B = \mathcal{E} \setminus \mathcal{E}_I.$$

We suppose that the partition is constructed so that:

(**A1**) the endpoints of each edge in $\mathcal{E}_B$ lie on Γ.

(**A2**) each element in $\mathcal{P}$ has at most one edge in $\mathcal{E}_B$.

In light of assumptions (**A1**) and (**A2**), we define the approximate domain boundary $\Gamma_{\mathcal{P}}$ to be $\bigcup_{\gamma \in \mathcal{E}_B} \gamma$. We let $\mathcal{E}_0$, $\mathcal{E}_+$ and $\mathcal{E}_-$ denote the subsets of edges defined by

$$\begin{aligned}
\mathcal{E}_0 &= \{\gamma \in \mathcal{E} : \ \gamma \in \mathcal{E}_I \text{ or } \gamma \subset \Gamma\}, \\
\mathcal{E}_+ &= \left\{\gamma \in \mathcal{E} : \ \gamma \notin \mathcal{E}_0 \text{ and } \gamma \subset \overline{\Omega}\right\}, \\
\mathcal{E}_- &= \{\gamma \in \mathcal{E} : \ \gamma \notin \mathcal{E}_0 \cup \mathcal{E}_+ \text{ and only the endpoints of } \gamma \text{ lie on } \Gamma\}.
\end{aligned}$$

For simplicity, we assume that the partition is such that

(**A3**) $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_+ \cup \mathcal{E}_-$.

Assumption (**A3**) means that the boundary of the true domain does not cross the edge of an element. This can always be achieved by applying suitable refinements or adjustments to the mesh. Consequently, we can partition $\mathcal{P}$ into three disjoint sets such that $\mathcal{P} = \mathcal{P}_+ \cup \mathcal{P}_- \cup \mathcal{P}_0$ where

$$\begin{aligned}
\mathcal{P}_+ &= \{K \in \mathcal{P} : \mathcal{E}_K \cap \mathcal{E}_+ \text{ is non-empty}\}, \\
\mathcal{P}_- &= \{K \in \mathcal{P} : \mathcal{E}_K \cap \mathcal{E}_- \text{ is non-empty}\}, \\
\mathcal{P}_0 &= \mathcal{P} \setminus (\mathcal{P}_+ \cup \mathcal{P}_-).
\end{aligned}$$

In general, the triangulated region $\Omega_{\mathcal{P}}$ differs from the true domain Ω. The "skin" between these domains is defined by

$$S = (\Omega \cap \Omega_{\mathcal{P}}^c) \cup (\Omega_{\mathcal{P}} \cap \Omega^c)$$

where $\cdot^c$ denotes the complement in $\mathbb{R}^2$. The skin S is the union of disconnected subsets which we shall refer to as "slivers". Each sliver S_K is associated with a unique element $K \in \mathcal{P}$ for which $\partial S_K \cap \partial K = \gamma_K \in \mathcal{E}_K$. We denote the (curved) edge of the sliver by $\Gamma_K = \partial S_K \setminus \gamma_K$. Evidently, the slivers are associated with elements K belonging to the curved portions of the boundary, i.e. elements $K \in \mathcal{P}_+ \cup \mathcal{P}_-$. Moreover,

$$\Omega = \{S_K : \ K \in \mathcal{P}_+\} \cup \Omega_{\mathcal{P}} \setminus \{S_K : \ K \in \mathcal{P}_-\}. \tag{2.3}$$

Figure 1 illustrates the two possible types of slivers.

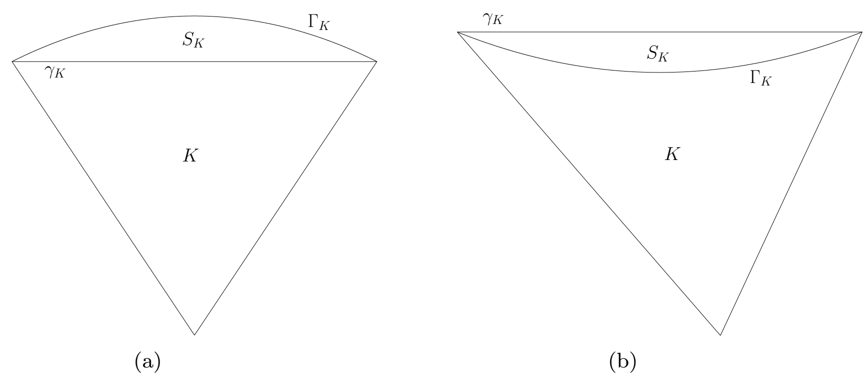

FIGURE 1. Examples of the two types of sliver S_K: (a) $K \in \mathcal{P}_+$ and (b) $K \in \mathcal{P}_-$.

With each linear triangle $K \in \mathcal{P}$, we can associate a (possibly) curvilinear triangle K^*, with at most one curved edge, as follows:

$$K^* = \begin{cases} K \cup S_K & \text{if } K \in \mathcal{P}_+ \\ K \setminus S_K & \text{if } K \in \mathcal{P}_- \\ K & \text{if } K \in \mathcal{P}_0. \end{cases} \tag{2.4}$$

The chief motivation behind the foregoing constructions lies in the fact that $\{K^* : \ K \in \mathcal{P}\}$ forms a partitioning of the true domain Ω:

$$\Omega = \bigcup_{K \in \mathcal{P}} K^*. \tag{2.5}$$

We shall need to impose some restrictions on the shape regularity of these curvilinear triangles:

(**A4**) there exists a positive constant C such that, for each $K \in \mathcal{P}_+ \cup \mathcal{P}_-$, there exists a point $\boldsymbol{x}_0 \in K^*$ such that K^* is star-shaped with respect to the ball $\{\boldsymbol{x} \in K^* : \ |\boldsymbol{x} - \boldsymbol{x}_0| < Ch_K\}$.

(**A5**) there exists a positive constant C such that, for each $K \in \mathcal{P}_+ \cup \mathcal{P}_-$, $\min_{\boldsymbol{x} \in \overline{\Gamma_K}} \boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{x}_{\Gamma_K}) \geq Ch_K$ where $\boldsymbol{x}_{\Gamma_K}$ is the vertex of element K opposite to the curved edge Γ_K.

The partition shown in Figure 2(b) violates (**A4**) since the K^* associated with the element $K \in \mathcal{P}_-$ fails to be star-shaped with respect to a ball in K^*. Likewise $\boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{x}_{\Gamma_K}) = 0$ at the points indicated meaning that the partition shown in Figure 2(b) also violates (**A5**). Observe that this issue does not go away be merely carrying out a refinement to obtain the mesh shown in Figure 2(c). However, the partition shown in Figure 2(d) does satisfy assumptions (**A4**) and (**A5**).

Let $K \in \mathcal{P}$ be any element for which the associated sliver S_K is non-empty and let $\boldsymbol{x} \in S_K$. The point may be written uniquely in the form $\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{t}_{\gamma_K}\hat{x} + \boldsymbol{n}_{\gamma_K}\hat{y}$ where the vertices $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ of S_K and the unit tangent vector $\boldsymbol{t}_{\gamma_K}$ and unit normal vector $\boldsymbol{n}_{\gamma_K}$ to edge γ_K of S_K are labelled and oriented as in Figure 3. Consequently, we may define a local $\hat{x} - \hat{y}$ coordinate system on S_K with the origin at $\boldsymbol{x}_1$ and the positive $\hat{x}$-axis aligned with γ_K. Our final assumption concerns the smoothness of the curvilinear triangle edges: for $\mu = 1$ or $\mu = 2$, we assume that

$(\mathbf{A6})^\mu$ Γ_K is locally the graph of a function ϕ, i.e.

$$\Gamma_K = \{(\hat{x}, \hat{y}) : \ \hat{y} = \phi(\hat{x}), \ \hat{x} \in (0, |\gamma_K|)\},$$

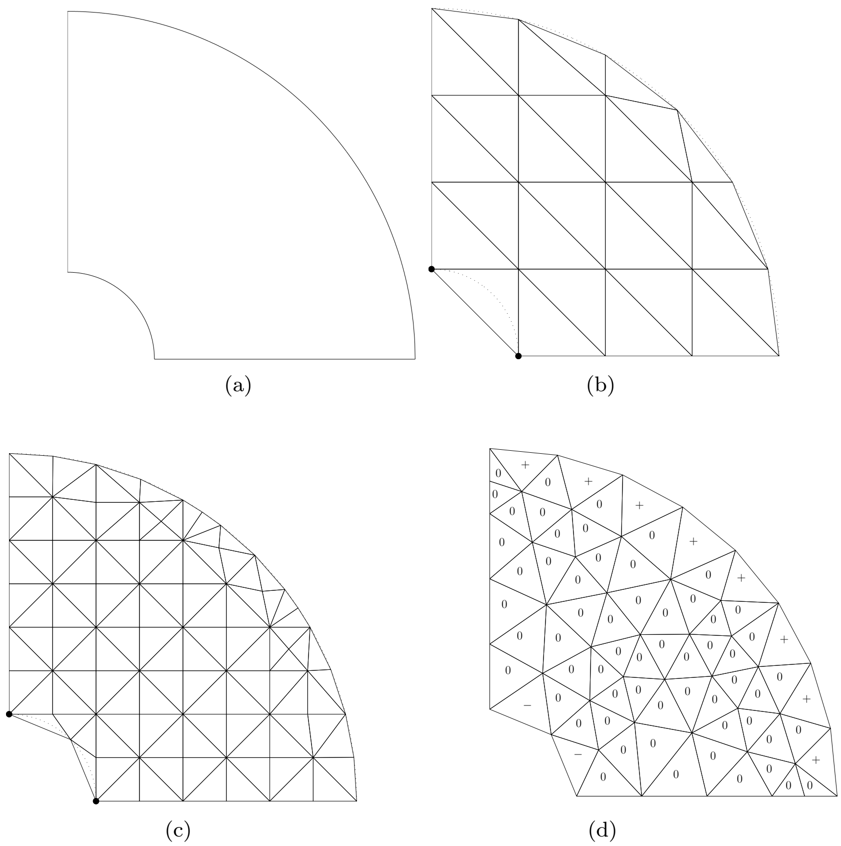

FIGURE 2. (a) Domain Ω and (b)-(d) three possible partitions. In (d) the elements in the set $\mathcal{P}_0$ contain a 0 and the elements in the sets $\mathcal{P}_\pm$ contain a $\pm$

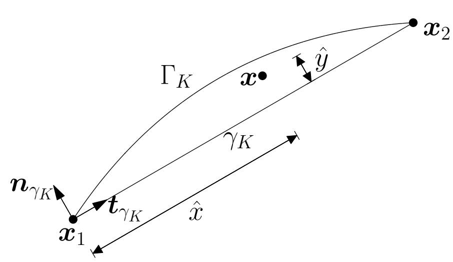

FIGURE 3. The position of the endpoints $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ and the labelling and orientation of the unit tangent vector $\boldsymbol{t}_{\gamma_K}$ and unit normal vector $\boldsymbol{n}_{\gamma_K}$ to edge γ_K of S_K used to define local $\hat{x} - \hat{y}$ coordinate system on S_K.

such that $\|\phi\|_{H^\mu(0,|\gamma_K|)} \leq C\,|\gamma_K|^{1/2}$.

Lemma 2.1. *Suppose that assumptions (**A1**)-(**A5**) and (**A6**)^1 hold. Then the length of the curved edge Γ_K satisfies*

$$ch_K \leq |\Gamma_K| \leq Ch_K. \tag{2.6}$$

Proof. We have that

$$\begin{aligned}|\Gamma_K| &= \int_0^{|\gamma_K|} \sqrt{1+\phi'(\hat{x})^2}\; d\hat{x} \\ &\leq |\gamma_K|^{1/2} \left(\int_0^{|\gamma_K|} 1+\phi'(\hat{x})^2 \; d\hat{x}\right)^{1/2} \\ &= |\gamma_K|^{1/2}\left(|\gamma_K| + \|\phi'\|^2_{L_2(0,|\gamma_K|)}\right)^{1/2} \leq C\,|\gamma_K|\end{aligned}$$

since $\|\phi'\|_{L_2(0,|\gamma_K|)} \leq \|\phi\|_{H^1(0,|\gamma_K|)} \leq C\,|\gamma_K|^{1/2}$. Hence, since it is immediate that $|\gamma_K| \leq |\Gamma_K|$, the desired result follows. □

Lemma 2.2. *Suppose that assumptions (**A1**)-(**A5**) and (**A6**)^2 hold. Then, in addition to* (2.6) *holding, the area of the sliver S_K satisfies*

$$|S_K| \leq Ch_K^3 \tag{2.7}$$

and the curvilinear triangle K^* satisfies

$$c\,|K| \leq |K^*| \leq C\,|K| \tag{2.8}$$

and

$$ch_K \leq h_{K^*} \leq Ch_K. \tag{2.9}$$

Proof. Since (**A1**) implies that $\phi(0) = \phi(|\gamma_K|) = 0$, integration by parts yields

$$\int_0^{|\gamma_K|} \frac{1}{2}\hat{x}\,(\hat{x}-|\gamma_K|)\,\phi''(\hat{x}) \; d\hat{x} = \int_0^{|\gamma_K|} \phi(\hat{x}) \; d\hat{x}.$$

Moreover, assumption (**A3**) means that $\phi(\hat{x})$ is positive for $\hat{x} \in (0, |\gamma_K|)$. Hence, the area of the sliver satisfies

$$|S_K| = \int_0^{|\gamma_K|} \phi(\hat{x}) \; d\hat{x} \leq \left\|\frac{1}{2}\hat{x}\,(\hat{x}-|\gamma_K|)\right\|_{L_2(0,|\gamma_K|)} \|\phi''\|_{L_2(0,|\gamma_K|)} \leq C\,|\gamma_K|^3$$

since $\left\|\frac{1}{2}\hat{x}\,(\hat{x}-|\gamma_K|)\right\|_{L_2(0,|\gamma_K|)} = \frac{\sqrt{17}}{\sqrt{240}}\,|\gamma_K|^{5/2}$ and $\|\phi''\|_{L_2(0,|\gamma_K|)} \leq \|\phi\|_{H^2(0,|\gamma_K|)} \leq C\,|\gamma_K|^{1/2}$. Consequently, $(\mathbf{A6})^2$ implies that, in addition to (2.6) holding, (2.7) holds. In turn, these estimates mean that (2.9) and (2.8) are satisfied. □

2.2. **Oscillation of the boundary.** Suppose that assumption $(\mathbf{A6})^\mu$ holds with $\mu \geq 1$. We introduce a measure to quantify the notion of the oscillation of Γ_K, $\operatorname{osc}(\Gamma_K)$, as follows:

$$\operatorname{osc}(\Gamma_K) = \left(\frac{1}{|\gamma_K|}\int_0^{|\gamma_K|} \frac{\left(\sqrt{1+\phi'(\hat{x})^2}-1\right)^2}{\sqrt{1+\phi'(\hat{x})^2}}\; d\hat{x}\right)^{1/2}. \tag{2.10}$$

If the boundary segment containing Γ_K is linear, then $\phi = 0$ and hence $\operatorname{osc}(\Gamma_K) = 0$. Conversely, if the boundary "wiggles" in the neighbourhood of Γ_K, then $|\phi'|$ will be large which, in turn, means the oscillation is large. We present three results which show how this oscillation measures how well quantities on the curvilinear entities are approximated by the corresponding quantity on the polygonal approximation.

Lemma 2.3. *Let the partition $\mathcal{P}$ satisfy assumptions (**A1**)-(**A5**) and (**A6**)$^{\mu}$ with $\mu \geq 1$. Let $K \in \mathcal{P}_{+}$ and let $w \in H^{1}(K^{*})$ be such that $(w,1)_{K^{*}} = 0$. Then*

$$\left|\frac{1}{|\Gamma_K|}\int_{\Gamma_K} w\ ds - \frac{1}{|\gamma_K|}\int_{\gamma_K} w\ ds\right| \leq \frac{1}{|\Gamma_K|}\left(|S_K|^{1/2} + \left(C^{K^*}_{\Gamma_K,K^*}|\gamma_K|^{1/2} + C^{K^*}_{\gamma_K,K}|\Gamma_K|^{1/2}\right)h^{1/2}_{K^*}\operatorname{osc}(\Gamma_K)\right)\|w\|_{K^*} \tag{2.11}$$

where $C^{K^*}_{\Gamma_K,K^*}$ and $C^{K^*}_{\gamma_K,K}$ are the constants in the Poincaré inequality (3.8).

Proof. This lemma is proved in Section 8.1. □

Lemma 2.4. *Let the partition $\mathcal{P}$ satisfy assumptions (**A1**)-(**A5**) and (**A6**)$^{\mu}$ with $\mu \geq 1$. For $K \in \mathcal{P}_{+}$, let $w \in H^{1}(K^{*})$, $F \in L_2(K^{*})$ and $G \in L_2(\Gamma_K)$. Then*

$$\begin{aligned}&\left(F, w - \langle w\rangle_{\gamma_K}\right)_{S_K} + \left(G, w - \langle w\rangle_{\gamma_K}\right)_{\Gamma_K} \leq \\ &\Bigg(C_{K^*}h_{K^*}\|F - \langle F\rangle_{K^*}\|_{L_2(S_K)} + \frac{C^{K^*}_{\gamma_K,K}h^{1/2}_{K^*}}{|\gamma_K|^{1/2}}|S_K|\left|\langle F\rangle_{S_K}\right| \\ &+ C^{K^*}_{\Gamma_K,K^*}h^{1/2}_{K^*}\left\|G - \langle G\rangle_{\Gamma_K}\right\|_{L_2(\Gamma_K)} \\ &+ \left(|S_K|^{1/2} + \left(C^{K^*}_{\Gamma_K,K^*}|\gamma_K|^{1/2} + C^{K^*}_{\gamma_K,K}|\Gamma_K|^{1/2}\right)h^{1/2}_{K^*}\operatorname{osc}(\Gamma_K)\right)\left|\langle G\rangle_{\Gamma_K}\right|\Bigg)\|w\|_{K^*}\end{aligned} \tag{2.12}$$

where C_{K^*} is the constant in the Poincaré inequality (3.7) *and $C^{K^*}_{\gamma_K,K}$ and $C^{K^*}_{\Gamma_K,K^*}$ are the constants in the Poincaré inequality* (3.8).

Proof. This lemma is proved in Section 8.2. □

Lemma 2.5. *Let the partition $\mathcal{P}$ satisfy assumptions (**A1**)-(**A5**) and (**A6**)$^{\mu}$ with $\mu \geq 2$. Then*

$$\operatorname{osc}(\Gamma_K) \leq C h_K. \tag{2.13}$$

Proof. Since $\phi(0) = \phi(|\gamma_K|) = 0$, we have for $\hat{x}, s \in (0, |\gamma_K|)$

$$\begin{aligned}|\phi'(\hat{x})| &= \left|\phi'(\hat{x}) - \frac{1}{|\gamma_K|}\int_0^{|\gamma_K|}\phi'(s)\ \ ds\right| \\ &= \frac{1}{|\gamma_K|}\left|\int_0^{|\gamma_K|}\phi'(\hat{x}) - \phi'(s)\ \ ds\right| \\ &= \frac{1}{|\gamma_K|}\left|\int_0^{|\gamma_K|}\int_s^{\hat{x}}\phi''(t)\ \ dt\ ds\right| \\ &\leq \frac{1}{|\gamma_K|}\int_0^{|\gamma_K|}\left|\int_s^{\hat{x}} 1\ dt\right|^{1/2}\left|\int_s^{\hat{x}}\phi''(t)^2\ \ dt\right|^{1/2}\ \ ds \\ &\leq \frac{1}{|\gamma_K|}\int_0^{|\gamma_K|}\left|\int_s^{\hat{x}} 1\ dt\right|^{1/2}\ \ ds\,\|\phi''\|_{L_2(0,|\gamma_K|)}.\end{aligned}$$

Moreover,

$$\frac{1}{|\gamma_K|}\int_0^{|\gamma_K|}\left|\int_s^{\hat{x}} 1\ dt\right|^{1/2}\ \ ds \leq \frac{1}{|\gamma_K|^{1/2}}\left(\int_0^{|\gamma_K|}|\hat{x} - s|\ \ ds\right)^{1/2}.$$

Direct computation then leads to the estimate

$$\|\phi'\|_{L_2(0,|\gamma_K|)} \leq \frac{|\gamma_K|}{\sqrt{3}} \|\phi''\|_{L_2(0,|\gamma_K|)} .$$

Consequently,

$$\begin{aligned} \operatorname{osc}(\Gamma_K) &\leq \left(\frac{1}{|\gamma_K|} \int_0^{|\gamma_K|} \left(\sqrt{1 + \phi'(\hat{x})^2} - 1 \right)^2 d\hat{x} \right)^{1/2} \\ &\leq \frac{1}{|\gamma_K|^{1/2}} \|\phi'\|_{L_2(0,|\gamma_K|)} \leq \frac{|\gamma_K|^{1/2}}{\sqrt{3}} \|\phi''\|_{L_2(0,|\gamma_K|)} \leq C |\gamma_K| \end{aligned}$$

since $\|\phi''\|_{L_2(0,|\gamma_K|)} \leq \|\phi\|_{H^2(0,|\gamma_K|)} \leq C |\gamma_K|^{1/2}$. Hence, (2.2) yields (2.13). □

3. Finite element discretisation

3.1. **Model problem.** Consider the model problem

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ \boldsymbol{n} \cdot \mathbf{grad}\, u &= g \text{ on } \Gamma, \end{aligned} \tag{3.1}$$

where Ω is an open domain in $\mathbb{R}^2$ with piecewise smooth, possibly curvilinear, boundary $\Gamma = \partial\Omega$ and $\boldsymbol{n}$ is the outward unit normal vector to Γ. We shall use the notation $(\cdot,\cdot)_\omega$ to denote the integral inner product over a region ω. The data satisfy $f \in L_2(\Omega)$ and $g \in L_2(\Gamma)$, along with the compatibility condition

$$(f,1)_\Omega + (g,1)_\Gamma = 0, \tag{3.2}$$

needed to ensure the existence of a solution to (3.1). We define

$$H^1_{\backslash\mathbb{R}}(\Omega) = \left\{ v \in H^1(\Omega) : \ (v,1)_\Omega = 0 \right\}$$

and let the energy norm over a region ω be denoted by

$$|||\cdot|||_\omega = (\mathbf{grad}\,\cdot, \mathbf{grad}\,\cdot)_\omega^{1/2} .$$

The variational form of (3.1) consists of finding $u \in H^1_{\backslash\mathbb{R}}(\Omega)$ such that

$$(\mathbf{grad}\, u, \mathbf{grad}\, v)_\Omega = (f,v)_\Omega + (g,v)_\Gamma \text{ for all } v \in H^1(\Omega) . \tag{3.3}$$

3.2. **Finite element approximation.** For $m \in \mathbb{N}_0$, let $\mathbb{P}_m(\omega)$ denote the space of polynomials on a region ω of total degree at most m and let $\mathbb{P}_m(\gamma)$ denote the space of polynomials on an edge $\gamma \in \mathcal{E}$ of total degree at most m (with respect to arc-length). For a triangle K and $v \in L_2(K)$, let $P_K v \in \mathbb{P}_1(K)$ denote the orthogonal projection defined by $(v - P_K v, p)_K = 0$ for all $p \in \mathbb{P}_1(K)$. Likewise, for an edge $\gamma \in \mathcal{E}$ and $v \in L_2(\gamma)$, $P_\gamma v \in \mathbb{P}_1(\gamma)$ denotes the orthogonal projection on the edge. For a two dimensional region ω and $v \in L_2(\omega)$, let $|\omega|$ denote the area of ω and let $\langle v \rangle_\omega = \frac{1}{|\omega|}(v,1)_\omega$. For a one dimensional region τ and $v \in L_2(\tau)$, let $|\tau|$ denote the length of τ and let $\langle v \rangle_\tau = \frac{1}{|\tau|}(v,1)_\tau$.

The finite element space $X^{\mathcal{P}}$ of first order on $\mathcal{P}$ is defined by

$$X^{\mathcal{P}} = \left\{ v \in C\left(\overline{\Omega_{\mathcal{P}}}\right) : \ v_{|K} \in \mathbb{P}_1(K) \text{ for all } K \in \mathcal{P} \right\}$$

along with the subspace

$$X^{\mathcal{P}}_{\backslash\mathbb{R}} = \left\{ v \in X^{\mathcal{P}} : \ (v,1)_{\Omega_{\mathcal{P}}} = 0 \right\} .$$

In order to define a finite element approximation of (3.3) we must construct a suitable approximation on $\mathcal{P}$ for each term appearing in (3.3). Moreover, we must ensure that the analogue of the compatibility condition (3.2) holds for the discrete scheme. The approximate domain boundary $\Gamma_{\mathcal{P}} = \bigcup_{\gamma \in \mathcal{E}_B} \gamma$ may be expressed in the alternative form $\Gamma_{\mathcal{P}} = \sum_{K\in\mathcal{P}} \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_B} \gamma$ which we shall use to discretise the boundary flux term in (3.3).

We define a finite element approximation $u_{\mathcal{P}} \in X^{\mathcal{P}}_{\backslash\mathbb{R}}$ of the solution u to problem (3.3) as follows:

$$(\mathbf{grad}\, u_{\mathcal{P}}, \mathbf{grad}\, v)_{\Omega_{\mathcal{P}}} = (f, v)_{\Omega_{\mathcal{P}}} + \sum_{K\in\mathcal{P}} \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_B} (g_{K,\gamma}, v)_\gamma \;\text{ for all } v \in X^{\mathcal{P}} \tag{3.4}$$

where the data f has been extended from Ω to $\Omega \cup \Omega_{\mathcal{P}}$ such that $f \in L_2(\Omega \cup \Omega_{\mathcal{P}})$ and the flux data $g_{K,\gamma} \in L_2(\gamma)$ is chosen so that the discrete compatibility condition

$$(f, 1)_{\Omega_{\mathcal{P}}} + \sum_{K\in\mathcal{P}} \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_B} (g_{K,\gamma}, 1)_\gamma = 0 \tag{3.5}$$

is satisfied. This condition does not uniquely determine the fluxes, and several reasonable choices are possible. We choose to define $g_{K,\gamma}$ by the rule

$$g_{K,\gamma} = \begin{cases} P_\gamma g & \text{if } \gamma \in \mathcal{E}_0 \cap \mathcal{E}_B, \\ \dfrac{1}{|\gamma|}\left((g,1)_{\Gamma_K} + (f,1)_{S_K}\right) & \text{if } \gamma \in \mathcal{E}_+, \\ \dfrac{1}{|\gamma|}\left((g,1)_{\Gamma_K} - (f,1)_{S_K}\right) & \text{if } \gamma \in \mathcal{E}_-. \end{cases} \tag{3.6}$$

Thanks to (2.3), it follows that

$$(f,1)_\Omega = (f,1)_{\Omega_{\mathcal{P}}} + \sum_{K\in\mathcal{P}_+} (f,1)_{S_K} - \sum_{K\in\mathcal{P}_-} (f,1)_{S_K},$$

and, as a consequence,

$$(f,1)_{\Omega_{\mathcal{P}}} + \sum_{K\in\mathcal{P}} \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_B} (g_{K,\gamma}, 1)_\gamma = (f,1)_\Omega + (g,1)_\Gamma = 0$$

and so the compatibility condition (3.5) holds for the choice (3.6). The important issue of the effect of the choice (3.6) on the accuracy of the resulting finite element approximation is deferred to Section 4.

The approximation $u_{\mathcal{P}}$ is defined on the polygonal domain $\Omega_{\mathcal{P}}$. It is desirable to have an approximation to u over the original domain Ω. To this end, for $K \in \mathcal{P}_+$, we extend $u_{\mathcal{P}}$ to the sliver S_K by requiring $u_{\mathcal{P}|K^*} \in \mathbb{P}_1(K^*)$. In other words, we extend $u_{\mathcal{P}}$ from K onto $K^* = K \cup S_K$ as an affine function by simply using the same rule used to define $u_{\mathcal{P}}$ on K, to define $u_{\mathcal{P}}$ on K^*. Adopting this convention means that the extended finite element approximation, which we again denote by $u_{\mathcal{P}}$, belongs to the space $\widetilde{X}^{\mathcal{P}}$ defined by

$$\widetilde{X}^{\mathcal{P}} = \left\{v :\ \Omega \cup \Omega_{\mathcal{P}} \to \mathbb{R},\ v_{|\Omega_{\mathcal{P}}} \in X^{\mathcal{P}} \text{ and } v_{|K^*} \in \mathbb{P}_1(K^*) \text{ for all } K \in \mathcal{P}_+\right\}.$$

It will be useful to define an associated subspace $\widetilde{X}^{\mathcal{P}}_{\backslash\mathbb{R}}$ as follows:

$$\widetilde{X}^{\mathcal{P}}_{\backslash\mathbb{R}} = \left\{v \in \widetilde{X}^{\mathcal{P}} :\ (v,1)_\Omega = 0\right\}.$$

3.3. **Poincaré inequalities.** Let ω be any two dimensional region which is star shaped with respect to a ball and let h_ω denote the diameter of ω. Then it is well-known, [20], that, for some appropriate choice of constant C_ω,

$$\|v - \langle v\rangle_\omega\|_{L_2(\omega)} \le C_\omega h_\omega \, |||v|||_\omega . \tag{3.7}$$

In a similar vein, if ω and $\widetilde{\omega}$ are two dimensional regions which are star-shaped with respect to a ball and $\omega \subset \widetilde{\omega}$ with $\tau \subset \partial\omega$, then there exists a constant $C^{\widetilde{\omega}}_{\tau,\omega}$ such that

$$\|v - \langle v \rangle_{\widetilde{\omega}}\|_{L_2(\tau)} \leq C^{\widetilde{\omega}}_{\tau,\omega} h^{1/2}_{\widetilde{\omega}} \|\!|v|\!\|_{\widetilde{\omega}} . \tag{3.8}$$

Proofs of these results, along with explicit *computable* expressions for the constants C_ω and $C^{\widetilde{\omega}}_{\tau,\omega}$ will be given in Section 7.2 for the cases where ω and $\tilde{\omega}$ are either an element $K \in \mathcal{P}$ or a curvilinear triangle K^* associated with K. In particular, whenever assumptions (**A4**) and (**A5**) are satisfied, the expressions for C_ω and $C^{\widetilde{\omega}}_{\tau,\omega}$ satisfy $C_\omega \leq C$ and $C^{\widetilde{\omega}}_{\tau,\omega} \leq C$, where C is a positive constant which is independent of the size of the elements in the mesh.

4. An a priori error estimate

We now return to the issue of the rate of convergence of the finite element approximation resulting from choosing the Neumann data on the approximate boundary according to the expression in (3.6). For this section only we shall assume that $\Omega_{\mathcal{P}} \subset \Omega$. Our a priori error estimate stems from the following extension of Cea's Lemma:

Lemma 4.1. *Let $\Omega_{\mathcal{P}} \subset \Omega$ and let the partition be such that assumptions (**A1**)-(**A5**) and (**A6**)$^\mu$ with $\mu \geq 2$ are satisfied. Then*

$$\begin{aligned}\|\!|u - u_{\mathcal{P}}|\!\|_\Omega \leq & C\Bigg(\inf_{p \in \widetilde{X}^{\mathcal{P}}} \|\!|u - p|\!\|_\Omega + \max_{K \in \mathcal{P}_+} h_K \\ & + \Bigg(\sum_{K \in \mathcal{P}_+} \Big(h_K \|\!|u|\!\|^2_{S_K} + h_K \left\| g - \langle g \rangle_{\Gamma_K} \right\|^2_{L_2(\Gamma_K)} \Big) \Bigg)^{1/2} \Bigg).\end{aligned} \tag{4.1}$$

Proof. This lemma is proved in Section 8.3. □

As usual in deriving an a priori rate of convergence estimate, we make an assumption $u \in H^2(\Omega)$ on the regularity of u on Ω. In addition, to bound the final term in (4.1), we shall make assumptions on the regularity of the true solution on the slivers.

Under the assumption that, for all $K \in \mathcal{P}_+$, $\mathbf{grad}\, u \in L_\infty(S_K)$ and $g \in H^1(\Gamma_K)$ we can say that

$$\|\!|u|\!\|_{S_K} \leq |S_K|^{1/2} \|\mathbf{grad}\, u\|_{L_\infty(S_K)} \leq C h_K^{3/2} \|\mathbf{grad}\, u\|_{L_\infty(S_K)}$$

by (2.7) and

$$\left\| g - \langle g \rangle_{\Gamma_K} \right\|_{L_2(\Gamma_K)} \leq C |\Gamma_K| \, |g|_{H^1(\Gamma_K)} \leq C h_K |g|_{H^1(\Gamma_K)}$$

by (2.6). Consequently,

$$\begin{aligned} & \Bigg(\sum_{K \in \mathcal{P}_+} \Big(h_K \|\!|u|\!\|^2_{S_K} + h_K \left\| g - \langle g \rangle_{\Gamma_K} \right\|^2_{L_2(\Gamma_K)} \Big) \Bigg)^{1/2} \\ \leq & C \Bigg(\max_{K \in \mathcal{P}_+} \Big(h_K^2 \|\mathbf{grad}\, u\|^2_{L_\infty(S_K)} \Big) \sum_{K \in \mathcal{P}_+} h_K^2 + \max_{K \in \mathcal{P}_+} h_K^2 \sum_{K \in \mathcal{P}_+} |g|^2_{H^1(\Gamma_K)} \Bigg)^{1/2} \\ \leq & C \max_{K \in \mathcal{P}_+} h_K \end{aligned}$$

since $\sum_{K\in\mathcal{P}_+} h_K^2 \le C \sum_{K\in\mathcal{P}_+} |K| \le C\,|\Omega| \le C$. Hence,

$$\|u - u_{\mathcal{P}}\|_\Omega \le C\left(\inf_{p\in\widetilde{X}^{\mathcal{P}}} \|u-p\|_\Omega + \max_{K\in\mathcal{P}_+} h_K\right). \tag{4.2}$$

Let $\Pi v \in \widetilde{X}^{\mathcal{P}}$ be such that $v = \Pi v$ at the three vertices of element K. Note that Πv differs from the standard interpolate in that Πv is taken from the extended finite element space. In essence, we simply extend the usual interpolate onto the slivers in the same way that the finite element approximation was extended to Ω from $\Omega_{\mathcal{P}}$. Then

$$\inf_{p\in\widetilde{X}^{\mathcal{P}}} \|u-p\|_\Omega^2 \le \|u-\Pi u\|_\Omega^2 = \sum_{K\in\mathcal{P}_0} \|u-\Pi u\|_K^2 + \sum_{K\in\mathcal{P}_+} \|u-\Pi u\|_{K^*}^2 . \tag{4.3}$$

For $K \in \mathcal{P}_+$, let $\widetilde{K}$ be a triangle obtained by extending the element K such that $K^* \subset \widetilde{K}$ with $\partial K \setminus \gamma_K \subset \partial\widetilde{K}$ and $h_{\widetilde{K}} \le C h_K$ (see Figure 4). From Theorem 5.6 in [20] we know that there exists an extension of u from $K^*\cap\Omega$ to $\widetilde{K}$ such that $|u|_{H^2(\tilde{K})} \le C\,|u|_{H^2(\tilde{K}\cap\Omega)}$.

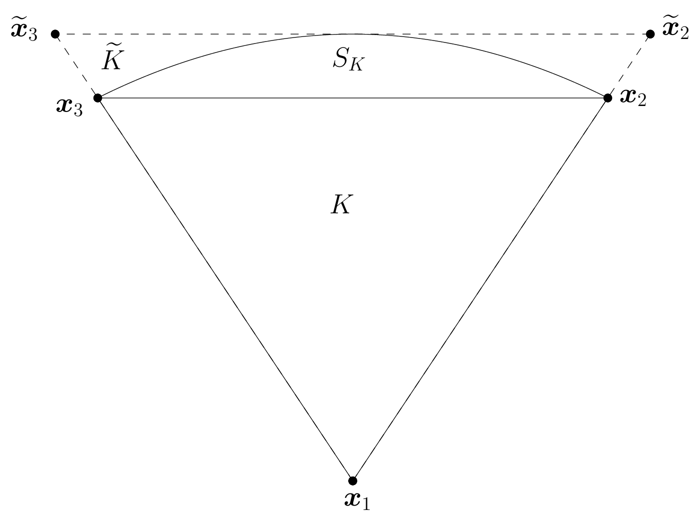

FIGURE 4. An example of an element K with vertices $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ and the corresponding extended triangle $\widetilde{K}$ with vertices $\boldsymbol{x}_1$, $\widetilde{\boldsymbol{x}}_2$ and $\widetilde{\boldsymbol{x}}_3$.

Let $\Pi_{\widetilde{K}} v \in \mathbb{P}_1\left(\widetilde{K}\right)$ be such that $v = \Pi_{\widetilde{K}} v$ at the three vertices of triangle $\widetilde{K}$. Now,

$$\|u-\Pi u\|_{K^*} \le \|u\|_{K^*} + \|\Pi u\|_{K^*} \le \|u\|_{K^*} + \|u\|_{K^*} \le C\,\|u\|_{K^*} .$$

Hence, since $\Pi(\Pi_{\tilde{K}} u) = \Pi_{\tilde{K}} u$ on K^* we can apply the above argument with u replaced by $u - \Pi_{\tilde{K}} u$ to get

$$\|u-\Pi u\|_{K^*} \le \|u-\Pi_{\tilde{K}} u\|_{K^*} \le \|u-\Pi_{\tilde{K}} u\|_{\tilde{K}} \le C h_K\,|u|_{H^2(\tilde{K})} .$$

where the final estimate is a standard interpolation estimate. Moreover,

$$h_{\tilde{K}}\,|u|_{H^2(\tilde{K})} \le C h_K\,|u|_{H^2(\tilde{K})} \le C h_K\,|u|_{H^2(\tilde{K}\cap\Omega)} .$$

since $h_{\tilde{K}} \leq Ch_K$ and $|u|_{H^2(\tilde{K})} \leq C\,|u|_{H^2(\tilde{K}\cap\Omega)}$. Applying this bound, along with the standard interpolation estimate $\|u - \Pi u\|_K \leq Ch_K^2\,|u|_{H^2(K)}$ for $K \in \mathcal{P}_0$, to (4.3) yields

$$\inf_{p\in\widetilde{X}^{\mathcal{P}}} \|u-p\|_\Omega^2 \leq C\left(\sum_{K\in\mathcal{P}_0} h_K^2\,|u|^2_{H^2(K)} + \sum_{K\in\mathcal{P}_+} h_K^2\,|u|^2_{H^2(\tilde{K}\cap\Omega)}\right) \leq C\max_{K\in\mathcal{P}} h_K^2\,|u|^2_{H^2(\Omega)} \leq C\max_{K\in\mathcal{P}} h_K^2. \tag{4.4}$$

Finally, combining (4.4) and (4.2) we obtain the following estimate showing that our choice of discrete flux gives the optimal rate of convergence:

Theorem 4.2. *Let $\Omega_{\mathcal{P}} \subset \Omega$ and let $u \in H^2(\Omega)$. Also, for all $K \in \mathcal{P}_+$, let $\mathbf{grad}\, u \in L_\infty(S_K)$ and $g \in H^1(\Gamma_K)$. Moreover, let the partition $\mathcal{P}$ be such that assumptions **(A1)**-**(A5)** and $(\boldsymbol{A6})^\mu$ with $\mu \geq 2$ are satisfied. Then, there exists a positive constant C, independent of the error $u - u_{\mathcal{P}}$ and the size of the elements in the mesh such that*

$$\|u - u_{\mathcal{P}}\|_\Omega \leq C\max_{K\in\mathcal{P}} h_K. \tag{4.5}$$

5. A posteriori estimation of the energy norm of the error

Let $u \in H^1(\Omega)$ be the true solution to (3.3) and $u_{\mathcal{P}} \in X^{\mathcal{P}} \subset H^1(\Omega_{\mathcal{P}})$ be the solution to (3.4) extended onto $\Omega \cup \Omega_{\mathcal{P}}$ as described in Section 3. Then the error e in the extended approximation is given by $e = u - u_{\mathcal{P}} \in H^1(\Omega)$. We now turn our attention to developing computable bounds for $\|e\|_\Omega$.

5.1. **Upper bound on the energy norm of the error.** Let $v \in H^1(\Omega)$, then thanks to (3.3),

$$(\mathbf{grad}\, e, \mathbf{grad}\, v)_\Omega = (f,v)_\Omega + (g,v)_\Gamma - (\mathbf{grad}\, u_{\mathcal{P}}, \mathbf{grad}\, v)_\Omega$$

and hence using (2.5) and (2.4):

$$(\mathbf{grad}\, e, \mathbf{grad}\, v)_\Omega = \sum_{K\in\mathcal{P}_0}\left((f,v)_K + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_B}(g,v)_\gamma - (\mathbf{grad}\, u_{\mathcal{P}}, \mathbf{grad}\, v)_K\right) + \sum_{K\in\mathcal{P}_+\cup\mathcal{P}_-}\left((f,v)_{K^*} + (g,v)_{\Gamma_K} - (\mathbf{grad}\, u_{\mathcal{P}}, \mathbf{grad}\, v)_{K^*}\right). \tag{5.1}$$

For $\gamma \in \mathcal{E}$, we suppose that $g_{K,\gamma} \in \mathbb{P}_1(\gamma)$ are equilibrated fluxes given by (3.6) on $\mathcal{E}_B$, and satisfying

$$g_{K,\gamma} + g_{K',\gamma} = 0 \text{ if } \gamma \in \mathcal{E}_K \cap \mathcal{E}_{K'} \text{ for } K, K' \in \mathcal{P}, K \neq K' \tag{5.2}$$

and

$$(f,p)_K + \sum_{\gamma\in\mathcal{E}_K}(g_{K,\gamma},p)_\gamma - (\mathbf{grad}\, u_{\mathcal{P}}, \mathbf{grad}\, p)_K = 0 \text{ for all } p \in \mathbb{P}_1(K) \tag{5.3}$$

for all $K \in \mathcal{P}$. A procedure which can be used to determine fluxes $g_{K,\gamma}$ satisfying these conditions will be given in Section 7.1. Now, (3.6) and (5.2) imply that

$$\sum_{K\in\mathcal{P}_0}\sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_B}(P_\gamma g, v)_\gamma = \sum_{K\in\mathcal{P}_0}\sum_{\gamma\in\mathcal{E}_K}(g_{K,\gamma},v)_\gamma + \sum_{K\in\mathcal{P}_+\cup\mathcal{P}_-}\sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_I}(g_{K,\gamma},v)_\gamma.$$

Consequently,

(5.4)
$$(\mathbf{grad}\, e, \mathbf{grad}\, v)_\Omega = \sum_{K\in\mathcal{P}_0}\left((f,v)_K + \sum_{\gamma\in\mathcal{E}_K}(g_{K,\gamma},v)_\gamma + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_B}(g-P_\gamma g,v)_\gamma - (\mathbf{grad}\, u_\mathcal{P},\mathbf{grad}\, v)_K\right) + \sum_{K\in\mathcal{P}_+\cup\mathcal{P}_-}\left((f,v)_{K^*} + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_I}(g_{K,\gamma},v)_\gamma + (g,v)_{\Gamma_K} - (\mathbf{grad}\, u_\mathcal{P},\mathbf{grad}\, v)_{K^*}\right).$$

The decomposition (5.4) consists of a contribution from the elements belonging to $\mathcal{P}_0$ which is precisely the usual expression for the error in the case when there is no approximation of the domain. However, for the case when the domain is curvilinear, (5.4) has an additional contribution from the elements in $\mathcal{P}_+\cup\mathcal{P}_-$. Our first task is to estimate the contributions from the elements in $\mathcal{P}_0$. This is familiar territory and our method of choice follows the approach outlined in [2] and references therein. We briefly outline the idea.

For $K\in\mathcal{P}$ and $\gamma\in\mathcal{E}_K$ we define the residuals

$$R_{K,\gamma} = g_{K,\gamma} - \boldsymbol{n}_\gamma^K\cdot\mathbf{grad}\, u_{\mathcal{P}|K} \tag{5.5}$$

where $\boldsymbol{n}_\gamma^K$ is the outward unit normal vector to edge γ of element K. For $K\in\mathcal{P}_+\cup\mathcal{P}_-$, we also define the residual

$$R_{K,\Gamma} = g - \boldsymbol{n}\cdot\mathbf{grad}\, u_{\mathcal{P}|K^*}. \tag{5.6}$$

For a triangle K and data $R_{K,\gamma}\in\mathbb{P}_1(\gamma)$ for $\gamma\in\mathcal{E}_K$ such that

$$(P_Kf,p)_K + \sum_{\gamma\in\mathcal{E}_K}(R_{K,\gamma},p)_\gamma = 0 \text{ for all } p\in\mathbb{P}_1(K), \tag{5.7}$$

let $\boldsymbol{\sigma}_K\in\mathbb{P}_2(K)\times\mathbb{P}_2(K)$ be any vector field satisfying

$$\boldsymbol{n}_\gamma^K\cdot\boldsymbol{\sigma}_K = R_{K,\gamma} \text{ on } K \text{ for all } \gamma\in\mathcal{E}_K \tag{5.8}$$

and

$$-\operatorname{div}\boldsymbol{\sigma}_K = P_Kf \text{ in } K. \tag{5.9}$$

Consequently,

$$(P_Kf,v)_K + \sum_{\gamma\in\mathcal{E}_K}(R_{K,\gamma},v)_\gamma = (\boldsymbol{\sigma}_K,\mathbf{grad}\, v)_K. \tag{5.10}$$

An explicit construction for a choice of $\boldsymbol{\sigma}_K$ satisfying (5.8) and (5.9) which also minimises $\|\boldsymbol{\sigma}_K\|_{\boldsymbol{L}_2(K)}$ is given in Section 7.3. The following result is by now quite standard.

Lemma 5.1. *Let $K\in\mathcal{P}_0$. Then*

(5.11)
$$(f,v)_K + \sum_{\gamma\in\mathcal{E}_K}(g_{K,\gamma},v)_\gamma + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_B}(g-P_\gamma g,v)_\gamma - (\mathbf{grad}\, u_\mathcal{P},\mathbf{grad}\, v)_K \le \eta_K \,|\!|\!| v|\!|\!|_K$$

where

(5.12)
$$\eta_K = \|\boldsymbol{\sigma}_K\|_{\boldsymbol{L}_2(K)} + C_Kh_K\|f-P_Kf\|_{L_2(K)} + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_B\cap\mathcal{E}_0} C_{\gamma,K}^K h_K^{1/2}\|g-P_\gamma g\|_{L_2(\gamma)}.$$

Proof. Note that

$$(\mathbf{grad}\, u_\mathcal{P},\mathbf{grad}\, v)_K = \sum_{\gamma\in\mathcal{E}_K}\left(\boldsymbol{n}_\gamma^K\cdot\mathbf{grad}\, u_{\mathcal{P}|K}, v\right)_\gamma.$$

We choose $R_{K,\gamma} = g_{K,\gamma} - \boldsymbol{n}_\gamma^K \cdot \mathbf{grad}\, u_{\mathcal{P}|K}$ in (5.8) and recall property (5.3). The proof then follows the standard approach for polygonal domains. □

When $K \in \mathcal{P}_+ \cup \mathcal{P}_-$, the treatment of the error on the curvilinear region K^* is less straightforward but again, use will be made of the lifting $\boldsymbol{\sigma}_K$. We begin by stating the analogue of Lemma 5.1 for elements $K \in \mathcal{P}_-$.

Lemma 5.2. *Let $K \in \mathcal{P}_-$ and let the partition $\mathcal{P}$ be such that assumptions (**A1**)-(**A5**) are satisfied. Then*

$$(f,v)_{K^*} + \sum_{\gamma \in \mathcal{E}_K \cap \mathcal{E}_I} (g_{K,\gamma}, v)_\gamma + (g,v)_{\Gamma_K} - (\mathbf{grad}\, u_{\mathcal{P}}, \mathbf{grad}\, v)_{K^*} \leq \eta_K \,|\!|\!| v |\!|\!| \tag{5.13}$$

where

$$\begin{aligned} \eta_K &= \|\boldsymbol{\sigma}_K\|_{\boldsymbol{L}_2(K^*)} + C_{K^*} h_{K^*} \|f - P_K f\|_{L_2(K^*)} \\ &\quad + C_{\Gamma_K, K^*}^{K^*} h_{K^*}^{1/2} \|R_{K,\Gamma} - \boldsymbol{n} \cdot \boldsymbol{\sigma}_K\|_{L_2(\Gamma_K)}. \end{aligned} \tag{5.14}$$

Lemma 5.2 is similar to Lemma 5.1 with the differences being in the oscillation terms. The case of elements $K \in \mathcal{P}_+$ is more involved. The analogue of Lemma 5.1 reads as follows:

Lemma 5.3. *Let $K \in \mathcal{P}_+$ and let the partition $\mathcal{P}$ be such that assumptions (**A1**)-(**A5**) and (**A6**)$^\mu$ with $\mu \geq 1$ are satisfied. Then*

$$(f,v)_{K^*} + \sum_{\gamma \in \mathcal{E}_K \cap \mathcal{E}_I} (g_{K,\gamma}, v)_\gamma + (g,v)_{\Gamma_K} - (\mathbf{grad}\, u_{\mathcal{P}}, \mathbf{grad}\, v)_{K^*} \leq \eta_K \,|\!|\!| v |\!|\!| \tag{5.15}$$

where

(5.16)

$$\begin{aligned} \eta_K &= \|\boldsymbol{\sigma}_K\|_{\boldsymbol{L}_2(K)} + C_K h_K \|f - P_K f\|_{L_2(K)} + C_{K^*} h_{K^*} \|f - \langle f \rangle_{K^*}\|_{L_2(S_K)} \\ &\quad + \frac{C_{\gamma_K,K}^{K^*} h_{K^*}^{1/2}}{|\gamma_K|^{1/2}} |S_K| \left|\langle f \rangle_{S_K}\right| + C_{\Gamma_K,K^*}^{K^*} h_{K^*}^{1/2} \left\| R_{K,\Gamma} - \langle R_{K,\Gamma} \rangle_{\Gamma_K} \right\|_{L_2(\Gamma_K)} \\ &\quad + \left(|S_K|^{1/2} + \left(C_{\Gamma_K,K^*}^{K^*} |\gamma_K|^{1/2} + C_{\gamma_K,K}^{K^*} |\Gamma_K|^{1/2} \right) h_{K^*}^{1/2} \operatorname{osc}(\Gamma_K) \right) \left| \langle R_{K,\Gamma} \rangle_{\Gamma_K} \right|. \end{aligned}$$

Proof. The lemma is proved in Section 8.5. □

The result is again similar to Lemma 5.1 but now includes additional terms measuring the size of the sliver and oscillation of the boundary.

Our main result is the following computable bound on the energy norm of the error which takes into account the approximation of the boundary:

Theorem 5.4. *Let the partition $\mathcal{P}$ be such that assumptions (**A1**)-(**A5**) and (**A6**)$^\mu$ with $\mu \geq 1$ are satisfied. Let η_K be defined by* (5.12) *when $K \in \mathcal{P}_0$,* (5.16) *when $K \in \mathcal{P}_+$ and* (5.14) *when $K \in \mathcal{P}_-$. Then*

$$|\!|\!| e |\!|\!|_\Omega \leq \left(\sum_{K \in \mathcal{P}} \eta_K^2 \right)^{1/2}. \tag{5.17}$$

Proof. Since $K \in \mathcal{P}_0$ implies $K^* = K$, the result follows at once from Lemmas 5.1-5.3 and (2.5). □

5.2. **Local lower bounds on the energy norm of the error.** The next result shows that the upper bound in Theorem 5.4 is efficient.

Lemma 5.5. *Let the partition $\mathcal{P}$ be such that assumptions (**A1**)-(**A5**) and (**A6**)$^\mu$ with $\mu \geq 2$ are satisfied. Let $\mathcal{P}_{\mathcal{V}(K)}$ denote the set containing element K and the elements in $\mathcal{P}$ which share a vertex with element K and let $\mathcal{E}_{\mathcal{V}(K)}$ denote the* *set containing the edges in $\mathcal{E}$ which have an endpoint at a vertex of element K. Moreover, let*

$$\begin{aligned}\Phi_{K'} = &\sum_{K\in\mathcal{P}_{\mathcal{V}(K')}\cap\mathcal{P}_0}\Bigg(\left\|\!\left|e\right|\!\right\|_K + h_K\left\|f-P_Kf\right\|_{L_2(K)} \\ &+\sum_{\gamma'\in\mathcal{E}_{\mathcal{V}(K')}\cap\mathcal{E}_B\cap\mathcal{E}_0} h_K^{1/2}\left\|g-P_{\gamma'}g\right\|_{L_2(\gamma')}\Bigg) \\ &+\sum_{K\in\mathcal{P}_{\mathcal{V}(K')}\cap(\mathcal{P}_+\cup\mathcal{P}_-)}\Bigg(\left\|\!\left|e\right|\!\right\|_{K^*} + h_K^{1/2}\left\|R_{K,\Gamma}-\langle R_{K,\Gamma}\rangle_{\Gamma_K}\right\|_{L_2(\Gamma_K)} \\ &+h_K\left\|f-\langle f\rangle_{K^*}\right\|_{L_2(K)} + h_K^{3/2}\left\|f-\langle f\rangle_{K^*}\right\|_{L_2(S_K)}\Bigg). \end{aligned} \tag{5.18}$$

There exists a positive constant C, independent of the error e and the size of the elements in the mesh, such that

$$\eta_K \le C\Phi_K. \tag{5.19}$$

Proof. The lemma is proved in Section 8.6. □

6. Numerical Examples

6.1. **Example 1.** We consider the problem $-\Delta u = -2$ in $\Omega = \{(x,y) : x^2+y^2<1\}$ with $\boldsymbol{n}\cdot\mathbf{grad}\,u = 1$ on Γ, with true solution $u = \frac{1}{2}\left(x^2+y^2\right)$. The initial mesh is shown in Figure 5(b). In this, and the following examples, the problem is solved using local mesh refinement where we used a bulk criterion to refine the mesh on the smallest number of elements such that the sum of the contributions from these elements to η^2 from Theorem 5.4 exceeded 50% of the value of η^2. The results obtained are shown in Figure 6 with adaptively refined meshes being shown in Figure 5. From Table 1 we saw that the estimator from [2] did not provide an upper bound on $\left\|\!\left|e\right|\!\right\|_\Omega$ owing to neglecting approximation of the domain. In contrast, the estimator from Theorem 5.4 takes the domain approximation into account and, as shown in Figure 6, produces an upper bound on $\left\|\!\left|e\right|\!\right\|_\Omega$ on all of the meshes. Asymptotically, the estimator tends to overestimate the true error by a factor of 1.1. Remarkably, even starting with an initial mesh such as the one in Figure 5(b) only results in over-estimation by a factor of at most 4.2.

6.2. **Example 2.** Consider the problem $-\Delta u = f$ in Ω where Ω is the domain shown in Figure 7(a) with $\boldsymbol{n}\cdot\mathbf{grad}\,u = g$ on Γ where f and g are such that the true solution to this problem is $u = \left(r^{2/3}-r^3\right)\sin\left(\frac{2}{3}\theta\right)$. The initial mesh is shown in Figure 7(b). The results obtained are shown in Figure 8 with adaptively refined meshes being shown in Figure 7. Figure 8 shows once again that the estimator provides an upper bound on $\left\|\!\left|e\right|\!\right\|_\Omega$ on all of the meshes, with over-estimation by a factor asymptotically of the order of 1.3. The over-estimation by a factor of up to 7.2 on the initial very coarse mesh stems from the high data oscillation arising from the source term and the boundary.

6.3. **Example 3.** Finally, consider $-\Delta u = f$ in Ω where Ω is the domain shown in Figure 9(a) with $\boldsymbol{n}\cdot\mathbf{grad}\,u = g$ on Γ where f and g are such that the true solution to this problem is $u = r^4\left(\cos(4\theta)-1\right)$ when $x\ge 0$ and $y\ge 0$ but $u = 0$ in the remainder of Ω. The problem is of interest because no refinement will be needed outside the first quadrant. Moreover, the solution grows rapidly near the outer boundary but near the inner boundary varies slowly. This means that minimal refinement is expected near the inner boundary beyond controlling domain

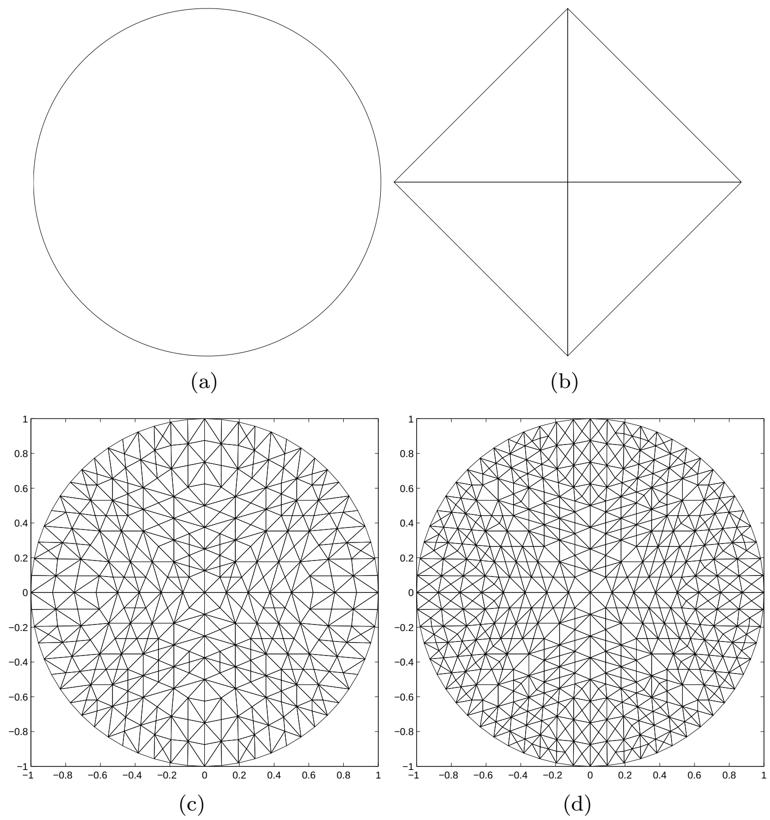

FIGURE 5. The (a) true domain Ω and (b) initial mesh for Example 1. Adaptively refined meshes for Example 1 containing (c) 648 and (d) 1110 elements.

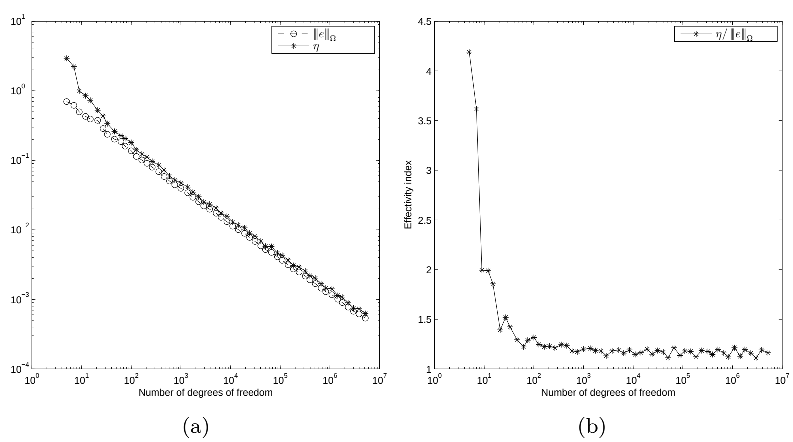

FIGURE 6. The (a) performance and (b) effectivity indices of the estimator for Example 1.

approximation. The initial mesh is shown in Figure 9(b). The results obtained are shown in Figure 10 with adaptively refined meshes being shown in Figure 9. Once again the estimator performs well both as an error estimator and in terms of guiding local refinements.

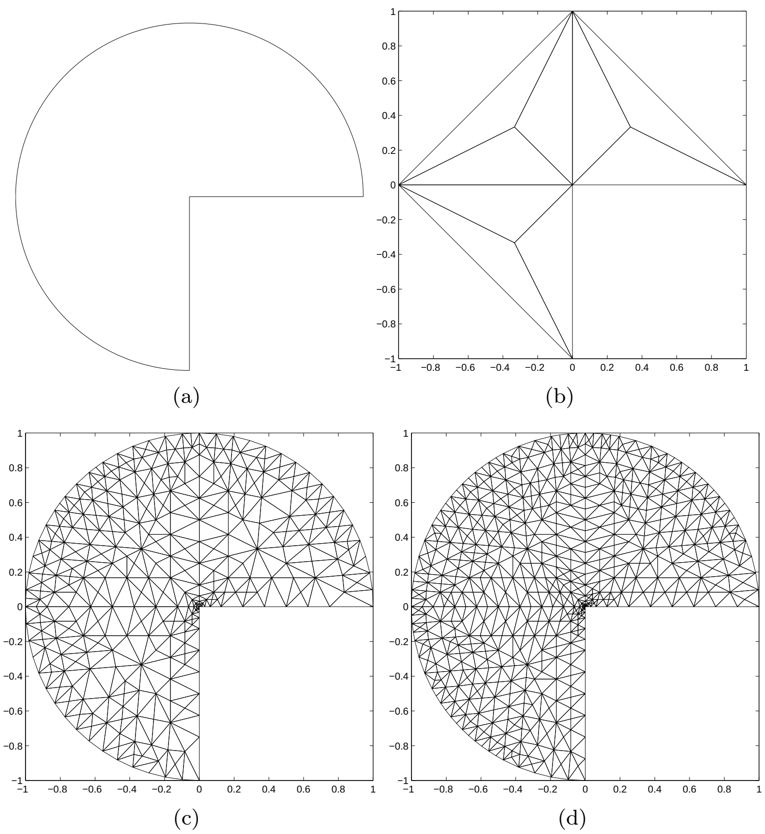

FIGURE 7. The (a) true domain Ω and (b) initial mesh for Example 2. Adaptively refined meshes containing (c) 608 and (d) 1050 elements.

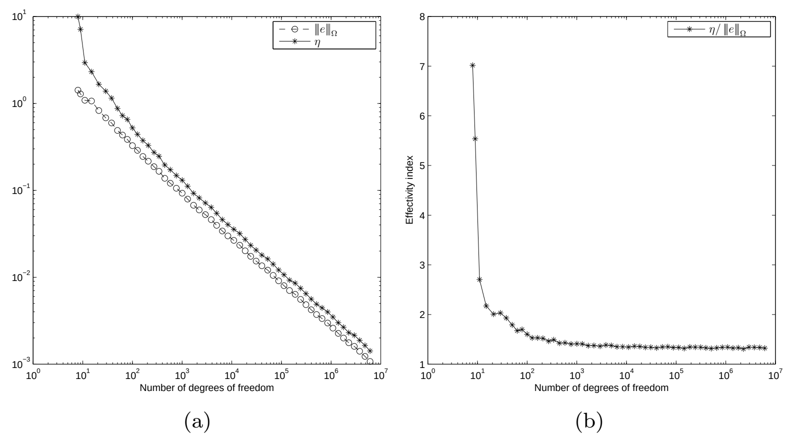

FIGURE 8. The (a) performance and (b) effectivity indices of the estimator for Example 2.

7. AUXILIARY RESULTS

7.1. **Equilibrated fluxes.** Let $\mathcal{V}$ index the set $\{\boldsymbol{x}_j\}_{j\in\mathcal{V}}$ of vertices of the elements in $\mathcal{P}$. For $j \in \mathcal{V}$, let $\mathcal{P}_j$ denote the set of elements in $\mathcal{P}$ that have a vertex at $\boldsymbol{x}_j$

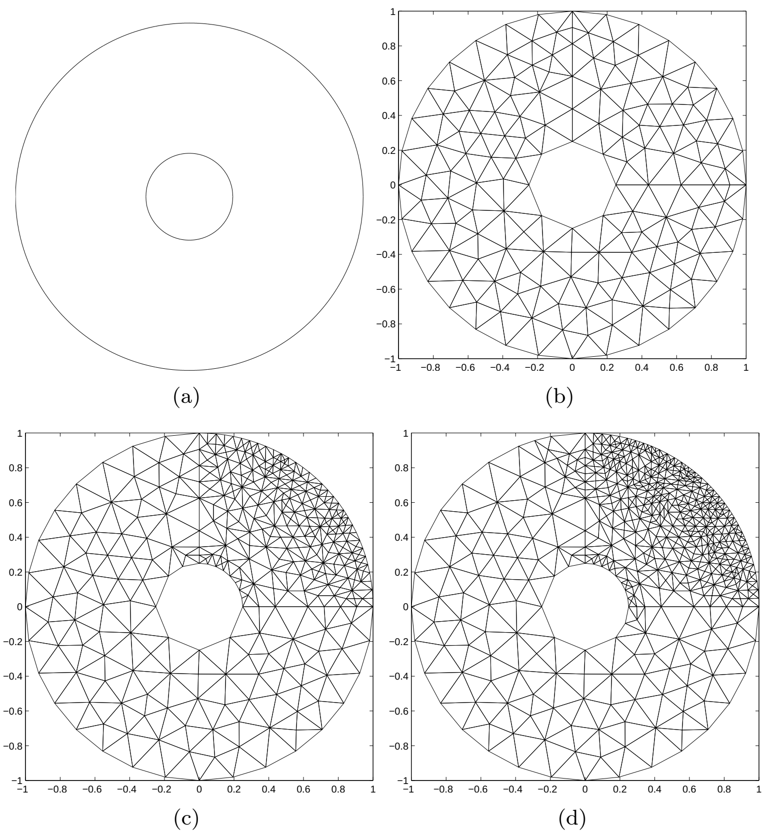

FIGURE 9. The (a) true domain Ω and (b) initial mesh for Example 3. Adaptively refined meshes containing (c) 608 and (d) 1002 elements.

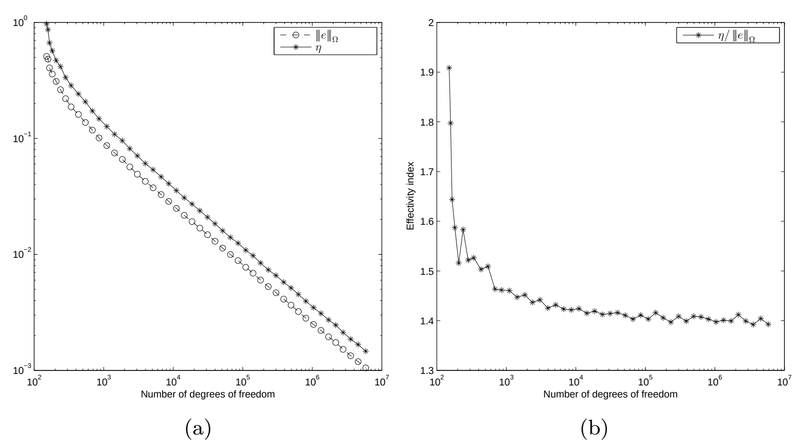

FIGURE 10. The (a) performance and (b) effectivity indices of the estimator for Example 3.

and let λ_j denote the function which is piecewise affine on $\mathcal{P}$ and vanishes at all the vertices in $\mathcal{P}$, except $\boldsymbol{x}_j$, where it takes the value one. Also, for $\gamma \in \mathcal{E}$, let $\mathcal{V}(\gamma)$ denote the subset of $\mathcal{V}$ which indexes the endpoints of edge γ.

The values of the two moments

$$\mu_{K,\gamma}^{(i)} = (g_{K,\gamma}, \lambda_i)_\gamma \text{ for } i \in \mathcal{V}(\gamma)$$

determine a unique function $g_{K,\gamma} \in \mathbb{P}_1(\gamma)$. Equally well, the flux $g_{K,\gamma}$ can be written as a linear combination of the moments $\mu_{K,\gamma}^{(i)}$. We now summarise a computational procedure from [3] which can be used to determine the moments $\mu_{K,\gamma}^{(i)}$ such that conditions (5.2) and (5.3) hold.

Let

$$A_\gamma^K = \begin{cases} \frac{1}{2}\boldsymbol{n}_\gamma^K \cdot \left(\mathbf{grad}\left(w_{n|K}\right) + \mathbf{grad}\left(w_{n|K'}\right)\right) & \text{if } \gamma \in \mathcal{E}_K \cap \mathcal{E}_{K'},\ K \neq K', \\ g_{K,\gamma} & \text{if } \gamma \in \mathcal{E}_K \cap \mathcal{E}_B, \end{cases}$$

where $\boldsymbol{n}_\gamma^K$ denotes the outward unit normal vector to edge γ of element K. We solve a system of linear equations for unknowns $\xi_{K,i}$:

$$\frac{1}{2} \sum_{K' \in \mathcal{P}_K \cap \mathcal{P}_i} (\xi_{K,i} - \xi_{K',i}) + \sum_{\gamma \in \mathcal{E}_K \cap \mathcal{E}_{\Gamma_\mathcal{P}} \cap \mathcal{E}_i} \xi_{K,i} = \Delta_K(\lambda_i) \text{ for all } K \in \mathcal{P}_i \tag{7.1}$$

where $\mathcal{P}_K$ denotes the set of elements that share an edge with element K, $\mathcal{E}_i$ denotes the set of edges that have an endpoint at $\boldsymbol{x}_i$ and

$$\Delta_K(\lambda_i) = (\mathbf{grad}\, u_\mathcal{P}, \mathbf{grad}\, \lambda_i)_K - (f, \lambda_i)_K - \sum_{\gamma \in \mathcal{E}_K} \left(A_\gamma^K, \lambda_i\right)_\gamma .$$

The moments $\mu_{K,\gamma}^{(i)}$ are then defined by

$$\mu_{K,\gamma}^{(i)} = \begin{cases} \frac{1}{2}(\xi_{K,i} - \xi_{K',i}) + \left(A_\gamma^K, \lambda_i\right)_\gamma & \text{if } \gamma \in \mathcal{E}_K \cap \mathcal{E}_{K'},\ K \neq K', \\ (g_{K,\gamma}, \lambda_i)_\gamma & \text{if } \gamma \in \mathcal{E}_K \cap \mathcal{E}_B. \end{cases} \tag{7.2}$$

The solvability and uniqueness of solutions of (7.1) is discussed in detail in [3], where it is also shown that (5.2) and (5.3) will hold. A key requirement of the fluxes is that they depend continuously on the local error and data oscillation. The following result extends Theorem 6.2 from [3] to the case when the domain approximation is taken into account.

Lemma 7.1. *Let the mesh be such that assumptions (**A1**)-(**A5**) and (**A6**)$^\mu$ with $\mu \geq 2$ are satisfied. There exists a positive constant C, independent of the error e and the size of the elements in the mesh, such that*

(7.3)

$$\begin{aligned} h_{K'}^{1/2} \|R_{K',\gamma}\|_{L_2(\gamma)} \leq & C \sum_{i \in \mathcal{V}(\gamma)} \left(\sum_{K \in \mathcal{P}_i \cap \mathcal{P}_0} \left(|\!|\!| e |\!|\!|_K + h_K \|f - P_K f\|_{L_2(K)} \right.\right. \\ & \left. + \sum_{\gamma' \in \mathcal{E}_i \cap \mathcal{E}_B \cap \mathcal{E}_0} h_K^{1/2} \|g - P_{\gamma'} g\|_{L_2(\gamma')} \right) \\ & + \sum_{K \in \mathcal{P}_i \cap (\mathcal{P}_+ \cup \mathcal{P}_-)} \left(|\!|\!| e |\!|\!|_{K^*} + h_K^{1/2} \left\| R_{K,\Gamma} - \langle R_{K,\Gamma} \rangle_{\Gamma_K} \right\|_{L_2(\Gamma_K)} \right. \\ & \left.\left. + h_K \|f - \langle f \rangle_{K^*}\|_{L_2(K)} + h_K^{3/2} \|f - \langle f \rangle_{K^*}\|_{L_2(S_K)} \right) \right) \end{aligned}$$

for $\gamma \in \mathcal{E}_{K'}$.

Proof. This lemma is proved in Section 8.7. □

7.2. **Explicit computable expressions for the constants in Poincaré inequalities.** Let ω be any two dimensional region which is star shaped with respect to a ball. From [14] we have that (3.7) holds with $C_\omega = \frac{1}{\pi}$ if ω is convex. Otherwise, if ω is star-shaped with respect to a point $\boldsymbol{x}_0 \in \omega$, from [18] (3.7) holds with

$$C_\omega = 2\left(\max\left(\frac{4\sqrt{6}}{3}\frac{4\left(\rho^2-1\right)+1}{\rho^2}+\frac{1}{2}\left(1-\frac{1}{\rho^2}\right), \frac{\rho^2-1}{2\rho^2}\ln\rho\right)\right)^{1/2}$$

where

$$\rho = \frac{\max\limits_{\boldsymbol{x}\in\partial\omega}|\boldsymbol{x}-\boldsymbol{x}_0|}{\min\limits_{\boldsymbol{x}\in\partial\omega}|\boldsymbol{x}-\boldsymbol{x}_0|}$$

and $\partial\omega$ denotes the boundary of ω.

Obviously, for $K \in \mathcal{P}$, $C_K \le C$ and $C_{K^*} \le C$ if K^* is convex. If K^* satisfies assumption (**A4**) then $\min\limits_{\boldsymbol{x}\in\partial K^*}|\boldsymbol{x}-\boldsymbol{x}_0| \ge ch_K$ and so $\rho \le C$ since (2.9) means that $\max\limits_{\boldsymbol{x}\in\partial K^*}|\boldsymbol{x}-\boldsymbol{x}_0| \le Ch_K$. Consequently, $C_{K^*} \le C$ when K^* is star shaped with respect to a ball.

We also require a bound for the constant in (3.8). We generalise the approach used in the appendix of [1] to the case of curvilinear triangles. Let ω and $\widetilde{\omega}$ be any star-shaped two dimensional regions such that $\omega \subset \widetilde{\omega}$. Let $\tau \subset \partial\omega$ and let $\boldsymbol{\theta}_\tau^\omega \in \boldsymbol{L}_\infty(\omega)$ be such that $\operatorname{div}\boldsymbol{\theta}_\tau^\omega \in \boldsymbol{L}_\infty(\omega)$ and $\boldsymbol{n}_{\partial\omega}\cdot\boldsymbol{\theta}_\tau^\omega > 0$ on $\overline{\tau}$ and $\boldsymbol{n}_{\partial\omega}\cdot\boldsymbol{\theta}_\tau^\omega = 0$ on $\partial\omega\setminus\tau$ where $\boldsymbol{n}_{\partial\omega}$ is the outward unit normal vector to $\partial\omega$. Define

$$m_\tau^\omega = \min_{\boldsymbol{x}\in\tau}\boldsymbol{n}_{\partial\omega}\cdot\boldsymbol{\theta}_\tau^\omega.$$

For $w \in H^1(\widetilde{\omega})$, we have that

$$\begin{aligned}
&\|w\|^2_{L_2(\tau)}\\
\le&\frac{1}{m_\tau^\omega}\int_\tau \boldsymbol{n}_{\partial\omega}\cdot\boldsymbol{\theta}_\tau^\omega w^2\,ds = \frac{1}{m_\tau^\omega}\int_{\partial\omega}\boldsymbol{n}_{\partial\omega}\cdot\boldsymbol{\theta}_\tau^\omega w^2\,ds = \frac{1}{m_\tau^\omega}\int_\omega \operatorname{div}\left(\boldsymbol{\theta}_\tau^\omega w^2\right)\,d\boldsymbol{x}\\
=&\frac{1}{m_\tau^\omega}\left(\left(\operatorname{div}\left(\boldsymbol{\theta}_\tau^\omega\right), w^2\right)_\omega + 2\left(w\boldsymbol{\theta}_\tau^\omega, \mathbf{grad}\, w\right)_\omega\right)\\
\le&\frac{1}{m_\tau^\omega}\|w\|_{L_2(\omega)}\left(\|\operatorname{div}\boldsymbol{\theta}_\tau^\omega\|_{L_\infty(\omega)}\|w\|_{L_2(\omega)} + 2\|\boldsymbol{\theta}_\tau^\omega\|_{L_\infty(\omega)}|\!|\!|w|\!|\!|_\omega\right)\\
\\
\le&\frac{1}{m_\tau^\omega}\|w\|_{L_2(\widetilde{\omega})}\left(\|\operatorname{div}\boldsymbol{\theta}_\tau^\omega\|_{L_\infty(\omega)}\|w\|_{L_2(\widetilde{\omega})} + 2\|\boldsymbol{\theta}_\tau^\omega\|_{L_\infty(\omega)}|\!|\!|w|\!|\!|_{\widetilde{\omega}}\right).
\end{aligned}$$

Hence, choosing $w = v - \langle v\rangle_{\widetilde{\omega}}$ and applying (3.7), we deduce that the constant in (3.8) may be chosen as

$$C_{\tau,\omega}^{\widetilde{\omega}} = \left(C_{\widetilde{\omega}}\left(\|\operatorname{div}\boldsymbol{\theta}_\tau^\omega\|_{L_\infty(\omega)}C_{\widetilde{\omega}}h_{\widetilde{\omega}} + \|\boldsymbol{\theta}_\tau^\omega\|_{L_\infty(\omega)}\right)\right)^{1/2}.$$

If K is a triangle and $\gamma \in \mathcal{E}_K$ then, following [1] we take

$$\boldsymbol{\theta}_\gamma^K = \frac{|\gamma|}{2|K|}\left(\boldsymbol{x}-\boldsymbol{x}_\gamma\right),$$

where $\boldsymbol{x}_\gamma$ is the vertex of K which is not an endpoint of γ. It is easy to verify that $\boldsymbol{\theta}_\gamma^K$ satisfies $\boldsymbol{n}_{\partial K}\cdot\boldsymbol{\theta}_\gamma^K = 1$ on γ and $\boldsymbol{n}_{\partial K}\cdot\boldsymbol{\theta}_\gamma^K = 0$ on $\partial K\setminus\gamma$. Hence, in this case $m_\gamma^K = 1$. Moreover, we have that $\left\|\operatorname{div}\boldsymbol{\theta}_\gamma^K\right\|_{L_\infty(K)} \le Ch_K^{-1}$ and $\left\|\boldsymbol{\theta}_\gamma^K\right\|_{L_\infty(K)} \le C$. Consequently, $C_{\gamma,K}^K \le C$ and $C_{\gamma,K}^{K^*} \le C$.

If $K \in \mathcal{P}_+\cup\mathcal{P}_-$, then in a similar vein, we take

$$\boldsymbol{\theta}_{\Gamma_K}^{K^*} = \frac{|\gamma_K|}{2|K|}\left(\boldsymbol{x}-\boldsymbol{x}_{\Gamma_K}\right),$$

where $\boldsymbol{x}_{\Gamma_K}$ is the vertex of K which is not an endpoint of Γ_K. The function again satisfies $\boldsymbol{n}_{\partial K} \cdot \boldsymbol{\theta}_{\Gamma_K}^{K^*} = 0$ on $\partial K^* \setminus \Gamma_K$. Assumption (**A5**) then means that $m_{\Gamma_K}^{K^*} \geq c$. Moreover, we have that $\left\|\operatorname{div} \boldsymbol{\theta}_{\Gamma_K}^{K^*}\right\|_{L_\infty(K^*)} \leq Ch_K^{-1}$ and $\left\|\boldsymbol{\theta}_{\Gamma_K}^{K^*}\right\|_{L_\infty(K^*)} \leq C$. Consequently, $C_{\Gamma_K, K^*}^{K^*} \leq C$.

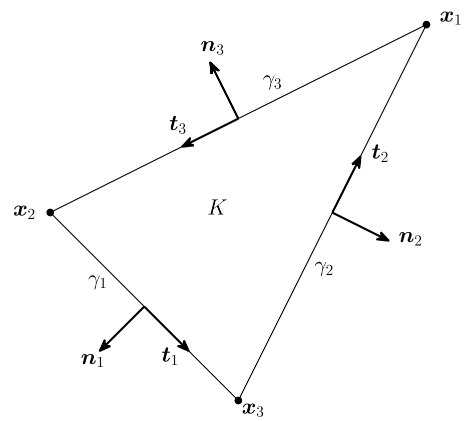

FIGURE 11. The labelling and orientation of the vertices, edges, tangent vectors and unit normal vectors of element K.

7.3. **An explicit expression for $\boldsymbol{\sigma}_K$.** Let K be any element and let the vertices, edges, tangent vectors and unit normal vectors of element K be labelled and oriented as shown in Figure 11, where we emphasise that the tangent vectors are such that $|\boldsymbol{t}_k| = |\gamma_k|$. Also, let $\lambda_k \in \mathbb{P}_1(K)$ be such that $\lambda_k = 1$ at vertex $\boldsymbol{x}_k$ of element K and vanishes at the remaining two vertices.

Lemma 7.2. *Let*

$$\begin{aligned}\boldsymbol{\sigma}_K^{\gamma_1} =& \frac{1}{2|K|}\Big((R_{K,\gamma_1}, \lambda_2)_{\gamma_1} ((2\lambda_3 - 3\lambda_2 - \lambda_1)\lambda_3 \boldsymbol{t}_2 + (4\lambda_2 - \lambda_3 - 7\lambda_1)\lambda_2 \boldsymbol{t}_3) \\ &+ (R_{K,\gamma_1}, \lambda_3)_{\gamma_1} ((-4\lambda_3 + \lambda_2 + 7\lambda_1)\lambda_3 \boldsymbol{t}_2 + (-2\lambda_2 + 3\lambda_3 + \lambda_1)\lambda_2 \boldsymbol{t}_3)\Big)\end{aligned}$$

with $\boldsymbol{\sigma}_K^{\gamma_2}$ and $\boldsymbol{\sigma}_K^{\gamma_3}$ being defined by permuting the indices, and

$$\boldsymbol{\sigma}_K^0 = \frac{1}{2|K|}\left((\lambda_2\lambda_3 - \lambda_3\lambda_1)\boldsymbol{t}_2 + (\lambda_2\lambda_3 - \lambda_1\lambda_2)\boldsymbol{t}_3\right).$$

Then

$$\boldsymbol{\sigma}_K = \sum_{i=1}^{3} \boldsymbol{\sigma}_K^{\gamma_i} - \frac{1}{(\boldsymbol{\sigma}_K^0, \boldsymbol{\sigma}_K^0)_K} \sum_{i=1}^{3} (\boldsymbol{\sigma}_K^{\gamma_i}, \boldsymbol{\sigma}_K^0)_K \boldsymbol{\sigma}_K^0 \tag{7.4}$$

satisfies (5.8) *and* (5.9) *and has minimal norm over* $\mathbb{P}_2(K) \times \mathbb{P}_2(K)$.

Proof. This lemma is proved in Section 8.8. □

An explicit computable expression for $\|\boldsymbol{\sigma}_K\|_{L_2(K)}$ is given in Section 9.4 of [4].

8. PROOFS

8.1. **Proof of Lemma 2.3.** Since $C\left(\overline{K^*}\right)$ is dense in $H^1\left(K^*\right)$, it suffices to prove the result for $w \in C\left(\overline{K^*}\right)$. We first observe that

$$\begin{aligned}
&\left|\frac{1}{|\Gamma_K|}\int_{\Gamma_K} w\ ds - \frac{1}{|\gamma_K|}\int_{\gamma_K} w\ ds\right| \\
&=\left|\frac{1}{|\Gamma_K|}\left(\int_{\Gamma_K} w\ ds - \int_{\gamma_K} w\ ds\right) + \left(\frac{1}{|\Gamma_K|} - \frac{1}{|\gamma_K|}\right)\int_{\gamma_K} w\ ds\right| \\
&\leq\frac{1}{|\Gamma_K|}\left|\int_{\Gamma_K} w\ ds - \int_{\gamma_K} w\ ds\right| + \frac{|\Gamma_K| - |\gamma_K|}{|\Gamma_K|\,|\gamma_K|}\left|\int_{\gamma_K} w\ ds\right|
\end{aligned} \tag{8.1}$$

By using the mapping and notation introduced at the end of Section 2.1 we can say that

$$\begin{aligned}
|\Gamma_K| - |\gamma_K| &= \int_0^{|\gamma_K|} \sqrt{1+\phi'\left(\hat{x}\right)^2} - 1\ d\hat{x} \\
&\leq |\gamma_K|^{1/2}\operatorname{osc}\left(\Gamma_K\right)\left(\int_0^{|\gamma_K|}\sqrt{1+\phi'\left(\hat{x}\right)^2}\ d\hat{x}\right)^{1/2} \\
&= |\gamma_K|^{1/2}\,|\Gamma_K|^{1/2}\operatorname{osc}\left(\Gamma_K\right)
\end{aligned}$$

on using the Cauchy–Schwarz inequality. Moreover, the Cauchy–Schwarz inequality and (3.8) give

$$\left|\int_{\gamma_K} w\ ds\right| \leq |\gamma_K|^{1/2}\left\|w\right\|_{L_2(\gamma_K)} \leq C^{K^*}_{\gamma_K,K} h^{1/2}_{K^*}\,|\gamma_K|^{1/2}\,|||w|||_{K^*}$$

on recalling $(w,1)_K = 0$. Consequently,

$$\frac{|\Gamma_K| - |\gamma_K|}{|\Gamma_K|\,|\gamma_K|}\left|\int_{\gamma_K} w\ ds\right| \leq \frac{1}{|\Gamma_K|} C^{K^*}_{\gamma_K,K}\,|\Gamma_K|^{1/2}\,h^{1/2}_{K^*}\operatorname{osc}\left(\Gamma_K\right)|||w|||_{K^*}\,. \tag{8.2}$$

Again using the mapping and notation introduced at the end of Section 2.1 we have that

$$\begin{aligned}
&\left|\int_{\Gamma_K} w\ ds - \int_{\gamma_K} w\ ds\right| \\
&=\left|\int_0^{|\gamma_K|}\hat{w}\left(\hat{x},\phi\left(\hat{x}\right)\right)\sqrt{1+\phi'\left(\hat{x}\right)^2}\ d\hat{x} - \int_0^{|\gamma_K|}\hat{w}\left(\hat{x},0\right)\ d\hat{x}\right| \\
&=\left|\int_0^{|\gamma_K|}\hat{w}\left(\hat{x},\phi\left(\hat{x}\right)\right) - \hat{w}\left(\hat{x},0\right)\ d\hat{x} + \int_0^{|\gamma_K|}\hat{w}\left(\hat{x},\phi\left(\hat{x}\right)\right)\left(\sqrt{1+\phi'\left(\hat{x}\right)^2} - 1\right)\ d\hat{x}\right| \\
&\leq\left|\int_0^{|\gamma_K|}\hat{w}\left(\hat{x},\phi\left(\hat{x}\right)\right) - \hat{w}\left(\hat{x},0\right)\ d\hat{x}\right| + \left|\int_0^{|\gamma_K|}\hat{w}\left(\hat{x},\phi\left(\hat{x}\right)\right)\left(\sqrt{1+\phi'\left(\hat{x}\right)^2} - 1\right)\ d\hat{x}\right|.
\end{aligned}$$

Now,

$$\begin{aligned}
&\left|\int_0^{|\gamma_K|}\hat{w}\left(\hat{x},\phi\left(\hat{x}\right)\right) - \hat{w}\left(\hat{x},0\right)\ d\hat{x}\right| = \left|\int_0^{|\gamma_K|}\int_0^{\phi(\hat{x})}\frac{\partial}{\partial\hat{y}}\hat{w}\left(\hat{x},\hat{y}\right)\ d\hat{y}\ d\hat{x}\right| \\
&\leq\left(\int_0^{|\gamma_K|}\int_0^{\phi(\hat{x})} 1\ d\hat{y}\ d\hat{x}\right)^{1/2}\left(\int_0^{|\gamma_K|}\int_0^{\phi(\hat{x})}\left(\frac{\partial}{\partial\hat{y}}\hat{w}\left(\hat{x},\hat{y}\right)\right)^2\ d\hat{y}\ d\hat{x}\right)^{1/2}
\end{aligned}$$

on applying the Cauchy–Schwarz inequality. Moreover,

$$\int_0^{|\gamma_K|}\int_0^{\phi(\hat{x})} 1\ d\hat{y}\ d\hat{x} = \int_0^{|\gamma_K|}\phi\left(\hat{x}\right)\ d\hat{x} = |S_K|$$

and

$$\left(\int_0^{|\gamma_K|}\int_0^{\phi(\hat{x})}\left(\frac{\partial}{\partial\hat{y}}\hat{w}\left(\hat{x},\hat{y}\right)\right)^2\ d\hat{y}\ d\hat{x}\right)^{1/2}\le \|w\|_{S_K}\le \|w\|_{K^*}.$$

Applying the Cauchy–Schwarz inequality also gives

$$\begin{aligned}&\left|\int_0^{|\gamma_K|}\hat{w}\left(\hat{x},\phi\left(\hat{x}\right)\right)\left(\sqrt{1+\phi'\left(\hat{x}\right)^2}-1\right)\ d\hat{x}\right|\\ &\le\left(\int_0^{|\gamma_K|}\left(\hat{w}\left(\hat{x},\phi\left(\hat{x}\right)\right)\right)^2\sqrt{1+\phi'\left(\hat{x}\right)^2}\ d\hat{x}\right)^{1/2}|\gamma_K|^{1/2}\operatorname{osc}\left(\Gamma_K\right)\\ &=\|w\|_{L_2(\Gamma_K)}|\gamma_K|^{1/2}\operatorname{osc}\left(\Gamma_K\right)\le C^{K^*}_{\Gamma_K,K^*}h^{1/2}_{K^*}|\gamma_K|^{1/2}\operatorname{osc}\left(\Gamma_K\right)\|w\|_{K^*}\end{aligned}$$

on applying (3.8). Consequently,

$$\begin{aligned}&\left|\int_{\Gamma_K}w\ ds-\int_{\gamma_K}w\ ds\right|\\ &\le\left(|S_K|^{1/2}+C^{K^*}_{\Gamma_K,K^*}|\gamma_K|^{1/2}h^{1/2}_{K^*}\operatorname{osc}\left(\Gamma_K\right)\right)\|w\|_{K^*}.\end{aligned}\tag{8.3}$$

The result then follows upon combining (8.1), (8.2) and (8.3).

8.2. **Proof of Lemma 2.4.** To bound the first term we first write

$$\left(F,v-\langle v\rangle_{\gamma_K}\right)_{S_K}=\left(F,v-\langle v\rangle_{K^*}\right)_{S_K}-\left(F,\langle v\rangle_{\gamma_K}-\langle v\rangle_{K^*}\right)_{S_K}.\tag{8.4}$$

The first term in (8.4) is then bounded by writing

$$\begin{aligned}\left(F,v-\langle v\rangle_{K^*}\right)_{S_K}&=\left(F-\langle F\rangle_{K^*},v-\langle v\rangle_{K^*}\right)_{S_K}\\ &\le\|F-\langle F\rangle_{K^*}\|_{L_2(S_K)}\|v-\langle v\rangle_{K^*}\|_{L_2(S_K)}\end{aligned}$$

and since

$$\|v-\langle v\rangle_{K^*}\|_{L_2(S_K)}\le\|v-\langle v\rangle_{K^*}\|_{L_2(K^*)}$$

we can use (3.7) to conclude that

$$\left(F,v-\langle v\rangle_{K^*}\right)_{S_K}\le C_{K^*}h_{K^*}\|F-\langle F\rangle_{K^*}\|_{L_2(S_K)}\|v\|_{K^*}.\tag{8.5}$$

For the second term in (8.4) we write

$$-\left(F,\langle v\rangle_{\gamma_K}-\langle v\rangle_{K^*}\right)_{S_K}=-\left(F,\langle v-\overline{v}_{K^*}\rangle_{\gamma_K}\right)_{S_K}=-\langle v-\overline{v}_{K^*}\rangle_{\gamma_K}\langle F\rangle_{S_K}|S_K|$$

and

$$\begin{aligned}\langle v-\overline{v}_{K^*}\rangle_{\gamma_K}=\frac{1}{|\gamma_K|}\int_{\gamma_K}v-\langle v\rangle_{K^*}\ ds&\le\frac{1}{|\gamma_K|^{1/2}}\|v-\langle v\rangle_{K^*}\|_{L_2(\gamma_K)}\\ &\le\frac{C^{K^*}_{\gamma_K,K}h^{1/2}_{K^*}}{|\gamma_K|^{1/2}}\|v\|_K\end{aligned}$$

by (3.8). Hence,

$$-\left(F,\langle v\rangle_{\gamma_K}-\langle v\rangle_{K^*}\right)_{S_K}\le\frac{C^{K^*}_{\gamma_K,K}h^{1/2}_{K^*}}{|\gamma_K|^{1/2}}|S_K|\left|\langle F\rangle_{S_K}\right|\|v\|_K.\tag{8.6}$$

To bound the second term we first write

$$\left(G,v-\langle v\rangle_{\gamma_K}\right)_{\Gamma_K}=\left(G,v-\langle v\rangle_{\Gamma_K}\right)_{\Gamma_K}+\left(G,\langle v\rangle_{\Gamma_K}-\langle v\rangle_{\gamma_K}\right)_{\Gamma_K}.$$

We can then say that

$$\begin{aligned}\left(G,v-\langle v\rangle_{\Gamma_K}\right)_{\Gamma_K}&=\left(G-\langle G\rangle_{\Gamma_K},v-\langle v\rangle_{\Gamma_K}\right)_{\Gamma_K}\\ &\le\left\|G-\langle G\rangle_{\Gamma_K}\right\|_{L_2(\Gamma_K)}\left\|v-\langle v\rangle_{\Gamma_K}\right\|_{L_2(\Gamma_K)}\end{aligned}$$

and since

$$\left\|v - \langle v\rangle_{\Gamma_K}\right\|_{L_2(\Gamma_K)} \leq \|v - \langle v\rangle_{K^*}\|_{L_2(\Gamma_K)} \tag{8.7}$$

we can use (3.8) to conclude that

$$\left(G, v - \langle v\rangle_{\Gamma_K}\right)_{\Gamma_K} \leq C^{K^*}_{\Gamma_K,K^*} h^{1/2}_{K^*} \left\|G - \langle G\rangle_{\Gamma_K}\right\|_{L_2(\Gamma_K)} \|v\|_{K^*}. \tag{8.8}$$

Moreover,

$$\left(G, \langle v\rangle_{\Gamma_K} - \langle v\rangle_{\gamma_K}\right)_{\Gamma_K} = \left|\langle v\rangle_{\Gamma_K} - \langle v\rangle_{\gamma_K}\right| \left|\langle G\rangle_{\Gamma_K}\right| |\Gamma_K|$$

and

$$\left|\langle v\rangle_{\Gamma_K} - \langle v\rangle_{\gamma_K}\right| = \left|\frac{1}{|\Gamma_K|}\int_{\Gamma_K} v - \langle v\rangle_{K^*}\ ds - \frac{1}{|\gamma_K|}\int_{\gamma_K} v - \langle v\rangle_{K^*}\ ds\right|.$$

Hence, on applying (2.11) we have that

$$\begin{aligned}&\left(G, \langle v\rangle_{\Gamma_K} - \langle v\rangle_{\gamma_K}\right)_{\Gamma_K} \\ &\leq \left(|S_K|^{1/2} + \left(C^{K^*}_{\Gamma_K,K^*}|\gamma_K|^{1/2} + C^{K^*}_{\gamma_K,K}|\Gamma_K|^{1/2}\right) h^{1/2}_{K^*}\operatorname{osc}(\Gamma_K)\right)\left|\langle G\rangle_{\Gamma_K}\right| \|v\|_{K^*}.\end{aligned} \tag{8.9}$$

Consequently, from (8.5), (8.6), (8.8) and (8.9) we have the bound claimed.

8.3. **Proof of Lemma 4.1.** Following [16] we define $\widetilde{u}_{\mathcal{P}} \in \widetilde{X}^{\mathcal{P}}_{\backslash\mathbb{R}}$ such that

$$(\mathbf{grad}\,\widetilde{u}_{\mathcal{P}}, \mathbf{grad}\,v)_\Omega = (f, v)_\Omega + (g, v)_\Gamma \text{ for all } v \in \widetilde{X}^{\mathcal{P}}. \tag{8.10}$$

First observe that

$$\|u - u_{\mathcal{P}}\|_\Omega \leq \|u - \widetilde{u}_{\mathcal{P}}\|_\Omega + \|\widetilde{u}_{\mathcal{P}} - u_{\mathcal{P}}\|_\Omega. \tag{8.11}$$

Now, (8.10) means that, for any $p \in \widetilde{X}^{\mathcal{P}}$,

$$\begin{aligned}(\mathbf{grad}\,(u - \widetilde{u}_{\mathcal{P}}), \mathbf{grad}\,(u - \widetilde{u}_{\mathcal{P}}))_\Omega &= (\mathbf{grad}\,(u - \widetilde{u}_{\mathcal{P}}), \mathbf{grad}\,(u - p))_\Omega \\ &\leq \|u - \widetilde{u}_{\mathcal{P}}\|_\Omega \|u - p\|_\Omega\end{aligned}$$

Consequently,

$$\|u - \widetilde{u}_{\mathcal{P}}\|_\Omega \leq \|u - p\|_\Omega \text{ for any } p \in \widetilde{X}_{\mathcal{P}}. \tag{8.12}$$

Moreover, since $\mathbf{grad}\,(\widetilde{u}_{\mathcal{P}} - u_{\mathcal{P}})$ is constant on K^*,

$$\|\widetilde{u}_{\mathcal{P}} - u_{\mathcal{P}}\|^2_{S_K} = |S_K|\left|\mathbf{grad}\left(\widetilde{u}_{\mathcal{P}|K^*} - u_{\mathcal{P}|K^*}\right)\right|^2 = \frac{|S_K|}{|K|}\|\widetilde{u}_{\mathcal{P}} - u_{\mathcal{P}}\|^2_K$$

Hence,

$$\begin{aligned}\|\widetilde{u}_{\mathcal{P}} - u_{\mathcal{P}}\|^2_\Omega &= \|\widetilde{u}_{\mathcal{P}} - u_{\mathcal{P}}\|^2_{\Omega_{\mathcal{P}}} + \sum_{K\in\mathcal{P}_+}\|\widetilde{u}_{\mathcal{P}} - u_{\mathcal{P}}\|^2_{S_K} \\ &= \|\widetilde{u}_{\mathcal{P}} - u_{\mathcal{P}}\|^2_{\Omega_{\mathcal{P}}} + \sum_{K\in\mathcal{P}_+}\frac{|S_K|}{|K|}\|\widetilde{u}_{\mathcal{P}} - u_{\mathcal{P}}\|^2_K \\ &\leq \left(1 + \max_{K\in\mathcal{P}_+}\frac{|S_K|}{|K|}\right)\|\widetilde{u}_{\mathcal{P}} - u_{\mathcal{P}}\|^2_{\Omega_{\mathcal{P}}}.\end{aligned} \tag{8.13}$$

Let $v \in \widetilde{X}^{\mathcal{P}}$. Then

$$\begin{aligned}&(\mathbf{grad}\,(\widetilde{u}_{\mathcal{P}} - u_{\mathcal{P}}), \mathbf{grad}\,v)_{\Omega_{\mathcal{P}}} \\ &= (\mathbf{grad}\,(\widetilde{u}_{\mathcal{P}} - u), \mathbf{grad}\,v)_{\Omega_{\mathcal{P}}} + (\mathbf{grad}\,(u - u_{\mathcal{P}}), \mathbf{grad}\,v)_{\Omega_{\mathcal{P}}}.\end{aligned} \tag{8.14}$$

Now,

$$\begin{aligned}(\mathbf{grad}\,(u - u_{\mathcal{P}}), \mathbf{grad}\,v)_{\Omega_{\mathcal{P}}} &= (\mathbf{grad}\,u, \mathbf{grad}\,v)_\Omega - (\mathbf{grad}\,u_{\mathcal{P}}, \mathbf{grad}\,v)_{\Omega_{\mathcal{P}}} \\ &\quad + (\mathbf{grad}\,u, \mathbf{grad}\,v)_{\Omega_{\mathcal{P}}} - (\mathbf{grad}\,u, \mathbf{grad}\,v)_\Omega\end{aligned}$$

where

$$\begin{aligned}(\mathbf{grad}\, u, \mathbf{grad}\, v)_{\Omega_{\mathcal{P}}} - (\mathbf{grad}\, u, \mathbf{grad}\, v)_{\Omega} &= -\sum_{K\in\mathcal{P}_+} (\mathbf{grad}\, u, \mathbf{grad}\, v)_{S_K} \\ &\leq \sum_{K\in\mathcal{P}_+} \|u\|_{S_K} \|v\|_{S_K} \\ &= \sum_{K\in\mathcal{P}_+} \|u\|_{S_K} \frac{|S_K|^{1/2}}{|K|^{1/2}} \|v\|_K\end{aligned}$$

and

$$\begin{aligned}&(\mathbf{grad}\, u, \mathbf{grad}\, v)_{\Omega} - (\mathbf{grad}\, u_{\mathcal{P}}, \mathbf{grad}\, v)_{\Omega_{\mathcal{P}}} \\ &= (f,v)_{\Omega} + (g,v)_{\Gamma} - (f,v)_{\Omega_{\mathcal{P}}} - \sum_{K\in\mathcal{P}} \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_B} (g_{K,\gamma}, v)_{\gamma} \\ &= \sum_{K\in\mathcal{P}_+} \left((f,v)_{S_K} + (g,v)_{\Gamma_K} - \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_B} (g_{K,\gamma}, v)_{\gamma_K} \right) \\ &= \sum_{K\in\mathcal{P}_+} \left(\left(f, v - \langle v\rangle_{\gamma_K}\right)_{S_K} + \left(g, v - \langle v\rangle_{\gamma_K}\right)_{\Gamma_K} \right)\end{aligned}$$

by (3.6). We can bound this term using Lemma 2.4 and the fact that $\|v\|_{K^*} = \frac{|K^*|^{1/2}}{|K|^{1/2}} \|v\|_K$ and $\sum_{K\in\mathcal{P}_+} \|v\|_K^2 \leq \|v\|_{\Omega_{\mathcal{P}}}^2$ to conclude that

$$(\mathbf{grad}\,(u - u_{\mathcal{P}}), \mathbf{grad}\, v)_{\Omega_{\mathcal{P}}} \leq \sum_{K\in\mathcal{P}_+} \Psi_K \|v\|_K \leq \left(\sum_{K\in\mathcal{P}_+} \Psi_K^2\right)^{1/2} \|v\|_{\Omega_{\mathcal{P}}} \tag{8.15}$$

where

$$\begin{aligned}\Psi_K =& \frac{|K^*|^{1/2}}{|K|^{1/2}} \Bigg(\frac{|S_K|}{|K|^{1/2}} \|u\|_{S_K} + C_{K^*} h_{K^*} \|f - \langle f\rangle_{K^*}\|_{L_2(S_K)} \\ &+ \frac{C^{K^*}_{\gamma_K,K} h^{1/2}_{K^*}}{|\gamma_K|^{1/2}} |S_K| \left|\langle f\rangle_{S_K}\right| + C^{K^*}_{\Gamma_K,K^*} h^{1/2}_{K^*} \left\|g - \langle g\rangle_{\Gamma_K}\right\|_{L_2(\Gamma_K)} \\ &+ \left(|S_K|^{1/2} + \left(C^{K^*}_{\Gamma_K,K^*} |\gamma_K|^{1/2} + C^{K^*}_{\gamma_K,K} |\Gamma_K|^{1/2} \right) h^{1/2}_{K^*} \operatorname{osc}(\Gamma_K) \right) \left|\langle g\rangle_{\Gamma_K}\right| \Bigg).\end{aligned}$$

Then, by letting $v = \widetilde{u}_{\mathcal{P}} - u_{\mathcal{P}}$ in (8.14), applying (8.15) and the Cauchy–Schwarz inequality, we can conclude that

$$\|\widetilde{u}_{\mathcal{P}} - u_{\mathcal{P}}\|^2_{\Omega_{\mathcal{P}}} \leq \|\widetilde{u}_{\mathcal{P}} - u\|_{\Omega_{\mathcal{P}}} \|\widetilde{u}_{\mathcal{P}} - u_{\mathcal{P}}\|_{\Omega_{\mathcal{P}}} + \left(\sum_{K\in\mathcal{P}_+} \Psi_K^2\right)^{1/2} \|\widetilde{u}_{\mathcal{P}} - u_{\mathcal{P}}\|_{\Omega_{\mathcal{P}}}$$

and hence

$$\|\widetilde{u}_{\mathcal{P}} - u_{\mathcal{P}}\|_{\Omega_{\mathcal{P}}} \leq \|\widetilde{u}_{\mathcal{P}} - u\|_{\Omega} + \left(\sum_{K\in\mathcal{P}_+} \Psi_K^2\right)^{1/2} \tag{8.16}$$

since $\|\widetilde{u}_{\mathcal{P}} - u\|_{\Omega_{\mathcal{P}}} \leq \|\widetilde{u}_{\mathcal{P}} - u\|_{\Omega}$.

Combining (8.11), (8.12), (8.13) and (8.16) then gives, for any $p \in \widetilde{X}^{\mathcal{P}}$,

(8.17)

$$\|u - u_{\mathcal{P}}\|_{\Omega} \leq \|u - p\|_{\Omega} + \left(1 + \max_{K\in\mathcal{P}_+} \frac{|S_K|}{|K|}\right)^{1/2} \left(\|u - p\|_{\Omega} + \left(\sum_{K\in\mathcal{P}_+} \Psi_K^2\right)^{1/2}\right).$$

Now, (2.1), (2.2), (2.13), (2.7), (2.8), (2.9) and (2.6) imply that

$$\begin{aligned}\Psi_K \leq & C\Bigg(h_K^{1/2} \|u\|_{S_K} + \left(C_{K^*} h_K + C_{\gamma_K,K}^{K^*} h_K^{3/2}\right) \|f\|_{L_2(S_K)} \\ & + C_{\Gamma_K,K^*}^{K^*} h_K^{1/2} \left\|g - \langle g\rangle_{\Gamma_K}\right\|_{L_2(\Gamma_K)} \\ & + \left(h_K + \left(C_{\Gamma_K,K^*}^{K^*} + C_{\gamma_K,K}^{K^*}\right) h_K^{3/2}\right) \|g\|_{L_2(\Gamma_K)}\Bigg)\end{aligned}$$

since $\left|\langle f\rangle_{S_K}\right| \leq |S_K|^{-1/2} \|f\|_{L_2(S_K)}$, $\left|\langle g\rangle_{\Gamma_K}\right| \leq |\Gamma_K|^{-1/2} \|g\|_{L_2(\Gamma_K)}$ and $\|f - \langle f\rangle_{K^*}\|_{L_2(S_K)} \leq \|f\|_{L_2(S_K)}$. Consequently, owing to the fact that assumptions (**A4**) and (**A5**) mean that $C_{K^*} \leq C$, $C_{\Gamma_K,K^*}^{K^*} \leq C$ and $C_{\gamma_K,K}^{K^*} \leq C$, we can say that

$$\begin{aligned}&\left(\sum_{K\in\mathcal{P}_+} \Psi_K^2\right)^{1/2} \\ \leq & C\Bigg(\sum_{K\in\mathcal{P}_+} \left(h_K \|u\|_{S_K}^2 + h_K \left\|g - \langle g\rangle_{\Gamma_K}\right\|_{L_2(\Gamma_K)}^2\right) \\ & + \max_{K\in\mathcal{P}_+} h_K^2 \|f\|_{L_2(\Omega)}^2 + \max_{K\in\mathcal{P}_+} h_K^2 \|g\|_{L_2(\Gamma)}^2\Bigg)^{1/2} \\ \leq & C\left(\left(\sum_{K\in\mathcal{P}_+} \left(h_K \|u\|_{S_K}^2 + h_K \left\|g - \langle g\rangle_{\Gamma_K}\right\|_{L_2(\Gamma_K)}^2\right)\right)^{1/2} + \max_{K\in\mathcal{P}_+} h_K\right)\end{aligned}$$

since $h_K^3 \leq C h_K^2$ since Ω is bounded. This also means that

$$1 + \max_{K\in\mathcal{P}_+} \frac{|S_K|}{|K|} \leq 1 + C \max_{K\in\mathcal{P}_+} h_K \leq C$$

by (2.1) and (2.7) and hence by substituting the above inequalities into (8.17) we arrive at the result claimed.

8.4. **Proof of Lemma 5.2.** Integration by parts allows us to say that

$$\begin{aligned}&(f, v)_{K^*} + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_I} (g_{K,\gamma}, v)_\gamma + (g, v)_{\Gamma_K} - (\mathbf{grad}\, u_{\mathcal{P}}, \mathbf{grad}\, v)_{K^*} \\ =& (f, v)_{K^*} + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_I} (R_{K,\gamma}, v)_\gamma + (R_{K,\Gamma}, v)_{\Gamma_K}\end{aligned}$$

where $R_{K,\gamma}$ is given by (5.5) and $R_{K,\Gamma}$ is given by (5.6).

Thanks to (5.3) we have that (5.7) holds and so $\boldsymbol{\sigma}_K$ satisfies (5.8) and (5.9). Integration by parts yields

$$(\boldsymbol{\sigma}_K, \mathbf{grad}\, v)_{K^*} = (\boldsymbol{n} \cdot \boldsymbol{\sigma}_K, v)_{\Gamma_K} + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_I} \left(\boldsymbol{n}_\gamma^K \cdot \boldsymbol{\sigma}_K, v\right)_\gamma - (\operatorname{div} \boldsymbol{\sigma}_K, v)_{K^*}$$

and so (5.8) and (5.9) mean that

$$(P_K f, v)_{K^*} + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_I} (R_{K,\gamma}, v)_\gamma = (\boldsymbol{\sigma}_K, \mathbf{grad}\, v)_{K^*} - (\boldsymbol{n} \cdot \boldsymbol{\sigma}_K, v)_{\Gamma_K}.$$

Therefore,

$$\begin{aligned}(f,v)_{K^*} &+ \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_I}(R_{K,\gamma},v)_\gamma + (R_{K,\Gamma},v)_{\Gamma_K}\\ &= (\boldsymbol{\sigma}_K,\mathbf{grad}\,v)_{K^*} + (R_{K,\Gamma}-\boldsymbol{n}\cdot\boldsymbol{\sigma}_K,v)_{\Gamma_K} + (f-P_Kf,v)_{K^*}.\end{aligned}$$

Now, since

$$(\mathbf{grad}\,u_{\mathcal{P}},\mathbf{grad}\,v)_{S_K} = \left(\boldsymbol{n}^K_{\gamma_K}\cdot\mathbf{grad}\,u_{\mathcal{P}},v\right)_{\gamma_K} - (\boldsymbol{n}\cdot\mathbf{grad}\,u_{\mathcal{P}},v)_{\Gamma_K}$$

we have that

$$\left(\boldsymbol{n}^K_{\gamma_K}\cdot\mathbf{grad}\,u_{\mathcal{P}},1\right)_{\gamma_K} = (\boldsymbol{n}\cdot\mathbf{grad}\,u_{\mathcal{P}},1)_{\Gamma_K}. \tag{8.18}$$

Moreover, since

$$(\boldsymbol{\sigma}_K,\mathbf{grad}\,v)_{S_K} = \left(\boldsymbol{n}^K_{\gamma_K}\cdot\boldsymbol{\sigma}_K,v\right)_{\gamma_K} - (\boldsymbol{n}\cdot\boldsymbol{\sigma}_K,v)_{\Gamma_K} - (\operatorname{div}\boldsymbol{\sigma}_K,v)_{S_K}$$

we have that

$$(\boldsymbol{n}\cdot\boldsymbol{\sigma}_K,1)_{\Gamma_K} = \left(\boldsymbol{n}^K_{\gamma_K}\cdot\boldsymbol{\sigma}_K,1\right)_{\gamma_K} - (\operatorname{div}\boldsymbol{\sigma}_K,1)_{S_K} = (R_{K,\gamma_K},1)_{\gamma_K} + (P_Kf,1)_{S_K}$$

by (5.8) and (5.9). The definitions of R_{K,γ_K} and P_Kf then allow us to say that

$$(\boldsymbol{n}\cdot\boldsymbol{\sigma}_K,1)_{\Gamma_K} = \left(g_{K,\gamma_K} - \boldsymbol{n}^K_{\gamma_K}\cdot\mathbf{grad}\,u_{\mathcal{P}},1\right)_{\gamma_K} + (f,1)_{S_K} = (g-\boldsymbol{n}\cdot\mathbf{grad}\,u_{\mathcal{P}},1)_{\Gamma_K}$$

upon using definition (3.6) and (8.18). Consequently, definition (5.6) means that

$$(R_{K,\Gamma}-\boldsymbol{n}\cdot\boldsymbol{\sigma}_K,1)_{\Gamma_K} = 0 \tag{8.19}$$

which allows us to say that

$$(R_{K,\Gamma}-\boldsymbol{n}\cdot\boldsymbol{\sigma}_K,v)_{\Gamma_K} = (R_{K,\Gamma}-\boldsymbol{n}\cdot\boldsymbol{\sigma}_K,v-\langle v\rangle_{K^*})_{\Gamma_K}.$$

Therefore,

$$\begin{aligned}&(f,v)_{K^*} + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_I}(R_{K,\gamma},v)_\gamma + (R_{K,\Gamma},v)_{\Gamma_K}\\ &= (\boldsymbol{\sigma}_K,\mathbf{grad}\,v)_{K^*} + (R_{K,\Gamma}-\boldsymbol{n}\cdot\boldsymbol{\sigma}_K,v-\langle v\rangle_{K^*})_{\Gamma_K} + (f-P_Kf,v-\langle v\rangle_{K^*})_{K^*}\\ &\le \|\boldsymbol{\sigma}_K\|_{\boldsymbol{L}_2(K^*)}\,|\!|\!|v|\!|\!|_{K^*} + \|R_{K,\Gamma}-\boldsymbol{n}\cdot\boldsymbol{\sigma}_K\|_{L_2(\Gamma_K)}\,\|v-\langle v\rangle_{K^*}\|_{L_2(\Gamma_K)}\\ &\quad + \|f-P_Kf\|_{L_2(K^*)}\,\|v-\langle v\rangle_{K^*}\|_{L_2(K^*)}\\ &\le \Big(\|\boldsymbol{\sigma}_K\|_{\boldsymbol{L}_2(K^*)} + C^{K^*}_{\Gamma_K,K^*}h^{1/2}_{K^*}\|R_{K,\Gamma}-\boldsymbol{n}\cdot\boldsymbol{\sigma}_K\|_{L_2(\Gamma_K)}\\ &\quad + C_{K^*}h_{K^*}\|f-P_Kf\|_{L_2(K^*)}\Big)\,|\!|\!|v|\!|\!|_{K^*}\end{aligned}$$

by (3.7) and (3.8) and so we have the bound claimed.

8.5. **Proof of Lemma 5.3.** Integration by parts allows us to say that

$$\begin{aligned}(f,v)_{K^*} &+ \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_I}(g_{K,\gamma},v)_\gamma + (g,v)_{\Gamma_K} - (\mathbf{grad}\,u_{\mathcal{P}},\mathbf{grad}\,v)_{K^*}\\ &= (P_Kf,v)_K + \sum_{\gamma\in\mathcal{E}_K}(R_{K,\gamma},v)_\gamma + (f-P_Kf,v)_K\\ &\quad + (f,v)_{S_K} - (R_{K,\gamma_K},v)_{\gamma_K} + (g-\boldsymbol{n}\cdot\mathbf{grad}\,u_{\mathcal{P}},v)_{\Gamma_K}\end{aligned}$$

where $R_{K,\gamma}$ is given by (5.5). Arguing as in the proof of Lemma 5.1 we obtain

$$(P_Kf,v)_K + \sum_{\gamma\in\mathcal{E}_K}(R_{K,\gamma},v)_\gamma = (\boldsymbol{\sigma}_K,\mathbf{grad}\,v)_K \le \|\boldsymbol{\sigma}_K\|_{\boldsymbol{L}_2(K)}\,|\!|\!|v|\!|\!|_K \tag{8.20}$$

and that

$$(f-P_Kf,v)_K \le C_Kh_K\,\|f-P_Kf\|_{L_2(K)}\,|\!|\!|v|\!|\!|_K \tag{8.21}$$

by (3.7). It remains to bound the contribution from the sliver. Since,

$$(\mathbf{grad}\, u_{\mathcal{P}}, \mathbf{grad}\, v)_{S_K} = (\boldsymbol{n}\cdot\mathbf{grad}\, u_{\mathcal{P}}, v)_{\Gamma_K} - \left(\boldsymbol{n}^K_{\gamma_K}\cdot\mathbf{grad}\, u_{\mathcal{P}}, v\right)_{\gamma_K}$$

and $\boldsymbol{n}^K_{\gamma_K}\cdot\mathbf{grad}\, u_{\mathcal{P}}$ is constant on γ_K we have that

$$-\boldsymbol{n}^K_{\gamma_K}\cdot\mathbf{grad}\, u_{\mathcal{P}} = -\frac{1}{|\gamma_K|}\left(\boldsymbol{n}\cdot\mathbf{grad}\, u_{\mathcal{P}}, 1\right)_{\Gamma_K} \text{ on } \gamma_K. \tag{8.22}$$

Now, (5.5), (3.6) and (8.22) imply

$$-\left(R_{K,\gamma_K}, v\right)_{\gamma_K} = -\left(\frac{1}{|\gamma_K|}\left((g-\boldsymbol{n}\cdot\mathbf{grad}\, u_{\mathcal{P}}, 1)_{\Gamma_K} + (f,1)_{S_K}\right), v\right)_{\gamma_K}$$

and hence

$$\begin{aligned}(f,v)_{S_K} &+ (g-\boldsymbol{n}\cdot\mathbf{grad}\, u_{\mathcal{P}}, v)_{\Gamma_K} - \left(R_{K,\gamma_K}, v\right)_{\gamma_K}\\ &= \left(f, v-\langle v\rangle_{\gamma_K}\right)_{S_K} + \left(R_{K,\Gamma}, v-\langle v\rangle_{\gamma_K}\right)_{\Gamma_K}\end{aligned} \tag{8.23}$$

where $R_{K,\Gamma}$ is given by (5.6).

Consequently, since $\|v\|_K \le \|v\|_{K^*}$, from (8.20), (8.21) and Lemma 2.4 we have the bound claimed.

8.6. **Proof of Lemma 5.5.** Let

$$J_\gamma = \begin{cases} \frac{1}{2}\left(\boldsymbol{n}^K_\gamma\cdot\mathbf{grad}\, u_{\mathcal{P}|K} + \boldsymbol{n}^{K'}_\gamma\cdot\mathbf{grad}\, u_{\mathcal{P}|K'}\right) & \text{if } \gamma\in\mathcal{E}_K\cap\mathcal{E}_{K'} \text{ for distinct } K,K'\in\mathcal{P}\\ g_{K,\gamma} - \boldsymbol{n}^K_\gamma\cdot\mathbf{grad}\, u_{\mathcal{P}|K} & \text{if } \gamma\in\mathcal{E}_K\cap\mathcal{E}_B \text{ for } K\in\mathcal{P}.\end{cases}$$

Integrating (5.1) by parts allows us to say that

(8.24)

$$\begin{aligned}(\mathbf{grad}\, e, \mathbf{grad}\, v)_\Omega = \sum_{K\in\mathcal{P}_0}&\left((f,v)_K + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_B}(g-P_\gamma g, v)_\gamma + \sum_{\gamma\in\mathcal{E}_K}(J_\gamma, v)_\gamma\right)\\ &+\sum_{K\in\mathcal{P}_+\cup\mathcal{P}_-}\left((f,v)_{K^*} + (R_{K,\Gamma}, v)_{\Gamma_K} + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_I}(J_\gamma, v)_\gamma\right)\end{aligned}$$

for all $v\in H^1(\Omega)$. Applying standard "bubble function" arguments [3, 19] yields

$$h_K\left\|P_K f\right\|_{L_2(K)} \le C\left(\|e\|_K + h_K\left\|f-P_K f\right\|_{L_2(K)}\right) \tag{8.25}$$

for $K\in\mathcal{P}_0\cup\mathcal{P}_+$. For $\gamma\in\mathcal{E}_I$, let

$$\mathcal{P}_\gamma = \{K\in\mathcal{P} : \gamma\in\mathcal{E}_K\}.$$

Applying these "bubble function" arguments but with K^* in place of K when $K\in\mathcal{P}_+\cup\mathcal{P}_-$ and Γ_K in place of γ when $\gamma\in\mathcal{E}_K\cap(\mathcal{E}_+\cup\mathcal{E}_-)$ yields

$$h_K\left\|\langle f\rangle_{K^*}\right\|_{L_2(K^*)} \le C\left(\|e\|_{K^*} + h_K\left\|f-\langle f\rangle_{K^*}\right\|_{L_2(K^*)}\right) \tag{8.26}$$

for $K\in\mathcal{P}_+\cup\mathcal{P}_-$;

$$\begin{aligned}h^{1/2}_{K'}\left\|J_\gamma\right\|_{L_2(\gamma)} \le C&\left(\sum_{K\in\mathcal{P}_\gamma\cap(\mathcal{P}_0\cup\mathcal{P}_+)}\left(\|e\|_K + h_K\left\|f-P_K f\right\|_{L_2(K)}\right)\right.\\ &\left.+\sum_{K\in\mathcal{P}_\gamma\cap\mathcal{P}_-}\left(\|e\|_{K^*} + h_K\left\|f-\langle f\rangle_{K^*}\right\|_{L_2(K^*)}\right)\right)\end{aligned} \tag{8.27}$$

for $\gamma\in\mathcal{E}_I$;

$$h^{1/2}_K\left\|J_\gamma\right\|_{L_2(\gamma)} \le C\left(\|e\|_K + h_K\left\|f-P_K f\right\|_{L_2(K)} + h^{1/2}_K\left\|g-P_\gamma g\right\|_{L_2(\gamma)}\right) \tag{8.28}$$

for $K \in \mathcal{P}_0$ and $\gamma \in \mathcal{E}_K \cap \mathcal{E}_B$; and

$$
\begin{aligned}
&h_K \left|\langle R_{K,\Gamma}\rangle_{\Gamma_K}\right|_{L_2(\Gamma_K)} \\
&\le C\left(\|e\|_{K^*} + h_K \|f - \langle f\rangle_{K^*}\|_{L_2(K^*)} + h_K^{1/2} \left\|R_{K,\Gamma} - \langle R_{K,\Gamma}\rangle_{\Gamma_K}\right\|_{L_2(\Gamma_K)}\right)
\end{aligned}
\tag{8.29}
$$

for $K \in \mathcal{P}_+ \cup \mathcal{P}_-$.

Observing that

$$\|\boldsymbol{\sigma}_K\|_{\boldsymbol{L}_2(K)} \le \left\| \sum_{\gamma\in\mathcal{E}_K} \boldsymbol{\sigma}_K^\gamma \right\|_{\boldsymbol{L}_2(K)}$$

we can apply standard inequalities and scaling arguments to conclude that

$$\|\boldsymbol{\sigma}_K\|_{\boldsymbol{L}_2(K)} \le C h_K^{1/2} \sum_{\gamma\in\mathcal{E}_K} \|R_{K,\gamma}\|_{L_2(\gamma)}.$$

Using Lemma 7.1 then gives

$$\|\boldsymbol{\sigma}_K\|_{\boldsymbol{L}_2(K)} \le C\Phi_K. \tag{8.30}$$

When $K \in \mathcal{P}_0$, (5.19) then follows by noting that $C_K \le C$ and $C^K_{\gamma,K} \le C$ and so

$$C_K h_K \|f - P_K f\|_{L_2(K)} + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_\Gamma\cap\mathcal{E}_0} C^K_{\gamma,K} h_K^{1/2} \|g - P_\gamma g\|_{L_2(\gamma)} \le C\Phi_K.$$

When $K \in \mathcal{P}_+$, applying (2.13), (2.7), (2.6) and (8.29) and using the fact that $C^{K^*}_{\Gamma_K,K^*} \le C$ and $C^{K^*}_{\gamma_K,K} \le C$ we have that

$$
\begin{aligned}
&\left(|S_K|^{1/2} + \left(C^{K^*}_{\Gamma_K,K^*}|\gamma_K|^{1/2} + C^{K^*}_{\gamma_K,K}|\Gamma_K|^{1/2}\right) h_{K^*}^{1/2}\operatorname{osc}(\Gamma_K)\right)\left|\langle R_{K,\Gamma}\rangle_{\Gamma_K}\right| \\
&\le C\left(h_K \|e\|_{K^*} + h_K^2 \|f - \langle f\rangle_{K^*}\|_{L_2(K^*)} + h_K^{3/2}\left\|R_{K,\Gamma} - \langle R_{K,\Gamma}\rangle_{\Gamma_K}\right\|_{L_2(\Gamma_K)}\right) \\
&\le C\left(\|e\|_{K^*} + h_K \|f - \langle f\rangle_{K^*}\|_{L_2(K^*)} + h_K^{1/2}\left\|R_{K,\Gamma} - \langle R_{K,\Gamma}\rangle_{\Gamma_K}\right\|_{L_2(\Gamma_K)}\right) \le C\Phi_K
\end{aligned}
\tag{8.31}
$$

since h_K must be bounded by the diameter of Ω. Moreover,

$$
\begin{aligned}
\left|\langle f\rangle_{S_K}\right| &= \frac{1}{|S_K|}\left|(\langle f\rangle_{K^*} + f - \langle f\rangle_{K^*}, 1)_{S_K}\right| \\
&\le \frac{1}{|S_K|^{1/2}}\left(\|\langle f\rangle_{K^*}\|_{S_K} + \|f - \langle f\rangle_{K^*}\|_{S_K}\right) \\
&\le \frac{1}{|S_K|^{1/2}}\left(\|\langle f\rangle_{K^*}\|_{K^*} + \|f - \langle f\rangle_{K^*}\|_{S_K}\right).
\end{aligned}
$$

Hence, applying (2.7), (2.9), (2.6) and (8.29) and using the fact that $C^{K^*}_{\gamma_K,K} \le C$ we have that

$$
\begin{aligned}
\frac{C^{K^*}_{\gamma_K,K} h_{K^*}^{1/2}}{|\gamma_K|^{1/2}} |S_K| \left|\langle f\rangle_{S_K}\right| &\le C\left(h_K^{1/2}\|e\|_{K^*} + h_K^{3/2}\|f - \langle f\rangle_{K^*}\|_{L_2(K^*)}\right) \\
&\le C\left(\|e\|_{K^*} + h_K\|f - \langle f\rangle_{K^*}\|_{L_2(K^*)}\right) \le C\Phi_K
\end{aligned}
\tag{8.32}
$$

since h_K must be bounded by the diameter of Ω. Consequently, we arrive at (5.19) when $K \in \mathcal{P}_+$ by using (8.30), (8.31) and (8.32) and noting that $C_K \le C$, $C_{K^*} \le C$ and $C^{K^*}_{\Gamma_K,K^*} \le C$ and so

$$
\begin{aligned}
&C_K h_K \|f - P_K f\|_{L_2(K)} + C_{K^*} h_{K^*} \|f - \langle f\rangle_{K^*}\|_{L_2(S_K)} \\
&\qquad + C^{K^*}_{\Gamma_K,K^*} h_{K^*}^{1/2}\left\|R_{K,\Gamma} - \langle R_{K,\Gamma}\rangle_{\Gamma_K}\right\|_{L_2(\Gamma_K)} \le C\Phi_K
\end{aligned}
$$

upon using (2.9).

When $K \in \mathcal{P}_-$, we have that

$$\begin{aligned}&\|R_{K,\Gamma} - \boldsymbol{n}_\Gamma \cdot \boldsymbol{\sigma}_K\|_{L_2(\Gamma_K)}\\ &\leq \left\|R_{K,\Gamma} - \langle R_{K,\Gamma}\rangle_{\Gamma_K}\right\|_{L_2(\Gamma_K)} + \left\|\boldsymbol{n}_\Gamma \cdot \boldsymbol{\sigma}_K - \langle R_{K,\Gamma}\rangle_{\Gamma_K}\right\|_{L_2(\Gamma_K)}.\end{aligned}$$

Now, (8.19) means that

$$\begin{aligned}\left\|\boldsymbol{n}_\Gamma \cdot \boldsymbol{\sigma}_K - \langle R_{K,\Gamma}\rangle_{\Gamma_K}\right\|_{L_2(\Gamma_K)} &= \left\|\boldsymbol{n}_\Gamma \cdot \boldsymbol{\sigma}_K - \langle \boldsymbol{n}_\Gamma \cdot \boldsymbol{\sigma}_K\rangle_{\Gamma_K}\right\|_{L_2(\Gamma_K)}\\ &\leq \|\boldsymbol{n}_\Gamma \cdot \boldsymbol{\sigma}_K\|_{L_2(\Gamma_K)}.\end{aligned}$$

Moreover,

$$\|\boldsymbol{n}_\Gamma \cdot \boldsymbol{\sigma}_K\|_{L_2(\Gamma_K)} \leq \|\boldsymbol{\sigma}_K\|_{\boldsymbol{L}_2(\Gamma_K)} \leq C h_{K^*}^{-1/2} \|\boldsymbol{\sigma}_K\|_{\boldsymbol{L}_2(K^*)}$$

by the equivalence of norms on finite dimensional spaces and scaling arguments. Consequently, upon using (2.9) and the fact that $C_{\Gamma_K,K^*}^{K^*} \leq C$ we have that

$$\begin{aligned}&C_{\Gamma_K,K^*}^{K^*} h_{K^*}^{1/2} \|R_{K,\Gamma} - \boldsymbol{n}_\Gamma \cdot \boldsymbol{\sigma}_K\|_{L_2(\Gamma_K)}\\ &\leq C\left(\|\boldsymbol{\sigma}_K\|_{\boldsymbol{L}_2(K)} + h_{K^*}^{1/2}\left\|R_{K,\Gamma} - \langle R_{K,\Gamma}\rangle_{\Gamma_K}\right\|_{L_2(\Gamma_K)}\right).\end{aligned} \tag{8.33}$$

Hence, we can arrive at (5.19) when $K \in \mathcal{P}_-$ by observing that $\|\boldsymbol{\sigma}_K\|_{\boldsymbol{L}_2(K^*)} \leq \|\boldsymbol{\sigma}_K\|_{\boldsymbol{L}_2(K)}$ and $C_{K^*} h_{K^*} \|f - P_K f\|_{L_2(K^*)} \leq C\Phi_K$ and using (8.30) and (8.33).

8.7. **Proof of Lemma 7.1.** For $K' \in \mathcal{P}$ and $\gamma \in \mathcal{E}_{K'}$, the definitions of $R_{K',\gamma}$, $A_\gamma^{K'}$ and J_γ mean that

$$\|R_{K',\gamma}\|_{L_2(\gamma)} = \left\|g_{K',\gamma} - A_\gamma^{K'} - J_\gamma\right\|_{L_2(\gamma)} \leq \left\|g_{K',\gamma} - A_\gamma^{K'}\right\|_{L_2(\gamma)} + \|J_\gamma\|_{L_2(\gamma)}.$$

Since $g_{K',\gamma} - A_\gamma^{K'}$ is affine we can say that

$$h_{K'}^{1/2}\left\|g_{K',\gamma} - A_\gamma^{K'}\right\|_{L_2(\gamma)} \leq C \sum_{i\in\mathcal{V}(\gamma)} \left|\left(g_{K',\gamma} - A_\gamma^{K'}, \lambda_i\right)_\gamma\right|$$

and as in [3] we have that

$$\left|\left(g_{K',\gamma} - A_\gamma^{K'}, \lambda_i\right)_\gamma\right| \leq C \sum_{K\in\mathcal{P}_i} |\Delta_K(\lambda_i)|.$$

Now, integration by parts yields

$$\begin{aligned}&|\Delta_K(\lambda_i)|\\ &= \left|\sum_{\gamma\in\mathcal{E}_K} \left(\boldsymbol{n}_\gamma^K \cdot \mathbf{grad}\, u_{\mathcal{P}} - A_\gamma^K, \lambda_i\right)_\gamma - (f, \lambda_i)_K\right|\\ &= \left|\sum_{\gamma\in\mathcal{E}_K} (J_\gamma, \lambda_i)_\gamma - (P_K f, \lambda_i)_K\right|\\ &\leq \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_i} \|J_\gamma\|_{L_2(\gamma)} \|\lambda_i\|_{L_2(\gamma)} + \|P_K f\|_{L_2(K)} \|\lambda_i\|_{L_2(K)}\\ &\leq C\left(\sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_i} h_K^{1/2} \|J_\gamma\|_{L_2(\gamma)} + h_K \|P_K f\|_{L_2(K)}\right).\end{aligned}$$

Now, for $K \in \mathcal{P}_+$, (3.6) and (8.22) allow us to write

$$J_{\gamma_K} = \frac{1}{|\gamma_K|}\left((R_{K,\Gamma}, 1)_{\Gamma_K} + (f, 1)_{S_K}\right).$$

Similarly for, for $K \in \mathcal{P}_-$, (3.6) and (8.18) allow us to write

$$J_{\gamma_K} = \frac{1}{|\gamma_K|}\left((R_{K,\Gamma}, 1)_{\Gamma_K} - (f, 1)_{S_K}\right).$$

Hence, we can say that, for $K \in \mathcal{P}_+ \cup \mathcal{P}_-$,

$$\begin{aligned} h_K^{1/2} \|J_\gamma\|_{L_2(\gamma_K)} \le & C\left(|\Gamma_K| \left|\langle R_{K,\Gamma}\rangle_{\Gamma_K}\right| + |S_K|^{1/2} \|f\|_{L_2(S_K)}\right) \\ \le & C\left(h_K \left|\langle R_{K,\Gamma}\rangle_{\Gamma_K}\right| + h_K^{3/2} \|f\|_{L_2(S_K)}\right) \end{aligned}$$

by (2.7) and (2.9). Moreover, since the boundedness of the domain Ω means that $h_K^3 \le C h_K^2$ which in turn means that $|S_K| \le C\,|K^*|$, we can say that

$$\begin{aligned} \|f\|_{L_2(S_K)} \le & \|\langle f\rangle_{K^*}\|_{L_2(S_K)} + \|f - \langle f\rangle_{K^*}\|_{L_2(S_K)} \\ \le & C\,\|\langle f\rangle_{K^*}\|_{L_2(K^*)} + \|f - \langle f\rangle_{K^*}\|_{L_2(S_K)} \end{aligned}$$

and similarly, (2.8) means that

$$\|P_K f\|_{L_2(K)} \le \|f\|_{L_2(K)} \le C\,\|\langle f\rangle_{K^*}\|_{L_2(K^*)} + \|f - \langle f\rangle_{K^*}\|_{L_2(K)}.$$

Consequently,

$$\begin{aligned} & h_{K'} \|R_{K',\gamma}\|_{L_2(\gamma)} \\ \le & C \sum_{i\in\mathcal{V}(\gamma)} \left(\sum_{K\in\mathcal{P}_i\cap\mathcal{P}_0} \left(h_K^{1/2} \sum_{\gamma'\in\mathcal{E}_K\cap\mathcal{E}_i} \|J_{\gamma'}\|_{L_2(\gamma')} + h_K \|P_K f\|_{L_2(K)} \right) \right. \\ & + \sum_{K\in\mathcal{P}_i\cap(\mathcal{P}_+\cup\mathcal{P}_-)} \left(h_K^{1/2} \sum_{\gamma'\in\mathcal{E}_K\cap\mathcal{E}_i\cap\mathcal{E}_I} \|J_{\gamma'}\|_{L_2(\gamma')} + h_K \left|\langle R_{K,\Gamma}\rangle_{\Gamma_K}\right| \right. \\ & \left.\left. + h_K \|\langle f\rangle_{K^*}\|_{L_2(K^*)} + h_K \|f - \langle f\rangle_{K^*}\|_{L_2(K)} + h_K^{3/2} \|f - \langle f\rangle_{K^*}\|_{L_2(S_K)} \right)\right). \end{aligned}$$

The result then follows upon using (8.25), (8.26), (8.27), (8.28) and (8.29).

8.8. **Proof of Lemma 7.2.** It is relatively straightforward to show that

$$\boldsymbol{n}_j \cdot \boldsymbol{\sigma}_K^{\gamma_i} = R_{K,\gamma_i}\delta_{ij} \text{ on } \gamma_j \text{ for } i, j = 1, 2, 3;$$

$$\boldsymbol{n}_j \cdot \boldsymbol{\sigma}_K^0 = 0 \text{ on } \gamma_j \text{ for } j = 1, 2, 3;$$

and

$$(\boldsymbol{\sigma}_K^{\gamma_i}, \mathbf{grad}\, p)_K = \left(\boldsymbol{\sigma}_K^0, \mathbf{grad}\, p\right)_K = 0 \text{ for all } p \in \mathbb{P}_1(K) \text{ for } i = 1, 2, 3.$$

It then follows that

$$\boldsymbol{\sigma}_K = \sum_{i=1}^{3} \boldsymbol{\sigma}_K^{\gamma_i} - \frac{1}{(\boldsymbol{\sigma}_K^0, \boldsymbol{\sigma}_K^0)_K} \sum_{i=1}^{3} \left(\boldsymbol{\sigma}_K^{\gamma_k}, \boldsymbol{\sigma}_K^0\right)_K \boldsymbol{\sigma}_K^0$$

satisfies

$$\boldsymbol{n}_i \cdot \boldsymbol{\sigma}_K = R_{K,\gamma_i} \text{ on } \gamma_i \text{ for } i = 1, 2, 3$$

and

$$-\operatorname{div} \boldsymbol{\sigma}_K = P_K f \text{ in } K$$

as desired. The fact that $\|\boldsymbol{\sigma}_K\|_{\boldsymbol{L}_2(K)}$ is minimised over $\mathbb{P}_2(K) \times \mathbb{P}_2(K)$ follows upon observing that any function whose normal components vanishes on all of the edges of K and whose divergence is zero in K must be a multiple of $\boldsymbol{\sigma}_K^0$ and that $\left(\boldsymbol{\sigma}_K, \boldsymbol{\sigma}_K^0\right)_K = 0$.

References

1. M. Ainsworth, *A posteriori error estimation for discontinuous Galerkin finite element approximation*, SIAM J. Numer. Anal. **45** (2007), no. 4, 1777–1798 (electronic). MR MR2338409
2. ———, *A framework for obtaining guaranteed error bounds for finite element approximations*, J. Comput. Appl. Math. **234** (2010), no. 9, 2618–2632.
3. M. Ainsworth and J. T. Oden, *A Posteriori Error Estimation in Finite Element Analysis*, Pure and Applied Mathematics, Wiley-Interscience, John Wiley & Sons, New York, 2000. MR 2003b:65001
4. M. Ainsworth and R. Rankin, *Guaranteed computable bounds on quantities of interest in finite element computations*, Internat. J. Numer. Methods Engrg. **89** (2012), no. 13, 1605–1634.
5. J. W. Barrett and C. M. Elliott, *A finite-element method for solving elliptic equations with Neumann data on a curved boundary using unfitted meshes*, IMA J. Numer. Anal. **4** (1984), no. 3, 309–325. MR 752608 (85h:65231)
6. ———, *A practical finite element approximation of a semi-definite Neumann problem on a curved domain*, Numer. Math. **51** (1987), no. 1, 23–36.
7. J. J. Blair, *Bounds for the change in the solutions of second order elliptic PDE's when the boundary is perturbed*, SIAM J. Appl. Math. **24** (1973), 277–285. MR 0317557 (47 #6104)
8. C. Carstensen and S. A. Sauter, *A posteriori error analysis for elliptic PDEs on domains with complicated structures*, Numer. Math. **96** (2004), no. 4, 691–721. MR 2036362 (2004k:65212)
9. R. Cottereau, P. Díez, and A. Huerta, *Strict error bounds for linear solid mechanics problems using a subdomain-based flux-free method*, Comput. Mech. **44** (2009), no. 4, 533–547. MR 2520189 (2010g:74083)
10. W. Dörfler and M. Rumpf, *An adaptive strategy for elliptic problems including a posteriori controlled boundary approximation*, Math. Comp. **67** (1998), no. 224, 1361–1382. MR 1489969 (99b:65141)
11. L. A. Oganesyan, *Convergence of difference schemes with improved boundary approximation*, U.S.S.R. Computational Math. and Math. Phys. **6** (1966), no. 6, 116–134.
12. N. Parés, J. Bonet, A. Huerta, and J. Peraire, *The computation of bounds for linear-functional outputs of weak solutions to the two-dimensional elasticity equations*, Comput. Methods Appl. Mech. Engrg. **195** (2006), no. 4-6, 406–429. MR 2186143 (2006i:74024)
13. N. Parés, H. Santos, and P. Díez, *Guaranteed energy error bounds for the Poisson equation using a flux-free approach: solving the local problems in subdomains*, Internat. J. Numer. Methods Engrg. **79** (2009), no. 10, 1203–1244. MR 2560964 (2010j:65240)
14. L. E. Payne and H. F. Weinberger, *An optimal Poincaré inequality for convex domains*, Arch. Rational Mech. Anal. **5** (1960), 286–292. MR MR0117419 (22 #8198)
15. A. M. Sauer-Budge, J. Bonet, A. Huerta, and J. Peraire, *Computing bounds for linear functionals of exact weak solutions to Poisson's equation*, SIAM J. Numer. Anal. **42** (2004), no. 4, 1610–1630 (electronic). MR MR2114293 (2006f:65109)
16. G. Strang and G. J. Fix, *An analysis of the finite element method*, Prentice-Hall Inc., Englewood Cliffs, N. J., 1973, Prentice-Hall Series in Automatic Computation. MR 0443377 (56 #1747)
17. V. Thomée, *Approximate solution of Dirichlet's problem using approximating polygonal domains*, Topics in numerical analysis (Proc. Roy. Irish Acad. Conf., University Coll., Dublin, 1972), Academic Press, London, 1973, pp. 311–328. MR 0349034 (50 #1528)
18. A. Thrun, *über die konstanten in poincaréschen ungleichungen*, Masters thesis, Ruhr University, 2003.
19. R. Verfürth, *A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques*, Wiley-Teubner, 1996.
20. J. Wloka, *Partial differential equations*, Cambridge University Press, Cambridge, 1987, Translated from the German by C. B. Thomas and M. J. Thomas. MR 895589 (88d:35004)

Division of Applied Mathematics, Brown University, 182 George Street, Providence, Rhode Island 02912

E-mail address: Mark_Ainsworth@brown.edu

Department of Computational and Applied Mathematics, Rice University, 6100 Main Street, Houston, Texas 77005

E-mail address: Richard.A.Rankin@rice.edu