On regularity properties of solutions to hysteresis-type problems *

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Abstract

We consider equations with the simplest hysteresis operator at the right-hand side. Such equations describe the so-called processes "with memory" in which various substances interact according to the hysteresis law.

We present some results concerning the optimal regularity of solutions. Our arguments are based on quadratic growth estimates for solutions near the free boundary.

1 Introduction.

In this paper we study the regularity properties of bounded solutions of the following parabolic free boundary problem:

$$H[u] = h[u] \quad \text{in} \quad Q = \mathcal{U}\times]0, T]. \tag{1}$$

Eq. (1) is understood in the weak (distributional) sence. Here $H = \Delta - \partial_t$ is the heat operator, $\mathcal{U}$ is a domain in $\mathbb{R}^n$, and h is a hysteresis-type operator acting from $C(\overline{Q})$ to $\{\pm 1\}$ which is defined as follows.

We fix two numbers α and β ($\alpha < \beta$) and consider a **multivalued** function

$$f(s) = \begin{cases} -1, & \text{for} \quad s \in]-\infty, \alpha], \\ 1, & \text{for} \quad s \in [\beta, +\infty[, \\ -1 \text{ or } 1, & \text{for} \quad s \in]\alpha, \beta[. \end{cases}$$

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For $u \in C(\overline{Q})$ we suppose that on the bottom of the cylinder Q the initial values of u as well as of $h[u](x,0) := f(u(x,0))$ are prescribed.

After that for every point $z = (x,t) \in Q$ the corresponding value of $h[u](z)$ is uniquely defined in the following manner. Let us denote by E a set of points

$$E := \{z \in Q : u(z) \leqslant \alpha\} \cup \{z \in Q : u(z) \geqslant \beta\} \cup \{\mathcal{U} \times \{0\}\}\,.$$

In other words, E is a set where $f(u(z))$ is well-defined.

If $z \in E$ then $h[u](z) = f(u(z))$. Otherwise, for $z = (x,t) \in Q$ such that $\alpha < u(z) < \beta$ we set

$$h[u](x,t) = h[u](x,\hat{t}(x)). \tag{2}$$

Here

$$\hat{t}(x) = \max_{[0,t]} \{s : (x,s) \in E\}$$

Roughly speaking, condition (2) means that the hysteresis function $h[u](x,t)$ takes for $u(x,t) \in (\alpha,\beta)$ the same value as "at the previous moment" (see Figure 1).

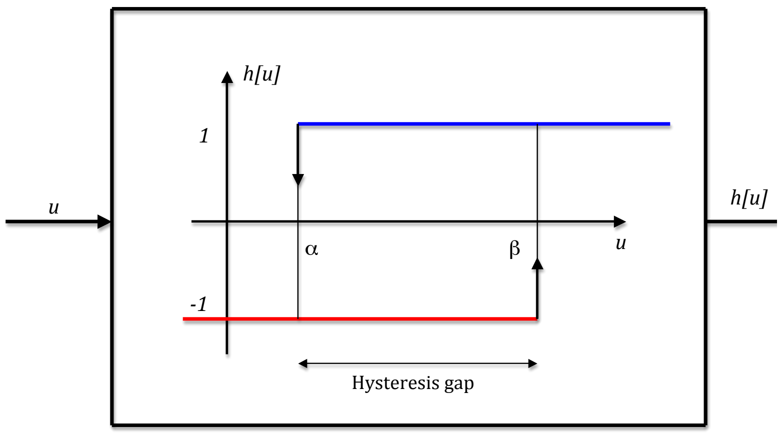

Figure 1: The hysteresis operator h

Let us emphasize that for fixed x a jump of $h[u](x,\cdot)$ can happen only on thresholds $\{u(x,t) = \alpha\}$ and $\{u(x,t) = \beta\}$. Moreover, **"jump down"** (from $h = 1$ to $h = -1$) is possble on $\{u(x,t) = \alpha\}$ only, whereas **"jump up"** (from $h = -1$ to $h = 1$) is possible on $\{u(x,t) = \beta\}$ only.

Thus, the cylinder Q consists of two disjoint regions where $h[u]$ assumes the values $+1$ and -1, respectively. If u is a solution of (1) then the interface between these two regions is apriori unknown and, therefore, may be considered as the free boundary.

We suppose also that

$$\sup_{Q} |u| \leqslant M \quad \text{with} \quad M > 1. \tag{3}$$

Since the right-hand side of (1) is bounded, the general parabolic theory (see, e.g. [LSU67]) implies for any $\epsilon > 0$ the estimates

$$\|\partial_t u\|_{q,Q^\epsilon} + \|D^2 u\|_{q,Q^\epsilon} \leqslant N_1(\epsilon, q, M) \quad \forall q < \infty, \tag{4}$$

where $Q^\epsilon = \mathcal{U}^\epsilon \times]\epsilon^2, T]$, $\mathcal{U}^\epsilon \subset \mathcal{U}$ and $dist\,\{\mathcal{U}^\epsilon, \partial\mathcal{U}\} \geqslant \epsilon$.

We note that if $\partial\mathcal{U}$ as well as the values of u on the parabolic boundary of Q are smooth then in the whole cylinder Q the corresponding estimates of L^q-norm for $\partial_t u$ and $D^2 u$ are true.

In particular, (4) implies that u satisfies (1) a.e. in Q and, consequently, the $(n+1)$-dimensional Lebesgue measure of the sets $\{u = \alpha\}$ and $\{u = \beta\}$ equals zero. In addition, functions u and Du are Hölder continuous in Q.

Equation of type (1) arises in various biological and chemical processes in which diffusive and nondiffusive substances interact according to hysteresis law (see, for instance, [Kop06], and references therein). Despite the large number of applications there are only several publications devoted to equations involving a spatially distributed hysteretic discontinuity. We are only aware of the results of [GST13] and [GT12], where the one-(space)-dimensional case were studied. In the paper [GST13] the authors proved the local existence of solutions of (1) under the assumption that the corresponding initial data are spatially transverse. This transversality property roughly speaking means that the solution has a nonvanishing spatial gradient on the free boundary. It was also shown in [GST13] that transversal solutions depend continuously on initial data. A theorem on the uniqueness of solutions was established in [GT12] under the similar assumption about transversality of solutions. Observe also that to our knowledge the regularity properties of solutions to equation (1) has not previously been studied.

In this paper we are interested in local L^∞-estimates for the derivatives $D^2 u$ and $\partial_t u$ of the function u satisfying (1). We do not suppose that our solutions have the transversality property.

The paper is organized as follows. In Section 2 we introduce notations used in this paper, describe the different components of the free boundary and formulate the main result of the paper: Theorem 2.3. In Section 3 we show the continuity of the time-derivative $\partial_t u$ across the special part of the free boundary where the spatial gradient Du does not vanish, and estimate $|\partial_t u|$ on this part unformly by a constant depending only on given quantities. Further, in Section 4 we verify that positive and negative parts of the space directional derivatives $D_e u$ for any direction $e \in \mathbb{R}^n$ are sub-caloric outside some "pathological" part of the free boundary. We use this information in Section 5 for proving the quadratic growth estimates which are crucial for the final estimates of the higher order derivatives. The uniform L^∞-estimates of $\partial_t u$ and $D^2 u$ depending on given quantities and on the distance to the "pathological" part of the free boundary are obtained in Section 6. Finally, in Section 7 we state and prove some preliminary facts which are used intensively for proving of almost all results in the previous sections.

2 Notation and Preliminaries.

Throughout this article we use the following notation:
$z = (x, t)$ are points in $\mathbb{R}^{n+1}_{x,t}$, where $x \in \mathbb{R}^n$, $n \geqslant 1$, and $t \in \mathbb{R}^1$;
$x = (x_1, x') = (x_1, x_2, \dots, x_n)$, if $n \geqslant 2$;
$|x|$ is the Euclidean norm of x;
$B_r(x^0)$ denotes the open ball in $\mathbb{R}^n$ with center x^0 and radius r;
$Q_r(z^0) = Q_r(x^0, t^0) = B_r(x^0) \times]t^0 - r^2, t^0 + r^2]$;
$Q_r^-(z^0) = Q_r(z^0) \cap \{t < t^0\}$.
When omitted, x^0 (or $z^0 = (x^0, t^0)$, respectively) is assumed to be the origin.
$\partial' Q_r(z^0)$ or $\partial' Q_r^-(z^0)$ denote the parabolic boundary of the corresponding cylinder, i.e., the topological boundary minus the top of the cylinder.

For a cylinder $Q = \mathcal{U} \times]0, T]$ and any $\epsilon > 0$ we define the corresponding cylinder Q^ϵ as

$$Q^\epsilon = \mathcal{U}^\epsilon \times]\epsilon^2, T],$$

where $\mathcal{U}^\epsilon \subset \mathcal{U}$ and *dist* $\{\mathcal{U}^\epsilon, \mathcal{U}\} \geqslant \varepsilon$.
$u_+ = \max\{u, 0\}$; $\qquad u_- = \max\{-u, 0\}$;
D_i denotes the differential operator with respect to x_i;
$D = (D_1, D') = (D_1, D_2, \dots, D_n)$ denotes the spatial gradient;
$D^2 u = D(Du)$ denotes the Hessian of u;
$\partial_t u = \dfrac{\partial u}{\partial t}$.
D_ν stands for the operator of differentiation along a direction $\nu \in \mathbb{R}^n$, i.e., $|\nu| = 1$ and $D_\nu u = \sum\limits_{i=1}^{n} \nu_i D_i u$.

We adopt the convention that the indices i, j, l always vary from 1 to n. We also adopt the convention regarding summation with respect to repeated indices.

$\|\cdot\|_{p,\mathcal{D}}$ denotes the norm in $L^p(\mathcal{D})$, $1 < p \leqslant \infty$;

$W^{2,1}_p(\mathcal{D})$ and $W^{1,0}_p(\mathcal{D})$ are anisotropic Sobolev spaces with the norms

$$\|u\|_{W^{2,1}_p(\mathcal{D})} = \|\partial_t u\|_{p,\mathcal{D}} + \|D^2 u\|_{p,\mathcal{D}} + \|u\|_{p,\mathcal{D}},$$
$$\|u\|_{W^{1,0}_p(\mathcal{D})} = \|Du\|_{p,\mathcal{D}} + \|u\|_{p,\mathcal{D}},$$

respectively.

For a cylinder $\mathcal{Q} = \mathcal{U}\times]T_1, T_2] \subset \mathbb{R}^n_x \times \mathbb{R}^1_t$ we denote by $V_2(\mathcal{Q})$ the Banach space consisting of all elements of $W^{1,0}_2(\mathcal{Q})$ with a finite norm

$$\|u\|_{V_2(\mathcal{Q})} = \sup_{T_1 < t \leqslant T_2} \|u\|_{2,\mathcal{U}} + \|Du\|_{2,\mathcal{Q}}.$$

$\fint_{\mathcal{D}} \dots$ stands for the average integral over the set $\mathcal{D}$, i.e.,

$$\fint_{\mathcal{D}} \dots = \frac{1}{\text{meas}\,\{\mathcal{D}\}} \int_{\mathcal{D}} \dots .$$

We say that $\xi = \xi(x,t)$ is a cut-off function for a cylinder $Q_r(\hat{z})$ if

$$\xi(x,t) = \xi_1(x)\xi_2(t),$$

where $\xi_i \geqslant 0$, $i = 1, 2$,

$$\xi_1 \in C^\infty_0\left(B_r(\hat{x})\right), \qquad \xi_1 \equiv 1 \quad \text{in} \quad B_{r/2}(\hat{x}),$$

while $\xi_2 \in C^\infty([\hat{t} - r^2/4, \hat{t}])$, $\xi_2(\hat{t} - r^2) = 0$ and $\xi_2(t) \equiv 1$ for $t \geqslant \hat{t} - r^2/4$.

We define the parabolic distance $dist_p$ from a point $z = (x,t)$ to a set $\mathcal{D} \subset \mathbb{R}^{n+1}$ by

$$dist_p\,(z, \mathcal{D}) := \sup\left\{r > 0 : Q^-_r(z) \cap \mathcal{D} = \emptyset\right\}.$$

We use letters M, N, C and c (with or without sub-indices) to denote various constants. To indicate that, say, C depends on some parameters, we list them in the parentheses: $C(\dots)$. We do not indicate the dependence of constants on n. In addition, we will write *sup* instead of *ess sup* and *inf* instead of *ess inf*.

We denote

$$\Omega_{\pm}(u) := \{z \in Q, \text{ where } h[u](z) = \pm 1\},$$
$$\Gamma(u) := \partial\Omega_+ \cap \partial\Omega_- \text{ is the free boundary.}$$

The latter means that $\Gamma(u)$ is the set where the function $h[u](z)$ has a jump.

We also introduce special notation for the different parts of $\Gamma(u)$

$$\Gamma_\alpha(u) := \Gamma(u) \cap \{u = \alpha\},$$
$$\Gamma_\beta(u) := \Gamma(u) \cap \{u = \beta\}.$$

By definition,

$$\{u \leqslant \alpha\} \subset \Omega_- \quad \text{and} \quad \{u \geqslant \beta\} \subset \Omega_+.$$

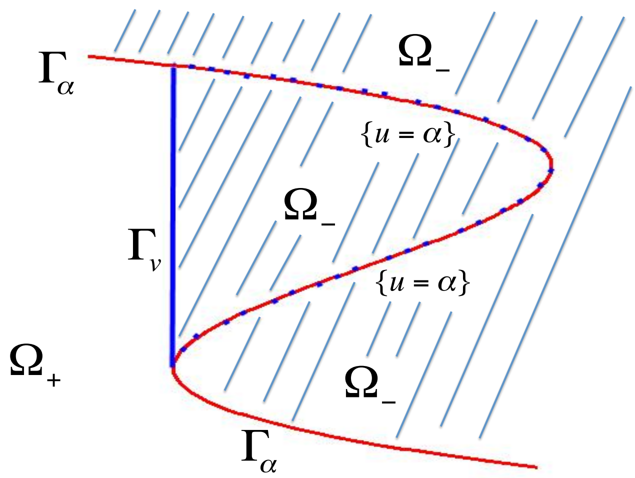

Figure 2: Structure of the free boundary for $n = 1$

It is also easy to see that the sets $\{u = \alpha\}$ and $\{u = \beta\}$ are separated from each other.

Remark 2.1. *In any cylinder Q^ϵ the distance from the level set $\{u = \alpha\}$ to the level set $\{u = \beta\}$ is estimated from below by a positive constant depending on M, ϵ and $\beta - \alpha$ only.*

Observe that the level sets $\{u = \alpha\}$ and $\{u = \beta\}$ are not alsways the parts of the free boundary $\Gamma(u)$. Indeed, if the level set $\{u = \alpha\}$ is locally not a t-graph, then a part of $\{u = \alpha\}$ may occur inside Ω_-. In this case $\Gamma(u)$ may contain several components of Γ_α connected by cylindrical surfaces with generatrixes parallel to t-axis (see Figure 2). Similar statement is true for the level set $\{u = \beta\}$. We will denote by Γ_v the set of all points z lying in such vertical parts of $\Gamma(u)$. It should be noted that Γ_v is, in general, not the level set $\{u = \alpha\}$ as well as not the level set $\{u = \beta\}$. This Γ_v is just the "pathological" part of the free boundary that we have mentioned in Introduction.

We will also distinguish the following parts of Γ:

$$\Gamma^0_\alpha(u) = \Gamma_\alpha(u) \cap \{|Du| = 0\}, \qquad \Gamma^*_\alpha(u) = \Gamma_\alpha(u) \setminus \Gamma^0_\alpha(u).$$

The sets Γ^0_β and Γ^*_β are defined analogously. In addition, we set

$$\Gamma^0(u) := \Gamma^0_\alpha(u) \cup \Gamma^0_\beta(u), \quad \Gamma^*(u) := \Gamma^*_\alpha(u) \cup \Gamma^*_\beta(u).$$

Remark 2.2. *It is obvious that $u \in C^\infty$ in the interior of the sets $\Omega_\pm$.*

Now we formulate the main result of the paper.

Theorem 2.3. *Let u satisfy (1), and let $z \in Q \setminus \Gamma(u)$. Then*

$$|\partial_t u(z)| + |D^2 u(z)| \leqslant C(\rho_0, \varepsilon, M, \beta - \alpha).$$

Here $\rho_0 := dist_p\{z, \Gamma_v\}$ and $\epsilon := dist_p\{z, \partial' Q\}$.

Proof. The proof of this statement follows from Lemmas 6.1 and 6.2. □

3 Estimates of $\partial_t u$ on $\Gamma^*(u)$

Lemma 3.1. *Let u be a solution of Eq. (1), and let $Q^-_{3\rho}(z^*)$ be an arbitrary cylinder contained in Q. Then we have the estimates*

$$\inf_{Q^-_\rho(z^*)} \partial_t u \geqslant -N, \qquad \textit{provided that} \quad Q^-_{3\rho}(z^*) \cap \Gamma_\beta = \emptyset, \tag{5}$$

$$\sup_{Q^-_\rho(z^*)} \partial_t u \leqslant N, \qquad \textit{provided that} \quad Q^-_{3\rho}(z^*) \cap \Gamma_\alpha = \emptyset. \tag{6}$$

Here $N = N(M, \rho)$.

Proof. Assume for the definiteness that z^* lyies in a neighborhood of Γ_β. Consider in $Q^-_{2\rho}(z^*)$ the difference quotient of u in the t-direction, i.e.,

$$u^{(\tau)}(x,t) = \frac{u(x,t) - u(x,t-\tau)}{\tau}$$

with some small positive τ. To prove (6) it is sufficient to get the corresponding estimate for $u^{(\tau)}$ uniformly with respect to τ.

Further, we observe that equation (1) and integration by parts provide for all test-finctions $\eta \in W^{1,1}_2(Q^-_{2\rho}(z^*))$ vanishing on $\partial B_{2\rho}(x^*) \times [t^* - 4\rho^2, t^*]$ the validity of the following integral identity

$$\int\limits_{Q^-_{2\rho}(z^*)} (\partial_t u \eta + DuD\eta)\, dxdt = - \int\limits_{Q^-_{2\rho}(z^*)} h[u]\eta dxdt. \tag{7}$$

Using the same reasonings as in deriving of (7) we get for all test-functions $\widetilde{\eta} \in W^{1,1}_2\left(Q^-_{2\rho}(x^*, t^* + \tau)\right)$ that are equal to zero on $\partial' Q^-_{2\rho}(x^*, t^* + \tau)$ the integral identity

$$\int\limits_{Q^-_{2\rho}(x^*,t^*+\tau)} (\partial_t u \widetilde{\eta} + DuD\widetilde{\eta})\, dxdt = - \int\limits_{Q^-_{2\rho}(x^*,t^*+\tau)} h[u]\widetilde{\eta} dxdt. \tag{8}$$

Putting in (8) $\widetilde{\eta}(x,t) = \eta(x, t+\tau)$ we obtain after elementary change of variables the relation

$$\begin{aligned} &\int\limits_{Q^-_{2\rho}(z^*)} [\partial_t u(x,t-\tau)\ \eta(x,t) + Du(x,t-\tau) D\eta(x,t)]\, dxdt \\ &\qquad = - \int\limits_{Q^-_{2\rho}(z^*)} h[u](x,t-\tau)\eta(x,t) dxdt. \end{aligned} \tag{9}$$

Now, we substract (9) from (7), divide the result by τ and integrate by parts. After these transformations we arrive at the equality

$$\begin{aligned} &\int\limits_{Q^-_{2\rho}(z^*)} \left[\partial_t u^{(\tau)} \eta + Du^{(\tau)} D\eta\right] dxdt \\ &\qquad = -\frac{1}{\tau} \int\limits_{Q^-_{2\rho}(z^*)} (h[u](x,t) - h[u](x,t-\tau))\, \eta dxdt. \end{aligned} \tag{10}$$

Setting in (10)

$$\eta(x,t) = \left(u^{(\tau)} - k\right)_+ \xi^2(x,t), \quad k \geqslant 0,$$

where ξ is a standard cut-off function for a cylinder $Q^-_{2\rho}(z^*)$ (see Notation), we can rewrite (10) in the form

$$\begin{aligned}\int\limits_{Q^-_{2\rho}(z^*)} &\left\{\partial_t u^{(\tau)} \left(u^{(\tau)} - k\right)_+ \xi^2 + Du^{(\tau)} D\left[\left(u^{(\tau)} - k\right)_+ \xi^2\right]\right\} dxdt \\ &= -\frac{1}{\tau} \int\limits_{Q^-_{2\rho}(z^*)} \left(h[u](x,t) - h[u](x,t-\tau)\right) \left(u^{(\tau)} - k\right)_+ \xi^2 dxdt.\end{aligned} \tag{11}$$

We claim that $h[u](x,t) - h[u](x,t-\tau) \geqslant 0$ in $Q^-_{2\rho}(z^*)$. Indeed, we have the relation

$$Q^-_{2\rho}(z^*) \cap \Gamma_\alpha = \emptyset.$$

Recall that by definition $h[u](x,t)$ may decrease in t only in a neighborhood of Γ_α. Therefore, in $Q^-_{2\rho}(z^*)$ the function $h[u]$ is either constant or increasing one. The latter means that for all $k \geqslant 0$ we have instead of (11) the inequality

$$\int\limits_{Q^-_{2\rho}(z^*)} \left\{\partial_t u^{(\tau)} \left(u^{(\tau)} - k\right)_+ \xi^2 + Du^{(\tau)} D\left[\left(u^{(\tau)} - k\right)_+ \xi^2\right]\right\} dxdt \leqslant 0. \tag{12}$$

Observe that we may take in (12) the cut-off fucntion ξ multiplied by the characteristic function of an interval $[t^* - 4\rho^2, t]$ with an arbitrary $t \in]t^* - 4\rho^2, t^*]$ instead of ξ. This leads to the inequalities

$$\int\limits_{t^*-4\rho^2}^{t} \int\limits_{B_{2\rho}(x^*)} \left\{\partial_t u^{(\tau)} \left(u^{(\tau)} - k\right)_+ \xi^2 + Du^{(\tau)} D\left[\left(u^{(\tau)} - k\right)_+ \xi^2\right]\right\} dxdt \leqslant 0,$$

$$\forall t \in]t^* - 4\rho^2, t^*].$$

Further arguments are rather standard. We leave the trivially nonnegative terms in the left-hand side of the above inequalities, while the rest terms are transferred to the right-hand side and estimated from above with the help of Young's inequality. As a consequence, for $k \geqslant 0$ we get

$$\begin{aligned}\sup_{t^*-4\rho^2 < t \leqslant t^*} \int\limits_{B_{2\rho}(x^*)} (u^{(\tau)} - k)^2_+ \xi^2 dx \Bigg|^t + \int\limits_{Q^-_{2\rho}(z^*)} \left[D\left((u^{(\tau)} - k)_+\right)\right]^2 \xi^2 dxdt \\ \leqslant \int\limits_{Q^-_{2\rho}(z^*)} \left(u^{(\tau)} - k\right)^2_+ \left[4|D\xi|^2 + 2\xi|\partial_t \xi|\right] dxdt.\end{aligned} \tag{13}$$

With inequalities (13) at hands we may apply succesively Fact 7.1 with $v = u^{(\tau)}$ and inequalities (4) which immediately imply the desired estimate (6).

It remains only to observe that the case of z^* lying near Γ_α is treated almost similarly. The only differences are that we should choose in (10)

$$\eta(x,t) = \left(u^{(\tau)} - k\right)_- \xi^2(x,t), \quad k \leqslant 0,$$

and then check the validity of the inequality $h[u](x,t) - h[u](x,t-\tau) \leqslant 0$ in the cylinder $Q^-_{2\rho}(z^*)$. □

Lemma 3.2. *Let u be a solution of Eq. (1) and let $z^* \in \Gamma^* \setminus \Gamma_v$.*

Then $\Gamma^* \setminus \Gamma_v$ is locally a C^1-surface and $\partial_t u$ is a continuous function in a neigborhood of z^*. In addition, the mixed second derivatives $D_i(\partial_t u)$ are L^2-functions near z^*.

Proof. Continuity of $\partial_t u$ across Γ^* can be proved by using the same arguments as in (the proof of) Lemma 7.1 [SUW09]. For the readers convenience we sketch the details.

Suppose for the definiteness that $z^* \in \Gamma^*_\alpha \setminus \Gamma_v$. Without restriction it may be assumed that $D_1 u(z^*) > 0$. Then, in a sufficiently small cylinder $Q_\rho(z^*)$ satisfying $Q_\rho(z^*) \cap \Gamma_v$ the function u is strictly increasing in x_1-direction.

Further, using the von Mises transformation, we introduce the new variables

$$(x_1, x', t) \to (y, x', t),$$

where $y := u(x,t) - \alpha$. We also introduce the function v such that

$$x_1 = v(y, x', t).$$

Transforming in $Q_\rho(z^*)$ Eq. (1) for u into terms of v we obtain the uniformly parabolic equation

$$\partial_t v - a^{ij}(\partial v)\, \partial_i(\partial_j v) = g(y)\partial_1 v,$$

where $\partial_1 v := \dfrac{\partial v}{\partial y} = \dfrac{1}{D_1 u} > 0$, $\partial_m v := \dfrac{\partial v}{\partial x_m} = D_m v = -\dfrac{D_m u}{D_1 u}$,

$$\partial v = (\partial_1 v, D'v) = \left(\frac{1}{D_1 u}, -\frac{D'u}{D_1 u}\right), \qquad \partial_t v := \frac{\partial v}{\partial t} = -\frac{\partial_t u}{D_1 u}, \tag{14}$$

$$g(y) = \begin{cases} 1, & \text{if } y > 0 \\ -1, & \text{if } y < 0 \end{cases},$$

and the coefficients a^{ij} are defined as follows

$$a^{11}(p) = \frac{1+|p'|^2}{p_1^2}, \quad a^{mm}(p) = 1, \quad a^{1m}(p) = a^{m1}(p) = -\frac{p_m}{p_1}, \\ a^{m\widetilde{m}}(p) = 0 \quad \text{if} \quad m \neq \widetilde{m} \tag{15}$$

(here the indices m and $\widetilde{m}$ vary from 2 to n, and $p \in \mathbb{R}^n$).

Elementary calculation shows that for the difference quotient in the t-direction

$$v^{(\tau)}(y, x', t) := \frac{v(y, x', t) - v(y, x', t-\tau)}{\tau}$$

we have

$$\partial_t v^{(\tau)} - a^{ij}\,(\partial v)\,\partial_i(\partial_j v^{(\tau)}) - b^k \partial_k v^{(\tau)} = g(y)\partial_1 v^{(\tau)}, \tag{16}$$

where $b^k := \dfrac{\partial a^{ij}(Z_\tau)}{\partial p_k}\partial_i\left(\partial_j v(y, x', t-\tau)\right)$,

$$Z_\tau = \vartheta(y, x', t)\partial v(y, x', t-\tau) - [1 - \vartheta(y, x', t)]\,\partial v(y, x', t)$$

and $\vartheta(y, x', t) \in [0, 1]$.

Observe that for the second derivatives of v we have the relations

$$\begin{aligned} \partial_1\left(\partial_1 v\right) = -\frac{D_{11}u}{(D_1u)^3}, \quad \partial_1\left(\partial_m v\right) &= \frac{D_{11}uD_mu}{|D_1u|^2} - \frac{D_{1m}u}{D_1u}, \\ \partial_m(\partial_{\tilde{m}} v) &= \frac{D_{11}uD_muD_{\tilde{m}}u}{|D_1u|^2}\left(\frac{1}{D_1u} - 2\right) \\ &\quad + \frac{D_{1m}uD_{\tilde{m}}u}{D_1u} + \frac{D_{1\tilde{m}}uD_mu}{D_1u} - \frac{D_{m\tilde{m}}u}{D_1u}. \end{aligned} \tag{17}$$

According to estimates (4) and formulas (14)-(15) and (17) we may conclude that in Eq. (16) the coefficients a^{ij} are Hölder continuous functions satisfying the ellipticity condition, whereas the coefficients b^k are elements of L^q with an arbitrary $q < \infty$. Therefore, the parabolic theory implies that $v^{(\tau)} \in C^\sigma$ for some $\sigma \in (0, 1)$ and $\partial v^{(\tau)}$ is locally an element of L^2-space. We note also that all the estimates of corresponding norms are uniformly bounded in τ. Hence we immediately conclude that $\partial_t u$ is also Hölder continuous with some exponent σ' satisfying $0 < \sigma' < \sigma$ and that the mixed derivatives $D_i(\partial_t u)$ belong locally to a class of L^2-functions. It is also evident that near z^* the free boundary Γ_α is a C^1-surface .

It remains only to observe that in the case $z^* \in \Gamma^*_\beta \setminus \Gamma_v$ we should choose the new variable y in von Mises transformation as $y := u(x, t) - \beta$ and repeat the above steps. □

Corollary 3.3. *Let u satisfy Eq. (1). Then for any cylinder $Q^\epsilon \subset Q$ we have*

$$\sup_{\Gamma^*\cap Q^\epsilon} |\partial_t u| \leqslant N_*(M,\epsilon,\beta-\alpha). \tag{18}$$

Proof. Consider for the definiteness the case $z^* \in \{\Gamma^*_\alpha \setminus \Gamma_v\} \cap Q^\epsilon$. Due to Lemma 3.2 a function $\partial_t u$ is continuous in a neighborhood of z^*.

Recall that by definition of Γ_α the function $h[u]$ has a jump in t-direction from $+1$ to -1 there. The latter means that if we cross the free boundary Γ^*_α in positive t-direction then the corresponding phases change from Ω_+ to Ω_-. Since $u(z^*) = \alpha$ and $u(x^*, t^* - \varepsilon) > \alpha$ for any $\varepsilon > 0$ we conclude that $\partial_t u(z^*) \leqslant 0$. Hence the inequality

$$\sup_{\Gamma^*_\alpha} \partial_t u \leqslant 0 \tag{19}$$

is valid.

Now , taking into account Remark 2.1, one may combine (19) with one-sided inequality (5). It gives the desired estimate (18).

The other case, i.e., $z^* \in \Gamma^*_\beta \setminus \Gamma_v$ is treated in a similar manner. It is necessary only to observe that if we cross the free boundary Γ^*_β in positive t-direction then the phases will change from Ω_- to Ω_+ and, consequently, $\partial_t u(z^*) \geqslant 0$ and the inequality

$$\sup_{\Gamma^*_\beta} \partial_t u \geqslant 0 \tag{20}$$

holds true. In view of Remark 2.1, the combination of (20) with one-sided estimate (6) finishes the proof. □

4 Sub-Caloricity of $D_e u$

Lemma 4.1. *Let $w \in C(\mathcal{D}) \cap W^{1,0}_{2,loc}(\mathcal{D})$ with $\mathcal{D}$ being a domain in $\mathbb{R}^{n+1}$, and let the inequality*

$$\int_{\mathcal{D}} (-w\partial_t \eta + DwD\eta)\, dz \leqslant 0 \tag{21}$$

hold for any nonnegative function $\eta \in C_0^\infty(\mathcal{D})$ with $supp\, \eta \subset \{w > 0\}$.

Then the function w_+ is sub-caloric in $\mathcal{D}$.

Proof. First, we take in (21) nonnegative functions $\eta \in C_0^\infty(\mathcal{D})$ with

$$supp\, \eta \subset \left\{ w \geqslant \frac{\delta}{2} > 0 \right\}. \tag{22}$$

Without loss of generality we may consider instead of w in (21) its mollifier w_ρ with sufficiently small parameter ρ. After integration by parts we arrive at

$$\int\limits_{\mathcal{D}} [\partial_t w_\rho \eta + Dw_\rho D\eta]\, dz \leqslant 0. \tag{23}$$

We set in (23) $\eta = \psi_\delta(w_\rho)\varphi$, where $\varphi \in C_0^\infty(\mathcal{D})$ is an arbitrary nonnegative test function, while

$$\psi_\delta(s) = \begin{cases} 0, & \text{if } s \leqslant \delta \\ \dfrac{(s-\delta)}{\delta}, & \text{if } \delta < s < 2\delta \\ 1, & \text{if } s \geqslant 2\delta \end{cases}.$$

Observe that such a choice of η is not restrictive, since due to definition of ψ_δ we have for sufficiently small ρ the evident inclusions

$$supp\, \eta \subset \{w_\rho > \delta\} \subset \left\{ w > \frac{\delta}{2} \right\}.$$

After substitution of η inequality (23) takes the form

$$\int\limits_{\mathcal{D}} \left[\partial_t w_\rho \psi_\delta(w_\rho)\varphi + |Dw_\rho|^2 \psi'_\delta(w_\rho)\varphi + Dw_\rho \psi_\delta(w_\rho) D\varphi\right] dz \leqslant 0. \tag{24}$$

Elementary calculation shows that $\partial_t w_\rho \psi_\delta(w_\rho) = \frac{d}{dt} F_\delta(w_\rho)$ where the function F_δ is defined as

$$F_\delta(s) = \int\limits_0^s \psi_\delta(\tau) d\tau = \begin{cases} 0, & \text{if } s \leqslant \delta \\ \dfrac{(s-\delta)^2}{2\delta}, & \text{if } \delta < s < 2\delta \\ s - (3/2)\delta, & \text{if } s \geqslant 2\delta \end{cases}.$$

So, again integrating by parts and taking into account that the second term in (24) is nonnegative we get the inequality

$$\int\limits_{\mathcal{D}} [-F_\delta(w_\rho)\partial_t \varphi + Dw_\rho \psi_\delta(w_\rho) D\varphi]\, dz \leqslant 0. \tag{25}$$

Tending in (25) $\rho \to 0$ and taking into account the definitions of ψ_δ and F_δ we arrive at

$$\int\limits_{\{w>2\delta\}} [-w\partial_t \varphi + DwD\varphi]\, dz \leqslant \int\limits_{\{\delta<w<2\delta\}} |DwD\varphi| dz + C\delta.$$

Letting $\delta \to 0$ in the above inequality provides the inequality

$$\int\limits_{\{w>0\}} [-w\partial_t\varphi + DwD\varphi]\, dz \leqslant 0. \tag{26}$$

It remains only to recall that φ in (26) is an arbitrary nonnegative test-function. This completes the proof. $\square$

Lemma 4.2. *Let u be a solution of Eq. (1). Then for any direction $e \in \mathbb{R}^n$ functions $(D_e u)_\pm$ are sub-caloric in $Q \setminus \Gamma_v$.*

Proof. Due to Lemma 4.1 it sufficies to check that for $w = D_e u$ inequality (21) holds true for any nonnegative function $\eta \in C_0^\infty(Q \setminus \Gamma_v)$ with $supp\, \eta \subset \{D_e u > 0\}$.

It follows from Eq. (1) that functions $D_e u$ satisfy in Q the equation

$$H\,[D_e u] = D_e\,(h[u]) \tag{27}$$

in the weak (distributional) sence. Hence we obtain

$$\int\limits_Q D_e u\,(\partial_t\eta + \Delta\eta)\, dz = -\int\limits_Q h[u] D_e\eta dz = -\int\limits_{\Omega_+} D_e\eta dz + \int\limits_{\Omega_-} D_e\eta dz$$

$$= 2\int\limits_{\Gamma^*} \eta \cos(\widehat{\mathbf{n}, \mathbf{e}}) d\mathcal{H}^n,$$

where $\mathbf{n} = \mathbf{n}(z)$ is the unit normal vector to Γ^* directed into Ω_+, $\mathbf{e} := (e, 0)$, and $\mathcal{H}^n$ stands for the n-dimensional Hausdorff measure.

It is easy to see that the normal vector $\mathbf{n}$ has on Γ^* the following representation

$$\mathbf{n}(z) = \left(\frac{Du(z)}{\sqrt{|Du(z)|^2 + (\partial_t u(z))^2}}, \frac{\partial_t u(z)}{\sqrt{|Du(z)|^2 + (\partial_t u(z))^2}} \right). \tag{28}$$

Indeed, since $u > \alpha$ in Ω_+ and $\Gamma_\alpha \subset \{u = \alpha\}$, the vector $Du(z)$ at $z \in \Gamma_\alpha^*$ is directed into Ω_+. In addition, we recall (see (19)) that $\partial_t u \leqslant 0$ on Γ_α^*. Therefore, the projection of $\mathbf{n}$ from formula (28) on the t-axis is also nonpositive. Because of Ω_+ is locally a subgraph of Γ_α in t-direction, we conclude that on Γ_α^* the whole vector $\mathbf{n}$ defined by (28) is directed into Ω_+. Similarly, we have $\{u < \beta\}$ in Ω_- and $\Gamma_\beta \subset \{u = \beta\}$. Therefore, the spatial gradient $Du(z)$ at $z \in \Gamma_\beta^*$ is directed into Ω_+. Moreover, on Γ_β^* we have

$\partial_t u \geqslant 0$ (see (20)) and Ω_+ is a t-epigraph of Γ^*_β. So, the vector $\mathbf{n}$ from formula (28) is again directed into Ω_+.

Now, taking into account the inclusion $supp\,\eta \subset \{D_e u > 0\}$ and representation (28) we conclude that

$$\eta \cos\left(\widehat{\mathbf{n}(\mathbf{z}), \mathbf{e}}\right) \geqslant 0 \qquad \forall z \in \Gamma^*$$

and complete the proof. $\square$

Remark 4.3. *We emphasize that $(D_e u)_\pm$ are, in general, not sub-caloric near Γ_v.*

5 Quadratic Growth Estimates

Lemma 5.1. *Let u satisfy (1), let $z^0 \in \Gamma^0$, and let*

$$dist_p\left\{z^0, \Gamma_v\right\} \geqslant \rho_0 > 0, \qquad dist_p\left\{z^0, \partial' Q\right\} \geqslant \rho_0.$$

There exists a positive constant C_0 completely defined by the values of ρ_0 and M such that

$$\underset{Q_r^-(z^0)}{osc}\ u \leqslant C_0 r^2 \qquad \text{for all} \quad r \leqslant \rho_0. \tag{29}$$

Proof. We verify inequality (29) for $z^0 \in \Gamma^0_\alpha$. The other case, i.e., $z^0 \in \Gamma^0_\beta$ can be proved by using similar arguments.

We argue by contradiction. Suppose (29) fails. Then there exist a sequence $r_k > 0$ as well as sequences u_k of solutions to (1) satisfying (3), and points $z^k \in \Gamma^0_\alpha(u_k)$ such that for all $k \in \mathbb{N}$ we have

$$dist_p\left(z^k, \Gamma_v(u_k)\right) \geqslant \rho_0, \qquad dist_p\left(z^k, \partial' Q\right) \geqslant \rho_0$$

and

$$\sup_{Q^-_{r_k}(z^k)} |u_k - \alpha| \geqslant k r_k^2. \tag{30}$$

Thanks to assumption (3) the left-hand side of (30) is bounded by $2M$ and, consequently, $r_k \to 0$ as $k \to \infty$. It is evident that we can choose r_k as the maximal value of r for which

$$\sup_{Q^-_r(z^k)} |u_k - \alpha| \geqslant k r^2.$$

In other words, we have the relations

$$\begin{cases} \mathcal{M}_r(z^k, u_k) := \sup\limits_{Q_r^-(z^k)} |u_k - \alpha| < kr^2 \quad \text{for all } r \in (r_k, \rho_0], \\ \mathcal{M}_{r_k}(z^k, u_k) = kr_k^2. \end{cases} \tag{31}$$

Next, we define a scaling $\tilde{u}_k$ as

$$\tilde{u}_k(x,t) = \frac{u_k(x^k + r_k x, t^k + r_k^2 t) - \alpha}{\mathcal{M}_{r_k}(z^k, u_k)}$$

for $(x,t) \in Q^-_{\rho_0/r_k}$. Then $\tilde{u}_k$ satisfies the following properties

$$\sup_{Q_1^-} |\tilde{u}_k| = 1, \tag{32}$$

$$\tilde{u}_k(0,0) = 0, \qquad |D\tilde{u}_k(0,0)| = 0, \tag{33}$$

$$\|H[\tilde{u}_k]\|_{\infty, Q^-_{1/r_k}} \leqslant \frac{r_k^2}{\mathcal{M}_{r_k}(z^k, u_k)} = \frac{1}{k} \to 0 \text{ as } k \to \infty. \tag{34}$$

In addition, due to (31) we have for $R \in (1, \rho_0/r_k]$ the inequality

$$\sup_{Q_R^-} |\tilde{u}_k| = \frac{\mathcal{M}_{r_k R}(z^k, u_k)}{\mathcal{M}_{r_k}(z^k, u_k)} < \frac{k\,(r_k R)^2}{k r_k^2} = R^2. \tag{35}$$

Now, by (32)-(35) we will have a subsequence of $\tilde{u}_k$ weakly converging in $W^{2,1}_{q,loc}\left(\mathbb{R}^{n+1}_{x,t} \cap \{t \leqslant 0\}\right)$, $q < \infty$, to a caloric function u_0 satisfying

$$\begin{aligned} &\sup_{Q_R^-} |u_0| \leqslant R^2 \qquad \forall R \geqslant 1, \\ &u_0(0,0) = |Du_0(0,0)| = 0, \\ &\sup_{Q_1^-} |u_0| = 1. \end{aligned} \tag{36}$$

According to the Liouville theorem (see, for example, Lemma 2.1 [ASU00]), there exist constants a^{ij} such that

$$u_0(x,t) = a^{ij} x_i x_j + 2\left(\sum_{i=1}^n a^{ii}\right) t \quad \text{in } \mathbb{R}^{n+1}_{x,t} \cap \{t \leqslant 0\}. \tag{37}$$

On the other hand, due to inequalities (4), Lemma 4.2 and Fact 7.3 we may conclude that for any direction $e \in \mathbb{R}^n$ and for all $k \in \mathbb{N}$ such that $r_k \leqslant \rho_0$

$$\Phi(r_k, (D_e u_k)_+, (D_e u_k)_-, \xi_{\rho_0, z^k}, z^k) \leqslant c(\rho_0), \tag{38}$$

where $c(\rho_0)$ is defined completely by the values of ρ_0 and M. More precisely, by $c(\rho_0)$ we may take a majorant of the right-hand side of inequality (52) calculated for $\theta_1 = (D_e u_k)_+$ and $\theta_2 = (D_e u_k)_-$. After simple rescaling (38) takes the form

$$\Phi(1, (D_e\tilde{u}_k)_+, (D_e\tilde{u}_k)_-, \zeta^k, 0, 0) \leqslant c(\rho_0)\left(\frac{r_k^2}{\mathcal{M}_{r_k}(z^k, u_k)}\right)^4 = \frac{c(\rho_0)}{k^4}, \qquad (39)$$

where for brevity we denote the corresponding cut-off function $\xi_{\rho_0/r_k,(0,0)}$ by ζ^k. Observe that $\zeta^k \equiv 1$ in $B_{\rho_0/(2r_k)}$. In addition, $B_{\rho_0/(2r_k)} \supset B_1$ if k is big enough, while for $\varepsilon > 0$ (small and fixed) we have

$$G(x, -t) \geqslant N(n, \varepsilon) > 0 \quad \text{for} \quad -1 < t < -\varepsilon, \quad x \in B_1.$$

Hence,

$$N(n, \varepsilon)\int_{-1}^{-\varepsilon}\int_{B_1} |\,(D_e\tilde{u}_k)_\pm\,|^2 dxdt \leqslant \int_{-1}^{0}\int_{\mathbb{R}^n} |D_e\left((\tilde{u}_k)_\pm\zeta^k\right)|^2 G(x, -t)dxdt. \qquad (40)$$

Next, using (40) and invoking the Poincare inequality we may reduce (39) to

$$\int_{-1}^{-\varepsilon}\int_{B_1} |\,(D_e\tilde{u}_k)_+ - m_+^k(t)|^2 dxdt \int_{-1}^{-\varepsilon}\int_{B_1} |\,(D_e\tilde{u}_k)_- - m_-^k(t)|^2 dxdt$$
$$\leqslant N^{-2}(n, \varepsilon)\frac{c(\rho_0)}{k^4},$$

where $m_\pm^k(t)$ denotes the corresponding average of $(D_e\tilde{u}_k)_\pm$ on t-sections over B_1.

Letting k tend to infinity (and then ε tend to zero), we obtain

$$\int_{Q_1^-} |\,(D_e u_0)_+ - m^+|^2 dxdt \int_{Q_1^-} |\,(D_e u_0)_- - m^-|^2 dxdt = 0, \qquad (41)$$

where $m^\pm$ is the corresponding average of $(D_e u_0)_\pm$ over B_1. Observe that, due to representation (37), $m^\pm$ do not depend on t.

Obviously, (41) implies that $D_e u_0$ does not change its sign in Q_1^-. Recall that e is an arbitrary direction in $\mathbb{R}^n$ and u_0 is a polinomial of the form (37). It means, in particulary, that $u_0 \equiv 0$ in Q_1^-. The latter contradicts (36) and complete the proof of (29). □

We will need the extension of Lemma 5.1 to the "upper half-cylinders" $Q_r(z^0) \cap [t^0, t^0 + r^2]$ as well.

Lemma 5.2. *Let all the assumptions of Lemma 5.1 be valid. Then*

$$\underset{Q_r(z^0)}{osc}\, u \leqslant C_1 r^2 \qquad \textit{for all} \quad r \leqslant \rho_0, \tag{42}$$

where ρ_0 is the same constant as in Lemma 5.1 and $C_1 = C_1(\rho_0, M)$.

Proof. To obtain estimate (42) for $\{t > t^0\}$ we consider the barrier function

$$w(x,t) = C'(\rho_0, M)\left\{|x - x^0|^2 + 2n(t - t^0)\right\} + (t - t^0),$$

where $C'(\rho_0, M) = \max\left\{C_0, M\rho_0^{-2}\right\}$ and $C_0 = C_0(\rho_0, M)$ is the constant from Lemma 5.1. Using (29) for $t = t^0$ and the comparison principle one can easily verify that

$$|u(x,t)| \leqslant w(x,t) \qquad \text{in} \quad B_{\rho_0}(x^0)\times]t^0, t^0 + r^2]. \tag{43}$$

Combination of (29) and (43) finishes the proof of (42). □

Lemma 5.3. *Let all the assumptions of Lemma 5.1 be valid. Then*

$$\sup_{Q_r(z^0)} |Du| \leqslant C_2 r \qquad \textit{for all} \quad r \leqslant \rho_0, \tag{44}$$

where $\rho_0 > 0$ is just the same as in Lemma 5.1, while C_2 is a positive constant completely defined by the values of M and ρ_0.

Proof. We verify (44) for $z^0 \in \Gamma^0_\alpha$. The case $z^0 \in \Gamma^0_\beta$ is treated in a similar manner.

Let us choose an arbitrary $r \leqslant \rho_0/2$ and consider a point $\tilde{z} \in Q_r(z^0)$. Further, we take identity (7) with $Q^-_{2\rho}(z^*)$ replaced by $Q^-_r(\tilde{z})$) and plug in this identity a test-function

$$\eta(x,t) = (u(x,t) - \alpha)\zeta^2(x)$$

where $\zeta \in C^\infty_0(B_r(\tilde{x}))$ satisfying $0 \leqslant \zeta \leqslant 1$ and $|D\zeta| \leqslant cr^{-1}$. After standard transformations we get the inequality

$$\begin{aligned} \int\limits_{B_r(\tilde{x})} (u-\alpha)^2\xi^2 dx\Big|^{\tilde{t}} + \int\limits_{Q^-_r(\tilde{z})} |Du|^2\xi^2 dxdt \leqslant \int\limits_{B_r(\tilde{x})} (u-\alpha)^2\xi^2 dx\Big|^{\tilde{t}-r^2} \\ + c\int\limits_{Q^-_r(\tilde{z})} (u-\alpha)^2|D\xi|^2 dxdt + c\int\limits_{Q^-_r(\tilde{z})} |u-\alpha|\xi^2 dxdt, \end{aligned} \tag{45}$$

where c stands for an absolute constant.

In view of (42) the right-hand side of (31) can be estimated from above by $2c\,C_1(\rho_0, M)r^{n+4}$ which guarantees

$$\int\limits_{Q_r^-(\tilde{z})} |Du|^2\xi^2 dxdt \leqslant 2c\,C_1 r^{n+4}.$$

It remains only to observe that combination of the latter inequality with Eq. (27) and Fact 7.2 implies the estimate

$$|Du(\tilde{z})| \leqslant \tilde{c}\,r$$

which completes the proof. □

6 Estimates of $\partial_t u$ and $D^2 u$ beyond Γ_v

In this section we obtain the estimates of $|\partial_t u(\hat{z})|$ and $|D^2 u(\hat{z})|$ in any $\hat{z}$ being a point of smoothness for u. We emphasize that these bounds do not depend on the parabolic distance from $\hat{z}$ to Γ^0 as well as to Γ^*. Unfortunately, we cannot remove the dependence of both bounds on the parabolic distance from $\hat{z}$ to Γ_v.

Lemma 6.1. *Let u satisfy (1), let $\hat{z} \in Q \setminus \Gamma(u)$, and let*

$$dist_p\{\hat{z}, \Gamma_v\} \geqslant \rho_0 > 0, \qquad dist_p\{\hat{z}, \partial' Q\} \geqslant \epsilon > 0.$$

There exists a positive constant C_3 depending only on ρ_0, ϵ,M and $\beta - \alpha$ such that

$$|\partial_t u(\hat{z})| \leqslant C_3. \tag{46}$$

Proof. Define $d_0 = d_0(\hat{z}) := \min\{dist_p\{\hat{z}, \Gamma^0\}, \rho_0, \epsilon/2\}$. It is obvious that for any $\delta > 0$

$$Q^-_{d_0/2}(\hat{x}, \hat{t} - \delta) \cap \left\{\Gamma^0 \cup \Gamma_v \cup \partial' Q\right\} = \emptyset.$$

However, $Q^-_{d_0/2}(\hat{x}, \hat{t} - \delta)$ may contain the points of $\Gamma^* \setminus \Gamma_v$.

1. First, we consider the case $d_0 = dist_p\{\hat{z}, \Gamma^0\}$.

   Using the same arguments as in the derivation of (10) in the proof of Lemma 3.1 we get for all test-functions $\eta \in W_2^{1,1}(Q^-_{d_0/2}(\tilde{x}, \tilde{t} - \delta))$

vanishing on $\partial' Q^-_{d_0/2}(\tilde{x}, \tilde{t}-\delta)$ the equality

$$\begin{aligned}\int\limits_{Q^-_{d_0/2}(\hat{x},\hat{t}-\delta)} & \left[\partial_t u^{(\tau)}\eta + Du^{(\tau)}D\eta\right] dxdt \\ &= -\frac{1}{\tau}\int\limits_{Q^-_{d_0/2}(\hat{x},\hat{t}-\delta)} \left(h[u](x,t) - h[u](x,t-\tau)\right)\eta dxdt,\end{aligned} \tag{47}$$

where $u^{(\tau)}$ denotes the difference quotient of u in the t-direction. Plugging in (47)

$$\eta(x,t) = (\partial_t u(x,t) - k)_+ \, \xi^2(x,t), \qquad k \geqslant 2N_*,$$

where ξ is a standard cut-off function for a cylinder $Q^-_{d_0/2}(\hat{x}, \hat{t}-\delta)$ (see Notation), and N_* is the constant from Corollary 3.3, we arrive at the relation

$$\begin{aligned}&\int\limits_{Q^-_{d_0/2}(\hat{x},\hat{t}-\delta)} \left\{\partial_t u^{(\tau)}\left(\partial_t u - k\right)_+ \xi^2 + Du^{(\tau)}D\left[\left(\partial_t u - k\right)_+ \xi^2\right]\right\} dxdt \\ &= -\frac{1}{\tau}\int\limits_{Q^-_{d_0/2}(\hat{x},\hat{t}-\delta)} \left\{h[u](x,t) - h[u](x,t-\tau)\right\}\left(\partial_t u - k\right)_+ \xi^2 dxdt.\end{aligned} \tag{48}$$

Observe that due to Corollary 3.3 the distance from the set $\{supp\,\eta\}$ to $\Gamma(u)$ is positive. Therefore, $\partial_t u$ is smooth on $\{supp\,\eta\}$ and, consequently, the right-hand side of (48) vanishes if τ is small enough. In addition, we make take in (48) the cut-off function ξ multiplied by the characteristic function of an interval $[\hat{t}-\delta-d_0^2/4, t]$ with an arbitrary $t \in]\hat{t}-\delta-d_0^2/4, \hat{t}-\delta]$. This leads for sufficiently small τ to the inequalities

$$\int\limits_{\hat{t}-\delta-d_0^2/4}^{t}\int\limits_{B_{d_0/2}(\hat{x})} \left\{\partial_t u^{(\tau)}\left(\partial_t u - k\right)_+ \xi^2 + Du^{(\tau)}D\left[\left(\partial_t u - k\right)_+ \xi^2\right]\right\} dxdt \leqslant 0$$
$$\forall t \in]\hat{t}-\delta-d_0^2/4, \hat{t}-\delta].$$

Now, we let in the latter inequalities $\tau \to 0$ and then tend $\delta \to 0$, leave the nonnegative terms in the left-hand side, transfer the rest terms to the right-hand side and estimate these rest terms from above via Young's inequality. As a consequence, for $k \geqslant 2N_*$ we get the inequalities

$$\sup_{\hat t - d_0^2/4 < t < \hat t} \int\limits_{B_{d_0/2}(\hat x)} (\partial_t u - k)_+ \, dx \Big|^t + \int\limits_{Q^-_{d_0/2}(\hat z) \cap \{\partial_t u > k\}} |D(\partial_t u)|^2 \xi^2 dxdt$$

$$\leqslant c \int\limits_{Q^-_{d_0/2}(\hat z)} (\partial_t u - k)_+ \left[|D\xi|^2 + 2\xi|\partial_t \xi| \right] dxdt.$$

Application of Fact 7.1 with $v = \partial_t u$ implies the estimate

$$\partial_t u(\hat z) \leqslant 2N_* + N_0 \sqrt{ \fint_{Q^-_{d_0/2}(\hat z)} |\partial_t u|^2 dxdt }. \tag{49}$$

In order to obtain a bound for the integral term on the right-hand side of (49) we take identity (7) with $Q^-_{2\rho}(z^*)$ replaced by $Q^-_{d_0}(\hat z)$ and plug in this identity a test-function

$$\eta(x,t) = \partial_t u(x,t) \zeta^2(x),$$

where ζ is a smooth cut-off function in $B_{d_0}(\hat x)$ that equals 1 in $B_{d_0/2}(\hat z)$ and vanishes outside of $B_{3d_0/4}(\hat x)$. After standard manipulations we end up with

$$\int\limits_{Q^-_{3d_0/4}(\hat z)} |\partial_t u|^2 \zeta^2 dxdt \leqslant c \int\limits_{Q^-_{3d_0/4}(\hat z)} \left(h^2[u]\zeta^2 + |Du|^2 |D\zeta|^2 \right) dxdt$$

$$\leqslant \tilde c \, (d_0)^{n+2} + \tilde c \, (d_0)^{-2} \int\limits_{Q^-_{3d_0/4}(\hat z)} |Du|^2 dxdt \tag{50}$$

$$\leqslant \tilde c \left\{ 1 + C_1^2 \right\} (d_0)^{n+2},$$

where the last inequality follows from Lemma 5.3.

Thus, combination of (49) and (50) provides the estimate

$$\partial_t u(\hat z) \leqslant 2N_* + N_0 \sqrt{\tilde c \{1 + C_1^2(\rho_0, \epsilon . M)\}}.$$

Observe that the constant on the right-hand side of the above inequality does not depend on d_0, i.e., on the parabolic distance from $\hat z$ to Γ^0.

2. Suppose now that $d_0 = \min\{\rho_0, \epsilon/2\}$. In this case we repeat all the above up to deriving (49). Then we estimate the integral term on the right-hand side of (49) with the help of inequalities (4) with $q = 2$. This gives us the bound

$$\int\limits_{Q^-_{d_0/2}(\hat{z})} |\partial_t u|^2 dxdt \leqslant N_1(\epsilon, 2, M)$$

which together with (49) implies

$$\partial_t u(\hat{z}) \leqslant 2N_* + N_0 N_1^{1/2} \left(\min\{\rho_0, \epsilon\}\right)^{-1-n/2}.$$

Again, the right-hand side of the latter bound is independent of the parabolic distance from $\hat{z}$ to Γ^0.

Repeating the above arguments for the function $-u$ instead of u we complete the proof. □

Lemma 6.2. *Let u satisfy the same assumptions as in Lemma 6.1. Then there exists a positive constant C_4 depending only on ρ_0, ϵ, M and $\beta - \alpha$ such that*

$$\left|D^2 u(\hat{z})\right| \leqslant C_4. \tag{51}$$

Proof. Let $\hat{z} \in Q \setminus \Gamma(u)$ be fixed, and let $\nu = Du(\hat{z})/|Du(\hat{z})|$. Suppose that e is an arbitrary direction in $\mathbb{R}^n$ if $|Du(\hat{z})| = 0$ and $e \perp \nu$ otherwise. We also define $d_0 = d_0(\hat{z}) := \min\{dist_p\{\hat{z}, \Gamma^0\}, \rho_0, \epsilon/2\}$.

In view of our choice of e we have $D_e u(\hat{z}) = 0$ and, consequently, we may apply Fact 7.4 to the sub-caloric functions $(D_e u)_\pm$ in $Q^-_{d_0}(\hat{z})$. From here, taking into account Lemma 5.3, we obtain the estimate

$$|D(D_e u)(\hat{z})| \leqslant C_4(\rho_0, \epsilon, M, \beta - \alpha),$$

where C_4 does not depend on d_0. Since e is an arbitrary direction in $\mathbb{R}^n$ satisfying $e \perp \nu$, the derivative $D_\nu(D_\nu u(\hat{z}))$ can now be estimated from Eq. (1). Thus, we proved the desired inequality (51). □

7 Appendix

For the readers convenience and for the references, we recall and explain several facts. Most of these auxiliary results are known, but probably not well known in the context used in this paper.

Fact 7.1. *Let $r_0 \in (0,1)$, and let $v \in V_2\left(Q^-_{r_0}(z^*)\right)$ satisfy the inequalities*

$$\sup_{t^*-r_0^2<t<t^*} \int\limits_{B_{r_0}(x^*)} (v-k)_+^2\, \xi^2 dx\Big|^t + \int\limits_{Q^-_{r_0}(z^*)} \left[D\left((v-k)_+\right)\right]^2 \xi^2 dz$$
$$\leqslant c \int\limits_{Q^-_{r_0}(z^*)} (v-k)_+^2 \left[|D\xi|^2 + \xi|\partial_t \xi|\right] dz$$

for all $k \geqslant k_0$ and all cut-off functions $\xi = \xi(x,t)$ defined in $Q^-_{r_0}(z^*)$ (see Notation). Here c stands for a positive constant.

Then there exists a positive constant $N_0 = N_0(c)$ such that

$$\sup_{Q^-_{r_0/2}(z^*)} v \leqslant k_0 + N_0 \sqrt{\fint\limits_{Q^-_{r_0}(z^*)} v^2(z) dz}.$$

Proof. For the proof of this assertion we refer the reader to (the proof of) Theorem 6.2, Chapter II [LSU67]. □

Fact 7.2. *Let $\mathcal{D}$ be a domain in $\mathbb{R}^{n+1}$, and let $g^i \in L^\infty(\mathcal{D})$, $i = 0, 1, \ldots, n$. Then if $v \in V_2(\mathcal{D})$ is a solution of the equation*

$$H[v] = div\, \vec{g} + g^0, \qquad \vec{g} = \left(g^1, \ldots, g^n\right)$$

in $\mathcal{D}$, we have, for any cylinder $Q^-_{2R}(z^0) \subset \mathcal{D}$,

$$\sup_{Q^-_R(z^0)} |v| \leqslant \hat{N}_0 \sqrt{\fint_{Q^-_{2R}(z^0)} v^2 dxdt} + \hat{N}_1 R \|\vec{g}\|_{\infty, Q^-_{2R}(z^0)} + \hat{N}_2 R^2 \|g^0\|_{\infty, Q^-_{2R}(z^0)}$$

Proof. The validity of Fact 7.2 follows from results of §6 Chapter II and §8 Chapter III [LSU67] (see also Theorem 6.17 in [Lie96]). □

We denote

$$I(r, v, z^*) = \int\limits_{t^*-r^2}^{t^*} \int\limits_{\mathbb{R}^n} |Dv(x,t)|^2 G(x - x^*, t^* - t) dxdt,$$

where $r \in]0, \rho_0]$, $z^* = (x^*, t^*)$ is a point in $\mathbb{R}^{n+1}$, a function v is defined n the strip $\mathbb{R}^n \times [t^* - \rho_0^2, t^*]$, and the heat kernel $G(x,t)$ is defined by

$$G(x,t) = \frac{\exp\left(-|x|^2/4t\right)}{(4\pi t)^{n/2}} \text{ for } t > 0 \text{ and } G(x,t) = 0 \text{ for } t \leqslant 0.$$

To prove the quadratic growth estimate for solutions of (1), we need the following local version of the famous Caffarelli monotonicity formula (see [CS05]) for pairs of disjointly supported subsolutions of the heat equation.

Fact 7.3. *Let $z^* = (x^*, t^*)$ be a point in $\mathbb{R}^{n+1}$, let $\xi_{\rho_0,x^*} := \xi_{\rho_0,x^*}(x)$ be a standard time-independent cut-off function belonging $C^2(\overline{B}_{\rho_0}(x^*))$, having support in $B_{\rho_0}(x^*)$, and satisfying $\xi_{\rho_0,x^*} \equiv 1$ in $B_{\rho_0/2}(x^*)$, and let θ_1, θ_2 be nonnegative, sub-caloric and continuous functions in $Q^-_{\rho_0}(z^*)$, satisfying*

$$\theta_1(x^*, t^*) = \theta_2(x^*, t^*) = 0, \qquad \theta_1(x,t) \cdot \theta_2(x,t) = 0 \quad \text{in} \quad Q^-_{\rho_0}(z^*).$$

Then, for $0 < r < \rho_0$ the functional

$$\Phi(r, \xi_{\rho_0,z^*}) := \Phi(r, \theta_1, \theta_2, \xi_{\rho_0,z^*}, z^*) = \frac{1}{r^4} I(r, \theta_1 \xi_{\rho_0,z^*}, z^*) I(r, \theta_2 \xi_{\rho_0,z^*}, z^*)$$

satisfies the inequality

$$\Phi(r, \xi_{\rho_0,z^*}) \leqslant \frac{\tilde{N}}{\rho_0^{2n+8}} \|\theta_1\|^2_{2,Q^-_{\rho_0}(z^*)} \|\theta_2\|^2_{2,Q^-_{\rho_0}(z^*)} \tag{52}$$

with an absolute positive constant $\tilde{N}$.

Proof. Using the same arguments as in the proof of Lemma 2.4 and Remark after that in [ASU00] (see also Fact 1.6 and Remark 1.7 in [AU13]) one can get the inequality

$$\Phi(r, \xi_{\rho_0,z^*}) \leqslant \Phi(\rho_0/2, \xi_{\rho_0,z^*}) + \frac{N'}{\rho_0^{2n+8}} \|\theta_1\|^2_{2,Q^-_{\rho_0}(z^*)} \|\theta_2\|^2_{2,Q^-_{\rho_0}(z^*)}, \tag{53}$$

where N' is an absolute positive constant.

We claim that the first term on the right-hand side of (53) can be estimated via the second term. Indeed, it is evident that

$$\Phi(\rho_0/2, \xi_{\rho_0,z^*}) \leqslant \frac{c}{\rho_0^4} I(\rho_0, \theta_1 \zeta_0, z^*) I(\rho_0, \theta_2 \zeta_0, z^*), \tag{54}$$

where $\zeta_0 = \zeta_0(x,t) = \xi_{\rho_0,z^*}(x) \varsigma_{\rho_0,z^*}(t)$, while ς_{ρ_0,z^*} stands for a nonnegative function belonging $C^2\left([t^* - \rho_0^2, t^*]\right)$, having support in $[t^* - 3\rho_0^2/4, t^*]$ and satisfiying $\varsigma_{\rho_0,z^*}(t) \equiv 1$ in $[t^* - \rho_0^2/4, t^*]$.

On the other hand, functions θ_i, $i = 1, 2$, are sub-caloric in $Q^-_{\rho_0}(z^*)$, i.e., $H[\theta_i] \geqslant 0$ in the sense of distributions. Since

$$|D\theta_i|^2 + \theta_i H[\theta_i] = \frac{1}{2} H[\theta_i^2]$$

we have

$$
\begin{aligned}
&\int_{t^*-\rho_0^2}^{t^*} \int_{\mathbb{R}^n} |D\theta_i(x,t)|^2 \zeta_0^2(x,t) G(x-x^*, t^*-t)\, dx dt \\
&\qquad \leqslant \frac{1}{2} \int_{t^*-r_0^2}^{t^*} \int_{\mathbb{R}^n} H[\theta_i^2(x,t)] \zeta_0^2(x,t) G(x-x^*, t^*-t)\, dx dt.
\end{aligned}
\tag{55}
$$

After successive integration the right-hand side of (55) by parts we get

$$
\begin{aligned}
\int_{t^*-\rho_0^2}^{t^*} \int_{\mathbb{R}^n} |D\theta_i|^2 \zeta_0^2 G dx dt &= \int_{t^*-\rho_0^2}^{t^*} \int_{B_{\rho_0}(x^*)} |D\theta_i|^2 \zeta_0^2 G dx dt \\
&\leqslant - \int_{B_{\rho_0}(x^*)} \left(\frac{\theta_i^2}{2} \zeta_0^2 G \right) dx \Big|_{t^*-\rho_0^2/4}^{t^*} \\
&+ \int_{t^*-\rho_0^2}^{t^*} \int_{B_{\rho_0}(x^*)} \frac{\theta_i^2}{2} \zeta_0^2 \left[\partial_t G + \Delta G \right] dx dt \\
&+ \int_{t^*-\rho_0^2}^{t^*} \int_{B_{\rho_0}(x^*)} \theta_i^2 \left[2\zeta_0 D\zeta_0 DG + G|D\zeta_0|^2 + G\zeta_0 \Delta\zeta_0 \right] dx dt \\
&+ \int_{t^*-\rho_0^2}^{t^*} \int_{B_{\rho_0}(x^*)} \theta_i^2 G \zeta_0 |\partial_t \zeta_0| dx dt \\
&=: J_1 + J_2 + J_3 + J_4.
\end{aligned}
$$

It is evident that due to our choice of ζ_0 we have $J_1 \leqslant 0$.

Further, taking into account the relation

$$
\partial_t G + \Delta G = \partial_t G(x-x^*, t^*-t) + \Delta G(x-x^*, t^*-t) = 0 \quad \text{for} \quad t < t^*,
$$

we conclude that $J_2 = 0$.

Finally, we observe that the integral in J_3 is really taken over the set $\mathcal{E} =]t^* - \rho_0^2, t^*] \times \left\{ B_{\rho_0}(x^*) \setminus B_{\rho_0/2}(x^*) \right\}$, while the integral in J_4 is taken over the set $\mathcal{E}' = [t^* - \rho_0^2, t^* - \rho_0^2/4] \times B_{\rho_0}(x^*)$. Therefore, in $\mathcal{E}$ we have the following estimates for functions involved into J_3

$$|G(x-x^*,t^*-t)| \leqslant \hat{c}\frac{e^{-\frac{\rho_0^2}{16(\rho_0^2-t)}}}{(\rho_0^2-t)^{n/2}} \leqslant \hat{c}\rho_0^{-n};$$

$$\begin{aligned}|DG(x-x^*,t^*-t)D\zeta_0(x,t)| &\leqslant \hat{c}|G(x-x^*,t^*-t)|\frac{|x-x^*|}{\rho_0\,(\rho_0^2-t)} \\ &\leqslant \hat{c}\frac{e^{-\frac{\rho_0^2}{16(\rho_0^2-t)}}}{(\rho_0^2-t)^{1+n/2}} \leqslant \hat{c}\rho_0^{-n-2}.\end{aligned}$$

Similarly, in $\mathcal{E}'$ we have

$$|G(x-x^*,t^*-t)| \leqslant \hat{c}\rho_0^{-n},$$

and, consequently,

$$J_3+J_4 \leqslant \tilde{c}\rho_0^{-n-2} \iint\limits_{Q^-_{\rho_0}(z^*)} \theta_i^2 dxdt \leqslant \tilde{c}\rho_0^{-n-2}\|\theta_i\|^2_{2,Q^-_{\rho_0}(z^*)}.$$

Thus, collecting all inequalities we get

$$\begin{aligned}I(\rho_0,\theta_i\zeta_0,z^*) &\leqslant 2\int\limits_{t^*-\rho_0^2}^{t^*}\int\limits_{B_{\rho_0}(x^*)} \left[|D\zeta_0|^2\theta_i^2+|D\theta_i|^2\zeta_0^2\right] Gdxdt \\ &\leqslant N''\rho_0^{-n-2}\|\theta_i\|^2_{2,Q^-_{\rho_0}(z^*)},\end{aligned} \tag{56}$$

where N'' denotes a positive absolute constant.

Now, combination of (53), (54) and (56) finishes the proof of (52). □

Fact 7.4. *Let a continuous function v in the cylinder $Q^-_R(z^0)$ satisfies the following conditions:*

$v(z^0)=0$;
v is differentiable at z^0;
$v_\pm$ are subcaloric in $Q^-_R(z^0)$.

Then

$$|Dv(z^0)| \leqslant \tilde{N}'\sqrt{R^{-2}\fint\limits_{Q_R(z^0)} v^2 dxdt}.$$

Proof. The above inequality follows directly from Fact 7.3. □

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