ON CONVERGENT SCHEMES FOR DIFFUSE INTERFACE MODELS FOR TWO-PHASE FLOW OF INCOMPRESSIBLE FLUIDS WITH GENERAL MASS DENSITIES

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ABSTRACT. We are concerned with convergence results for fully discrete finite-element schemes suggested in [Grün, Klingbeil, ArXiv e-prints (2012), arXiv:1210.5088]. They were developed for the diffuse-interface model in [Abels, Garcke, Grün, M3AS, 2012, DOI:10.1142/S0218202511500138] which is to describe two-phase flow of immiscible, incompressible viscous fluids. We formulate general conditions on discretization spaces and projection operators which allow to prove compactness of discrete solutions with respect to both time and space and which hence permit to establish convergence of the scheme to a generalized solution. We identify a simple quantitative and physical criterion to decide whether this generalized solution is in fact a weak solution. In this case, our analysis provides another pathway to establish existence of weak solutions to the aforementioned model in two and in three space dimensions. Our argument is particularly based on higher regularity results for discrete solutions to convective Cahn-Hilliard equations and on discrete versions of Sobolev's embedding theorem.

1. INTRODUCTION

In this paper, we prove convergence of a fully discrete finite-element scheme for a recently suggested diffuse interface model for two-phase flow of incompressible, viscous fluids with different mass densities. The model was introduced by Abels, Garcke, and the author of this paper in [4]. To the best of our knowledge, it is the only model so far which complies with physical principles like consistency with thermodynamics and frame-indifference and which allows at the same time for a solenoidal velocity field. It reads as follows.

$$\bar{\rho}(\varphi)\partial_t\mathbf{v} + \left(\left(\bar{\rho}(\varphi)\mathbf{v} + \frac{\partial\bar{\rho}(\varphi)}{\partial\varphi}\mathbf{j}\right)\cdot\nabla\right)\mathbf{v} - \nabla\cdot(2\eta(\varphi)\mathbf{D}\mathbf{v}) + \nabla p = \mu\nabla\varphi + \mathbf{k}_{\text{grav}}, \quad (1.1a)$$

$$\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi - \nabla\cdot(M(\varphi)\nabla\mu) = 0, \quad (1.1b)$$

$$\mu = \sigma(-\Delta\varphi + F'(\varphi)), \quad (1.1c)$$

$$\nabla\cdot\mathbf{v} = 0 \quad \text{in } \Omega\times(0,T). \quad (1.1d)$$

As boundary conditions, no-slip conditions for $\mathbf{v}$ and zero normal derivatives of φ and of μ on $\partial\Omega\times(0,T)$ are imposed.
Note that system (1.1) couples a hydrodynamic momentum equation with a Cahn-Hilliard type phase-field equation. F is a double-well potential with minima in ±1 - representing the pure phases $\varphi\equiv\pm1$. The parameter σ is the surface tension coefficient, which is assumed to be $\sigma=1$ in this paper. The term μ stands for the so called chemical potential, and the order parameter φ stands for the difference of the volume fractions u_2-u_1 where $u_i(x,t) := \frac{\rho_i(x,t)}{\tilde{\rho}_i}$ with $\tilde{\rho}_i$ the specific (constant) density of fluid i in a unmixed setting.

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Denoting the individual velocities by $\mathbf{v}_i$, $i = 1, 2$, we write $\mathbf{v} := u_1\mathbf{v}_1 + u_2\mathbf{v}_2$ for the volume averaged velocity. Assuming $\tilde{\rho}_2 \geq \tilde{\rho}_1$, the density of the total mass $\bar{\rho}(\varphi)$ is given by

$$\bar{\rho}(\varphi) = \frac{\tilde{\rho}_2 + \tilde{\rho}_1}{2} + \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2}\varphi, \tag{1.2}$$

and $\mathbf{Dv}$ denotes the symmetrized gradient. The term $\mathbf{k}_{\text{grav}}$ stands for the density of external volume forces. Finally, the flux $\mathbf{j}$ is defined by $\mathbf{j} := -M(\varphi)\nabla\mu$ where $M(\varphi)$ is the mobility.
System (1.1) is consistent with thermodynamics in the sense that the total energy (i.e. the sum of the kinetic and the interfacial energy) at a time $t_2 > t_1$ is bounded by the sum of the total energy at time t_1 and the work done by external forces during the time interval (t_1, t_2). More precisely,

$$\begin{aligned} &\tfrac{1}{2}\int_\Omega \bar{\rho}(\varphi(t_2))\left|\mathbf{v}\right|^2(t_2) + \tfrac{1}{2}\int_\Omega \left|\nabla\varphi\right|^2(t_2) + \int_\Omega F(\varphi(t_2)) \\ &\quad + \int_{t_1}^{t_2}\int_\Omega M(\varphi)\left|\nabla\mu\right|^2 + \int_{t_1}^{t_2}\int_\Omega 2\eta(\varphi)\left|\mathbf{Dv}\right|^2 \\ &\qquad = \tfrac{1}{2}\int_\Omega \bar{\rho}(\varphi(t_1))\left|\mathbf{v}\right|^2(t_1) + \tfrac{1}{2}\int_\Omega \left|\nabla\varphi\right|^2(t_1) + \int_\Omega F(\varphi(t_1)) + \int_{t_1}^{t_2}\int_\Omega \langle \mathbf{k}_{grav}, \mathbf{v}\rangle. \end{aligned} \tag{1.3}$$

The model (1.1) has already been the subject of further mathematical investigations. In [2], Abels, Depner, and Garcke prove existence of weak solutions for the case of logarithmic potentials, in [3], they consider mobilities $M(\varphi)$ which degenerate in $\varphi = \pm 1$. In [22], Klingbeil and the author of the present paper suggest a numerical scheme for (1.1) which is discretely consistent with thermodynamics in the sense that in the absence of external forces the discrete counterpart of the total energy is decreasing in time. Various numerical experiments underline the full practicality of this approach – see [22]. In the benchmark paper [5] on Taylor-Flow, the method of [22] was successfully validated by comparison with physical experiments and different numerical approaches.

Diffuse interface models for two-phase flow of incompressible viscous fluids began to interest mathematicians some ten years ago while the basic concept of coupling momentum equations with the Cahn-Hilliard equation had been suggested much earlier – see the famous "Model H" of Halperin and Hohenberg [25]. Two advantages of diffuse interface models compared to other approaches like sharp-interface models or volume-of-fluid-methods are well known. First, no artificial additional conditions are necessary to model topology changes or to guarantee conservation of individual masses. Secondly, in many cases it is possible to prove global existence of solutions and to formulate convergent numerical schemes.
Let us concentrate on the numerical aspects of diffuse interface models – for an overview of analytical results, we refer the reader to [1], [2], [3], and the references therein. Many authors contributed already to the numerics of diffuse interface models for two-phase flow in the special case that the two fluids share the same mass density. In this case, one has to deal essentially with a coupling of the Navier-Stokes system with a Cahn-Hilliard equation. To obtain a first impression of the numerical approaches suggested so far, we refer to [16], [9], [26], [27], [28], and the references therein.
Concerning numerical analysis, we mention the papers by Feng [16] and by Kay, Styles, and Welford [27]. The former one focuses on P_2P_0-elements, assumes a double-well potential $F(\varphi) := (1-\varphi^2)^2$, and establishes convergence of discrete solutions to the Navier-Stokes-Cahn-Hilliard system in two and three space dimensions. The latter one studies

P_1-ISO-$P_2 - P_1$-elements and obtains comparable convergence results – assuming the same smooth double-well potential as Feng [16]. Note in particular that in both papers neither discrete nor continuous solutions are confined to the interval $[-1, 1]$. This is due to the choice of the double-well potential and due to the fact that degenerate mobilities are not considered.

The case of different mass densities to be studied in this paper is conceptually much different. Various models were proposed to extend model H also to the case of mass density contrast (see [8] and the references therein). Lowengrub and Truskinovsky proposed in [31] for the first time a diffuse-interface model consistent with thermodynamics. The gross velocity field is obtained by mass averaging of individual velocities. As a consequence, it is not divergence free, and the pressure p enters the model as an essential unknown. However, no energy estimates are available to control p. Moreover, the pressure enters the chemical potential and is hence strongly coupled to the phase-field equation. This intricate coupling may be one reason why so far it has not been possible to formulate numerical schemes for model [31].

Ding et al. [15] suggested to define the gross velocity field by volume averaging. Prohibiting in addition volume changes due to mixing ("simple mixture assumption"), the gross velocity field is solenoidal. To the best of our knowledge, however, all attempts failed to establish energy inequalities and to show that the model in [15] is consistent with thermodynamics.

In [32], Shen and Yang propose an extension of the model [15] which allows for energy estimates. Their modeling ansatz is to add a multiple of the term $\rho_t + \operatorname{div}(\rho \mathbf{v})$ in the momentum equation. They justify this idea by the assertion that the continuity equation $\rho_t + \operatorname{div}(\rho \mathbf{v}) = 0$ were valid and therefore this term were zero. Nevertheless, the phase-field equation $\varphi_t + \operatorname{div}(\varphi \mathbf{v}) - \operatorname{div} \mathbf{j} = 0$ is also part of their model, and ρ depends in an affine-linear way on φ.

A third strategy was pursued by Boyer [9], allowing also for solenoidal vector fields, but apparently not for energy estimates.

The papers [9], [15], [32] present numerical simulations, too. Kim and Lowengrub [29] suggest numerical schemes for multi-phase flow, and Aland and Voigt [6] present first results on the comparison of different diffuse interface models.

In all these papers, numerical analysis of the proposed schemes has not been performed. As discrete counterparts of an energy estimate seem to be a prerequisite for convergence results, we concentrate here on the fully discrete finite-element scheme which was introduced by Klingbeil and the author in [22], formula (3.2), and which allows for such an estimate.

It is the scope of this paper to prove the convergence of discrete solutions obtained by the scheme in [22] in two and in three space dimensions. This way, a different pathway to the existence of solutions in the continuous setting is suggested as well. It is important to emphasize that our approach is different from the methods of [2] and of [3]. Indeed, both papers rely on the Leray-Schauder principle and on discretizations only with respect to time. Therefore, in this setting the coupling term $(\mathbf{j} \cdot \nabla)\mathbf{v}$ in (1.1a) does not cause such intricacies related to compactness in time and to the identification of weak limits as we will encounter them in the fully discrete setting. This is one reason why the numerical analysis in the sequel is confined to the case of a constant mobility $M(\varphi)$ and of a double-well potential F with p-growth – where p can be chosen in $[1, \infty)$ for the case of two space dimensions and in $[1, 4)$ for the case of three space dimensions. It is worth mentioning that the papers [16] and [27] devoted to the case of identical mass densities assume in three dimensions comparable and in two dimensions even stricter conditions on

the growth of F. In particular, they do not study degenerate mobilities, neither.
Moreover, it is important to mention that due to (1.2) a mechanism is needed which bounds $\bar\rho$ strictly away from zero. Since the Cahn-Hilliard equation is fourth-order parabolic, comparison principles do not hold. Therefore, bounds on φ rely on integral estimates. In the presence of external forces, however, the total energy is in general not expected to decrease in time. Hence, regularizations of degenerate mobilities (cf. [23]) or regularizations of singular (logarithmic) potentials would have to be chosen depending on the applied external forces.
Another approach is to modify the φ-dependency of $\bar\rho$, which will be pursued in this paper – see (H4) and Remark 2.1. It is interesting to note that for instance in [15] and in [32] the issue of definiteness of $\bar\rho$ does not seem to be addressed at all. This may be due to the fact that it is not expected to become relevant in many practical computations, as long as the Atwood number $\frac{\tilde\rho_2-\tilde\rho_1}{\tilde\rho_1+\tilde\rho_2}$ is not chosen too large, see also Remark 2.1 and Corollary 5.5.

The outline of the paper is as follows. In Section 2, we introduce the scheme of [22] and we set the frame for the subsequent analysis. In particular, Subsection 2.2 is devoted to the formulation of general conditions on discrete function spaces and projection operators which will be needed for the convergence proof. Examples of admissible finite elements are P_2P_0-elements and Taylor-Hood elements.
In Section 3, we prove a discrete version of the energy estimate and we establish existence of discrete solutions. Section 4 is the core of the paper. We prove for discrete solutions $(\varphi_{\tau h}, \mu_{\tau h}, \mathbf{v}_{\tau h})$ that $\partial_\tau^-\varphi_{\tau h}$, the backward difference quotient with respect to time, is uniformly bounded in $L^2(\Omega_T)$ and that the discrete Laplacians $\Delta_h\varphi_{\tau h}$ and $\Delta_h\mu_{\tau h}$ are uniformly bounded in $L^\infty((0,T);L^2(\Omega))$ and in $L^2(\Omega_T)$, respectively. Combining these results with appropriate discrete versions of Sobolev's embedding theorem (see Theorem 6.4), we succeed to prove that $\mathcal{R}_h(\rho_{\tau h}\mathbf{v}_{\tau h})$ strongly converges to $P_\sigma(\rho\mathbf{v})$ in $\mathbf{L}^2(\Omega_T)$ where P_σ is the Helmholtz projection and $\mathcal{R}_h$ is the orthogonal L^2-projection onto the space $\mathbf{W}_h$ of discretely divergence free velocity fields. Section 5 is devoted to the proof of the convergence of appropriate subsequences of $(\varphi_{\tau h}, \mu_{\tau h}, \mathbf{v}_{\tau h})$ to a generalized solution in the continuous setting. In particular, Corollary 5.4 shows that the generalized solution obtained is a weak solution to (1.1) if the phase-field φ in the continuous setting stays sufficiently close to $[-1,1]$. With a grain of salt, a sufficient condition is given by the requirement that the modulus of φ stays bounded by the inverse Atwood number, i.e. by $\frac{\tilde\rho_1+\tilde\rho_2}{\tilde\rho_2-\tilde\rho_1}$. In particular, for given initial data there is always a regime of Atwood numbers such that this condition is satisfied on appropriate time intervals, see Corollary 5.5.

Notation. We consider the two-phase problem on a bounded, convex polygonal (or polyhedral, respectively) domain $\Omega\subset\mathbb{R}^d$ in spatial dimensions $d\in\{2,3\}$. By $\langle\cdot,\cdot\rangle$, we denote the Euclidean scalar product on $\mathbb{R}^d$, and $(\cdot,\cdot)$ is used for the scalar product in $L^2(\Omega)$. Sometimes, we write Ω_T for the space-time cylinder $\Omega\times(0,T)$. By $W^{k,p}(\Omega)$, we denote the space of k-times weakly differentiable functions with weak derivatives in $L^p(\Omega)$. The symbol $W_0^{k,p}(\Omega)$ stands for the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. Corresponding spaces of vector-valued functions are denoted in boldface. Moreover, we use the function spaces $\mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega) := \{\mathbf{v}\in\mathbf{W}^{1,2}_0(\Omega)|\operatorname{div}\mathbf{v}=0\}$, $L^2_0(\Omega) := \{v\in L^2(\Omega)|\int_\Omega v=0\}$, $H^s(\Omega):=W^{s,2}(\Omega)$, and $H^1_*(\Omega):=H^1(\Omega)\cap L^2_0(\Omega)$.
For a Banach space X and a time interval I, the symbol $L^p(I;X)$ stands for the parabolic space of L^p-integrable functions on I with values in X. P_σ denotes the Helmholtz-projection from $\mathbf{L}^2(\Omega)$ onto the space of solenoidal vectorfields $\mathbf{H}_2(\Omega)$ which is obtained as the closure of the solenoidal smooth vector fields with compact support (see [18]). We

recall that P_σ is an orthogonal projection. We also write $v \in L^{p-}(S)$ as a short form meaning that $v \in L^q(S)$ for all $1 \le q < p$. The notation $\|v\|_{L^{p-}(S)}$ stands for $\|v\|_{L^{p-\varepsilon}(S)}$ for an arbitrary, but fixed $\varepsilon > 0$. Similarly, $\|v\|_{L^{p+}(S)}$ is a short form for $\|v\|_{L^{p+\varepsilon}(S)}$ for a sufficiently small, but fixed $\varepsilon > 0$. For further notation related to the discretization, we refer the reader to Subsection 2.2.

2. The scheme

2.1. **Discretization in space and time.** We assume $\mathcal{T}_h$ to be a regular and admissible triangulation of Ω with simplicial elements in the sense of [12]. Let us suppose in addition that the discretization is *rectangular* in the sense, that

(T1) for each simplicial element $E \in \mathcal{T}_h$, a vertex $x_0(E)$ exists such that the edges connecting $x_0(E)$ with vertices $x_i(E)$ and $x_j(E)$ are perpendicular to each other for $i, j \in \{1, \cdots, d\}$, $i \neq j$.

We will take advantage of (T1) in the proof of compactness in time, see Theorem 4.2. Note that (T1) does not exclude the applicability of standard strategies for local mesh refinement.

Concerning discretization with respect to time, we assume that

(T2) the time interval $I := [0, T)$ is subdivided in intervals $I_k = [t_k, t_{k+1})$ with $t_{k+1} = t_k + \tau_k$ for time increments $\tau_k > 0$ and $k = 0, \cdots, N-1$. For simplicity, we take $\tau_k \equiv \tau$ for $k = 0, \cdots, N-1$.

2.2. **Discrete function spaces and projection operators.** For the approximation of both the phase-field φ and the chemical potential μ, we introduce the space U_h of continuous, piecewise linear finite element functions on $\mathcal{T}_h$. The expression $\mathcal{I}_h$ stands for the nodal interpolation operator from $C^0(\Omega)$ to U_h defined by $\mathcal{I}_h u := \sum_{j=1}^{\dim U_h} u(x_j)\theta_j$, where the functions θ_j form a dual basis to the nodes x_j, i.e. $\theta_i(x_j) = \delta_{ij}$, $i, j = 1, \ldots, \dim U_h$. Let us furthermore introduce the well–known lumped masses scalar product corresponding to the integration formula

$$(\Theta, \Psi)_h := \int_\Omega \mathcal{I}_h(\Theta\Psi).$$

The diagonal, positive definite lumped masses matrix is given by $(M_h)_{ij} = (\varphi_i, \varphi_j)_h$. We recall the following well known estimates:

$$|(u_h, v_h) - (u_h, v_h)_h| \le C h^{1+l} \|u_h\|_l \|v_h\|_1 \qquad \text{for all } u_h, v_h \in U_h, \quad l = 0, 1, \tag{2.1}$$

where (u, v) denotes the L^2-scalar product on Ω. In the same spirit, there exist positive constants c, C such that we have for $|\cdot|_h := \sqrt{(\cdot,\cdot)_h}$:

$$c\,|\cdot|_h^2 \le (\cdot,\cdot) \le C\,|\cdot|_h^2\,. \tag{2.2}$$

We will use the Ritz projection $\mathcal{P}_h : H^1(\Omega) \to U_h$, defined by

$$\int_\Omega \langle \nabla \mathcal{P}_h v, \nabla \theta_j \rangle = \int_\Omega \langle \nabla v, \nabla \theta_j \rangle, \qquad j = 1, \cdots, \dim U_h.$$

We note the existence of a positive constant C such that

$$\|\mathcal{P}_h v - v\|_{L^2(\Omega)} + h \|\nabla(\mathcal{P}_h v - v)\|_{L^2(\Omega)} \le C h^j \|v\|_{H^j(\Omega)} \tag{2.3}$$

for $j = 1, 2$ and any $v \in H^j(\Omega)$.

For the discretization of the velocity field $\mathbf{v}$ and the pressure p, we use function spaces $\mathbf{W}_h \subset \mathbf{X}_h \subset \mathbf{W}_0^{1,2}(\Omega)$ and $S_h \subset L_0^2(\Omega) := \{v \in L^2(\Omega) | \int_\Omega v = 0\}$ such that the following conditions hold.

(S1) $\mathbf{W}_h := \{\mathbf{v}_h \in \mathbf{X}_h | \int_\Omega q_h \operatorname{div} \mathbf{v}_h = 0 \quad \forall q_h \in S_h\}$.

(S2) The Babuška-Brezzi condition is satisfied, i.e. a positive constant β exists such that

$$\sup_{\mathbf{v}_h \in \mathbf{X}_h} \frac{(q_h, \operatorname{div} \mathbf{v}_h)}{\|\mathbf{v}_h\|_{\mathbf{W}_0^{1,2}(\Omega)}} \geq \beta \|q_h\|_{L^2(\Omega)}$$

for all $q_h \in S_h$.

(S3) The orthogonal L^2-projection $\mathcal{R}_h : \mathbf{L}^2(\Omega) \to \mathbf{W}_h$ is H^1-stable, i.e. a positive constant C exists such that

$$\|\nabla \mathcal{R}_h \mathbf{v}\|_{\mathbf{L}^2(\Omega)} \leq C \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega)} \tag{2.4}$$

for all $\mathbf{v} \in \mathbf{W}_0^{1,2}(\Omega)$. Moreover,

$$\lim_{h \to 0} \|\mathcal{R}_h \mathbf{v} - \mathbf{v}\|_{\mathbf{L}^2(\Omega)} = 0 \tag{2.5}$$

for all $\mathbf{v} \in \mathbf{W}_{0,\mathrm{div}}^{1,2}(\Omega)$.

(S4) A projection operator $\mathbf{Q}_{\mathrm{div}}^h : \mathbf{W}_{0,\mathrm{div}}^{1,2}(\Omega) \to \mathbf{W}_h$ exists such that

$$\left\|\mathbf{Q}_{\mathrm{div}}^h \mathbf{v} - \mathbf{v}\right\|_{\mathbf{L}^2(\Omega)} + h \left\|\nabla(\mathbf{Q}_{\mathrm{div}}^h \mathbf{v} - \mathbf{v})\right\|_{\mathbf{L}^2(\Omega)} \leq C h^j \|\mathbf{v}\|_{\mathbf{H}^j(\Omega)} \tag{2.6}$$

for all $\mathbf{v} \in \mathbf{H}^j(\Omega) \cap \mathbf{W}_{0,\mathrm{div}}^{1,2}(\Omega)$, $j = 1, 2$.

(S5) The orthogonal L^2-projection $\mathcal{Q}_h : \mathbf{W}_0^{1,2}(\Omega) \to \mathbf{X}_h$ is H^1-stable.

(S6) The orthogonal L^2-projection $\mathcal{S}_h : L^2(\Omega) \to S_h$ satisfies

$$\lim_{h \to 0} \|q - \mathcal{S}_h q\|_{L^2(\Omega)} = 0 \tag{2.7}$$

for all $q \in L^2(\Omega)$.

Examples of finite-element spaces $\mathbf{X}_h, S_h$ which comply with (S1) –(S6) are P_2P_1-elements (the so called Taylor-Hood elements) and P_2P_0-elements. In both examples, $\mathbf{X}_h$ is given as

$$\mathbf{X}_h := \left\{\mathbf{w} \in (\mathbf{C}_0^0(\bar{\Omega})) : (\mathbf{w})_j|_K \in P_2(K), K \in \mathcal{T}_h, j = 1, \ldots, d\right\}, \quad d = 2, 3.$$

For Taylor-Hood elements, $S_h := U_h \cap L_0^2(\Omega)$. In the case of P_2P_0-elements,

$$S_h := \{q_h \in L_0^2(\Omega) : q_h|_K \equiv const. \quad \forall K \in \mathcal{T}_h\}.$$

Following the exposition in [16] and [17], using in particular error estimates in [24], we note that P_2P_0-elements satisfy the conditions (S2), (S4)–(S6). Observe that in (S4) the orthogonal projection $\mathcal{R}_h$ may be chosen for $\mathbf{Q}_{\mathrm{div}}^h$. Concerning (S3), we refer to Lemma 6.5 in the Appendix where we prove that (2.4) is satisfied by both P_2P_0- and Taylor-Hood elements. Moreover, we note that (S2), (S4)–(S6) hold for Taylor-Hood elements as well, see for instance [19] and [27]. In particular, the Stokes projection $\mathbf{Q}_{Stokes} : \mathbf{W}_{0,\mathrm{div}}^{1,2}(\Omega) \to \mathbf{W}_h$, defined by

$$\int_\Omega \nabla \mathbf{Q}_{Stokes} \mathbf{v} : \nabla \chi = \int_\Omega \nabla \mathbf{v} : \nabla \chi \qquad \forall \chi \in \mathbf{W}_h,$$

is a possible choice for $\mathbf{Q}_{\mathrm{div}}^h$ in (S4). Using finally the best-approximation property of $\mathcal{R}_h$ with respect to the L^2-norm, (2.5) follows from (2.6) for $j = 1$.

We conclude this subsection by introducing some notation. Given a time increment $\tau > 0$ (cf. (T2)), we will denote the backward (and forward) difference quotients with respect to time by ∂_τ^- (or ∂_τ^+, respectively). Given a subdivision of the time interval $I := [0, T)$ with intervals $I_k := [t_k, t_{k+1})$ as in (T2), we introduce $S^{0,-1}([0, T); X)$ associated with a Banach space X as the space of functions $v : [0, T) \to X$ which are constant on each I_k,

$k = 0, \cdots, N-1$. Given a function $v \in S^{0,-1}([0,T);X)$, we abbreviate $v^k(\cdot) := v(\cdot, t_k)$. In particular, we have

$$\tau \sum_{k=0}^{N-1} v^k(\cdot) = \int_0^T v(\cdot, t)dt. \tag{2.8}$$

In general, we denote functions in $S^{0,-1}(I;U_h)$, $S^{0,-1}(I;\mathbf{W}_h)$, $S^{0,-1}(I;\mathbf{X}_h)$ by an index τh. We often abbreviate $f^k(\cdot) := f_{\tau h}(\cdot, t_k)$.

2.3. **The discrete scheme.** We decompose the double-well potential $F(\cdot) =: F_+(\cdot) + F_-(\cdot)$ and we make the following assumptions on the data.

(H1) $F_+ : \mathbb{R} \to \mathbb{R}_0^+$ is convex and of class C^2, the second derivatives are convex on $\mathbb{R}$, too, and they satisfy a growth estimate

$$F_+''(x) \leq C(1 + |x|^{\bar{q}})$$

with $\bar{q}$ in $[1,2)$, if $d = 3$, and in $[1,\infty)$, if $d = 2$.

(H2) $F_- : \mathbb{R} \to \mathbb{R}$ is concave and of class C^2 with bounded second derivatives on $\mathbb{R}$.

(H3) Let initial data $\Phi_0 \in H^2(\Omega;[-1,1])$ and $\mathbf{V}_0 \in \mathbf{W}^{1,2}_{0,\text{div}}(\Omega)$ be given such that we have for discrete initial data $\varphi_{0h} := \mathcal{I}_h \Phi_0$ and $\mathbf{v}_{0h} := \mathcal{R}_h \mathbf{V}_0$ uniformly in $h > 0$ that

$$\int_\Omega |\mathbf{v}_{0h}|^2 \leq C < \infty$$

and that

$$\int_\Omega |\Delta_h \varphi_{0h}|^2 + \frac{1}{2}\int_\Omega |\nabla \varphi_{0h}|^2 + \int_\Omega \mathcal{I}_h F(\varphi_{0h}) \leq const..$$

Here, the discrete Laplacian $\Delta_h w \in U_h \cap H^1_*(\Omega)$ is defined by

$$(\Delta_h w, \Theta)_h = -\int_\Omega \langle \nabla w, \nabla \Theta \rangle \quad \forall \Theta \in U_h. \tag{2.9}$$

(H4) Given mass densities $0 < \tilde{\rho}_1 \leq \tilde{\rho}_2 \in \mathbb{R}$ of the fluids involved and an arbitrary, but fixed regularization parameter $\bar{\varphi} \in \left(\frac{\tilde{\rho}_1}{\tilde{\rho}_2 - \tilde{\rho}_1}, \frac{2\tilde{\rho}_1}{\tilde{\rho}_2 - \tilde{\rho}_1}\right)$, we define the regularized mass density of the two-phase fluid by a smooth, increasing, strictly positive function ρ of the phase-field φ which satisfies

$$\rho(\varphi)|_{(-1-\bar{\varphi}, 1+\bar{\varphi})} = \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2}\varphi + \frac{\tilde{\rho}_1 + \tilde{\rho}_2}{2} \tag{2.10}$$

$$\rho(\varphi)|_{(-\infty, -1-\frac{2\tilde{\rho}_1}{\tilde{\rho}_2 - \tilde{\rho}_1})} \equiv const. \tag{2.11}$$

$$\rho(\varphi)|_{(1+\frac{2\tilde{\rho}_1}{\tilde{\rho}_2 - \tilde{\rho}_1}, \infty)} \equiv const. \tag{2.12}$$

Remark 2.1. *In the continuous setting (assuming in particular a mechanism which confines the values of the phase-field function to the interval $[-1,1]$, for instance by choosing a degenerate mobility or a logarithmic potential F), $\bar{\rho}$ depends linearly on φ via* (2.10) *and is therefore bounded from below by a positive constant by definition. In the discrete setting, however, it is not possible to mimic singular or degenerate behaviour – regularization is indispensable. Hence, strict inclusions $\varphi \in [-1,1]$ for discrete solutions φ cannot be expected in general. Bounds on solutions can only be obtained via integral estimates as the phase-field equation is fourth-order parabolic and therefore comparison principles do not hold. However, the energy of the system is not necessarily decreasing in time due to the work done by external forces. As a consequence, bounds on φ always will depend on the special choice of external forces. Therefore, we use the cut-off mechanism of (H4) to guarantee definiteness of ρ and hence definiteness of the density $\rho|\mathbf{v}|^2$ of the kinetic*

energy as well.
Note in particular that the upper bound on $\bar\varphi$ translates to the condition that $1+\bar\varphi$ is bounded by the inverse of the Atwood number $\frac{\tilde\rho_2-\tilde\rho_1}{\tilde\rho_1+\tilde\rho_2}$. Hence, the inverse Atwood number controls the regime of values of φ for which ρ linearly depends on φ.

Now, we are in the position to introduce the scheme to be analyzed in this paper. For its derivation, we refer the reader to [22][1]. Note that this scheme was formulated under the assumption that φ stays in the regime for which the ρ-dependency is linear. To stress this fact, we use the notation

$$\frac{\delta\rho}{\delta\varphi} := \frac{\tilde\rho_2-\tilde\rho_1}{2}. \tag{2.13}$$

We emphasize that (H1)-(H4), (T1)-(T2), (S1)-(S6) are assumed to hold. For the ease of presentation, we assume external forces $\mathbf{k}_{grav}$ to be zero as these forces are given quantities which enter the system linearly. Hence, they do not have a qualitative effect on estimates and results. Moreover, we assume $M(\varphi)\equiv 1$ and $\sigma=1$, and for initial data, we skip the index h.

For given functions $(\varphi^0,\mathbf{v}^0)\in U_h\times\mathbf{W}_h$ and $k=0,...,N-1$ we have to find functions $(\varphi^{k+1},\mu^{k+1},\mathbf{v}^{k+1},p^{k+1})\in U_h\times U_h\times\mathbf{W}_h\times S_h$ such that

$$\begin{aligned}
\int_\Omega \big\langle \partial_\tau^-(\rho^{k+1}\mathbf{v}^{k+1}),\mathbf{w}\big\rangle &- \tfrac12\int_\Omega \partial_\tau^-\rho^{k+1}\big\langle\mathbf{v}^{k+1},\mathbf{w}\big\rangle \\
&- \tfrac12\int_\Omega \rho^k\big\langle\mathbf{v}^k,(\nabla\mathbf{w})^T\mathbf{v}^{k+1}\big\rangle + \tfrac12\int_\Omega\rho^k\big\langle\mathbf{v}^k,(\nabla\mathbf{v}^{k+1})^T\mathbf{w}\big\rangle \\
&+ \tfrac12\int_\Omega\frac{\delta\rho}{\delta\varphi}\big\langle\mathbf{j}^{k+1},(\nabla\mathbf{v}^{k+1})^T\mathbf{w}\big\rangle - \tfrac12\int_\Omega\frac{\delta\rho}{\delta\varphi}\big\langle\mathbf{j}^{k+1},(\nabla\mathbf{w})^T\mathbf{v}^{k+1}\big\rangle \\
&+ \int_\Omega 2\eta(\varphi^k)\mathbf{D}\mathbf{v}^{k+1}:\mathbf{D}\mathbf{w} - \int_\Omega p^{k+1}\operatorname{div}\mathbf{w} \\
&= -\int_\Omega\varphi^k\big\langle\nabla\mu^{k+1},\mathbf{w}\big\rangle \quad \forall\mathbf{w}\in\mathbf{X}_h,
\end{aligned} \tag{2.14a}$$

$$\int_\Omega \psi\operatorname{div}\mathbf{v}^{k+1} = 0 \quad \forall\psi\in S_h, \tag{2.14b}$$

$$\big(\partial_\tau^-\varphi^{k+1},\psi\big)_h - \int_\Omega\big\langle\mathbf{v}^{k+1},\nabla\psi\big\rangle\varphi^k + \int_\Omega\big\langle\nabla\mu^{k+1},\nabla\psi\big\rangle = 0 \quad \forall\psi\in U_h, \tag{2.14c}$$

$$\big(\mu^{k+1},\psi\big)_h = \int_\Omega\big\langle\nabla\varphi^{k+1},\nabla\psi\big\rangle + \int_\Omega\mathcal{I}_h\big(\big(F'_+(\varphi^{k+1})+F'_-(\varphi^k)\big)\psi\big) \quad \forall\psi\in U_h. \tag{2.14d}$$

Here, we use the abbreviation $\rho^{k+1}:=\rho(\varphi^{k+1})$. Moreover, we define $\mathbf{j}^{k+1}:=-\nabla\mu^{k+1}$.

Remark 2.2. *The scheme studied in* [22] *differs from* (2.14) *in such a way that the last term in* (2.14a) *and the second term in* (2.14c) *are replaced by $\int_\Omega\mu^{k+1}\langle\nabla\varphi^k,\mathbf{w}\rangle$ and by $\int_\Omega\langle\mathbf{v}^{k+1},\nabla\varphi^k\rangle\psi$, respectively. Such a substitution is possible as long as the corresponding version of* (2.14c) *guarantees conservation of mass for the phase-field – or equivalently, if the subset in U_h of functions with zero mean is contained in S_h. This holds true for Taylor-Hood elements which were studied in* [22], *but e.g. not for P_2P_0-elements. Note that the proofs presented in the present paper can easily be modified to cover that case – in particular, the convergence results need not to be changed at all.*

[1] Corollary 5.4 may also serve as an explanation in which way weak formulations of (1.1a) and of (1.1b) have to be combined in order to get the counterpart of (2.14a) in the continuous setting.

Concerning existence of discrete solutions, we have the following result.

Lemma 2.3. *For given functions* $(\varphi^k, \mathbf{v}^k) \in U_h \times \mathbf{W}_h$ *, there exists a quadruple* $(\varphi^{k+1}, \mu^{k+1}, \mathbf{v}^{k+1}, p^{k+1}) \in U_h \times U_h \times \mathbf{W}_h \times S_h$ *which solves the discrete system* (2.14).

Proof. First, we prove for given $(\varphi^k, \mathbf{v}^k)$ the existence of functions $(\varphi^{k+1}, \mu^{k+1}, \mathbf{v}^{k+1}) \in U_h \times U_h \times \mathbf{W}_h$ which solve a modified version of (2.14) where (2.14a) is replaced by

$$\begin{aligned}\int_\Omega \left\langle \partial_\tau^-(\rho^{k+1}\mathbf{v}^{k+1}), \mathbf{w}\right\rangle - \tfrac{1}{2}\int_\Omega \partial_\tau^- \rho^{k+1}\left\langle \mathbf{v}^{k+1}, \mathbf{w}\right\rangle \\ - \tfrac{1}{2}\int_\Omega \rho^k \left\langle \mathbf{v}^k, (\nabla \mathbf{w})^T \mathbf{v}^{k+1}\right\rangle + \tfrac{1}{2}\int_\Omega \rho^k\left\langle \mathbf{v}^k, (\nabla \mathbf{v}^{k+1})^T\mathbf{w}\right\rangle \\ + \tfrac{1}{2}\int_\Omega \frac{\delta\rho}{\delta\varphi}\left\langle \mathbf{j}^{k+1}, (\nabla \mathbf{v}^{k+1})^T \mathbf{w}\right\rangle - \tfrac{1}{2}\int_\Omega \frac{\delta\rho}{\delta\varphi}\left\langle \mathbf{j}^{k+1}, (\nabla \mathbf{w})^T\mathbf{v}^{k+1}\right\rangle \\ + \int_\Omega 2\eta(\varphi^k)\mathbf{D}\mathbf{v}^{k+1} : \mathbf{D}\mathbf{w} = -\int_\Omega \varphi^k\left\langle \nabla\mu^{k+1}, \mathbf{w}\right\rangle \end{aligned} \tag{2.15}$$

for all $\mathbf{w} \in \mathbf{W}_h$.
For given $\mathbf{v}^k$ and $z^k := \varphi^k - \alpha$ with $\alpha := \frac{1}{|\Omega|}\int_\Omega \varphi^k$, we are looking for a pair $(\mathbf{v}^{k+1}, z^{k+1})$ in $\mathbf{W}_h \times U_h$ such that $(\mathbf{v}^{k+1}, \varphi^{k+1})$ satisfies the system of equations (2.15) and (2.14c) - (2.14d). Here, $\varphi^{k+1} = z^{k+1} + \alpha$, and the function μ^{k+1} is obtained from $(\mathbf{v}^{k+1}, \varphi^{k+1})$ by (2.14d). Denoting the nodal basis of U_h by $\{\theta_1, \cdots, \theta_{\dim U_h}\}$ and taking $\{\mathbf{w}_1, \cdots, \mathbf{w}_{\dim \mathbf{W}_h}\}$ to be a basis of $\mathbf{W}_h$, we expand $z = \sum_{i=1}^{\dim U_h} Z_i\theta_i$ and $\mathbf{v} = \sum_{i=1}^{\dim \mathbf{W}_h} V_i \mathbf{w}_i$ for given elements $z \in U_h$ and $\mathbf{v} \in \mathbf{W}_h$, respectively. We introduce the stiffness and lumped mass matrices $(L_h)_{ij} := \int_\Omega \langle \nabla\theta_i, \nabla\theta_j\rangle$ and $(M_h)_{ij} := \int_\Omega \mathcal{I}_h(\theta_i\theta_j)$ for $i,j \in \{1,\cdots,\dim U_h\}$, respectively. Moreover, we use the notation

$$(M(\rho(z)))_{ij} := \int_\Omega \tfrac{\rho(z^k+\alpha)+\rho(z+\alpha)}{2}\langle \mathbf{w}_i, \mathbf{w}_j\rangle, \qquad i,j \in \{1,\cdots,\dim \mathbf{W}_h\}.$$

for a weighted mass matrix on $\mathbf{W}_h$ corresponding to a function $z \in U_h$ which has mean-value zero. Due to (H4), the associated symmetric bilinear form defines a norm which is equivalent to the L^2-norm for vectorfields on Ω.
We have to solve the nonlinear system $\begin{pmatrix} G_1(Z,V) \\ G_2(Z,V)\end{pmatrix} = 0$ of $q = \dim U_h + \dim \mathbf{W}_h$ equations given by

$$\begin{aligned} G_1(Z,V) :=& Z - Z^k + \tau M_h^{-1}B_1(V) \\ &+ \tau M_h^{-1}L_h\left(M_h^{-1}L_h Z + F_+'(z+\alpha) + F_-'(z^k+\alpha)\right)\end{aligned} \tag{2.16}$$

and

$$G_2(Z,V) := M(\rho(z))(V - V^k) + \tau\left(B_2(Z,V) + B_3(V) + B_4(Z)\right). \tag{2.17}$$

Here, we emphasize that Z and V are the coefficient vectors for the still unknown functions $z \in U_h$ and $\mathbf{v} \in \mathbf{W}_h$. Within this proof, we shall assume in general that capital letters are used to denote coefficient vectors of elements in U_h and $\mathbf{W}_h$. Moreover, with a slight misuse of notation we write $F_+'(z+\alpha)$ for the coefficent vector corresponding to $\mathcal{I}_h(F_+'(z+\alpha))$. In addition, we have introduced the following new terms.

$$(B_1(V))_j := -\int_\Omega z^k\langle \mathbf{v}, \nabla\theta_j\rangle\,, \qquad j = 1,\cdots,\dim U_h,$$

$$(B_2(Z,V))_j := -\tfrac{1}{2\tau}\int_\Omega \big(\rho(z+\alpha)-\rho(z^k+\alpha)\big)\,\langle \mathbf{v},\mathbf{w}_j\rangle$$
$$+\tfrac12\int_\Omega \rho(z^k+\alpha)\langle \mathbf{v}^k,(\nabla\mathbf{v})^T\mathbf{w}_j\rangle - \tfrac12\int_\Omega \rho(z^k+\alpha)\langle \mathbf{v}^k,(\nabla\mathbf{w}_j)^T\mathbf{v}\rangle$$
$$+\tfrac12\int_\Omega \frac{\delta\rho}{\delta\varphi}\langle \mathbf{j},(\nabla\mathbf{v})^T\mathbf{w}_j\rangle - \tfrac12\int_\Omega \frac{\delta\rho}{\delta\varphi}\langle \mathbf{j},(\nabla\mathbf{w}_j)^T\mathbf{v}\rangle\,,\qquad j=1,\cdots,\dim \mathbf{W}_h,$$

where $\mathbf{j} := -\sum_{i=1}^{\dim U_h}\mathfrak{M}_i\nabla\theta_i$ with $\mathfrak{M}=\mathfrak{M}(Z)\subset\mathbb{R}^{\dim U_h}$ being defined by

$$\mathfrak{M} := M_h^{-1}\left(L_h Z + M_h\left(F_+'(z+\alpha)+F_-'(z^k+\alpha)\right)\right).$$

Moreover,

$$(B_3(V))_j := 2\int_\Omega \eta(z^k+\alpha)D\mathbf{v}:D\mathbf{w}_j\,,\qquad j=1,\cdots,\dim \mathbf{W}_h,$$

and

$$(B_4(Z))_j := -\int_\Omega\sum_{l=1}^{\dim U_h}\mathfrak{M}_l\theta_l\langle\nabla z,\mathbf{w}_j\rangle\,,\qquad j=1,\cdots,\dim \mathbf{W}_h.$$

Let us introduce a new bilinear form on $\mathbb{R}^{\dim U_h}\times\mathbb{R}^{\dim \mathbf{W}_h}$ by

$$\left\langle\left\langle\begin{pmatrix}Z_1\\V_1\end{pmatrix},\begin{pmatrix}Z_2\\V_2\end{pmatrix}\right\rangle\right\rangle := Z_1^T L_h Z_2 + \int_\Omega\left\langle\sum_{l=1}^{\dim \mathbf{W}_h}(V_1)_l\,\mathbf{w}_l,\sum_{m=1}^{\dim \mathbf{W}_h}(V_2)_m\,\mathbf{w}_m\right\rangle. \tag{2.18}$$

Obviously, this form is a scalar product on $K^\perp\times\mathbb{R}^{\dim \mathbf{W}_h}$ where $K^\perp\subset\mathbb{R}^{\dim U_h}$ is defined by $K^\perp := \{W\in\mathbb{R}^{\dim U_h} : (M_hW)^T\mathbb{1}=0\}$ with $\mathbb{1}:=(1,\cdots,1)^T$. By $\|\cdot\|_{new}$, we denote $\langle\langle\cdot,\cdot\rangle\rangle^{\frac12}$. It is easily verified that $Z^k\in K^\perp$ and that G_1 maps $K^\perp$ onto itself.
Let us now argue by contradiction. To this purpose, we assume that for a positive number R to be specified later on, a root $(\hat Z,\hat V)^T$ of $(G_1,G_2)^T$ did not exist on $\overline{B_R(0)}$ where $B_R(0)$ here denotes the ball of Radius R around the origin in the $\|\cdot\|_{new}$-norm. Then, due to Brouwer's fixed-point theorem (see [36]), the mapping $H:\overline{B_R(0)}\to\overline{B_R(0)}$ defined by

$$H(Z,V) := -R\frac{(G_1(Z,V),G_2(Z,V))^T}{\|(G_1(Z,V),G_2(Z,V))\|_{new}}$$

would have a fixed-point $(\bar Z,\bar V)\in K^\perp\times\mathbf{W}_h$ satisfying $\left\|(\bar Z,\bar V)\right\|_{new}=R$.
Following the ideas of [21], we introduce $Y\in\mathbb{R}^{\dim U_h}$ to be the solution of

$$L_hY := \mathcal{P}_{K^\perp}\left\{M_h\left\{F_+'(z+\alpha)+F_-'(z^k+\alpha)\right\}\right\}$$

where $\mathcal{P}_{K^\perp}$ denotes the orthogonal projection onto $K^\perp$. Observe that

$$\begin{aligned}\left\langle\left\langle\begin{pmatrix}\bar Z\\\bar V\end{pmatrix},\begin{pmatrix}\bar Z+Y\\\bar V\end{pmatrix}\right\rangle\right\rangle &= \left(L_h\bar Z\right)^T\bar Z + \left(\bar Z+\alpha\mathbb{1}-\alpha\mathbb{1}\right)^T M_h\left(F_+'(\bar z+\alpha)-F_+'(\alpha)\mathbb{1}\right)\\
&\quad+\left(\bar Z\right)^T M_hF_-'(z^k+\alpha)+F_+'(\alpha)\bar Z^TM_h\mathbb{1}+\int_\Omega|\bar{\mathbf{v}}|^2\\
&\ge \left(L_h\bar Z\right)^T\bar Z+\int_\Omega|\bar{\mathbf{v}}|^2\\
&\quad-\tfrac{\varepsilon}{2}\bar Z^T\bar Z-\tfrac{1}{2\varepsilon}\left(M_hF_-'(z^k+\alpha)\right)^TM_hF_-'(z^k+\alpha).\end{aligned} \tag{2.19}$$

Here, we used the monotonicity of F_+' as well as the fact that M_h is a diagonal matrix. Taking into account the equivalence of norms on finite dimensional spaces, there exists $R_1 > 0$ such that

$$\left\langle\!\!\left\langle \begin{pmatrix} \bar{Z} \\ \bar{V} \end{pmatrix}, \begin{pmatrix} \bar{Z} + Y \\ \bar{V} \end{pmatrix} \right\rangle\!\!\right\rangle > 0 \tag{2.20}$$

provided $\left\|(\bar{Z}, \bar{V})\right\|_{new} \geq R_1$.
In a similar fashion, we show the existence of a number $R_2 > 0$ such that

$$\left\langle\!\!\left\langle \begin{pmatrix} G_1(\bar{Z}, \bar{V}) \\ G_2(\bar{Z}, \bar{V}) \end{pmatrix}, \begin{pmatrix} \bar{Z} + Y \\ \bar{V} \end{pmatrix} \right\rangle\!\!\right\rangle > 0 \tag{2.21}$$

provided $\left\|(\bar{Z}, \bar{V})\right\|_{new} \geq R_2$. Note that (2.21) is a direct consequence of the energy estimate in the discrete setting – see (3.1). In fact, we have chosen $\langle\langle \cdot, \cdot \rangle\rangle$, G_2, and Y in such a way that (2.21) is just the coefficient version of testing the momentum equation by $\mathbf{v}^{k+1}$, the phase-field equation by μ^{k+1}, and the equation for the chemical potential by $\partial_\tau \varphi^{k+1}$. Hence,

$$\left\langle\!\!\left\langle \begin{pmatrix} \bar{Z} \\ \bar{V} \end{pmatrix}, \begin{pmatrix} \bar{Z} + Y \\ \bar{V} \end{pmatrix} \right\rangle\!\!\right\rangle = -\frac{R}{\left\| \begin{pmatrix} G_1(\bar{Z}, \bar{V}) \\ G_2(\bar{Z}, \bar{V}) \end{pmatrix} \right\|_{new}} \left\langle\!\!\left\langle \begin{pmatrix} G_1(\bar{Z}, \bar{V}) \\ G_2(\bar{Z}, \bar{V}) \end{pmatrix}, \begin{pmatrix} \bar{Z} + Y \\ \bar{V} \end{pmatrix} \right\rangle\!\!\right\rangle < 0$$

for $R = max\{R_1, R_2\}$ which is a contradiction to (2.20). Hence, a discrete solution exists. To obtain the existence of a pressure p^{k+1} and that way to justify (2.14a), we proceed as follows. Equation (2.14a) defines a linear functional $\mathcal{F} : \mathbf{X}_h \to \mathbb{R}$ which vanishes on $\mathbf{W}_h$. Using Lemma 4.1 in [19] together with the stability condition (S2), the existence of a pressure $p^{k+1} \in S_h$ is readily established. The lemma is proven. □

Remark 2.4. *Note that no further assumptions, e.g. on the size of time-increments or on the grid size, are necessary to prove existence of discrete solutions.*

3. Compactness in space

In this section, we show that the discrete counterpart of the physical energy - i.e. the sum of the kinetic and the interfacial energies - acts as a discrete Lyapunov-functional provided no external forces are applied. We start with a local result.

Theorem 3.1. *Assume that the triple* $\left(\varphi^{k+1}, \mu^{k+1}, \mathbf{v}^{k+1}, p^{k+1}\right)$ *solves the system* (2.14) *for given* $\left(\varphi^k, \mu^k, \mathbf{v}^k, p^k\right)$. *Then,*

$$\begin{aligned} \frac{1}{2\tau}\left[\int_\Omega \rho^{k+1}\left|\mathbf{v}^{k+1}\right|^2 - \int_\Omega \rho^k \left|\mathbf{v}^k\right|^2 + \int_\Omega \rho^k \left|\mathbf{v}^{k+1} - \mathbf{v}^k\right|^2\right] \\ + \frac{1}{2\tau}\left[\int_\Omega \left|\nabla\varphi^{k+1}\right|^2 - \int_\Omega \left|\nabla\varphi^k\right|^2 + \int_\Omega \left|\nabla\varphi^{k+1} - \nabla\varphi^k\right|^2\right] \\ + \frac{1}{\tau}\int_\Omega \mathcal{I}_h\big(F(\varphi^{k+1}) - F(\varphi^k)\big) + \int_\Omega \left|\mathbf{j}^{k+1}\right|^2 + \int_\Omega 2\eta(\varphi^k)\left|\mathbf{D}\mathbf{v}^{k+1}\right|^2 \leq 0. \end{aligned} \tag{3.1}$$

Proof. Choosing $\psi := \partial_\tau^- \varphi^{k+1}$ in (2.14d) and $\psi := \mu^{k+1}$ in (2.14c), we infer - using the convexity of F_+ and of $(-F_-)$ (see (H1) and (H2)) - that

$$\begin{aligned} \frac{1}{2\tau}\left[\int_\Omega \left|\nabla\varphi^{k+1}\right|^2 - \int_\Omega \left|\nabla\varphi^k\right|^2 + \int_\Omega \left|\nabla\varphi^{k+1} - \nabla\varphi^k\right|^2\right] + \int_\Omega \left|\mathbf{j}^{k+1}\right|^2 \\ + \frac{1}{\tau}\int_\Omega \mathcal{I}_h\big(F(\varphi^{k+1}) - F(\varphi^k)\big) - \int_\Omega \langle \mathbf{v}^{k+1}, \nabla\mu^{k+1}\rangle \varphi^k \leq 0\,. \end{aligned} \tag{3.2}$$

Testing (2.14a) by $\mathbf{w} = \mathbf{v}^{k+1}$ and using the identity

$$(a_1 b_1 - a_0 b_0) b_1 - \frac{1}{2}(a_1 - a_0) b_1^2 = \frac{1}{2}\big(a_1 b_1^2 - a_0 b_0^2 + a_0 (b_1 - b_0)^2\big),$$

which holds for $a_i, b_i \in \mathbb{R},\ i = 0, 1$, gives

$$\frac{1}{2\tau}\left[\int_\Omega \rho^{k+1}\left|\mathbf{v}^{k+1}\right|^2 - \int_\Omega \rho^k \left|\mathbf{v}^k\right|^2 + \int_\Omega \rho^k \left|\mathbf{v}^{k+1} - \mathbf{v}^k\right|^2\right] + \int_\Omega 2\eta(\varphi^k)\left|\mathbf{D}\mathbf{v}^{k+1}\right|^2 = -\int_\Omega \varphi^k \langle \mathbf{v}^k, \nabla \mu^{k+1}\rangle .$$

By summation, (3.1) follows. □

We immediately obtain the following global result.

Corollary 3.2. *For every $1 \le l \le N$ we have*

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \rho^l \left|\mathbf{v}^l\right|^2 + \frac{1}{2}\int_\Omega \left|\nabla\varphi^l\right|^2 + \int_\Omega \mathcal{I}_h\big(F(\varphi^l)\big) + \frac{1}{2}\sum_{m=0}^{l-1}\int_\Omega \rho^m \left|\mathbf{v}^{m+1} - \mathbf{v}^m\right|^2 \\
&\quad + \frac{1}{2}\sum_{m=0}^{l-1}\int_\Omega \left|\nabla\varphi^{m+1} - \nabla\varphi^m\right|^2 + \tau\sum_{m=0}^{l-1}\int_\Omega \left|\mathbf{j}^{m+1}\right|^2 + \tau\sum_{m=0}^{l-1}\int_\Omega 2\eta(\varphi^m)\left|\mathbf{D}\mathbf{v}^{m+1}\right|^2 \\
&\qquad \le \frac{1}{2}\int_\Omega \rho^0 \left|\mathbf{v}^0\right|^2 + \frac{1}{2}\int_\Omega \left|\nabla\varphi^0\right|^2 + \int_\Omega \mathcal{I}_h\big(F(\varphi^0)\big). \qquad (3.3)
\end{aligned}$$

4. Compactness in time

We begin this section with an auxiliary result.

Lemma 4.1. *Let $(\varphi_{\tau h}, \mu_{\tau h}, \mathbf{v}_{\tau h}, p_{\tau h})$ be a discrete solution of (2.14) on $(0,T)$. Then if $d = 2$ and $1 \le q < \infty$ or if $d = 3$ and $1 \le q < 4$, a positive constant C exists such that*

$$\|\varphi_{\tau h}\|_{L^q((0,T);C^0(\bar\Omega))} \le C(1+T)\left\|\varphi^0\right\|_{H^1(\Omega)} . \qquad (4.1)$$

Proof. Let us prove first that $\Delta_h \varphi_{\tau h} \in L^2(\Omega_T)$. Note the identity

$$-\big(\Delta_h \varphi^{k+1}, \theta\big)_h = \big(\mu^{k+1} - F'_+(\varphi^{k+1}) - F'_-(\varphi^k), \theta\big)_h \quad \forall \theta \in U_h \qquad (4.2)$$

which is a consequence of (2.14d) and the definition (2.9) of the discrete Laplacian. Starting from this identity and choosing $\theta = -\Delta_h \varphi^{k+1}$, we have

$$\begin{aligned}
(\mu^{k+1}, \mu^{k+1})_h = &\left\|\Delta_h \varphi^{k+1}\right\|_h^2 + \left\|F'_+(\varphi^{k+1}) + F'_-(\varphi^k)\right\|_h^2 \\
&- 2(\Delta_h \varphi^{k+1}, F'_+(\varphi^{k+1}))_h - 2(\Delta_h\varphi^{k+1}, F'_-(\varphi^k))_h.
\end{aligned}$$

For the third term on the right-hand side, we get

$$\big(\Delta_h \varphi^{k+1}, F'_+(\varphi^{k+1})\big)_h = -\int_\Omega \big\langle \nabla\varphi^{k+1}, \nabla \mathcal{I}_h F'_+(\varphi^{k+1})\big\rangle \le 0 \qquad (4.3)$$

according to Lemma 4.3 in [21], using in particular assumption (T1) on the triangulation. Hence,

$$\left\|\Delta_h \varphi^{k+1}\right\|_h^2 \le \left\|\mu^{k+1}\right\|_h^2 + \frac{1}{2}\left\|\Delta_h\varphi^{k+1}\right\|_h^2 + \left\|F'_-(\varphi^k)\right\|_h^2 . \qquad (4.4)$$

By the linear growth of F'_-, absorption, and the energy estimate (3.3), we find

$$\left\|\Delta_h\varphi^{k+1}\right\|_h^2 \le 2\left\|\mu^{k+1}\right\|_h^2 + \left\|\varphi^0\right\|_{H^1(\Omega)}^2 .$$

Hence,

$$\|\Delta_h \varphi_{\tau h}\|^2_{L^2((0,T);L^2(\Omega))} \le CT \left\|\varphi^0\right\|^2_{H^1(\Omega)}, \tag{4.5}$$

again using (3.3). By Theorem 6.4, formula (6.4),

$$\|\nabla \varphi_{\tau h}\|_{L^2((0,T);L^{\frac{2d}{d-2}-}(\Omega))} \le CT \left\|\varphi^0\right\|_{H^1(\Omega)}. \tag{4.6}$$

By interpolation, using the estimate

$$\|\nabla \varphi_{\tau h}\|_{L^\infty((0,T);L^2(\Omega))} \le C \left\|\varphi^0\right\|_{H^1(\Omega)},$$

which follows from (3.3), we find in space dimension $d = 3$ for every $1 \le q < 4$ an exponent $p > 3$ such that

$$\|\nabla \varphi_{\tau h}\|_{L^q((0,T);L^p(\Omega))} \le C(1+T) \left\|\varphi^0\right\|_{H^1(\Omega)}.$$

Therefore, estimate (4.1) follows using Sobolev embedding and the conservation of the mean value of $\varphi_{\tau h}$. The argumentation in $d = 2$ dimensions is analogous. □

The next step is to establish both L^2-regularity of difference quotients in time and higher regularity of the discrete Laplacian for the phase-field.

Theorem 4.2. *Let $(\varphi_{\tau h}, \mu_{\tau h}, \mathbf{v}_{\tau h}, p_{\tau h})$ be a discrete solution of (2.14) on $[0,T]$. Assuming $N := \frac{T}{\tau}$, we have the following.*

(1) *Positive constants $C_1 = C_1(\|\Delta_h\varphi^0\|_h, \|\varphi^0\|_{H^1(\Omega)}, \|\mathbf{v}_0\|_{L^2(\Omega)})$ and C_2 exist such that*

$$\begin{aligned}\sup_{k\in\{1,\dots,N\}} \left\|\Delta_h \varphi^k\right\|_h^2 \le \left(\left\|\Delta_h\varphi^0\right\|_h^2 + C_1\right) \exp\Bigg(C_2\Bigg(T + \int_\tau^{T+\tau} \|\mathbf{v}_{\tau h}\|^2_{H^1}\, ds \\ + \left((T+\tau)\left\|\varphi^0\right\|_{H^1(\Omega)}\right)^{2\bar q}\Bigg)\Bigg)\end{aligned} \tag{4.7}$$

where $\bar q$ is defined in (H1).

(2) *A positive constant $C_3 = C_3(T, \|\Delta_h\varphi^0\|_h, \|\varphi^0\|_{H^1}, \|\mathbf{v}^0\|_{L^2})$ exists such that*

$$\sup_{k\in\{1,2,\dots,N\}} \left\|\Delta_h\varphi^k\right\|_h^2 + \tfrac12 \int_0^T \left\|\partial_\tau^- \varphi_{\tau h}\right\|_h^2 \le C_3 \tag{4.8}$$

Proof. Using (2.9) to define the discrete Laplacians $\Delta_h\varphi^{k+1}$ and $\Delta_h\varphi^k$ in $U_h \cap H^1_*(\Omega)$, subtracting the corresponding weak formulations from each other and dividing by τ, we have

$$-\left(\partial_\tau^-\Delta_h\varphi^{k+1}, \theta\right)_h = \int_\Omega \left\langle \nabla\partial_\tau^-\varphi^{k+1}, \nabla\theta\right\rangle \quad \forall\theta\in U_h.$$

Choosing $\theta = \mu^{k+1}$ and using (2.14c) entails

$$-\left(\partial_\tau^-\Delta_h\varphi^{k+1}, \mu^{k+1}\right)_h + \left(\partial_\tau^-\varphi^{k+1}, \partial_\tau^-\varphi^{k+1}\right)_h = \int_\Omega \left\langle \mathbf{v}^{k+1}, \partial_\tau^-\nabla\varphi^{k+1}\right\rangle\varphi^k.$$

By (4.2), we expand μ^{k+1} to obtain

$$\begin{aligned}&\left(\partial_\tau^-\Delta_h\varphi^{k+1}, \Delta_h\varphi^{k+1}\right)_h + \left\|\partial_\tau^-\varphi^{k+1}\right\|_h^2 \\ &= -\int_\Omega \left\{\left\langle \mathbf{v}^{k+1}, \nabla\varphi^k\right\rangle\partial_\tau^-\varphi^{k+1} + \varphi^k\partial_\tau^-\varphi^{k+1}\operatorname{div}\mathbf{v}^{k+1}\right\} + \left(\partial_\tau^-\Delta_h\varphi^{k+1}, F_+'(\varphi^{k+1}) + F_-'(\varphi^k)\right)_h \\ &= R_1^k + R_2^k\end{aligned} \tag{4.9}$$

Discrete integration by parts with respect to time gives – after a discrete time-integration $\tau\sum_{k=0}^{N-1}$ of (4.9) – for R_2^k

$$\tau\sum_{k=0}^{N-1} R_2^k = -\tau\sum_{k=0}^{N-2}\big(\partial_\tau^+\big(F_+'(\varphi^{k+1})+F_-'(\varphi^k)\big),\Delta_h\varphi^{k+1}\big)_h-\big(\Delta_h\varphi^0,F_+'(\varphi^1)+F_-'(\varphi^0)\big)_h$$
$$+\big(\Delta_h\varphi^N,F_+'(\varphi^N)+F_-'(\varphi^{N-1})\big)_h = R_{21}^k+R_{22}^k+R_{23}^k. \quad (4.10)$$

Here, we used the well-known formula of integration by parts

$$\sum_{i=0}^{N}\big(\partial_\tau^- v^i\big)w^i = -\sum_{i=0}^{N}\big(\partial_\tau^+ w^i\big)v^i-\frac{v^{-1}w^0}{\tau}+\frac{v^Nw^N}{\tau}. \quad (4.11)$$

For the first term in R_{23}^k, we get by a similar argument as in (4.3)

$$\big(\Delta_h\varphi^N,F_+'(\varphi^N)\big)_h = -\int_\Omega\big\langle\nabla\varphi^N,\nabla\mathcal{I}_hF_+'(\varphi^N)\big\rangle\le 0 \quad (4.12)$$

According to (H1), we may estimate

$$\big|\big(\Delta_h\varphi^0,F_+'(\varphi^1)\big)_h\big|\le C\big(\|\Delta_h\varphi^0\|,\|\nabla\varphi^0\|_{L^2}\big). \quad (4.13)$$

By an analogous argument for the term involving F_-' (using (H2)), all the boundary terms can be controlled by $C=C(\|\Delta_h\varphi^0\|,\|\nabla\varphi^0\|_{L^2})$.
For R_{21}^k, we estimate using (H2) and (H1) (in particular the convexity of F_+'')

$$\begin{aligned}\big|R_{21}^k\big| &\le \tau\sum_{k=0}^{N-2}\big(\|F_+''(\varphi^{k+1})\|_{C^0}+\|F_+''(\varphi^{k+2})\|_{C^0}\big)\|\partial_\tau^+\varphi^{k+1}\|_h\|\Delta_h\varphi^{k+1}\|_h\\ &\qquad + C\tau\sum_{k=0}^{N-2}\|\partial_\tau^+\varphi^k\|_h\|\Delta_h\varphi^{k+1}\|_h \qquad (4.14)\\ &\le \varepsilon\tau\sum_{k=1}^{N-1}\|\partial_\tau^-\varphi^{k+1}\|_h^2+C_\varepsilon\tau\sum_{k=0}^{N-2}\Big(1+\|\varphi^{k+1}\|_{C^0}^{2\bar q}+\|\varphi^{k+2}\|_{C^0}^{2\bar q}\Big)\|\Delta_h\varphi^{k+1}\|_h^2.\end{aligned}$$

Finally,

$$\left|\tau\sum_{k=0}^{N-1}R_1^k\right|\le\varepsilon\tau\sum_{k=0}^{N-1}\|\partial_\tau^-\varphi^{k+1}\|_h^2+C_\varepsilon\tau\sum_{k=0}^{N-1}\|\mathbf{v}^{k+1}\|_{H^1(\Omega)}^2(1+\|\Delta_h\varphi^k\|_h^2) \quad (4.15)$$

where we have used the estimate $\|\nabla\varphi^k\|_{L^q(\Omega)}+\|\varphi^k\|_{C^0(\Omega)}\le C(1+\|\Delta_h\varphi^k\|_h)$ which holds for all $1\le q<\frac{2d}{d-2}$ according to Theorem 6.4, formula (6.4). Note that we also took advantage of the fact that $\|\cdot\|_h$ and $\|\cdot\|_{L^2}$ are equivalent norms on U_h with constants uniform in h, see (2.2). Collecting (4.9), (4.12)-(4.14), (4.15) and taking the aforementioned boundedness of the boundary terms with respect to time into account, we get

$$\begin{aligned}
&\frac{1}{2}\left\|\Delta_h\varphi^N\right\|_h^2-\frac{1}{2}\left\|\Delta_h\varphi^0\right\|_h^2+\frac{1}{2}\sum_{k=0}^{N-1}\left\|\Delta_h\varphi^{k+1}-\Delta_h\varphi^k\right\|_h^2\\
&\qquad+\frac{1}{2}\tau\sum_{k=0}^{N-1}\left\|\partial_\tau^-\varphi^{k+1}\right\|_h^2+\int_\Omega\left\langle\nabla\varphi^N,\nabla\mathcal{I}_hF_+'(\varphi^N)\right\rangle\\
&\leq C_1\big(\left\|\Delta_h\varphi^0\right\|_h,\left\|\nabla\varphi^0\right\|_{L^2},\left\|\mathbf{v}_0\right\|_{L^2}\big)+C_2\tau\sum_{k=1}^{N-1}\Big(1+\left\|\varphi^k\right\|_{C^0}^{2\bar{q}}+\left\|\varphi^{k+1}\right\|_{C^0}^{2\bar{q}}\Big)\left\|\Delta_h\varphi^k\right\|_h^2\\
&\qquad+C_3\tau\sum_{k=0}^{N-1}\left\|\mathbf{v}^{k+1}\right\|_{H^1(\Omega)}^2\left\|\Delta_h\varphi^k\right\|_h^2.
\end{aligned}\tag{4.16}$$

In the next step, we apply a discrete version of Gronwall's lemma (cf. e.g. [35, Lemma 4.2.3.]):

Lemma. *Let $\varepsilon_j,\eta_j, j=0,...,m$, be non-negative real numbers with $\eta_0\leq\eta_1\leq...\leq\eta_m$. For $\delta>0$ and $h=(h_0,...,h_{m-1})\in(\mathbb{R}_0^+)^m$, assume the estimates*

$$\varepsilon_0\leq\eta_0\quad\textit{and}\quad\varepsilon_{j+1}\leq\delta\sum_{i=0}^{j}h_i\varepsilon_i+\eta_{j+1}\tag{4.17}$$

to hold. Then,

$$\varepsilon_j\leq\eta_j\exp\left(\delta\sum_{i=0}^{j-1}h_i\right),\quad j=0,1,...,m.\tag{4.18}$$

From equation (4.16), we infer

$$\begin{aligned}
\left\|\Delta_h\varphi^j\right\|_h^2&\leq\Big(\left\|\Delta_h\varphi^0\right\|_h^2+C_1\Big)\cdot\exp\left(C\tau\sum_{l=0}^{j-1}\Big(1+\left\|\mathbf{v}^{l+1}\right\|_{H^1(\Omega)}^2+\left\|\varphi^l\right\|_{C^0}^{2\bar{q}}+\left\|\varphi^{l+1}\right\|_{C^0}^{2\bar{q}}\Big)\right)\\
&\leq\Big(\left\|\Delta_h\varphi^0\right\|_h^2+C_1\Big)\cdot\exp\left(C\left(t_j+\int_\tau^{t_j+\tau}\left\|\mathbf{v}_{\tau h}\right\|_{H^1}^2ds+\Big((t_j+\tau)\left\|\varphi^0\right\|_{H^1}\Big)^{2\bar{q}}\right)\right)
\end{aligned}$$

for all $1\leq j\leq N$. Note that we used (4.1) in the last line, too. Hence, (4.7) holds true. Estimate (4.8) immediately follows by combination of (4.7) and of (4.16). ☐

For the passage to the limit in the fifth term of (2.14a) and in order to prove strong convergence of $\mathbf{v}_{\tau h}$ in $\mathbf{L}^2(\Omega_T)$, we need results on improved integrability of $\mathbf{j}_{\tau h}$. Note that so far $\mathbf{j}_{\tau h}$ and $\nabla\mathbf{v}_{\tau h}$ are only known to be square-integrable with respect to time. For the passage to the limit $\tau,h\to0$, we need L^p-integrability with an exponent $p>1$ for the product. This requires higher regularity of $\mathbf{j}_{\tau h}$ with respect to both space and time. With the perspective of a discrete analogon of compensated compactness, we look for estimates of the discrete Laplacian of $\mu_{\tau h}$ uniformly in (τ,h). This will give higher integrability of $\mathbf{j}_{\tau h}$ with respect to space, too – see Corollary 4.4.

Lemma 4.3. *Let $(\varphi_{\tau h},\mu_{\tau h},\mathbf{v}_{\tau h},p_{\tau h})$ be a discrete solution on $[0,T]$ and $T>0$ be arbitrary, but fixed. Let $w_{\tau h}(\cdot,t)\in U_h\cap H^1_*(\Omega)$ be defined as the negative discrete Laplacian of $\mu_{\tau h}(\cdot,t)$, i.e.*

$$\left(w_{\tau h}(\cdot,t),\theta\right)_h:=\left(\nabla\mu_{\tau h}(\cdot,t),\nabla\theta\right)\qquad\forall\theta\in U_h.\tag{4.19}$$

Then a positive constant $C = C(T, \|\Delta\varphi^0\|_h, \|\varphi^0\|_{H^1}, \|\mathbf{v}^0\|_{L^2})$ *exists such that*

$$\int_0^T \|w_{\tau h}(\cdot,t)\|^2_{L^2(\Omega)} \le \tilde{C} \int_0^T \|w_{\tau h}(\cdot,t)\|_h^2 \le C \tag{4.20}$$

Moreover, for every $1 \le p < \frac{2d}{d-2}$,

$$\mathbf{j}_{\tau h} := -\nabla \mu_{\tau h} \text{ is uniformly bounded in } L^2((0,T); L^p(\Omega)). \tag{4.21}$$

Corollary 4.4. *Under the assumptions of Lemma 4.3, the functions* $\mu_{\tau h}$ *are uniformly bounded in* $L^2\big((0,T); \mathcal{C}^\beta(\Omega)\big)$ *with* $\beta < 2 - \frac{d}{2}$.

Proof of Lemma 4.3. Choose $\theta = w^{k+1} = w_{\tau h}(\cdot, t_{k+1})$ as the test function in (4.19). Hence,

$$\begin{aligned}\big(w^{k+1}, w^{k+1}\big)_h &= \big(\nabla \mu^{k+1}, \nabla w^{k+1}\big)\\ &\overset{(2.14c)}{=} -\big(\partial_\tau^- \varphi^{k+1}, w^{k+1}\big)_h - \int_\Omega \langle \mathbf{v}^{k+1}, \nabla\varphi^k\rangle w^{k+1} - \int_\Omega w^{k+1}\varphi^k \operatorname{div}\mathbf{v}^{k+1}.\end{aligned}$$

Using Theorem 6.4, Theorem 4.2, and the energy estimate (3.3), we may estimate

$$\begin{aligned}\tau\sum_{k=0}^{N-1}\big(w^{k+1}, w^{k+1}\big)_h &\le \frac{1}{2}\tau\sum_{k=0}^{N-1}\big\|\partial_\tau^-\varphi^{k+1}\big\|_h^2 + \frac{1}{2}\tau\sum_{k=0}^{N-1}\big\|w^{k+1}\big\|_h^2\\ &\quad + \tau\sum_{k=0}^{N-1}\big\|\mathbf{v}^{k+1}\big\|_{H^1(\Omega)}\big(1 + \big\|\Delta_h\varphi^k\big\|_h\big)\big\|w_{\tau h}^{k+1}\big\|_h .\end{aligned}$$

The last term on the right-hand side is bounded by

$$\overline{C}\tau\sum_{k=0}^{N-1}\big\|\mathbf{v}^{k+1}\big\|^2_{H^1(\Omega)} + \frac{1}{4}\tau\sum_{k=0}^{N-1}\big\|w^{k+1}\big\|_h^2$$

with a constant $\overline{C} = \overline{C}(T, \|\Delta\varphi^0\|_h, \|\varphi^0\|_{H^1}, \|\mathbf{v}^0\|_{L^2})$. By absorption, (4.20) follows. The uniform boundedness in (4.21) is a consequence of (4.20), (4.19), and Theorem 6.4. □

Proof of Corollary 4.4. This result follows by combining (4.21) with the uniform boundedness of $\varphi_{\tau h}$ in space-time which follows by (4.8) and Theorem 6.4. □

The following lemma provides the aforementioned higher integrability with respect to time for $\mathbf{j}_{\tau h}$. It is a straightforward consequence of Lemma 4.3, interpolation, and the uniform boundedness of $\mu_{\tau h}$ in $L^\infty((0,T); L^2(\Omega))$, the latter of which follows by a combination of (4.8) with Theorem 6.4.

Lemma 4.5. *For arbitrary, but fixed* $T > 0$, *there is a constant* $C = C(T, \|\Delta\varphi^0\|_h, \|\varphi^0\|_{H^1}, \|\mathbf{v}^0\|_{L^2})$ *such that*

$$\|\mathbf{j}_{\tau h}\|_{L^2(L^{6-})} + \|\mathbf{j}_{\tau h}\|_{L^4(L^2)} + \|\mathbf{j}_{\tau h}\|_{L^{\frac{8}{3}-}(L^3)} \le C. \tag{4.22}$$

Let us turn to the velocity field and let us prove compactness with respect to time for the orthogonal L^2-projection of $\rho_{\tau h}\mathbf{v}_{\tau h}$ onto $\mathbf{W}_h$ in appropriate dual Sobolev spaces.

Lemma 4.6. *Let* $(\varphi_{\tau h}, \mu_{\tau h}, \mathbf{v}_{\tau h}, p_{\tau h})$ *be a solution of* (2.14). *Then for every* $T > 0$ *and for every* $1 \le p < \frac{8}{7}$, *a positive constant*

$$C = C\big(T, p, \big\|\Delta\varphi^0\big\|_h, \big\|\varphi^0\big\|_{H^1}, \big\|\mathbf{v}^0\big\|_{L^2}\big) < \infty$$

exists such that

$$\big\|\partial_\tau^- \mathcal{R}_h(\rho_{\tau h}\mathbf{v}_{\tau h})\big\|_{L^p\left((0,T);\left(\mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega)\right)'\right)} \le C\,. \tag{4.23}$$

Proof. We take $\mathbf{w} \in S^{0,-1}\big([0,T); \mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega)\big)$ arbitrarily. Recall that for $T = N\tau$ we identify $\tau \sum_{k=0}^{N-1} \mathbf{w}^k$ with the integral $\int_0^T \mathbf{w} dt$. Choosing $\mathcal{R}_h \mathbf{w}$ as the test function in the discrete version of the momentum equation and using L^2-orthogonality, we have

$$\begin{aligned}
0 = &\int_0^T \int_\Omega \big\langle \partial_\tau^- \mathcal{R}_h(\rho_{\tau h}\mathbf{v}_{\tau h}), \mathbf{w}\big\rangle - \frac{1}{2}\int_0^T \int_\Omega \partial_\tau^- \rho_{\tau h} \langle \mathbf{v}_{\tau h}(\cdot, \cdot+\tau), \mathcal{R}_h \mathbf{w}\rangle \\
&- \frac{1}{2}\int_0^T \int_\Omega \rho_{\tau h} \Big\langle \mathbf{v}_{\tau h}, (\nabla \mathcal{R}_h \mathbf{w})^T \mathbf{v}_{\tau h}(\cdot, \cdot+\tau)\Big\rangle + \frac{1}{2}\int_0^T \rho_{\tau h}\Big\langle \mathbf{v}_{\tau h}, (\nabla \mathbf{v}_{\tau h}(\cdot,\cdot+\tau))^T \mathcal{R}_h \mathbf{w}\Big\rangle \\
&+\frac{1}{2}\int_\tau^{T+\tau}\int_\Omega \frac{\delta\rho}{\delta\varphi}\Big\langle \mathbf{j}_{\tau h}, (\nabla \mathbf{v}_{\tau h})^T \mathcal{R}_h \mathbf{w}(\cdot, \cdot - \tau)\Big\rangle - \frac{1}{2}\int_\tau^{T+\tau}\int_\Omega \frac{\delta\rho}{\delta\varphi}\Big\langle \mathbf{j}_{\tau h}, (\nabla \mathcal{R}_h \mathbf{w}(\cdot,\cdot-\tau))^T \mathbf{v}_{\tau h}\Big\rangle \\
&+ \int_\tau^{T+\tau}\int_\Omega 2\eta(\varphi_{\tau h}(\cdot, \cdot-\tau))\mathbf{D}\mathbf{v}_{\tau h} : \mathbf{D}\mathcal{R}_h \mathbf{w} + \int_0^T \int_\Omega \varphi_{\tau h}\langle \nabla \mu_{\tau h}(\cdot, \cdot+\tau), \mathcal{R}_h \mathbf{w}\rangle \\
&=: L_1 + \ldots + L_8. \quad (4.24)
\end{aligned}$$

In the sequel, we sometimes do not indicate domains of integration if they are identical with $(0,T)$ or with Ω, and we estimate

$$|L_2| \le C \left\|\partial_\tau^- \rho_{\tau h}\right\|_{L^2(L^2)} \cdot \|\mathbf{v}_{\tau h}(\cdot, \cdot+\tau)\|_{L^{\frac{10}{3}}\left((\tau, T+\tau); L^{\frac{10}{3}}(\Omega)\right)} \cdot \|\mathcal{R}_h \mathbf{w}\|_{L^5(W^{1,2})}.$$

By the energy estimate (3.3) and interpolation between $L^\infty(L^2)$ and $L^2(W^{1,2})$ (see e.g. Proposition 3.3 in [14]), we have $\mathbf{v}_{\tau h} \in L^{\frac{8}{d}}(L^4)$. Hence,

$$|L_3| \le C \|\mathbf{v}_{\tau h}\|_{L^{\frac{8}{d}}(L^4)} \|\mathbf{v}_{\tau h}(\cdot, \cdot+\tau)\|_{L^{\frac{8}{d}}((\tau, T+\tau); L^4(\Omega))} \|\nabla \mathcal{R}_h \mathbf{w}\|_{L^{\frac{8}{8-2d}}(L^2)}.$$

To estimate L_5, we use (4.22) and obtain

$$|L_5| \le C \|\mathbf{j}_{\tau h}\|_{L^{\frac{8}{3}-}((\tau, T+\tau); L^3(\Omega))} \cdot \|\nabla \mathbf{v}_{\tau h}\|_{L^2((\tau, T+\tau); \Omega)} \cdot \|\mathcal{R}_h \mathbf{w}\|_{L^{8+}(W^{1,2})}.$$

The remaining terms can be estimated in the same spirit, and we get

$$|L_1| \le C \cdot \|\mathcal{R}_h \mathbf{w}\|_{L^{8+}(W^{1,2})} \overset{\text{(due to (2.4))}}{\le} C \cdot \|\mathbf{w}\|_{L^{8+}(W^{1,2})},$$

which gives the assertion. □

Lemma 4.7. *There is a subsequence $(\tau, h) \to 0$ such that $\mathcal{R}_h(\rho_{\tau h}\mathbf{v}_{\tau h})$ strongly converges in $L^2(\Omega_T)$ to a limit function $\mathbf{Z}$.*

Proof. Note that

$$\begin{aligned}
\int_0^T \|\nabla(\rho_{\tau h}\mathbf{v}_{\tau h})\|_{L^2}^2 \le &C\left(\sup_{t\in(0,T)} \|\Delta \varphi_{\tau h}\|_{L^2}^2\right) \cdot \int_0^T \|\mathbf{v}_{\tau h}\|_{L^6}^2 \\
&+ C\int_0^T \|\nabla \mathbf{v}_{\tau h}\|_{L^2}^2 \le const.
\end{aligned} \quad (4.25)$$

by (4.7), (6.4) and the boundedness of $\rho_{\tau h}$.
As $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \hookrightarrow \big(\mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega)\big)'$, the assertion of the lemma follows in a standard way by combination of (4.25), (4.23), and Theorem 6.1. □

Let us prove now that there is a subsequence $(\mathbf{v}_{\tau h})_{(\tau,h)\searrow 0}$ such that $\mathbf{v}_{\tau h}$ strongly converges to $\mathbf{v} \in L^2(\Omega_T)$ with respect to the L^2-norm. In particular, the limit function is contained in $L^2\big((0,T); \mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega)\big)$.

Lemma 4.8. *Under the assumptions*

$$\mathbf{v}_{\tau h} \in L^2((0,T);\mathbf{W}_h) \quad \text{for all } h>0, \tau>0, \tag{4.26}$$

$$\mathcal{R}_h(\rho_{\tau h}\mathbf{v}_{\tau h}) \to \mathbf{Z} \quad \text{strongly in } L^2(\Omega_T), \tag{4.27}$$

$$\mathbf{v}_{\tau h} \rightharpoonup \mathbf{v} \quad \text{weakly} * \text{ in } L^\infty\big((0,T);L^2\big) \cap L^2\big((0,T);H^1\big), \tag{4.28}$$

$$\rho_{\tau h} \rightharpoonup \rho \quad \text{weakly} * \text{ in } L^\infty\big((0,T);H^1\big) \cap H^1\Big(\big(H^1\big)'\Big), \tag{4.29}$$

$$\rho_{\tau h} \to \rho \quad \text{strongly in } L^2(\Omega_T), \tag{4.30}$$

the following is true for a subsequence $(\tau,h)\to 0$*:*

$$\mathbf{Z} = P_\sigma(\rho\mathbf{v}), \tag{4.31}$$

$$\int_0^T\int_\Omega \rho_{\tau h}\,|\mathbf{v}_{\tau h}|^2 \to \int_0^T\int_\Omega \rho\,|\mathbf{v}|^2, \tag{4.32}$$

$$\operatorname{div}\mathbf{v} = 0, \tag{4.33}$$

$$\mathbf{v}_{\tau h} \to \mathbf{v} \text{ strongly in } L^2(\Omega_T)\,. \tag{4.34}$$

Proof. The proof will be divided into three steps.

Step 1: Let $(\mathbf{y}_h)_{h\searrow 0}$ be a bounded sequence in $L^2\big((0,T);\mathbf{W}_0^{1,2}(\Omega)\big)$ and assume that

$$\mathcal{R}_h\mathbf{y}_h \rightharpoonup \mathbf{y} \qquad \text{in } L^2\big((0,T);\mathbf{W}_0^{1,2}(\Omega)\big). \tag{4.35}$$

Then,

$$\operatorname{div}\mathbf{y} = 0 \qquad \text{a.e. in } \Omega\times(0,T). \tag{4.36}$$

Proof of Step 1: Take $q\in L^2((0,T);H^1(\Omega))$ arbitrarily. Hence,

$$\iint_{\Omega_T} q\operatorname{div}\mathbf{y} = \iint_{\Omega_T}(q-\mathcal{S}_hq)\operatorname{div}\mathbf{y} + \iint_{\Omega_T}\mathcal{S}_hq(\operatorname{div}\mathbf{y}-\operatorname{div}\mathcal{R}_h\mathbf{y}_h) + \iint_{\Omega_T}\mathcal{S}_hq\operatorname{div}\mathcal{R}_h\mathbf{y}_h = I(h)+II(h)+III(h)\,.$$

By (2.7), $\mathcal{S}_hq\to q$ strongly in $L^2(\Omega_T)$ (using Lebesque's convergence theorem) and therefore $\lim_{h\to 0} I(h)=0$. For $II(h)$, we observe that $\operatorname{div}\mathcal{R}_h\mathbf{y}_h \rightharpoonup \operatorname{div}\mathbf{y}$ in $L^2(\Omega_T)$ and that $\mathcal{S}_hq$ strongly converges to q in $L^2(\Omega_T)$. Therefore, this term vanishes in the limit, too. Finally, $III(h)$ is always zero due to (2.14b). Now use that $L^2((0,T);H^1)$ is dense in $L^2(\Omega_T)$ and (4.36) is established. □

Step 2: We have

$$\mathbf{Z} = \lim_{h\to 0}\mathcal{R}_h(\rho_{\tau h}\mathbf{v}_{\tau h}) = P_\sigma(\rho\mathbf{v}). \tag{4.37}$$

Proof of Step 2: By the identity $\mathcal{R}_h\mathbf{v}_{\tau h}=\mathbf{v}_{\tau h}$ and the orthogonality of the L^2-projection $\mathcal{R}_h$, we have

$$\iint_{\Omega_T}\langle\rho_{\tau h}\mathbf{v}_{\tau h},\mathbf{v}_{\tau h}\rangle = \iint_{\Omega_T}\langle\mathcal{R}_h(\rho_{\tau h}\mathbf{v}_{\tau h}),\mathbf{v}_{\tau h}\rangle\,.$$

By (4.27) and (4.28), we infer

$$\lim_{h\to 0}\iint_{\Omega_T}\langle\rho_{\tau h}\mathbf{v}_{\tau h},\mathbf{v}_{\tau h}\rangle = \iint_{\Omega_T}\langle\mathbf{Z},\mathbf{v}\rangle.$$

Let us prove that $\mathbf{Z} = P_\sigma(\rho\mathbf{v})$. Take $\mathbf{\Sigma} \in L^2\big((0,T); \mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega)\big)$ arbitrarily. We have

$$\iint_{\Omega_T} \langle \mathcal{R}_h(\rho_{\tau h}\mathbf{v}_{\tau h}), \mathcal{R}_h\mathbf{\Sigma}\rangle = \iint_{\Omega_T} \langle \rho_{\tau h}\mathbf{v}_{\tau h}, \mathcal{R}_h\mathbf{\Sigma}\rangle .$$

By estimate (2.5) and Lebesgue's theorem, we have the strong convergence $\mathcal{R}_h\mathbf{\Sigma} \to \mathbf{\Sigma}$ in $\mathbf{L}^2(\Omega)$. Since also $\rho_{\tau h} \to \rho$ strongly, we have

$$\begin{aligned}\iint_{\Omega_T} \langle \mathbf{Z}, \mathbf{\Sigma}\rangle = \lim_{h\to 0}\iint_{\Omega_T} \langle \mathcal{R}_h(\rho_{\tau h}\mathbf{v}_{\tau h}), \mathcal{R}_h\mathbf{\Sigma}\rangle &= \lim_{h\to 0}\iint_{\Omega_T} \langle \rho_{\tau h}\mathbf{v}_{\tau h}, \mathcal{R}_h\mathbf{\Sigma}\rangle \\ &= \iint_{\Omega_T} \langle \rho\mathbf{v}, \mathbf{\Sigma}\rangle\end{aligned}$$

for all $\mathbf{\Sigma} \in L^2(I; \mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega))$. Therefore, $P_\sigma(\rho\mathbf{v}) = P_\sigma(\mathbf{Z})$. Step 1 implies that $\operatorname{div}\mathbf{Z} = 0$ and therefore $P_\sigma(\rho\mathbf{v}) = \mathbf{Z}$. □

Step 3: $\mathbf{v}_{\tau h} \to \mathbf{v}$ strongly in $L^2(\Omega_T)$.

Here, we translate ideas of [2] to the discrete setting. Introducing $\sigma_{\tau h} := \rho_{\tau h}^{\frac{1}{2}}\mathbf{v}_{\tau h}$, we find

$$\sigma_{\tau h} \rightharpoonup \rho^{\frac{1}{2}}\mathbf{v} \qquad \text{in } L^2(\Omega_T).$$

At the same time

$$\iint_{\Omega_T} |\sigma_{\tau h}|^2 = \iint_{\Omega_T} \langle \rho_{\tau h}\mathbf{v}_{\tau h}, \mathbf{v}_{\tau h}\rangle \overset{(\mathbf{v}_{\tau h} = \mathcal{R}_h\mathbf{v}_{\tau h})}{=} \iint_{\Omega_T} \langle \mathcal{R}_h(\rho_{\tau h}\mathbf{v}_{\tau h}), \mathbf{v}_{\tau h}\rangle$$

converges to $\iint_{\Omega_T} \langle P_\sigma(\rho\mathbf{v}), \mathbf{v}\rangle$. By Step 1 and assumption (4.28), $\mathbf{v}$ is solenoidal. Hence,

$$\iint_{\Omega_T} \langle P_\sigma(\rho\mathbf{v}), \mathbf{v}\rangle = \iint_{\Omega_T} \langle \rho\mathbf{v}, \mathbf{v}\rangle = \iint_{\Omega_T} \left|\rho^{\frac{1}{2}}\mathbf{v}\right|^2 .$$

Therefore, $\sigma_{\tau h} \to \rho^{\frac{1}{2}}\mathbf{v}$ strongly in $L^2(\Omega_T)$. To obtain (4.34), use that $\rho_{\tau h}$ is bounded from below by a positive constant (see (H4)). The lemma is proven.

□

5. Passage to the limit $(\tau, h) \to 0$

Let us begin this section by stating some boundedness and convergence results not explicitly mentioned before. By (4.7) and (6.4) in Theorem 6.4, we observe that

$$(\varphi_{\tau h}) \text{ is uniformly bounded in } L^\infty\big(W^{1,p}(\Omega)\big) \text{ for all } p < \frac{2d}{d-2}. \tag{5.1}$$

Moreover, a subsequence $(\tau, h) \to 0$ exists such that

$$(\mathbf{v}_{\tau h}) \text{ strongly to } \mathbf{v} \text{ in } L^2(\Omega_T)\,, \tag{5.2}$$

$$(\mathcal{R}_h(\rho_{\tau h}\mathbf{v}_{\tau h})) \text{ strongly to } P_\sigma(\rho\mathbf{v}) \text{ in } L^2(\Omega_T)\,, \tag{5.3}$$

$$(\varphi_{\tau h}) \text{ strongly to } \varphi \text{ in } L^2(\Omega_T)\,, \tag{5.4}$$

$$(\mu_{\tau h}) \text{ weakly to } \mu \text{ in } L^2((0,T); H^1(\Omega)). \tag{5.5}$$

For a proof of (5.2) and (5.3), see Lemma 4.8, in particular formula (4.37). Moreover, (5.4) and (5.5) follow from (4.8) combined with the energy estimate (3.3) and the compactness result of Simon (see Theorem 6.1).

The most critical term for the passage to the limit in (2.14) is the fifth term in equation (2.14a). At present, we only know $\mathbf{j}_{\tau h}$ and $\nabla\mathbf{v}_{\tau h}$ weakly to converge in certain L^p-spaces of space and time to $\mathbf{j} = -\nabla\mu$ or to $\nabla\mathbf{v}$, respectively. On a formal level, one might argue that the divergence of $\mathbf{j}_{\tau h}$ is controlled. On a rigorous level, we are only able to bound the discrete Laplacian of $\mu_{\tau h}$. But it turns out that this is already sufficient due to the $L^4(L^2)\cap L^2(L^{6-})$-regularity of $\mathbf{j}_{\tau h}$. The following lemma allows to identify weak limits.

Lemma 5.1. *There is a subsequence $(\tau,h)\to 0$ such that $\nabla\mathbf{v}_{\tau h}\mathbf{j}_{\tau h}$ weakly converges in $L^q((0,T);L^{\frac{6}{5}}(\Omega))$, $1\le q<\frac{5}{4}$, to $-\nabla\mathbf{v}\nabla\mu$.*

Proof. Combining (3.3) with (4.22) yields uniform boundedness of $(\nabla\mathbf{v}_{\tau h}\mathbf{j}_{\tau h})_{(\tau,h)\to 0}$ in the space $L^q((0,T);L^{\frac{6}{5}}(\Omega))$ for any $q<\frac{5}{4}$. Hence, a weakly convergent subsequence exists with limit $\mathbf{Z}$ in that space. Let us identify $\mathbf{Z}=-\nabla\mathbf{v}\nabla\mu$. First, we note the existence of a set $\mathcal{E}\subset(0,T)$ with $\mu_1(\mathcal{E})=T$, such that $-\nabla\mathbf{v}_{\tau h}\nabla\mu_{\tau h}=\nabla\mathbf{v}_{\tau h}\mathbf{j}_{\tau h}$ weakly converges to $\mathbf{Z}$ for all $t\in\mathcal{E}$. Next, we show that for $t\in\mathcal{E}$

$$\mathbf{Z}(\cdot,t) = -\nabla\mathbf{v}(\cdot,t)\nabla\mu(\cdot,t)\,.$$

To this scope, we consider for $t\in\mathcal{E}$ the auxiliary problems

$$-\Delta\mathcal{M}_{\tau h}(\cdot,t) = w_{\tau h}(\cdot,t) \qquad \text{in } \Omega, \tag{5.6a}$$

$$\frac{\partial}{\partial\nu}\mathcal{M}_{\tau h}(\cdot,t) = 0 \qquad \text{on } \partial\Omega, \tag{5.6b}$$

$$\int_\Omega \mathcal{M}_{\tau h}(x,t)\mathrm{d}x = \int_\Omega \mu_{\tau h}(x,t)\mathrm{d}x, \tag{5.6c}$$

where $w_{\tau h}\in H^1_*(\Omega)\cap U_h$ is the negative discrete Laplacian of $\mu_{\tau h}$ (cf. (4.19)). By (6.3), (5.6c), and the mean value Poincaré inequality, we have the existence of a positive constant C such that

$$\|(\mu_{\tau h}-\mathcal{M}_{\tau h})(\cdot,t)\|_{H^1(\Omega)} \le Ch\,\|w_{\tau h}(\cdot,t)\|_{L^2(\Omega)} \tag{5.7}$$

for all $t\in\mathcal{E}$. By elliptic regularity theory, we have

$$\|\mathcal{M}_{\tau h}(\cdot,t)\|_{W^{1,6}(\Omega)} \le C\,\|w_{\tau h}(\cdot,t)\|_{L^2} \tag{5.8}$$

uniformly in $\mathcal{E}$ and for $(\tau,h)\to 0$. From (5.7), we infer

$$\int_\Omega |(\nabla\mathbf{v}_{\tau h}\nabla\mathcal{M}_{\tau h}-\nabla\mathbf{v}_{\tau h}\nabla\mu_{\tau h})(\cdot,t)| \le \|\nabla(\mathcal{M}_{\tau h}-\mu_{\tau h})(\cdot,t)\|_{L^2}\cdot\|\nabla\mathbf{v}_{\tau h}(\cdot,t)\|_{L^2} \overset{h\to 0}{\to} 0$$

Due to the L^6-regularity of $\nabla\mathcal{M}_{\tau h}$, we have $\nabla\mathbf{v}_{\tau h}(\cdot,t)\nabla\mathcal{M}_{\tau h}(\cdot,t)\rightharpoonup -\mathbf{Z}(\cdot,t)$ in $L^{\frac{3}{2}-}(\Omega)$. Now observe the following identity for arbitrary $\mathbf{\Sigma}\in C^1(\Omega;\mathbb{R}^3)$

$$\begin{aligned}\int_\Omega \mathbf{Z}(\cdot,t)\mathbf{\Sigma} &= -\lim_{(\tau,h)\to 0}\int_\Omega \nabla\mathcal{M}_{\tau h}(\cdot,t)(\nabla\mathbf{v}_{\tau h}(\cdot,t))^T\mathbf{\Sigma}\\ &= \lim_{(\tau,h)\to 0}\left\{\int_\Omega(\Delta\mathcal{M}_{\tau h}\mathbf{v}_{\tau h})(\cdot,t)\mathbf{\Sigma}+\int_\Omega\nabla\mathcal{M}_{\tau h}(\cdot,t)(\nabla\mathbf{\Sigma})^T\mathbf{v}_{\tau h}(\cdot,t)\right\}\\ &= \int_\Omega w\langle\mathbf{v}(\cdot,t),\mathbf{\Sigma}\rangle+\int_\Omega\nabla\mathcal{M}(\nabla\mathbf{\Sigma})^T\mathbf{v} = -\int_\Omega\langle\nabla\mu,(\nabla\mathbf{v})^T\mathbf{\Sigma}\rangle\,.\end{aligned} \tag{5.9}$$

Here, we used in particular that $w_{\tau h}\rightharpoonup w$ in $L^2(\Omega_T)$, and that $\mu_{\tau h}\rightharpoonup\mu$ in $L^2(\Omega_T)$ and w.l.o.g. pointwise in $t\in\mathcal{E}$ for an appropriate subsequence. In particular,

$$\int_\Omega\langle\nabla\mu,\nabla\theta\rangle = \int_\Omega w\theta$$

for all $\theta \in H^1(\Omega)$. Since $C^1(\Omega;\mathbb{R}^3)$ is dense in $\mathbf{L}^p(\Omega)$ for all $1 \le p < \infty$, we have the identity

$$\mathbf{Z}(\cdot,t) = (-(\nabla\mathbf{v})\nabla\mu)(\cdot,t)$$

for all $t \in \mathcal{E}$. This gives the assertion of the lemma. □

Now we are in the position to state a first convergence result.

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a convex polyhedral domain and let initial data Φ_0 and $\mathbf{V}_0$ be given. Let $I = (0,T)$. Assume that (H1)-(H4), (S1)-(S6), and (T1), (T2) are satisfied and that $(\varphi_{\tau h}, \mu_{\tau h}, \mathbf{v}_{\tau h})$ is a sequence of discrete solutions to the system* (2.14)*. Then functions*

$$\begin{aligned}
&\mathbf{v} \in L^\infty\big(I;L^2(\Omega)\big) \cap L^2\big(I;\mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega)\big)\,,\\
&\varphi \in L^\infty\big(I;H^2(\Omega)\big) \cap H^1\big(I;L^2(\Omega)\big)\,, \quad \varphi(\cdot,0) = \Phi_0(\cdot),\\
&\mu \in L^\infty\big(I;L^2(\Omega)\big) \cap L^2\big(I;W^{1,6-}(\Omega)\big) \cap L^4\big(I;W^{1,2}(\Omega)\big)
\end{aligned}$$

exist which solve the system (1.1) *in the generalized sense that*

$$\begin{aligned}
-\int_0^T\!\!\int_\Omega \langle \rho\mathbf{v} - \rho(\Phi_0)\mathbf{V}_0, \partial_t\mathbf{w}\rangle - \frac12\int_0^T\!\!\int_\Omega \partial_t\rho\langle\mathbf{v},\mathbf{w}\rangle - \frac12\int_0^T\!\!\int_\Omega \rho\Big\langle \mathbf{v}, (\nabla\mathbf{w})^T\mathbf{v}\Big\rangle\\
+\frac12\int_0^T\!\!\int_\Omega \rho\Big\langle \mathbf{v}, (\nabla\mathbf{v})^T\mathbf{w}\Big\rangle + \frac12\int_0^T\!\!\int_\Omega \frac{\delta\rho}{\delta\varphi}\Big\langle \mathbf{j}, (\nabla\mathbf{v})^T\mathbf{w}\Big\rangle - \frac12\int_0^T\!\!\int_\Omega \frac{\delta\rho}{\delta\varphi}\Big\langle \mathbf{j}, (\nabla\mathbf{w})^T\mathbf{v}\Big\rangle\\
+\int_0^T\!\!\int_\Omega 2\eta(\varphi)\mathbf{D}\mathbf{v} : \mathbf{D}\mathbf{w} = \int_0^T\!\!\int_\Omega \mu\langle\nabla\varphi,\mathbf{w}\rangle
\end{aligned} \tag{5.10}$$

for all $\mathbf{w} \in C^1\big(I;\mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega)\big)$ satisfying $\mathbf{w}(\cdot,T) = 0$,

$$\int_0^T\!\!\int_\Omega \partial_t\varphi\theta + \int_0^T\!\!\int_\Omega \langle\nabla\varphi,\mathbf{v}\rangle\theta + \int_0^T\!\!\int_\Omega \langle\nabla\mu,\nabla\theta\rangle = 0 \tag{5.11}$$

for all $\theta \in L^2(I;H^1(\Omega))$,

$$\mu(\cdot,t) = -\Delta\varphi(\cdot,t) + F'(\varphi(\cdot,t)) \tag{5.12}$$

for almost all $t \in I$. Moreover, for a subsequence $(\tau,h) \to 0$ the following convergence results hold true:

- $\mathbf{v}_{\tau h} \to \mathbf{v}$ *strongly in* $L^2(\Omega_T)$,
- $\mathcal{R}_h(\rho_{\tau h}\mathbf{v}_{\tau h}) \to P_\sigma(\rho\mathbf{v})$ *strongly in* $L^2(\Omega_T)$,
- $\varphi_{\tau h} \to \varphi$ *strongly in* $L^2(\Omega_T)$ *and in* $L^p\big(I;C^\beta(\Omega)\big)$ *for any* $p < \infty$ *and any* $\beta < 2 - \frac{d}{2}$,
- $\mu_{\tau h} - \mathcal{I}_h F'_+(\varphi_{\tau h}) - \mathcal{I}_h F'_-(\varphi_{\tau h}(\cdot,\cdot-\tau)) \rightharpoonup -\Delta\varphi$ *weakly∗ in the space* $L^\infty(I;L^2(\Omega)) \cap L^2(I;W^{1,6-}(\Omega)) \cap L^4(I;W^{1,2}(\Omega))$,
- $\mathcal{I}_h F'_-(\varphi_{\tau h}(\cdot,\cdot-\tau)) \to F'_-(\varphi)$ *strongly in* $L^p(\Omega_T)$ *for any* $1 \le p < \infty$,
- $\mathcal{I}_h F'_+(\varphi_{\tau h}) \to F'_+(\varphi)$ *strongly in* $L^p(\Omega_T)$ *for any* $1 \le p < \infty$.

Remark 5.3. *1. For the pressure, we obtain similar results as in the case of equal mass densities – see e.g.* [16]*. Choosing an arbitrary test function $\mathbf{w} \in X_h$ in* (2.14b)*, summing up over subintervals in time and using (S2), it follows that $\int_0^t p_{\tau h}(.,s)ds$ is uniformly bounded in $L^\infty(I;L^2(\Omega))$. Hence, a weak∗-limit exists for $(\tau,h) \to 0$, which may be used in a very weak solution concept in the sense of distributions allowing for non-solenoidal test functions in the momentum equation. For details, see the equal density case* [16].
2. Solutions constructed in Theorem 5.2 are generalized solutions to the system (1.1) *since it is a priori not possible to identify $\frac{\delta\rho}{\delta\varphi}$ with $\rho'(\varphi)$. In Corollary 5.4, we will show that they*

are in fact weak solutions as soon as φ attains values only in the interval $[-1-\bar{\varphi}, 1+\bar{\varphi}]$ with $\bar{\varphi}$ given in (H4). In this case, conservation of individual masses is guaranteed, too. Therefore, the question whether solutions constructed in Theorem 5.2 are in fact weak solutions of (1.1) *is reduced to the problem of finding optimal L^∞-bounds for the solutions in* (5.11). *See Corollary 5.5 for a first result in that direction.*

Proof. Let us begin with a discussion of the convergence results. The first two of them were already obtained in (5.2) and (5.3). The strong convergence of $\varphi_{\tau h} \to \varphi$ in $L^p(I; C^\beta(\Omega))$ for any $1 \le p < \infty$ and any $0 < \beta < 2 - \frac{d}{2}$ is a consequence of (4.8) combined with (5.1), Simon's Theorem 6.1 and the compactness of the embedding $W^{1,q}(\Omega) \hookrightarrow\hookrightarrow C^\beta(\Omega)$. The strong convergence of the terms $\mathcal{I}_h F'_-(\varphi_{\tau h}(\cdot, \cdot - \tau)$ and $\mathcal{I}_h F'_+(\varphi_{\tau h})$ is implied by the uniform boundedness of $\varphi_{\tau h}$ (see (5.1)) and its strong convergence in $L^p(I; C^\beta(\Omega))$ combined with Lemma 6.3. Finally, note that $\mu_{\tau h} - \mathcal{I}_h F'_+(\varphi_{\tau h}) - \mathcal{I}_h F'_-(\varphi_{\tau h}(\cdot, \cdot - \tau)) = -\Delta_h \varphi_{\tau h}$. By (4.7), a weakly convergent subsequence of $\Delta_h \varphi_{\tau h}$ exists, and by duality its limit can easily be identified with $-\Delta\varphi$ (using the weak convergence of $\nabla\varphi_{\tau h}$ in $L^\infty(I; L^p(\Omega))$ for all $p < \frac{2d}{d-2}$). Hence, (5.12) is established as well.
It remains to prove that (5.10)-(5.11) hold true. Let us begin with (5.10). Writing $N := \frac{T}{\tau}$, we take the sum $\tau \sum_{k=0}^{N-1}$ in the discrete equation (2.14a). Hence,

$$
\begin{aligned}
&\tau \sum_{k=0}^{N-1} \int_\Omega \left\langle \partial_\tau^- \mathcal{R}_h(\rho^{k+1}\mathbf{v}^{k+1}), \mathbf{w}^{k+1} \right\rangle - \frac{\tau}{2} \sum_{k=0}^{N-1} \int_\Omega \partial_\tau^- \rho^{k+1} \left\langle \mathbf{v}^{k+1}, \mathbf{w}^{k+1} \right\rangle \\
&\quad - \frac{\tau}{2} \sum_{k=0}^{N-1} \int_\Omega \rho^k \left\langle \mathbf{v}^k, (\nabla \mathbf{w}^{k+1})^T \mathbf{v}^{k+1} \right\rangle + \frac{\tau}{2} \sum_{k=0}^{N-1} \int_\Omega \rho^k \left\langle \mathbf{v}^k, (\nabla \mathbf{v}^{k+1})^T \mathbf{w}^{k+1} \right\rangle \\
&\quad + \frac{\tau}{2} \sum_{k=0}^{N-1} \int_\Omega \frac{\delta\rho}{\delta\varphi} \left\langle \mathbf{j}^{k+1}, (\nabla \mathbf{v}^{k+1})^T \mathbf{w}^{k+1} \right\rangle - \frac{\tau}{2} \sum_{k=0}^{N-1} \int_\Omega \frac{\delta\rho}{\delta\varphi} \left\langle \mathbf{j}^{k+1}, (\nabla \mathbf{w}^{k+1})^T \mathbf{v}^{k+1} \right\rangle \\
&\quad + \tau \sum_{k=0}^{N-1} \int_\Omega 2\eta(\varphi^{k+1}) \mathbf{D}\mathbf{v}^{k+1} : \mathbf{D}\mathbf{w}^{k+1} = -\tau \sum_{k=0}^{N-1} \int_\Omega \varphi^k \left\langle \nabla \mu^{k+1}, \mathbf{w}^{k+1} \right\rangle \qquad (5.13)
\end{aligned}
$$

for all step functions $\mathbf{w} \in S^{0,-1}(I; \mathbf{W}_h)$. Using (4.11), the first term can be rewritten

$$
\begin{aligned}
&\tau \sum_{k=0}^{N-1} \int_\Omega \left\langle \partial_\tau^- \mathcal{R}_h(\rho^{k+1}\mathbf{v}^{k+1}), \mathbf{w}^{k+1} \right\rangle \\
&= -\tau \sum_{k=0}^{N-1} \int_\Omega \left\langle \partial_\tau^+ \mathbf{w}^k, \mathcal{R}_h(\rho^k \mathbf{v}^k) - \mathcal{R}_h(\rho(\varphi_{0h})\mathbf{v}_{0h}) \right\rangle + \int_\Omega \left\langle \mathbf{w}^N, \mathcal{R}_h(\rho^N \mathbf{v}^N) - \mathcal{R}_h(\rho(\varphi_{0h})\mathbf{v}_{0h}) \right\rangle.
\end{aligned}
\qquad (5.14)
$$

Now choose $\mathbf{\Sigma} \in C^1([0,T]; \mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega)) \cap C^1([0,T]; \mathbf{H}^2(\Omega))$ with $\mathbf{\Sigma}(\cdot, T) = 0$ arbitrarily, but fixed. Take $\mathbf{\Sigma}_{\tau h}|_{I_k} := \mathbf{Q}^h_{\mathrm{div}} \mathbf{\Sigma}(\cdot, t_k)$. Recalling (2.8) and using $\mathbf{\Sigma}_{\tau h}(\cdot, T) = 0$, (5.13) may be rewritten as

$$\begin{aligned}
&-\int_0^T \int_\Omega \langle \mathcal{R}_h(\rho_{\tau h}\mathbf{v}_{\tau h}) - \mathcal{R}_h(\rho(\varphi_{0h})\mathbf{v}_{0h}), \partial_\tau^+ \mathbf{\Sigma}_{\tau h}\rangle \\
&\qquad -\frac{1}{2}\int_0^{T-\tau}\int_\Omega \partial_\tau^- \rho_{\tau h}\langle \mathbf{v}_{\tau h}(\cdot,\cdot+\tau), \mathbf{\Sigma}_{\tau h}(\cdot,\cdot+\tau)\rangle \\
&\qquad -\frac{1}{2}\int_0^{T-\tau}\int_\Omega \rho_{\tau h}\Big\langle \mathbf{v}_{\tau h}, (\nabla \mathbf{\Sigma}_{\tau h}(\cdot,\cdot+\tau))^T \mathbf{v}_{\tau h}(\cdot,\cdot+\tau)\Big\rangle \\
&\qquad +\frac{1}{2}\int_0^{T-\tau}\int_\Omega \rho_{\tau h}\Big\langle \mathbf{v}_{\tau h}, (\nabla \mathbf{v}_{\tau h}(\cdot,\cdot+\tau))^T \mathbf{\Sigma}_{\tau h}(\cdot,\cdot+\tau)\Big\rangle \\
&\quad +\frac{1}{2}\int_\tau^T\int_\Omega \frac{\delta\rho}{\delta\varphi}\Big\langle \mathbf{j}_{\tau h}, (\nabla \mathbf{v}_{\tau h})^T \mathbf{\Sigma}_{\tau h}\Big\rangle - \frac{1}{2}\int_\tau^T\int_\Omega \frac{\delta\rho}{\delta\varphi}\Big\langle \mathbf{j}_{\tau h}, (\nabla \mathbf{\Sigma}_{\tau h})^T \mathbf{v}_{\tau h}\Big\rangle \\
&+\int_\tau^T\int_\Omega 2\eta(\varphi_{\tau h}(\cdot,\cdot-\tau))\mathbf{D}\mathbf{v}_{\tau h} : \mathbf{D}\mathbf{\Sigma}_{\tau h} - \int_\tau^T\int_\Omega \varphi_{\tau h}(\cdot,\cdot-\tau)\langle \nabla\mu_{\tau h}, \mathbf{\Sigma}_{\tau h}\rangle = 0 \quad (5.15)
\end{aligned}$$

By (S4), formula (2.6), we infer the strong convergence of $\mathbf{\Sigma}_{\tau h}$ to $\mathbf{\Sigma}$ in $L^2(I;\mathbf{H}^1(\Omega))$ as well as the strong convergence of $\partial_\tau^+\mathbf{\Sigma}_{\tau h}$ to $\partial_t\mathbf{\Sigma}$ in $L^2(I;\mathbf{H}^1(\Omega))$. From (5.3), it follows that $\mathcal{R}_h(\rho_{\tau h}\mathbf{v}_{\tau h})$ strongly converges to $P_\sigma(\rho\mathbf{v})$ in $L^2(\Omega_T)$. In particular, $\mathcal{R}_h(\rho(\varphi_{0h})\mathbf{v}_{0h})$ strongly converges to $P_\sigma(\rho(\Phi_0)\mathbf{V}_0)$ in $L^2(\Omega)$ by arguments similar to those used in the proof of Lemma 4.8.

As the Helmholtz projection P_σ is orthogonal, we may identify the limit of the first term in (5.15) with the first term in (5.10). To discuss the second and third term in (5.15), we employ the weak convergence of $\partial_\tau^-\rho_{\tau h}$ towards $\partial_t\rho$, which is a direct consequence of the $L^2(\Omega_T)$-boundedness of $\partial_\tau^-\varphi_{\tau h}$ as well as of the results on strong convergence for $\mathbf{v}_{\tau h}$, $\nabla\mathbf{\Sigma}_{\tau h}$ and $\mathbf{\Sigma}_{\tau h}$ (see (5.2), (4.8), and (2.6)).

A similar reasoning also applies to the fourth, sixth and seventh term, this time taking advantage of the uniform boundedness (3.3), (4.22). For the fifth term, we use that $\nabla\mathbf{v}_{\tau h}\mathbf{j}_{\tau h}$ weakly converges to $-\nabla\mathbf{v}\nabla\mu$ in $L^q((0,T);L^{\frac{6}{5}}(\Omega))$ for all $1 \le q < \frac{5}{4}$, see Lemma 5.1. Together with the strong convergence of $\mathbf{\Sigma}_{\tau h}$, also this limit is readily identified. To discuss the last term, we observe that $\mu_{\tau h}$ weakly converges in $L^2(I;W^{1,6-}(\Omega)) \cap L^4(I;W^{1,2}(\Omega))$ to μ. Together with the strong convergence of $\varphi_{\tau h}$ in $L^p(I;C^\beta(\Omega))$ for any $1 \le p < \infty$, we find that this term converges to $-\int_0^T\int_\Omega \varphi\langle\nabla\mu,\mathbf{w}\rangle$. Integration by parts, using the solenoidality of $\mathbf{w}$, gives the result. Let us discuss (5.11). Similarly as in (5.10), we take the sum $\tau\sum_{k=0}^{N-1}$ in the discrete equation (2.14c). Replacing $\tau\sum_{k=0}^{N-1}$ by $\int_0^T dt$ and choosing $\psi \in C^0([0,T];H^2(\Omega))$ arbitrarily, but fixed, we have for $\psi_{\tau h}$ defined by the Ritz projection $\psi_{\tau h}|_{I_k} := \mathcal{P}_h\psi(\cdot,t_k)$ that

$$\int_0^T \big(\partial_\tau^-\varphi_{\tau h}, \psi_{\tau h}\big)_h - \int_0^T \langle \mathbf{v}_{\tau h}(\cdot,\cdot+\tau), \nabla\psi_{\tau h}\rangle\varphi_{\tau h} + \int_0^T \langle \nabla\mu_{\tau h}(\cdot,\cdot+\tau), \nabla\psi_{\tau h}\rangle = 0\,. \quad (5.16)$$

Combining (2.1) and (4.8) with the fact that $\mathcal{P}_h\psi(\cdot,\cdot)$ converges to ψ in $H^1(\Omega)$ for $h \to 0$ (see (2.3)), we find that the first term in (5.16) converges to the first term in (5.11) as (τ,h) tends to $(0,0)$. The remaining terms may be discussed in a standard way, using (5.2), (4.22), (5.1) as well as the approximation properties of $\mathcal{P}_h$. By a density argument and integration by parts in the second term, (5.11) follows. This proves the theorem. □

Corollary 5.4. *Consider the solution* $(\varphi, \mu, \mathbf{v})$ *obtained in Theorem 5.2 and assume that in a time-interval* $[0, \bar{T}]$ *the phase-field* $\varphi(\cdot, t)$ *attains only values in* $(-1-\bar{\varphi}, 1+\bar{\varphi})$ *with the parameter* $\bar{\varphi}$ *as in (H4). Then,* $(\varphi, \mu, \mathbf{v})$ *solves the system* (1.1) *on* $(0, \bar{T})$ *in the sense that*

$$\begin{aligned} -\int_0^{\bar{T}}\int_\Omega \langle \bar{\rho}\mathbf{v} - \bar{\rho}(\Phi_0)\mathbf{V}_0, \partial_t \mathbf{w}\rangle - \int_0^{\bar{T}}\int_\Omega \partial_t\bar{\rho}\langle \mathbf{v}, \mathbf{w}\rangle + \int_0^{\bar{T}}\int_\Omega \bar{\rho}\Big\langle \mathbf{v}, (\nabla \mathbf{v})^T\mathbf{w}\Big\rangle \\ + \int_0^{\bar{T}}\int_\Omega \frac{\partial\bar{\rho}}{\partial\varphi}\Big\langle \mathbf{j}, (\nabla\mathbf{v})^T\mathbf{w}\Big\rangle + \int_0^{\bar{T}}\int_\Omega 2\eta(\varphi)\mathbf{D}\mathbf{v} : \mathbf{D}\mathbf{w} = \int_0^{\bar{T}}\int_\Omega \mu\langle\nabla\varphi, \mathbf{w}\rangle \end{aligned} \tag{5.17}$$

for all $\mathbf{w} \in C^1\big((0, \bar{T}); \mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega)\big)$ *satisfying* $\mathbf{w}\big(\cdot, \bar{T}\big) = 0$,

$$\int_0^{\bar{T}}\int_\Omega \partial_t\varphi\theta + \int_0^{\bar{T}}\int_\Omega \langle\nabla\varphi, \mathbf{v}\rangle\theta + \int_0^{\bar{T}}\int_\Omega \langle\nabla\mu, \nabla\theta\rangle = 0 \tag{5.18}$$

for all $\theta \in L^2\big((0, \bar{T}); H^1(\Omega)\big)$,

$$\mu(\cdot, t) = -\Delta\varphi(\cdot, t) + F'(\varphi(\cdot, t)) \tag{5.19}$$

for almost all $t \in (0, \bar{T})$*. In particular,* (5.17) *is a weak formulation of* (1.1a).

Proof. Observe that (5.11) implies for $\varphi \in (-1-\bar{\varphi}, 1+\bar{\varphi})$ the identity

$$\begin{aligned} -\frac{1}{2}\int_0^{\bar{T}}\int_\Omega \partial_t\rho\langle\mathbf{v}, \mathbf{w}\rangle = \\ -\int_0^{\bar{T}}\int_\Omega \partial_t\rho\langle\mathbf{v}, \mathbf{w}\rangle + \frac{1}{2}\int_0^{\bar{T}}\int_\Omega \rho\Big\langle\mathbf{v}, (\nabla\mathbf{w})^T\mathbf{v}\Big\rangle + \frac{1}{2}\int_0^{\bar{T}}\int_\Omega \rho\Big\langle\mathbf{v}, (\nabla\mathbf{v})^T\mathbf{w}\Big\rangle \\ + \frac{1}{2}\int_0^{\bar{T}}\int_\Omega \frac{\delta\rho}{\delta\varphi}\Big\langle\mathbf{j}, (\nabla\mathbf{v})^T\mathbf{w}\Big\rangle + \frac{1}{2}\int_0^{\bar{T}}\int_\Omega \frac{\delta\rho}{\delta\varphi}\Big\langle\mathbf{j}, (\nabla\mathbf{w})^T\mathbf{v}\Big\rangle. \end{aligned} \tag{5.20}$$

Inserting (5.20) in (5.10) and using that $\bar{\rho} \equiv \rho$ on $(-1-\bar{\varphi}, 1+\bar{\varphi})$ gives the result. □

We are already in the position to formulate a criterion which guarantees that the assumption of Corollary 5.4 is satisfied. With a grain of salt, the condition $\bar{\varphi} < \frac{2\tilde{\rho}_1}{\tilde{\rho}_2 - \tilde{\rho}_1}$, see (H4), translates to the condition that the modulus of φ is bounded by the inverse Atwood number. The Atwood number $\frac{\tilde{\rho}_2-\tilde{\rho}_1}{\tilde{\rho}_1+\tilde{\rho}_2}$ itself is a measure for the density contrast. Obviously, it attains values close to zero for a small density contrast and values close to one for a large density contrast. Hence, combining a continuous counterpart of the estimate (4.7) with Corollary 5.4, we have the following result.

Corollary 5.5. *For given initial data* $\Phi_0, \mathbf{V}_0$*, there is a number* $\alpha \in (0,1)$ *such that for all Atwood numbers* $\frac{\tilde{\rho}_2-\tilde{\rho}_1}{\tilde{\rho}_1+\tilde{\rho}_2} < \alpha$ *there exists a positive time* $\hat{T}$ *such that the generalized solution* (5.10) – (5.12) *is a weak solution to* (1.1).

Let us conclude the paper with some remarks related to practical computations. For given grid parameters τ and h, given external forces $\mathbf{k}_{grav}$, and a given tolerance $\varepsilon > 0$, it is always possible to find a regularization of a logarithmic potential which guarantees that discrete solutions $\varphi_{\tau h}$ corresponding to τ, h are confined to the interval $(-1-\varepsilon, 1+\varepsilon)$ – for a related reasoning in the case of the thin-film equation with singular potentials, see [21], Section 5. It remains an open problem, however, whether there is a regularization depending on h such that the limit process $h \to 0$ can be performed.
Finally, numerical experiments: This scheme has been implemented in two and in three

space dimensions – see [22] and [30]. Numerical experiments – ranging from Rayleigh-Taylor instabilities to rising drop experiments and comparisons with other modeling approaches (see [22]) – show the full practicality of this approach. In particular, in characteristic 2D simulations, an experimental order of convergence of $EOC = 1.9$ has been obtained.

6. Appendix

In the appendix, we collect a number of results frequently used in the paper. We begin with statements on relative compactness.

Theorem 6.1 (Simon [33, page 84])**.** *Let $X \subset B \subset Y$ be Banach spaces with compact embedding $X \hookrightarrow B$ and $1 \leq p \leq \infty$. If $F \subset L^p(I;X)$ is bounded and*

$$\|f(\cdot, \cdot + h) - f(\cdot, \cdot)\|_{L^p(0,T-h;Y)} \to 0$$

uniformly for $f \in F$ as $h \to 0$, then F is relatively compact in $L^p(I;B)$.

Theorem 6.2 (Fréchet-Kolmogorov [7])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. For $1 \leq p < \infty$, a set $A \subset L^p(\Omega)$ is relatively compact if and only if*

i) $\sup_{f\in A} \|f\|_{L^p} \leq C < \infty$

ii) $\sup_{f\in A} \|f(\cdot + h) - f\|_{L^p} \to 0$ *for* $|h| \to 0$.

We also need the following straight forward consequence of Theorem 6.2.

Lemma 6.3. *Given a sequence $(\tau_n, h_n)_{n\in\mathbb{N}} \to 0$, assume ρ_{τ_n,h_n} to converge strongly in the space $L^2(0,T;L^2(\Omega))$ to ρ. Then the sequence $\big(\rho_{\tau_n,h_n}(\cdot, \cdot + \tau_n) \cdot \chi_{[0,T-\tau_n)}\big)_{n\in\mathbb{N}}$ strongly converges to ρ in the space $L^2((0,T);L^2(\Omega))$.*

For the reader‘s convenience, we reformulate regularity results for the discrete Laplacian associated with the lumped masses scalar product which were part of the proof of Theorem 6.1 in [21]. Note in particular that (6.4) is a discrete version of Sobolev's embeddding theorem.

Theorem 6.4. *Let Ω be a convex, polyhedral domain in $\mathbb{R}^d$, $d \in \{2,3\}$. Related to $f_h \in U_h \cap H^1_*(\Omega)$, consider*

i) *$\phi \in H^1_*(\Omega)$ which solves the variational equation*

$$\int_\Omega \langle \nabla\phi, \nabla\psi\rangle = \int_\Omega f_h \psi \tag{6.1}$$

for all $\psi \in H^1(\Omega)$,

ii) *$\Phi_h \in U_h \cap H^1_*(\Omega)$ which solves the discrete variational equation*

$$\int_\Omega \langle \nabla\Phi_h, \nabla\Psi_h\rangle = (f_h, \Psi_h)_h \tag{6.2}$$

for all $\Psi_h \in U_h$.

Then,

i) *there is a positive constant C such that*

$$\|\nabla\Phi_h - \nabla\phi\|_{L^2(\Omega)} \leq C \cdot h\|f_h\|_{L^2(\Omega)}, \tag{6.3}$$

ii) *for any $1 \leq p < \frac{2d}{d-2}$, there is a positive constant $C(p)$ such that*

$$\|\nabla\Phi_h\|_{L^p(\Omega)} \leq C(p)\|f_h\|_{L^2(\Omega)}. \tag{6.4}$$

Our last result guarantees H^1-stability of the L^2-projection $\mathcal{R}_h$ onto the space of discretely divergence free vector fields.

Lemma 6.5. *Let $\mathcal{T}_h$ be a quasi-uniform triangulation of the polyhedral domain Ω and let the triple $(\mathbf{X}_h, \mathbf{W}_h, S_h)$ denote the function spaces corresponding to P_2P_0-elements or to Taylor-Hood elements. Then, there is a constant C, such that*

$$\|\nabla \mathcal{R}_h \mathbf{v}\|_{\mathbf{L}^2(\Omega)} \leq C \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega)} \tag{6.5}$$

for all $\mathbf{v} \in \mathbf{W}_0^{1,2}(\Omega)$.

Proof. Our argument is based on the explicit formulae for the orthogonal L^2-projection from $\mathbf{X}_h$ onto $\mathbf{W}_h$ to be found in Section A.5.3 of [20]. For the ease of presentation, we consider here only the case $d = 3$, the case $d = 2$ being analogous. Writing N_v for the degrees of freedom of each component of the velocity field, we have for an arbitrary element of $\mathbf{X}_h$

$$\mathbf{u}_h = \sum_{i=1}^{N_v} \begin{pmatrix} u_i \\ v_i \\ w_i \end{pmatrix} \phi_i^h$$

with real coefficients u_i, v_i, w_i and finite-element basis functions $\phi_i^h \in H_0^1(\Omega)$. Similarly, we write N_p for the degrees of freedom of the pressure and have

$$\lambda_h = \sum_{k=1}^{N_p} \lambda_k \psi_k^h \in U_h^0$$

for a generic element in the space of pressure functions.
Following the presentation in [20] (which is restricted to the two-dimensional case), we introduce matrices

$$M^h \in \mathbb{R}^{N_v \times N_v}, \qquad M_{ij}^h := \int_\Omega \phi_i^h(x) \phi_j^h(x)$$

and $C_x^h, C_y^h, C_z^h \in \mathbb{R}^{N_v \times N_p}$ defined by

$$\begin{aligned}
(C_x^h)_{ik} &:= -\int_\Omega \psi_k^h \frac{\partial}{\partial x} \phi_i^h, \\
(C_y^h)_{ik} &:= -\int_\Omega \psi_k^h \frac{\partial}{\partial y} \phi_i^h, \\
(C_z^h)_{ik} &:= -\int_\Omega \psi_k^h \frac{\partial}{\partial z} \phi_i^h,
\end{aligned}$$

$i, j = 1, \cdots, N_v$, $k = 1, \cdots, N_p$.
Given an element

$$\tilde{\mathbf{u}}_h := \sum_{i=1}^{N_v} \begin{pmatrix} \tilde{u}_i \\ \tilde{v}_i \\ \tilde{w}_i \end{pmatrix} \phi_i^h \in \mathbf{X}_h,$$

the orthogonal projection

$$\mathcal{R}_h \tilde{\mathbf{u}}_h =: \sum_{i=1}^{N_v} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \phi_i^h \in \mathbf{W}_h$$

is obtained by solving the linear system

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} I - (M^h)^{-1} C_x^h A_h^{-1} (C_x^h)^T & -(M^h)^{-1} C_x^h A_h^{-1} (C_y^h)^T & -(M^h)^{-1} C_x^h A_h^{-1} (C_z^h)^T \\ -(M^h)^{-1} C_y^h A_h^{-1} (C_x^h)^T & I - (M^h)^{-1} C_y^h A_h^{-1} (C_y^h)^T & -(M^h)^{-1} C_y^h A_h^{-1} (C_z^h)^T \\ -(M^h)^{-1} C_z^h A_h^{-1} (C_x^h)^T & -(M^h)^{-1} C_z^h A_h^{-1} (C_y^h)^T & I - (M^h)^{-1} C_z^h A_h^{-1} (C_z^h)^T \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} \tag{6.6}$$

where

$$A_h := (C_x^h)^T (M^h)^{-1} C_x^h + (C_y^h)^T (M^h)^{-1} C_y^h + (C_z^h)^T (M^h)^{-1} C_z^h.$$

Concerning well-posedness of this system, we refer to [34] and [20].
In order to prove H^1-stability of $\mathcal{R}_h$, we have to discuss two issues. First, we show that the matrix in (6.6) scales as h^0 in the case of a quasi-uniform triangulation $\mathcal{T}_h$. Indeed, M^h scales like h^3 and the matrices C^h_x, C^h_y, C^h_z scale like h^2 – using the fact that the volume of elements $T \in \mathcal{T}_h$ scale in three spatial dimensions like h^3. Hence, A_h scales like h and the matrices of the generic type $(M^h)^{-1}C^h_x A_h^{-1}(C^h_x)^T$ scale like $h^0 = 1$. As a consequence, H^1-stability follows for the restriction of $\mathcal{R}_h$ on $\mathbf{X}_h$.
To show also H^1-stability of $\mathcal{R}_h : \mathbf{W}^{1,2}_0(\Omega) \to \mathbf{W}_h$, we note that

$$\mathcal{R}_h = \mathcal{R}_h \circ \mathcal{Q}_h \tag{6.7}$$

on $\mathbf{W}^{1,2}_0(\Omega)$ where $\mathcal{Q}_h : \mathbf{W}^{1,2}_0(\Omega) \to \mathbf{X}_h$ is the orthogonal L^2-projection which is well-known to be H^1-stable, see [10], [11], and [13] . For a proof of (6.7), we refer to Section A.5.6 in [20]. This proves the second assertion of the lemma. □

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