On a Cahn-Hilliard type phase field system related to tumor growth

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Abstract

The paper deals with a phase field system of Cahn-Hilliard type. For positive viscosity coefficients, the authors prove an existence and uniqueness result and study the long time behavior of the solution by assuming the nonlinearities to be rather general. In a more restricted setting, the limit as the viscosity coefficients tend to zero is investigated as well.

Key words: phase field model, tumor growth, viscous Cahn-Hilliard equations, well posedness, long-time behavior, asymptotic analysis

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1 Introduction

In this article, we study the coupled system of partial differential equations

$$\alpha\partial_t\mu + \partial_t u - \Delta\mu = p(u)(\sigma - \gamma\mu) \tag{1.1}$$

$$\mu = \alpha\partial_t u - \Delta u + \mathcal{W}'(u) \tag{1.2}$$

$$\partial_t\sigma - \Delta\sigma = -p(u)(\sigma - \gamma\mu) \tag{1.3}$$

in a domain $\Omega \times (0, \infty)$, together with the boundary conditions

$$\partial_\nu \mu = \partial_\nu u = \partial_\nu \sigma = 0 \quad \text{on the boundary } \Gamma \times (0, \infty) \tag{1.4}$$

and the initial conditions

$$\mu(0) = \mu_0, \quad u(0) = u_0 \quad \text{and} \quad \sigma(0) = \sigma_0\,. \tag{1.5}$$

Each of the partial differential equations (1.1)–(1.3) is meant to hold in a three-dimensional bounded domain Ω endowed with a smooth boundary Γ and for every positive time, and ∂_ν in (1.4) stands for the normal derivative. Moreover, α and γ are positive constants. Furthermore, p is a nonnegative function and $\mathcal{W}$ is a nonnegative double well potential. Finally, μ_0, u_0, and σ_0 are given initial data defined in Ω.

The physical context is that of a tumor-growth model which has been derived from a general mixture theory [16, 12]. We also refer to [4], [15] and [6] for overview articles and to [7] and [5] for the study of related sharp interface models.

We point out that the unknown function u is an order parameter which is close to two values in the regions of nearly pure phases, say $u \simeq 1$ in the tumorous phase and $u \simeq -1$ in the healthy cell phase; the second unknown μ is the related chemical potential, specified by (1.2) as in the case of the viscous Cahn-Hilliard or Cahn-Hilliard equation depending on whether $\alpha > 0$ or $\alpha = 0$ (see [3, 9, 10]); the third unknown σ stands for the nutrient concentration, typically $\sigma \simeq 1$ in a nutrient-rich extracellular water phase and $\sigma \simeq 0$ in a nutrient-poor extracellular water phase.

In the case that the parameter α is strictly positive, the problem (1.1)–(1.5) is a generalized phase field model, while it becomes of pure Cahn-Hilliard type in the case that $\alpha = 0$. On the other hand, the presence of the term $\alpha\mu_t$ in (1.1) gives, in the case $\alpha > 0$, a parabolic structure to equation (1.1) with respect to μ. Let us note that the meaning of the coefficient α here differs from the one in (1.2): in (1.1) α is not a viscosity coefficient since it enters in the natural Lyapunov functional of the system, which reads (cf. [13])

$$E(u, \mu, \sigma) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \mathcal{W}(u) + \frac{\alpha}{2}\mu^2 + \frac{1}{2}\sigma^2 \right).$$

However, the fact that the coefficients are taken equal in the system (1.1)–(1.3) is somehow related to the limiting problem obtained by formal asymptotics on α. Indeed, we refer to the forthcoming article [13] for a formal study of the relation between these models and the corresponding sharp interfaces limits.

We remark that the original model deals with functions $\mathcal{W}$ and p that are precisely related to each other. Namely, we have

$$p(u) = 2p_0\sqrt{\mathcal{W}(u)} \quad \text{if } |u| \le 1 \quad \text{and} \quad p(u) = 0 \quad \text{otherwise} \tag{1.6}$$

where p_0 is a positive constant and where $\mathcal{W}(u) := -\int_0^u f(s)\,ds$ is the classical Cahn–Hilliard double well free-energy density. However, this relation is not used in this paper, whose first aim is proving the well-posedness of the initial-boundary value problem (1.1)–(1.5) in the case of a positive parameter α. In this setting, we can allow $\mathcal{W}$ to be even a singular potential (the reader can see the later Remark 2.1).

Actually, we prove the existence of a unique strong solution to the system (1.1)–(1.5) under very general conditions on p and $\mathcal{W}$, as well as we study the long time behavior of the solution; in particular, we can characterize the omega limit set and deduce an interesting property in a special physical case (cf. the later Corollary 2.5). Next, in a more restricted setting for the double-well potential $\mathcal{W}$, we investigate the asymptotic behavior of the problem as the coefficient α tends to zero and find the convergence of subsequences to weak solutions of the limiting problem. Moreover, under a smoothness condition on the initial value u_0 we are able to show uniqueness for the limit problem and consequently also the convergence of the entire family as $\alpha \searrow 0$ (see Theorem 2.6).

Our paper is organized as follows. In Section 2, we state our assumptions and results on the mathematical problem. The forthcoming sections are devoted to the corresponding proofs. In Section 3, we prove the uniqueness of the solution. After presenting a priori estimates in Section 4, we prove the existence of the solution on an arbitrary time interval in Section 5, while we study its large time behavior in Section 6; Section 7 is devoted to the study of the limit of the phase field model (1.1)-(1.5) to the corresponding Cahn-Hilliard problem as $\alpha \to 0$.

2 Statement of the mathematical problem

In this section, we make precise assumptions and state our results. First of all, we assume Ω to be a bounded connected open set in $\mathbb{R}^3$ (lower-dimensional cases could be considered with minor changes) whose boundary Γ is supposed to be smooth. As in the Introduction, the symbol ∂_ν denotes the (say, outward) normal derivative on Γ. As the first aim of our analysis is to study the well-posedness on any finite time interval, we fix a final time $T \in (0, +\infty)$ and let

$$Q := \Omega \times (0,T) \quad \text{and} \quad \Sigma := \Gamma \times (0,T). \tag{2.1}$$

Moreover, we set for convenience

$$V := H^1(\Omega), \quad H := L^2(\Omega) \quad \text{and} \quad W := \{v \in H^2(\Omega) : \ \partial_\nu v = 0 \text{ on } \Gamma\} \tag{2.2}$$

and endow the spaces (2.2) with their standard norms, for which we use a self-explanatory notation like $\|\cdot\|_V$. If $p \in [1, +\infty]$, it will be useful to write $\|\cdot\|_p$ for the usual norm in $L^p(\Omega)$. In the sequel, the same symbols are used for powers of the above spaces. It is understood that $H \subset V^*$ as usual, i.e., in order that $\langle u, v\rangle = \int_\Omega uv$ for every $u \in H$ and $v \in V$, where $\langle \cdot \, , \cdot \rangle$ stands for the duality pairing between V^* and V.

As far as the structure of the system is concerned, we are given two constants α and γ and three functions p, $\widehat{\beta}$ and λ satisfying the conditions listed below

$$\alpha \in (0,1) \quad \text{and} \quad \gamma > 0 \tag{2.3}$$

$$p : \mathbb{R} \to \mathbb{R} \text{ is nonnegative, bounded and Lipschitz continuous} \tag{2.4}$$

$$\widehat{\beta} : \mathbb{R} \to [0, +\infty] \text{ is convex, proper, lower semicontinuous} \tag{2.5}$$

$$\lambda \in C^1(\mathbb{R}) \text{ is nonnegative and } \lambda' \text{ is Lipschitz continuous.} \tag{2.6}$$

We define the potential $\mathcal{W}\colon \mathbb{R} \to [0,+\infty]$ and the graph β in $\mathbb{R}\times\mathbb{R}$ by

$$\mathcal{W} := \widehat{\beta} + \lambda \quad \text{and} \quad \beta := \partial\widehat{\beta} \tag{2.7}$$

and note that β is maximal monotone. In the sequel, we write $D(\widehat{\beta})$ and $D(\beta)$ for the effective domains of $\widehat{\beta}$ and β, respectively, and we use the same symbol β for the maximal monotone operators induced on L^2 spaces.

Remark 2.1. Note that lots of potentials $\mathcal{W}$ fit our assumptions. Typical examples are the classical double well potential and the logarithmic potential defined by

$$\mathcal{W}_{cl}(r) := \tfrac{1}{4}(r^2-1)^2 = \tfrac{1}{4}((r^2-1)^+)^2 + \tfrac{1}{4}((1-r^2)^+)^2 \quad \text{for } r\in\mathbb{R} \tag{2.8}$$

$$\mathcal{W}_{log}(r) := (1-r)\ln(1-r) + (1+r)\ln(1+r) + \kappa(1-r^2)^+ \quad \text{for } |r|<1 \tag{2.9}$$

where the decomposition $\mathcal{W} = \widehat{\beta} + \lambda$ as in (2.7) is written explicitely. More precisely, in (2.9), κ is a positive constant which does or does not provide a double well depending on its value, and the definition of the logarithmic part of $\mathcal{W}_{log}$ is extended by continuity at ±1 and by $+\infty$ outside $[-1,1]$. Moreover, another possible choice is the following

$$\mathcal{W}(r) := I(r) + ((1-r^2)^+)^2 \quad \text{where } I \text{ is the indicator function of } [-1,1] \tag{2.10}$$

taking the value 0 in $[-1,1]$ and $+\infty$ elsewhere. Clearly, if β is multi-valued like in the case (2.10), the precise statement of problem (1.1)–(1.5) has to introduce a selection ξ of $\beta(u)$. We also remark that our assumptions do not include the relationship (1.6) between $\mathcal{W}$ and p, and one can wonder whether what we have required is compatible with (1.6). This is the case if $\widehat{\beta}$ and λ satisfy suitable conditions, in addition. For instance, we can assume the following: $D(\widehat{\beta})$ includes the interval $[-1,1]$ and $\widehat{\beta}$ vanishes there; λ is strictly positive in $(-1,1)$ and $\lambda(\pm1) = \lambda'(\pm1) = 0$. In such a case, $\mathcal{W}$ presents two minima with quadratic behavior at ±1 and the function p given by (1.6) actually satisfies (2.4). We note that this excludes the case of the logarithmic potential, while it includes both (2.8) and (2.10).

As far as the initial data of our problem are concerned, we assume that

$$\mu_0, u_0, \sigma_0 \in V \quad \text{and} \quad \widehat{\beta}(u_0) \in L^1(\Omega) \tag{2.11}$$

while the regularity properties which we obtain for the solution are the following

$$\mu, u, \sigma \in H^1(0,T;H) \cap L^2(0,T;W) \subset C^0([0,T];V) \tag{2.12}$$

$$\xi \in L^2(0,T;H) \quad \text{and} \quad \xi \in \beta(u) \quad \text{a.e. in } Q. \tag{2.13}$$

At this point, the problem we want to investigate consists in looking for a quadruplet (μ, u, σ, ξ) satisfying the above regularity and the following boundary value problem

$$\alpha\partial_t\mu + \partial_t u - \Delta\mu = R \quad \text{where} \quad R = p(u)(\sigma - \gamma\mu) \qquad \text{a.e. in } Q \tag{2.14}$$

$$\mu = \alpha\partial_t u - \Delta u + \xi + \lambda'(u) \qquad \text{a.e. in } Q \tag{2.15}$$

$$\partial_t\sigma - \Delta\sigma = -R \qquad \text{a.e. in } Q \tag{2.16}$$

$$\partial_\nu\mu = \partial_\nu u = \partial_\nu\sigma = 0 \qquad \text{a.e. on } \Sigma \tag{2.17}$$

$$\mu(0) = \mu_0, \quad u(0) = u_0 \quad \text{and} \quad \sigma(0) = \sigma_0 \qquad \text{in } \Omega\,. \tag{2.18}$$

We note once and for all that adding (2.14) and (2.16) yields

$$\partial_t\bigl(\alpha\mu + u + \sigma\bigr) - \Delta(\mu+\sigma) = 0 \quad \text{a.e. in } Q. \tag{2.19}$$

Here is our well-posedness result.

Theorem 2.2. *Assume* (2.3)–(2.7) *and* (2.11). *Then, there exists a unique quadruplet* (μ, u, σ, ξ) *satisfying* (2.12)–(2.13) *and solving problem* (2.14)–(2.18).

Remark 2.3. By starting from the regularity requirements (2.12)–(2.13) and owing to the regularity theory of elliptic and parabolic equations, one can easily derive further properties of the solution. For instance, as $R \in L^\infty(0,T;H)$ one can show that

$$\sigma \in W^{1,p}(0,T;H) \cap L^p(0,T;W) \quad \text{for every } p \in [1,+\infty) \tag{2.20}$$

provided that σ_0 is smooth enough (see, e.g., [8, Thm. 2.3]).

Once we know that there exists a unique solution on any finite time interval, we can address its long time behavior. Our next result deals with the omega limit of an arbitrary initial datum satisfying (2.11). Even though the possible topologies are several, by recalling (2.12) we choose

$$\Phi := V \times V \times V \tag{2.21}$$

as a phase space and set

$$\omega = \omega(\mu_0, u_0, \sigma_0) := \Bigl\{ (\mu_\omega, u_\omega, \sigma_\omega) \in \Phi : \ (\mu, u, \sigma)(t_n) \to (\mu_\omega, u_\omega, \sigma_\omega) \\ \text{strongly in } \Phi \text{ for some } \{t_n\} \nearrow +\infty \Bigr\}. \tag{2.22}$$

Our result reads as follows

Theorem 2.4. *Assume* (2.3)–(2.7) *and* (2.11). *Then, the omega limit set ω is non-empty. Moreover, if $(\mu_\omega, u_\omega, \sigma_\omega)$ is any element of ω, then μ_ω and σ_ω are constant and their constant values μ_s, σ_s and the function u_ω are related to each other by*

$$p(u_\omega)(\sigma_s - \gamma\mu_s) = 0 \quad \textit{and} \quad -\Delta u_\omega + \beta(u_\omega) + \lambda'(u_\omega) \ni \mu_s \quad \textit{a.e. in } \Omega, \\ \partial_\nu u_\omega = 0 \quad \textit{a.e. on } \Gamma. \tag{2.23}$$

We deduce an interesting consequence in a special case which, however, is significant.

Corollary 2.5. *Assume that $D(\widehat{\beta}) = [-1,1]$ and that p is strictly positive in $(-1,1)$ and vanishes at ± 1. Then, we have*

$$\textit{either} \quad \sigma_s = \gamma\mu_s \quad \textit{or} \quad u_\omega = -1 \quad \textit{a.e. in } \Omega \quad \textit{or} \quad u_\omega = 1 \quad \textit{a.e. in } \Omega\,.$$

Indeed, if $\sigma_s \neq \gamma\mu_s$, (2.23) implies $u_\omega \in D(\widehat{\beta})$ and $p(u_\omega) = 0$ a.e. in Ω. On the other hand, we have $u_\omega \in V$, whence the conclusion. We also remark that, in the case of the potential (2.10), the inclusion in (2.23) reduces to $\beta(u_\omega) \ni \mu_s$. In particular, $u_\omega = -1$ if $\mu_s < 0$ and $u_\omega = 1$ if $\mu_s > 0$.

Our final result regards the asymptotic analysis as the viscosity coefficient α tends to zero and the study of the limit problem. We can deal with this by restricting ourselves to a particular class of potentials. Namely, we also assume that

$$D(\widehat{\beta}) = \mathbb{R} \quad \text{and} \quad \mathcal{W} = \widehat{\beta} + \lambda \quad \text{is a } C^3 \text{ function on } \mathbb{R} \tag{2.24}$$

$$\mathcal{W}(r) \geq \delta_0 |r| - c_0 \,, \quad |\mathcal{W}'(r)| \leq c_1(|r|^3 + 1) \,, \quad |\mathcal{W}''(r)| \leq c_2(|r|^2 + 1) \tag{2.25}$$

for any $r \in \mathbb{R}$ and some positive constants δ_0 and $c_0\,, c_1\,, c_2$. We have written both the last two conditions in (2.25) for convenience even though the latter implies the former. We also remark that the classical potential (2.8) fulfils such assumptions. Here is our result.

Theorem 2.6. *Assume* (2.3)–(2.7), (2.11), *and* (2.24)–(2.25) *in addition. Moreover, let* $(\mu_\alpha, u_\alpha, \sigma_\alpha, \xi_\alpha)$ *be the unique solution to problem* (2.14)–(2.18) *given by Theorem 2.2. Then, we have: i) the following convergence holds*

$$\mu_\alpha \to \mu \qquad \textit{weakly in } L^2(0,T;V) \tag{2.26}$$

$$u_\alpha \to u \qquad \textit{weakly star in } L^\infty(0,T;V) \cap L^2(0,T;W) \tag{2.27}$$

$$\sigma_\alpha \to \sigma \qquad \textit{weakly in } H^1(0,T;H) \cap L^2(0,T;W) \tag{2.28}$$

$$\partial_t(\alpha\mu_\alpha + u_\alpha) \to \partial_t u \quad \textit{weakly in } L^2(0,T;V^*) \tag{2.29}$$

at least for a subsequence; ii) every limiting triplet (μ, u, σ) *satisfies*

$$\langle \partial_t u, v \rangle + \int_\Omega \nabla\mu \cdot \nabla v = \int_\Omega R\, v \qquad \forall\, v \in V, \ \ \textit{a.e. in } (0,T) \tag{2.30}$$

$$R = p(u)(\sigma - \gamma\mu) \qquad \textit{a.e. in } Q \tag{2.31}$$

$$\mu = -\Delta u + \mathcal{W}'(u) \qquad \textit{a.e. in } Q \tag{2.32}$$

$$\partial_t\sigma - \Delta\sigma = -R \qquad \textit{a.e. in } Q \tag{2.33}$$

$$\partial_\nu u = \partial_\nu \sigma = 0 \qquad \textit{a.e. on } \Sigma \tag{2.34}$$

$$u(0) = u_0 \quad \textit{and} \quad \sigma(0) = \sigma_0 \qquad \textit{in } \Omega\,; \tag{2.35}$$

iii) if

$$u_0 \in W \tag{2.36}$$

then every solution to problem (2.30)–(2.35) *satisfying the regularity given by* (2.26)–(2.29) *also satisfies the further regularity*

$$\mu \in L^\infty(0,T;H) \cap L^2(0,T;W) \tag{2.37}$$

$$u \in H^1(0,T;H) \cap L^\infty(0,T;W) \subset L^\infty(Q)\,. \tag{2.38}$$

Moreover, such a solution is unique.

Remark 2.7. We observe that even further regularity for σ could be derived on account of the regularity of R induced by (2.37), provided that σ_0 is smoother. It must be pointed out that a uniqueness result for the limit problem has been proved in [11] by a different argument. In the same paper, a slightly different regularity result is shown as well. Finally, we remark that the uniqueness property implies that the whole family $(\mu_\alpha, u_\alpha, \sigma_\alpha)$ converges to (μ, u, σ) in the sense of (2.26)–(2.29) as $\alpha \searrow 0$.

The rest of the section is devoted to list some facts and to fix some notations. We recall that Ω is bounded and smooth. So, throughout the paper, we owe to some well-known embeddings of Sobolev type, namely $V \subset L^p(\Omega)$ for $p \in [1,6]$, together with the related Sobolev inequality

$$\|v\|_p \leq C\|v\|_V \quad \text{for every } v \in V \text{ and } 1 \leq p \leq 6 \tag{2.39}$$

and $W^{1,p}(\Omega) \subset C^0(\overline{\Omega})$ for $p > 3$, together with

$$\|v\|_\infty \leq C_p\|v\|_{W^{1,p}(\Omega)} \quad \text{for every } v \in W^{1,p}(\Omega) \text{ and } p > 3. \tag{2.40}$$

In (2.39), C only depends on Ω, while C_p in (2.40) also depends on p. In particular, the continuous embedding $W \subset W^{1,6}(\Omega) \subset C^0(\overline{\Omega})$ holds. Some of the previous embeddings are in fact compact. This is the case for $V \subset L^4(\Omega)$ and $W \subset C^0(\overline{\Omega})$. We note that also the embeddings $W \subset V$, $V \subset H$ and $H \subset V^*$ are compact. Moreover, we often account for the well-known Poincaré inequality

$$\|v\|_V \leq C\Bigl(\|\nabla v\|_H + \bigl|\textstyle\int_\Omega v\bigr|\Bigr) \quad \text{for every } v \in V \tag{2.41}$$

where C depends only on Ω. Furthermore, we repeatedly make use of the notation

$$Q_t := \Omega \times (0,t) \quad \text{for } t \in [0,T] \tag{2.42}$$

and of well-known inequalities, namely, the Hölder inequality and the elementary Young inequality:

$$ab \leq \delta a^2 + \frac{1}{4\delta}\, b^2 \quad \text{for every } a,b \geq 0 \text{ and } \delta > 0. \tag{2.43}$$

Next, we introduce a tool that is generally used in the context of problems related to the Cahn-Hilliard equations. We define $\mathcal{N} : \operatorname{dom}\mathcal{N} \subset V^* \to V$ as follows:

$$\operatorname{dom}\mathcal{N} := \{v^* \in V^* : \ \langle v^*, 1\rangle = 0\} \tag{2.44}$$

and,

$$\begin{aligned} &\text{for } v^* \in \operatorname{dom}\mathcal{N}, \quad \mathcal{N}v^* \text{ is the unique } w \in V \text{ such that} \\ &\int_\Omega \nabla w \cdot \nabla v = \langle v^*, v\rangle \quad \text{for every } v \in V \quad \text{and} \quad \int_\Omega w = 0\,. \end{aligned} \tag{2.45}$$

Note that problem (2.45) actually has a unique solution $w \in V$ since Ω is also connected and that w solves the homogeneous Neumann problem for $-\Delta w = v^*$ in the special case $v^* \in H$. It is easily checked that

$$\langle z^*, \mathcal{N}v^*\rangle = \langle v^*, \mathcal{N}z^*\rangle = \int_\Omega (\nabla\mathcal{N}z^*)\cdot(\nabla\mathcal{N}v^*) \quad \text{for } z^*, v^* \in \operatorname{dom}\mathcal{N} \tag{2.46}$$

$$v^* \mapsto \|v^*\|_* := \|\nabla\mathcal{N}v^*\|_H = \langle v^*, \mathcal{N}v^*\rangle^{1/2} \quad \text{is a norm on } \operatorname{dom}\mathcal{N} \tag{2.47}$$

$$\|v^*\|_{V^*} \leq C\|v^*\|_* \quad \text{for some constant } C \text{ and every } v^* \in \operatorname{dom}\mathcal{N} \tag{2.48}$$

$$\begin{aligned} &2\langle \partial_t v^*(t), \mathcal{N}v^*(t)\rangle = \frac{d}{dt}\int_\Omega |\nabla\mathcal{N}v^*(t)|^2 = \frac{d}{dt}\,\|v^*(t)\|_*^2 \quad \text{for a.a. } t \in (0,T) \\ &\quad \text{and for every } v^* \in H^1(0,T;V^*) \text{ satisfying } \langle v^*(t), 1\rangle = 0 \text{ for all } t \in [0,T]. \end{aligned} \tag{2.49}$$

Inequality (2.48) (where one could check that $C = 1$ actually is suitable) essentially says that $\|\cdot\|_*$ and the usual dual norm $\|\cdot\|_{V^*}$ are equivalent on $\operatorname{dom}\mathcal{N}$ (the opposite inequality is straightforward, indeed). Finally, throughout the paper, we use a small-case italic c without any subscript for different constants, that may only depend on Ω, the constant γ, the shape of the nonlinearities p, β and λ, and the norms of the initial data related to assumption (2.11). We point out that c does not depend on α nor on the final time T nor on the parameter ε we introduce in a forthcoming section. For any parameter δ, a notation like c_δ or $c(\delta)$ signals a constant that depends also on the parameter δ. This holds, in particular, if δ is either α, or T, or the pair (α, T). The reader should keep in mind that the meaning of c and c_δ might change from line to line and even in the same chain of inequalities, whereas those constants we need to refer to are always denoted by different symbols, e.g., by a capital letter like in (2.39) or by a letter with a proper subscript as in (2.25).

3 Uniqueness

In this section, we prove the uniqueness part of Theorem 2.2, that is, we pick two solutions $(\mu_i, u_i, \sigma_i, \xi_i)$, $i = 1, 2$, and show that they are the same. As both α and T are fixed, we avoid to stress the dependence of the constants on such parameters. Moreover, as the solutions we are considering are fixed as well, we can allow the values of c to depend on them, in addition. So, we write (2.19) and some of the equations of (2.14)–(2.18) for both solutions and take the difference. If we set $\mu := \mu_1 - \mu_2$ for brevity and analogously define u, σ and ξ, we have

$$\partial_t\bigl(\alpha\mu + u + \sigma\bigr) - \Delta(\mu + \sigma) = 0 \tag{3.1}$$

$$\mu = \alpha\partial_t u - \Delta u + \xi + \lambda'(u_1) - \lambda'(u_2) \tag{3.2}$$

$$\partial_t\sigma - \Delta\sigma = R_2 - R_1. \tag{3.3}$$

We note that (3.1) implies $\int_\Omega(\alpha\mu + u + \sigma)(t) = \int_\Omega(\alpha\mu + u + \sigma)(0) = 0$ for every t, so that $\mathcal{N}(\alpha\mu + u + \sigma)$ is meaningful and we can use it to test (3.1). Then, we test (3.2) by $-u$ and sum the resulting equalities. Using the properties (2.45), (2.47) and (2.49) of $\mathcal{N}$, we have for every $t \in [0, T]$

$$\frac{1}{2}\|\alpha\mu(t) + u(t) + \sigma(t)\|_*^2 + \int_{Q_t}(\mu + \sigma)(\alpha\mu + u + \sigma)$$
$$- \int_{Q_t}\mu u + \frac{\alpha}{2}\int_\Omega |u(t)|^2 + \int_{Q_t}|\nabla u|^2 + \int_{Q_t}\xi u + \int_{Q_t}\bigl(\lambda'(u_1) - \lambda'(u_2)\bigr)\, u = 0$$

We note that the term involving ξ is nonnegative since β is monotone. So, we rearrange and estimate the right-hand side we obtain by accounting for (2.48) and the Lipschitz continuity of λ' as follows

$$\begin{aligned}
&\frac{1}{2}\|(\alpha\mu+u+\sigma)(t)\|_*^2 + \alpha\int_{Q_t}|\mu|^2 + \frac{\alpha}{2}\int_\Omega |u(t)|^2 + \int_{Q_t}|\nabla u|^2 \\
&\leq -\int_{Q_t}\mu\sigma - \int_0^t \langle(\alpha\mu+u+\sigma)(s),\sigma(s)\rangle\,ds + c\int_{Q_t}|u|^2 \\
&\leq \delta\int_{Q_t}|\mu|^2 + c_\delta\int_{Q_t}|\sigma|^2 + c\int_{Q_t}|u|^2 \\
&\quad + \delta\int_0^t \|\sigma(s)\|_V^2\,ds + c_\delta\int_0^t\|(\alpha\mu+u+\sigma)(s)\|_*^2\,ds
\end{aligned} \tag{3.4}$$

for every $\delta > 0$. Next, we test (3.3) by σ and get

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\sigma(t)|^2 + \int_{Q_t}|\nabla\sigma|^2 = \int_{Q_t}(R_2-R_1)\sigma \\
&= \int_{Q_t}\bigl(p(u_2)-p(u_1)\bigr)(\sigma_2-\gamma\mu_2)\sigma - \int_{Q_t}p(u_1)(\sigma-\gamma\mu)\sigma \\
&\leq c\int_{Q_t}|\sigma_2-\gamma\mu_2|\,|u|\,|\sigma| + c\int_{Q_t}|\sigma-\gamma\mu|\,|\sigma|
\end{aligned} \tag{3.5}$$

since p is Lipschitz continuous and bounded. Now, we estimate the last two integrals, separately. By the regularity (2.12) of μ_2 and σ_2 and the Sobolev inequality (2.39), we have

$$\begin{aligned}
&\int_{Q_t}|\sigma_2-\gamma\mu_2|\,|u|\,|\sigma| \leq \int_0^t \|(\sigma_2-\gamma\mu_2)(s)\|_4\,\|u(s)\|_2\,\|\sigma(s)\|_4\,ds \\
&\leq c\int_0^t\|u(s)\|_2\,\|\sigma(s)\|_V\,ds \leq \delta\int_{Q_t}\bigl(|\sigma|^2+|\nabla\sigma|^2\bigr) + c_\delta\int_{Q_t}|u|^2
\end{aligned}$$

for every $\delta > 0$. On the other hand

$$\int_{Q_t}|\sigma-\gamma\mu|\,|\sigma| \leq \int_{Q_t}|\sigma|^2 + c\int_{Q_t}|\mu|\,|\sigma| \leq \delta\int_{Q_t}|\mu|^2 + c_\delta\int_{Q_t}|\sigma|^2.$$

At this point, we combine the last two estimates with (3.5) and sum to (3.4). Then, we take δ small enough in order to absorb the corresponding terms in the left-hand side. By applying the Gronwall lemma, we obtain $\mu = 0$, $u = 0$ and $\sigma = 0$. By comparison in (3.2) we also deduce $\xi = 0$, and the proof is complete.

4 A priori estimates

In this section, we introduce an approximating problem and prove a number of a priori estimates on its solution. Some of the bounds we find may depend on α and T, while other ones are independent of such parameters. The notation we use follows the general rule explained at the end of Section 2. The estimates we obtain will be used in the subsequent sections in order to prove our results.

For $\varepsilon \in (0,1)$, the approximation to problem (2.12)–(2.18) is obtained by simply replacing (2.13) by

$$\xi = \beta_\varepsilon(u) \tag{4.1}$$

where β_ε and the related functions $\widehat{\beta}_\varepsilon$ and $\mathcal{W}_\varepsilon$ are defined on the whole of $\mathbb{R}$ as follows

$$\widehat{\beta}_\varepsilon(r) := \min_{s\in\mathbb{R}} \Bigl(\tfrac{1}{2\varepsilon}\,(s-r)^2 + \widehat{\beta}(s) \Bigr), \quad \beta_\varepsilon(r) := \frac{d}{dr}\widehat{\beta}_\varepsilon(r), \quad \mathcal{W}_\varepsilon(r) := \widehat{\beta}_\varepsilon(r) + \lambda(r). \tag{4.2}$$

It turns out that $\widehat{\beta}_\varepsilon$ is a well-defined C^1 function and that β_ε, the Yosida regularization of β, is Lipschitz continuous. Moreover the following properties

$$0 \le \widehat{\beta}_\varepsilon(r) \le \widehat{\beta}(r) \quad \text{and} \quad \widehat{\beta}_\varepsilon(r) \nearrow \widehat{\beta}(r) \text{ monotonically as } \varepsilon \searrow 0 \tag{4.3}$$

hold true for every $r \in \mathbb{R}$ (see, e.g., [2, Prop. 2.11, p. 39]). Our starting point is the result stated below.

Proposition 4.1. *Under the assumptions of Theorem 2.2, the approximating problem has a unique global solution.*

Uniqueness is already proved as a special case of the uniqueness part of Theorem 2.2. As far as existence is concerned, we avoid a detailed proof and just say that a Faedo-Galerkin method (obtained by taking a base of V, e.g., the base of the eigenfunctions of the Laplace operator with homogeneous Neumann boundary conditions) and some a priori estimates very close to the ones we are going to perform in the present section lead to the existence of a solution. We also remark that the system of ordinary differential equations given by the Faedo-Galerkin scheme has a unique global solution since β_ε is Lipschitz continuous and the function $(\mu, u, \sigma) \mapsto p(u)(\sigma - \gamma\mu)$ on $\mathbb{R}^3$ is smooth (since p is so) and sublinear (since p is bounded).

From now on, $(\mu, u, \sigma) = (\mu_\varepsilon, u_\varepsilon, \sigma_\varepsilon)$ stands for the solution to the approximating problem. Accordingly, we define R_ε by (2.14). However, we explicitely write the subscript ε only at the end of each calculation, for brevity.

First a priori estimate. We multiply (2.14) by μ, (2.15) by $-\partial_t u$ and (2.16) by σ/γ. Then, we add all the equalities we obtain to each other, integrate over Q_t, where $t \in (0,T)$ is arbitrary, and we take advantage of the boundary conditions (2.17). We have

$$\begin{aligned}
&\frac{\alpha}{2}\int_\Omega |\mu(t)|^2 - \frac{\alpha}{2}\int_\Omega |\mu_0|^2 + \int_{Q_t} \partial_t u\,\mu + \int_{Q_t} |\nabla\mu|^2 \\
&\quad + \frac{1}{2\gamma}\int_\Omega |\sigma(t)|^2 - \frac{1}{2\gamma}\int_\Omega |\sigma_0|^2 + \frac{1}{\gamma}\int_{Q_t} |\nabla\sigma|^2 - \int_{Q_t} \mu\partial_t u + \alpha\int_{Q_t} |\partial_t u|^2 \\
&\quad + \frac{1}{2}\int_\Omega |\nabla u(t)|^2 - \frac{1}{2}\int_\Omega |\nabla u_0|^2 + \int_\Omega \mathcal{W}_\varepsilon(u(t)) - \int_\Omega \mathcal{W}_\varepsilon(u_0) \\
&= \int_{Q_t} \bigl(R\mu - R(\sigma/\gamma)\bigr) = -\int_{Q_t} p(u)\bigl(\gamma^{1/2}\mu - \gamma^{-1/2}\sigma\bigr)^2
\end{aligned} \tag{4.4}$$

and notice that two integrals cancel out. Moreover, we point out that

$$\int_\Omega \mathcal{W}_\varepsilon(u_0) \le \int_\Omega \widehat{\beta}(u_0) + \int_\Omega \lambda(u_0) \le c$$

thanks to (4.2)–(4.3), (2.11) and (2.6). By rearranging in (4.4) and using (2.11) and (4.3), we thus deduce

$$\frac{\alpha}{2}\int_\Omega |\mu(t)|^2 + \int_{Q_t} |\nabla\mu|^2 + \frac{1}{2\gamma}\int_\Omega |\sigma(t)|^2 + \frac{1}{\gamma}\int_{Q_t} |\nabla\sigma|^2 \\ + \alpha\int_{Q_t} |\partial_t u|^2 + \frac{1}{2}\int_\Omega |\nabla u(t)|^2 + \int_\Omega \mathcal{W}_\varepsilon(u(t)) + \int_{Q_t} p(u)\bigl(\gamma^{1/2}\mu - \gamma^{-1/2}\sigma\bigr)^2 \leq c\,. \tag{4.5}$$

On the other hand, by simply integrating (2.19) over Ω for a.a. $t \in (0,T)$ and using (2.17), we get

$$\int_\Omega \Bigl(\alpha\mu(t) + u(t) + \sigma(t)\Bigr) = \int_\Omega \Bigl(\alpha\mu_0 + u_0 + \sigma_0\Bigr) = c$$

whence immediately (here $|\Omega|$ is the Lebesgue measure of Ω)

$$\Bigl|\int_\Omega u(t)\Bigr|^2 \leq \bigl(c + \alpha\|\mu(t)\|_1 + \|\sigma(t)\|_1\bigr)^2 \leq 3c^2 + 3\alpha^2\|\mu(t)\|_1^2 + 3\|\sigma(t)\|_1^2 \\ \leq c + 3|\Omega|\bigl(\alpha^2\|\mu(t)\|_2^2 + \|\sigma(t)\|_2^2\bigr) \leq c + D\bigl(\alpha\|\mu(t)\|_2^2 + \tfrac{1}{\gamma}\|\sigma(t)\|_2^2\bigr)$$

where $D := 3|\Omega|\max\{1,\gamma\}$. At this point, we multiply the above inequality by $1/(4D)$, sum what we obtain to (4.5) and rearrange. Using also the Poincaré inequality (2.41), we conclude that

$$\alpha^{1/2}\|\mu_\varepsilon\|_{L^\infty(0,T;H)} + \|\nabla\mu_\varepsilon\|_{L^2(0,T;H)} \\ + \alpha^{1/2}\|\partial_t u_\varepsilon\|_{L^2(0,T;H)} + \|u_\varepsilon\|_{L^\infty(0,T;V)} + \|\mathcal{W}_\varepsilon(u_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \\ + \|\sigma_\varepsilon\|_{L^\infty(0,T;H)} + \|\nabla\sigma_\varepsilon\|_{L^2(0,T;H)} \\ + \|(p(u_\varepsilon))^{1/2}(\gamma^{1/2}\mu - \gamma^{-1/2}\sigma)\|_{L^2(0,T;H)} \leq c\,. \tag{4.6}$$

By recalling the definition of R_ε in (2.14) (this is the notation for the approximating problem, indeed), we have

$$R_\varepsilon = p(u_\varepsilon)(\sigma_\varepsilon - \gamma\mu_\varepsilon) = -(p(u_\varepsilon))^{1/2}\gamma^{1/2}\cdot(p(u_\varepsilon))^{1/2}(\gamma^{1/2}\mu_\varepsilon - \gamma^{-1/2}\sigma_\varepsilon).$$

As p is bounded, from (4.6) we infer that

$$\|R_\varepsilon\|_{L^2(0,T;H)} \leq c\,. \tag{4.7}$$

Second a priori estimate. We write (2.14) in the form $\alpha\partial_t\mu - \Delta\mu = R - \partial_t u$ and multiply this equation by $\partial_t\mu$. By integrating over Q_t, we obtain

$$\alpha\int_{Q_t} |\partial_t\mu|^2 + \frac{1}{2}\int_\Omega |\nabla\mu(t)|^2 - \frac{1}{2}\int_\Omega |\nabla\mu_0|^2 \\ = \int_{Q_t} (R - \partial_t u)\partial_t\mu \leq \|R - \partial_t u\|_{L^2(0,T;H)}\|\partial_t\mu\|_{L^2(0,T;H)}\,.$$

Thanks to (4.6) and (4.7), we conclude that

$$\|\partial_t\mu_\varepsilon\|_{L^2(0,T;H)} + \|u_\varepsilon\|_{L^\infty(0,T;V)} \leq c_\alpha\,. \tag{4.8}$$

In the same way, we derive the analogue for σ_ε by using (2.16): however, in this case we obtain the better estimate

$$\int_{Q_t} |\partial_t \sigma|^2 + \frac{1}{2} \int_\Omega |\nabla \sigma(t)|^2 \le \frac{1}{2} \int_\Omega |\nabla \sigma_0|^2 + \|R\|_{L^2(0,T;H)} \|\partial_t \sigma\|_{L^2(0,T;H)} \,.$$

which implies, along with (4.6),

$$\|\partial_t \sigma_\varepsilon\|_{L^2(0,T;H)} + \|\sigma_\varepsilon\|_{L^\infty(0,T;V)} \le c. \tag{4.9}$$

Third a priori estimate. By writing our system in the form

$$\alpha \partial_t \mu - \Delta \mu = R - \partial_t u, \quad -\Delta u + \beta_\varepsilon(u) = \mu - \alpha \partial_t u - \lambda'(u), \quad \partial_t \sigma - \Delta \sigma = -R$$

and accounting for the bounds already found, it is straightforward to derive the following estimates (multiply by $-\Delta v$ with $v = \mu, u, \sigma$, respectively, and use $\beta'_\varepsilon \ge 0$)

$$\|\Delta \mu\|_{L^2(0,T;H)} \le c_\alpha \quad \|\Delta u\|_{L^2(0,T;H)} \le c_{\alpha,T} \quad \|\Delta \sigma\|_{L^2(0,T;H)} \le c\,. \tag{4.10}$$

By the elliptic regularity theory, (2.15) and the boundary conditions (2.17), we conclude that

$$\|\mu_\varepsilon\|_{L^2(0,T;W)} \le c_{\alpha,T} \quad \text{and} \quad \|\sigma_\varepsilon\|_{L^2(0,T;W)} \le c_T \tag{4.11}$$

$$\|u_\varepsilon\|_{L^2(0,T;W)} + \|\beta_\varepsilon(u_\varepsilon)\|_{L^2(0,T;H)} \le c_{\alpha,T}\,. \tag{4.12}$$

5 Existence on a finite time interval

In this section, we conclude the proof of Theorem 2.2 by showing the existence of a solution. As α and T are fixed now, we can account for all of the estimates of the previous section. Owing to standard weak compactness arguments as well as by the strong compactness result in [18, Sect. 8, Cor. 4], it turns out that the following convergence holds

$$\begin{aligned} &\mu_\varepsilon \to \mu, \quad u_\varepsilon \to u, \quad \sigma_\varepsilon \to \sigma \\ &\qquad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \\ &\qquad \text{strongly in } C^0([0,T];H) \cap L^2(0,T;V) \text{ and a.e. in } Q \end{aligned} \tag{5.1}$$

$$R_\varepsilon \to \rho \quad \text{and} \quad \beta_\varepsilon(u_\varepsilon) \to \xi \quad \text{weakly in } L^2(0,T;H) \tag{5.2}$$

at least for a subsequence. The strong convergence in $C^0([0,T];H)$ entails that the limiting functions satisfy the initial conditions (2.18). Moreover, the pointwise convergence a.e. and assumption (2.4) imply that R_ε converges to $R := p(u)(\sigma - \gamma\mu)$ a.e. in Q, so that $\rho = R$. Furthermore, as λ' is Lipschitz continuous, $\lambda'(u_\varepsilon)$ converges to $\lambda'(u)$ strongly in $L^2(0,T;H)$. Finally, a standard monotonicity argument (see, e.g., [1, Lemma 1.3, p. 42]) based on the weak convergence $u_\varepsilon \to u$, $\beta_\varepsilon(u_\varepsilon) \to \xi$ in $L^2(Q)$ and on the property

$$\limsup_{\varepsilon \searrow 0} \int_Q \beta_\varepsilon(u_\varepsilon) u_\varepsilon = \lim_{\varepsilon \searrow 0} \int_Q \beta_\varepsilon(u_\varepsilon) u_\varepsilon = \int_Q \xi u$$

(easily following from (5.1)–(5.2)) yields $\xi \in \beta(u)$ a.e. in Q. Therefore, the quadruplet (μ, u, σ, ξ) satisfies (2.12)–(2.13) and solves (2.14)–(2.18).

We conclude this section by recovering the uniform estimates for the solution (μ, u, σ, ξ) to the problem (2.12)–(2.18). First of all, we can speak of a unique solution on the time half line $[0, +\infty)$. For such a solution, the estimates we found for the approximating problem still hold, i.e., we have

$$\begin{aligned}
&\alpha^{1/2}\|\mu\|_{L^\infty(0,\infty;H)} + \|\nabla\mu\|_{L^2(0,\infty;H)} \\
&\quad + \alpha^{1/2}\|\partial_t u\|_{L^2(0,\infty;H)} + \|u\|_{L^\infty(0,\infty;V)} + \|\mathcal{W}(u)\|_{L^\infty(0,\infty;L^1(\Omega))} \\
&\quad + \|\partial_t \sigma\|_{L^2(0,\infty;H)} + \|\sigma\|_{L^\infty(0,\infty;V)} + \|\nabla\sigma\|_{L^2(0,\infty;H)} + \|\Delta\sigma\|_{L^2(0,\infty;H)} \\
&\quad + \|R\|_{L^2(0,\infty;H)} \leq c \qquad (5.3) \\
&\|\partial_t\mu\|_{L^2(0,\infty;H)} + \|\mu\|_{L^\infty(0,\infty;V)} + \|u\|_{L^\infty(0,\infty;V)} + \|\sigma\|_{L^\infty(0,\infty;V)} \leq c_\alpha \qquad (5.4) \\
&\|\mu\|_{L^2(0,T;W)} + \|u\|_{L^2(0,T;W)} + \|\sigma\|_{L^2(0,T;W)} + \|\xi\|_{L^2(0,T;H)} \leq c_{\alpha,T} \qquad (5.5)
\end{aligned}$$

the last one for every $T \in (0, +\infty)$. This simply follows from the weak semicontinuity of the norms for all the inequalities but the one involving $\mathcal{W}(u)$. As far as the latter is concerned, we note that, for all $t \geq 0$, $u_\varepsilon(t) \to u(t)$ strongly in H. Then, using (2.6) and the mean value theorem, it is not difficult to check that

$$|\lambda(u_\varepsilon(t)) - \lambda(u(t))| \leq c\,|u_\varepsilon(t) - u(t)|\,(1 + |u_\varepsilon(t)| + |u(t)|)$$

and consequently $\lambda(u_\varepsilon(t))$ converges to $\lambda(u(t))$ strongly in $L^1(\Omega)$. Hence, in view of (2.7) and (4.2) it suffices to prove that

$$\int_\Omega \widehat{\beta}(u(t)) \leq \liminf_{\varepsilon\searrow 0} \int_\Omega \widehat{\beta}_\varepsilon(u_\varepsilon(t)). \qquad (5.6)$$

To this end, we fix $\varepsilon' > 0$ for a while. By accounting for the lower semicontinuity of $\widehat{\beta}_{\varepsilon'}$ and the inequality $\widehat{\beta}_{\varepsilon'}(s) \leq \widehat{\beta}_\varepsilon(s)$ which holds for every $s \in \mathbb{R}$ and $\varepsilon \in (0, \varepsilon')$ (see (4.2)), we obtain

$$\int_\Omega \widehat{\beta}_{\varepsilon'}(u(t)) \leq \liminf_{\varepsilon\searrow 0} \int_\Omega \widehat{\beta}_{\varepsilon'}(u_\varepsilon(t)) \leq \liminf_{\varepsilon\searrow 0} \int_\Omega \widehat{\beta}_\varepsilon(u_\varepsilon(t)). \qquad (5.7)$$

Now, we let ε' vary and recall (4.3) in terms of ε'. Thus, the Beppo Levi monotone convergence theorem implies that

$$\int_\Omega \widehat{\beta}(u(t)) = \lim_{\varepsilon'\searrow 0} \int_\Omega \widehat{\beta}_{\varepsilon'}(u(t)) \qquad (5.8)$$

and combining (5.8) and (5.7) yields (5.6). Therefore, (5.3) follows.

6 Long time behavior

In this section, we prove Theorem 2.4. From (5.4) we see that the omega limit ω we are interested in is non-empty. It remains to characterize its elements as in the statement. So, we fix $(\mu_\omega, u_\omega, \sigma_\omega) \in \omega$ and a sequence $\{t_n\}$ according to definition (2.22). We set for convenience

$$v_n(t) := v(t + t_n) \quad \text{for } t \geq 0 \quad \text{with} \quad v = \mu,\ u,\ \sigma,\ \xi,\ R \qquad (6.1)$$

and study the behavior of such functions in a fixed finite time interval $(0,T)$. First of all, we notice that the quadruplet $(\mu_n, u_n, \sigma_n, \xi_n)$ solves the problem obtained from problem (2.14)–(2.18) by replacing the initial condition (2.18) by the following one

$$\mu_n(0) = \mu(t_n), \quad u_n(0) = u(t_n) \quad \text{and} \quad \sigma_n(0) = \sigma(t_n). \tag{6.2}$$

On the other hand, by (5.4), the new initial data are bounded in V and (5.3) provides a uniform L^1-estimate for $\mathcal{W}(u_n(0)) = \mathcal{W}(u(t_n))$. Therefore, the dependence of the constants on the norms of the initial data just mentioned leads to a dependence only on the norms involved in our assumptions (2.11). Moreover, we observe that, for every Banach space Z, if some function v belongs to $L^2(0,\infty;Z)$ and v_n is related to v as in (6.1), we trivially have

$$\lim_{n\to\infty} \int_0^T \|v_n(t)\|_Z^2 \, dt \le \lim_{n\to\infty} \int_{t_n}^\infty \|v(t)\|_Z^2 \, dt = 0$$

so that $v_n \to 0$ strongly in $L^2(0,T;Z)$. At this point, we can derive the estimates and the convergence we need from (5.3)–(5.5). We infer that

$$\begin{aligned}
&\|\mu_n\|_{L^\infty(0,\infty;V)} + \|u_n\|_{L^\infty(0,\infty;V)} + \|\sigma_n\|_{L^\infty(0,\infty;V)} \le c_\alpha \\
&\|\mu_n\|_{L^2(0,T;W)} + \|u_n\|_{L^2(0,T;W)} + \|\sigma_n\|_{L^2(0,T;W)} + \|\xi_n\|_{L^2(0,T;H)} \le c_{\alpha,T} \\
&\nabla\mu_n \to 0 \quad \text{and} \quad \nabla\sigma_n \to 0 \quad \text{strongly in } (L^2(0,T;H))^3 \\
&\partial_t\mu_n \to 0, \quad \partial_t u_n \to 0, \quad \partial_t\sigma_n \to 0 \quad \text{and} \quad R_n \to 0 \quad \text{strongly in } L^2(0,T;H).
\end{aligned}$$

Therefore, at least for a subsequence, we also have

$$\begin{aligned}
&\mu_n \to \mu_\infty, \quad u_n \to u_\infty \quad \text{and} \quad \sigma_n \to \sigma_\infty \\
&\qquad \text{weakly star in } L^\infty(0,T;V) \cap L^2(0,T;W) && (6.3) \\
&\mu_n \to \mu_\infty \quad u_n \to u_\infty \quad \text{and} \quad \sigma_n \to \sigma_\infty \quad \text{strongly in } L^2(0,T;H) && (6.4) \\
&\xi_n \to \xi_\infty \quad \text{weakly in } L^2(0,T;H) && (6.5)
\end{aligned}$$

the strong convergence (6.4) being a consequence of (6.3) and of the bounds for the time derivatives. In particular, thanks to the Lipschitz continuity of p, we derive that R_n converges to $p(u_\infty)(\sigma_\infty - \gamma\mu_\infty)$ strongly in $L^1(Q)$, whence

$$p(u_\infty)(\sigma_\infty - \gamma\mu_\infty) = 0 \quad \text{a.e. in } Q. \tag{6.6}$$

Furthermore, μ_∞ and σ_∞ are constant functions and u_∞ is time independent. We denote the constant values of μ_∞ and σ_∞ by μ_s and σ_s, respectively, and set $u_s := u_\infty(t)$ for $t \in (0,T)$. By taking the limit in (2.15) written for μ_n and u_n, we see that the pair (u_∞, ξ_∞) solves the following problem

$$\mu_s = -\Delta u_\infty + \xi_\infty + \lambda'(u_\infty) \quad \text{a.e. in } Q \quad \text{and} \quad \partial_\nu u_\infty = 0 \quad \text{a.e. on } \Sigma.$$

In particular, ξ_∞ also is time independent, $\xi_\infty(t) = \xi_s$ for $t \in (0,T)$, and the above boundary value problem and (6.6) become

$$\begin{aligned}
&\mu_s = -\Delta u_s + \xi_s + \lambda'(u_s) \quad \text{and} \quad p(u_s)(\sigma_s - \gamma\mu_s) = 0 \quad \text{a.e. in } \Omega, \\
&\qquad\qquad \partial_\nu u_s = 0 \quad \text{a.e. on } \Gamma. && (6.7)
\end{aligned}$$

Now, as in Section 5, we derive both $\xi_\infty \in \beta(u_\infty)$ a.e. in Q, i.e., $\xi_s \in \beta(u_s)$ a.e. in Ω, and the convergence

$$\mu_n \to \mu_\infty\,, \quad u_n \to u_\infty, \quad \sigma_n \to \sigma_\infty \quad \text{strongly in } C^0([0,T];H) \cap L^2(0,T;V).$$

It follows that $\mu(t_n) = \mu_n(0)$ converges to $\mu_\infty(0) = \mu_s$ in H. As $\mu(t_n)$ converges to μ_ω in V, we infer that $\mu_\omega = \mu_s$. In the same way we obtain $u_\omega = u_s$ and $\sigma_\omega = \sigma_s$. Therefore, we also have from (6.7)

$$\mu_\omega \in -\Delta u_\omega + \beta(u_\omega) + \lambda'(u_\omega) \quad \text{and} \quad p(u_\omega)(\sigma_s - \gamma\mu_s) = 0 \quad \text{a.e. in } \Omega$$

and the proof is complete.

Remark 6.1. Even though we have to confine ourselves to study the omega limit of an initial datum satisfying (2.11), we could take a phase space Φ that is larger than (2.21) and is endowed with a weaker topology. This may lead to further properties of ω. For instance, if we choose $\Phi = (L^2(\Omega))^3$ with the strong topology, estimate (5.3) implies that the whole trajectory of the initial datum is relatively compact in Φ, so that general results (see, e.g., [19, Lemma 6.3.2, p. 239]) ensure that ω is invariant, compact and connected in the L^2 topology.

7 Asymptotics and limit problem

In this section, we perform the proof of Theorem 2.6. As T is fixed, we avoid stressing the dependence of the constants on T.

i) As in the statement, $(\mu_\alpha, u_\alpha, \sigma_\alpha, \xi_\alpha)$ (where $\xi_\alpha = \beta(u_\alpha)$ since $\widehat{\beta}$ is smooth) is the solution to problem (2.14)–(2.18) and we define R_α accordingly. We recall that (5.3) implies $\|u_\alpha\|_{L^\infty(0,T;V)} \le c$, whence also $\|u_\alpha\|_{L^\infty(0,T;L^6(\Omega))} \le c$ due to the Sobolev inequality (2.39). By the assumption (2.25), we infer that

$$\|\beta(u_\alpha) + \lambda'(u_\alpha)\|_{L^\infty(0,T;H)} = \|\mathcal{W}'(u_\alpha)\|_{L^\infty(0,T;H)} \le c\bigl(\|u_\alpha\|^3_{L^\infty(0,T;L^6(\Omega))} + 1\bigr) \le c\,. \tag{7.1}$$

Now, we integrate (2.15) over Ω and use the homogeneous Neumann boundary condition for u_α. Then, we square and integrate over $(0,T)$ with respect to time. We obtain

$$\int_0^T \Bigl|\int_\Omega \mu_\alpha(t)\Bigr|^2 dt = \int_0^T \Bigl|\int_\Omega \bigl(\alpha\,\partial_t u_\alpha(t) + \mathcal{W}'(u_\alpha(t))\bigr)\Bigr|^2 dt$$
$$\le 2\,|\Omega|\,\alpha^2 \int_Q |\partial_t u_\alpha|^2 + 2\,|\Omega| \int_Q |\mathcal{W}'(u_\alpha)|^2 \le c$$

the last inequality following from (5.3) and (7.1). Then, recalling the estimate for $\nabla\mu_\alpha$ in (5.3) and owing to the Poincaré inequality (2.41), we conclude that

$$\|\mu_\alpha\|_{L^2(0,T;V)} \le c\,. \tag{7.2}$$

By comparison in (2.15), we deduce $\|\Delta u_\alpha\|_{L^2(0,T;H)} \le c$, whence also

$$\|u_\alpha\|_{L^2(0,T;W)} \le c \tag{7.3}$$

by elliptic regularity. Now, we test (2.14) by an arbitrary $v \in L^2(0,T;V)$ and get

$$\Bigl| \int_Q \partial_t(\alpha\mu_\alpha + u_\alpha)\, v \Bigr| = \Bigl| \int_Q \bigl(R_\alpha v - \nabla\mu_\alpha \cdot \nabla v \bigr) \Bigr| \leq c \|v\|_{L^2(0,T;V)}$$

by the estimate (5.3) of R_α and (7.2). This means that

$$\|\partial_t(\alpha\mu_\alpha + u_\alpha)\|_{L^2(0,T;V^*)} \leq c\,. \tag{7.4}$$

Next, still from (5.3) it follows that

$$\|\sigma_\alpha\|_{H^1(0,T;H)\cap L^2(0,T;W)} \leq c\,. \tag{7.5}$$

At this point, we can use weak and weak star compactness and conclude that

$$\mu_\alpha \to \mu \quad \text{weakly in } L^2(0,T;V) \tag{7.6}$$
$$u_\alpha \to u \quad \text{weakly star in } L^\infty(0,T;V) \cap L^2(0,T;W) \tag{7.7}$$
$$\sigma_\alpha \to \sigma \quad \text{weakly in } H^1(0,T;H) \cap L^2(0,T;W) \tag{7.8}$$
$$\partial_t(\alpha\mu_\alpha + u_\alpha) \to \zeta \quad \text{weakly in } L^2(0,T;V^*) \tag{7.9}$$

at least for a subsequence. This proves the part *i*) of the statement but (2.29), which is more precise than (7.9) and is justified in the next step.

ii) Take any triplet (μ, u, σ) satisfying the above convergence (note that (7.9) is weaker than (2.29)): we prove that it solves the limit problem (2.30)–(2.35) and that $\zeta = \partial_t u$ (so that (2.29) holds). First of all, we notice that (7.8) implies $\sigma(0) = \sigma_0$. Next, as (7.6)–(7.7) imply that $\alpha\mu_\alpha + u_\alpha \to u$ weakly in $L^2(0,T;V)$ and (7.9) holds, we can apply the Lions-Aubin theorem (see, e.g., [14, Thm. 5.1, p. 58]) and deduce that

$$\alpha\mu_\alpha + u_\alpha \to u \quad \text{strongly in } L^2(0,T;H).$$

On the other hand $\alpha\mu_\alpha \to 0$ in $L^2(0,T;V)$ by (7.6). Hence

$$u_\alpha \to u \quad \text{strongly in } L^2(0,T;H) \quad \text{and} \quad \zeta = \partial_t u. \tag{7.10}$$

By a standard argument (the same as in Section 5), we can identify the limit of R_α as $p(u)(\sigma - \gamma\mu)$ and the limits of the other nonlinear terms. Thus, we conclude that (μ, u, σ) solves (2.30)–(2.34) (in fact, one proves an equivalent integrated version of (2.30) rather than (2.30) itself). It remains to check the first condition in (2.35). By also accounting for (7.9), we see that $\alpha\mu_\alpha + u_\alpha$ converges to u weakly in $C^0([0,T];V^*)$. This implies that

$$\alpha\mu_0 + u_0 = (\alpha\mu_\alpha + u_\alpha)(0) \to u(0) \quad \text{weakly in } V^*.$$

On the other hand, $\alpha\mu_0 + u_0 \to u_0$ strongly in V. Therefore, $u(0) = u_0$, and the proof of *ii*) is complete.

iii) A formal estimate that leads to (2.37)–(2.38) could be obtained by testing (2.30) by $\partial_t u$, differentiating (2.32) with respect to time, testing the obtained equality by μ and adding up. Here we perform the correct procedure, namely, the discrete version of the formal one, by introducing a time step $h \in (0,1)$. For simplicity, we allow the (variable) value of the constant c to depend on the norm $\|u_0\|_W$ involved in (2.36) and on the solution we are considering (which is fixed). Of course, c does not depend on h. First of all, we introduce a notation. For $v \in L^2(-1,T;H)$ and $h \in (0,1)$, we define the mean $\bar v_h \in L^2(0,T;H)$ and the difference quotient $\delta_h v \in L^2(0,T;H)$ by setting for $t \in (0,T)$

$$\bar v_h(t) := \frac{1}{h}\int_{t-h}^t v(s)\,ds = \int_0^1 v(t-h\tau)\,d\tau \quad \text{and} \quad \delta_h v(t) := \partial_t \bar v_h(t) = \frac{v(t)-v(t-h)}{h} \tag{7.11}$$

and we do the same if $v \in (L^2(-1,T;H))^3$ in order to treat gradients. We notice that

$$\|\bar v_h\|_{L^2(0,T;H)} \le \|v\|_{L^2(-1,T;H)} \tag{7.12}$$

as we show at once. We have indeed

$$\begin{aligned}
\|\bar v_h\|^2_{L^2(0,T;H)} &\le \int_\Omega\int_0^T\int_0^1 |v(x,t-h\tau)|^2\,d\tau\,dt\,dx = \int_0^1\int_\Omega\int_0^T |v(x,t-h\tau)|^2\,dt\,dx\,d\tau \\
&= \int_0^1\int_\Omega\int_{-h\tau}^{T-h\tau} |v(x,s)|^2\,ds\,dx\,d\tau \le \int_0^1\int_\Omega\int_{-1}^{T} |v(x,s)|^2\,ds\,dx\,d\tau = \|v\|^2_{L^2(-1,T;H)}\,.
\end{aligned}$$

As we are going to apply (7.12) to μ and R, we need to extend such functions to the whole of $\Omega\times(-1,T)$. Our tricky construction is based on assumption (2.36) on u_0 and involves u as well. We first solve a backward variational problem with the help of the theory of linear abstract equations. We set

$$H_0 := \Big\{ v \in H : \ \textstyle\int_\Omega v = 0 \Big\} \quad \text{and} \quad W_0 := W \cap H_0$$

and construct the Hilbert triplet (W_0, H_0, W_0^*), where W_0^* is the dual space of W_0, by embedding H_0 into W_0^* in the standard way. In the sequel, the symbol $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the duality pairing between W_0^* and W_0. We introduce the continuous bilinear form $a : W_0 \times W_0 \to \mathbb{R}$ by setting

$$a(z,v) := \int_\Omega (\Delta z)\,(\Delta v) \quad \text{for } z, v \in W_0$$

and observe that $a(v,v) + \|v\|_H^2 \ge \alpha\|v\|_W^2$ for some $\alpha > 0$ and every $v \in W_0$, thanks to the elliptic regularity theory. We also notice that $\Delta u_0 \in H_0$ since $u_0 \in W$. Therefore, as is well known (e.g., [17, Prop. 2.3 p. 112]), there exists a unique z satisfying

$$z \in H^1(-1,0;W_0^*) \cap C^0([-1,0];H_0) \cap L^2(-1,0;W_0) \tag{7.13}$$

$$-\langle \partial_t z(t), v\rangle + a(z(t),v) = 0 \quad \text{for every } v \in W_0 \text{ and for a.a. } t \in (-1,0) \tag{7.14}$$

$$z(0) = -\Delta u_0 \tag{7.15}$$

and we also have

$$\|z\|_{H^1(-1,0;W_0^*)\cap L^\infty(-1,0;H)\cap L^2(-1,0;W)} \le c\|z(0)\|_H \le c\|u_0\|_W = c\,. \tag{7.16}$$

As $z \in C^0([-1,0];H_0)$, for every $t \in [-1,0]$ we have that $z(t) \in \operatorname{dom}\mathcal{N}$ (see (2.44)). Hence, we can define a function w by setting

$$w(t) := \mathcal{N}(z(t)) \quad \text{for every } t \in [-1,0] \tag{7.17}$$

and it turns out that $w \in C^0([-1,0];W)$: the restriction of $\mathcal{N}$ to H_0 is an isomorphism from H_0 onto W_0, indeed. Moreover, w is even smoother. Namely, from (7.16) we have that

$$\|w\|_{L^\infty(-1,0;W)} \le c \quad \text{and} \quad \|\partial_t w\|_{L^2(-1,0;H)} \le c\,. \tag{7.18}$$

Here, the former is due to the above argument and we now prove the latter. Clearly, an estimate on the difference quotients is sufficient to conclude. We observe that the operator $-\Delta : W_0 \to H_0$ is a well-defined isomorphism. Thus, the same property is enjoyed by its adjoint operator $(-\Delta)^* : H_0 \to W_0^*$ given by

$$\langle (-\Delta)^* y, v\rangle = \int_\Omega y(-\Delta v) \quad \text{for every } y \in H_0 \text{ and } v \in W_0.$$

Hence, for every $w^* \in W_0^*$ there exists a unique $y \in H_0$ such that $(-\Delta)^* y = w^*$, i.e.,

$$\int_\Omega y(-\Delta v) = \langle w^*, v\rangle \quad \text{for every } v \in W_0 \tag{7.19}$$

and the estimate $\|y\|_H \le C\|w^*\|_{W_0^*}$ holds true with C depending only on Ω. Assume now $h \in (0,1)$ and $t \in (-1{+}h,0)$. From the definition (7.17) of w we immediately derive that $y = \delta_h w(t)$ belongs to H_0 and satisfies

$$\int_\Omega y(-\Delta v) = \int_\Omega (-\Delta \delta_h w(t))v = \int_\Omega (\delta_h z(t))v \quad \text{for every } v \in W_0$$

i.e., it fulfils (7.19) with $w^* = \delta_h z(t)$. Therefore, we have $\|\delta_h w(t)\|_H \le c\|\delta_h z(t)\|_{W_0^*}$ for every $t \in (-1{+}h,0)$, and (7.16) immediately implies

$$\int_{-1+h}^0 \|\delta_h w(t)\|_H^2\,dt \le c\int_{-1+h}^0 \|\delta_h z(t)\|_{W_0^*}^2\,dt \le c \quad \text{for every } h \in (0,1).$$

This proves the second estimate in (7.18). Once (7.18) is completely established, we go on. We term $(u_0)_\Omega$ the mean value of u_0 and notice that $u_0 - (u_0)_\Omega$ coincides with $w(0)$ given by (7.17) and (7.15). Therefore, we are suggested to define u in $(-1,0)$ by setting

$$u(t) := w(t) + (u_0)_\Omega \quad \text{for } t \in (-1,0). \tag{7.20}$$

By doing that, we have both the estimates

$$\|u\|_{L^\infty(-1,0;W)} \le c \quad \text{and} \quad \|\partial_t u\|_{L^2(-1,0;H)} \le c \tag{7.21}$$

and the fact that $u(t) \to u_0$ (e.g., in H) as $t \nearrow 0$. This implies that the extended function $u \in L^2(-1,T;H)$ (which is continuous in $[-1,0]$ and $[0,T]$, separately) does not jump at $t=0$. Thus, $u \in C^0([-1,T];H)$ and its time derivative $\partial_t u$ is a V^*-valued function (rather than a distribution) on the whole of $(-1,T)$, so that (2.30) will hold in the whole of $(-1,T)$ whenever we properly extend μ and R. As the former is concerned, we set

$$\mu(t) := z(t) + \mathcal{W}'(u(t)) = -\Delta u(t) + \mathcal{W}'(u(t)) \quad \text{for } t \in (-1,0). \tag{7.22}$$

Now, in order to properly extend R, we check that $\Delta\mu$ is a well defined function. Indeed, $\Delta z \in L^2(-1,0;H)$ by (7.16). On the other hand

$$\Delta\mathcal{W}'(u) = \mathcal{W}'''(u)|\nabla u|^2 + \mathcal{W}''(u)\Delta u \in L^\infty(-1,0;H)$$

on account of (7.21), assumption (2.25) on $\mathcal{W}$ and the Sobolev embedding $W \subset W^{1,4}(\Omega)$. Hence, we can set

$$R(t) := \partial_t w(t) - \Delta\mu(t) = \partial_t u(t) - \Delta\mu(t) \quad \text{for } t \in (-1,0). \tag{7.23}$$

We obtain

$$\|\mu\|_{L^2(-1,0;W)} \le c \quad \text{and} \quad \|R\|_{L^2(-1,0;H)} \le c\,. \tag{7.24}$$

Notice that it is confirmed that (2.30) holds for every $v \in V$ and a.e. in $(-1,T)$. Moreover, (2.32) is satisfied a.e. in $\Omega \times (-1,T)$ (while (2.31) still holds only in Q). By collecting (7.21), (7.24) and the regularity of the original functions, we deduce, in particular, the following estimates

$$\|u\|_{L^\infty(-1,T;V)} \le c\,, \quad \|\mu\|_{L^2(-1,T;V)} \le c \quad \text{and} \quad \|R\|_{L^2(-1,T;H)} \le c\,. \tag{7.25}$$

At this point, we come back to (7.25) and apply it to the extended μ and R. By also observing that $\nabla\bar\mu_h = (\overline{\nabla\mu})_h$, we obtain the estimates

$$\|\bar\mu_h\|_{L^2(0,T;V)} \le \|\mu\|_{L^2(-1,T;V)} \le c \quad \text{and} \quad \|\bar R_h\|_{L^2(0,T;H)} \le \|R\|_{L^2(-1,T;H)} \le c\,. \tag{7.26}$$

Now, we are ready to perform our procedure that leads to the desired regularity of the solution. Clearly, in order to show that $\partial_t u \in L^2(0,T;H)$ and that $\mu \in L^\infty(0,T;H)$, it suffices to prove estimates for the proper norms of the different quotient $\delta_h u$ and of the mean $\bar\mu_h$, respectively. To this end, we remind the reader that (2.30) and (2.32) have been extended up to $t=-1$. So, we can integrate (2.30) with respect to time over $(s-h,s)$, with $s \in (0,T)$, and divide by h. We obtain

$$\int_\Omega \delta_h u(s)\,v + \int_\Omega \nabla\bar\mu_h(s)\cdot\nabla v = \int_\Omega \bar R_h(s) v \quad \text{for almost every } s \in (0,T) \text{ and every } v \in V$$

and choose $v = \delta_h u(s)$. Then, we integrate over $(0,t) \subset (0,T)$ with respect to s and have

$$\int_{Q_t} |\delta_h u|^2 + \int_{Q_t} \nabla\bar\mu_h \cdot \nabla\delta_h u = \int_{Q_t} \bar R_h\,\delta_h u. \tag{7.27}$$

At the same time, we derive from (2.32)

$$\delta_h\mu = -\Delta\delta_h u + \delta_h\mathcal{W}'(u) \quad \text{a.e. in } Q.$$

We multiply this equation by $\bar\mu_h$ and integrate over Q_t. We have

$$\int_{Q_t} \partial_t\bar\mu_h\,\bar\mu_h = \int_{Q_t} \nabla\delta_h u \cdot \nabla\bar\mu_h + \int_{Q_t} \delta_h\mathcal{W}'(u)\,\bar\mu_h\,. \tag{7.28}$$

Finally, we sum (7.27) and (7.28). Two integrals cancel out and we obtain

$$\int_{Q_t} |\delta_h u|^2 + \frac12\int_\Omega |\bar\mu_h(t)|^2 = \frac12\int_\Omega |\bar\mu_h(0)|^2 + \int_{Q_t} \bar R_h\,\delta_h u + \int_{Q_t} \delta_h\mathcal{W}'(u)\,\bar\mu_h\,. \tag{7.29}$$

Now, we estimate the integrals on the right-hand side, separately. The first one is bounded by the first inequality in (7.24). Moreover, the second condition in (7.26) implies

$$\int_{Q_t} \bar R_h\,\delta_h u \le \frac14\int_{Q_t} |\delta_h u|^2 + \int_Q |\bar R_h|^2 \le \frac14\int_{Q_t} |\delta_h u|^2 + c\,.$$

For the last term of (7.29) some more work has to be done. We notice that (2.24) and the mean value theorem yield $\delta_h \mathcal{W}'(u) = \mathcal{W}''(\tilde{u})\delta_h u$, where $\tilde{u}$ is some measurable function taking values in between u and of $u(\,\cdot - h)$. In particular, we have $|\tilde{u}| \le |u| + |u(\,\cdot - h)|$ a.e. in Q, so that our assumption (2.25), the Sobolev inequality (2.39) and (7.25) imply for $s \in (0,T)$

$$\begin{aligned}
&\|\mathcal{W}''(\tilde{u}(s))\|_3^3 \le c \int_\Omega \bigl(1 + |\tilde{u}(s)|^6\bigr) \le c \int_\Omega \bigl(1 + |u(s)|^6 + |u(s-h)|^6\bigr) \\
&\le c\bigl(1 + \|u(s)\|_V^6 + \|u(s-h)\|_V^6\bigr) \le c\bigl(1 + \|u\|_{L^\infty(-1,T;V)}^6\bigr) = c\,.
\end{aligned}$$

Hence, owing to (7.26) as well, we have

$$\begin{aligned}
&\int_{Q_t} \delta_h \mathcal{W}'(u)\, \bar{\mu}_h \le \int_{Q_t} |\mathcal{W}''(\tilde{u})|\, |\delta_h u|\, |\bar{\mu}_h| \\
&\le \int_0^t \|\mathcal{W}''(\tilde{u})\|_3\, \|\delta_h u(s)\|_2\, \|\bar{\mu}_h(s)\|_6\, ds \le c \int_0^t \|\delta_h u(s)\|_2\, \|\bar{\mu}_h(s)\|_V\, ds \\
&\le \frac{1}{4} \int_0^t \|\delta_h u(s)\|_2^2\, ds + c \int_0^T \|\bar{\mu}_h(s)\|_V^2\, ds \le \frac{1}{4} \int_{Q_t} |\delta_h u|^2 + c\,.
\end{aligned}$$

By combining all this and (7.29), we conclude that the estimate

$$\int_{Q_t} |\delta_h u|^2 + \int_\Omega |\bar{\mu}_h(t)|^2 \le c$$

holds true for h small. This proves that $\partial_t u \in L^2(0,T;H)$ and $\mu \in L^\infty(0,T;H)$. Now, by comparison in (2.30) and on account of (2.31), we infer that $\Delta\mu \in L^2(0,T;H)$. By elliptic regularity we deduce that $\mu \in L^2(0,T;W)$. Finally, by writing (2.32) in the form

$$-\Delta u + \beta(u) = \mu - \lambda'(u) \in L^\infty(0,T;H)$$

and using a standard argument, we see that $\Delta u \in L^\infty(0,T;H)$. Therefore also the regularity $u \in L^\infty(0,T;W)$ holds true and the proof of (2.37)–(2.38) is complete. To conclude *iii*), we show uniqueness, i.e., we pick two solutions (μ_i, u_i, σ_i), $i = 1,2$, to the limit problem and prove that they are the same. We set for convenience

$$\mu := \mu_1 - \mu_2\,, \quad u := u_1 - u_2 \quad \text{and} \quad \sigma := \sigma_1 - \sigma_2.$$

Let us write equations (2.30)–(2.33) for both solutions and exploit the regularity result just proved in the course of the proof. Taking the difference we obtain a.e. in Q

$$\partial_t u - \Delta\mu = R \tag{7.30}$$

$$R = \bigl(p(u_1) - p(u_2)\bigr)(\sigma_1 - \gamma\mu_1) + p(u_2)(\sigma - \gamma\mu) \tag{7.31}$$

$$\mu = -\Delta u + \mathcal{W}'(u_1) - \mathcal{W}'(u_2) \tag{7.32}$$

$$\partial_t \sigma - \Delta\sigma = -R \tag{7.33}$$

as well as the homogeneous initial and boundary conditions. Next, we multiply equations (7.30), (7.32) and (7.33) by u, μ and σ, respectively, integrate over Q_t with $t \in (0,T)$ and sum them up. As two integrals cancel out (thanks to boundary conditions), we have

$$\begin{aligned}
&\frac{1}{2} \int_\Omega |u(t)|^2 + \int_{Q_t} |\mu|^2 + \frac{1}{2} \int_\Omega |\sigma(t)|^2 + \int_{Q_t} |\nabla\sigma|^2 \\
&= \int_{Q_t} R\,(u - \sigma) + \int_{Q_t} \bigl(\mathcal{W}'(u_1) - \mathcal{W}'(u_2)\bigr)\mu
\end{aligned} \tag{7.34}$$

and we can estimate the term on the right-hand side, separately, as follows. In the sequel, the (variable) values of c are allowed to depend also on the solutions we are considering, since they are fixed. Accounting for (7.31) and for the boundedness and the Lipschitz continuity of p, we deduce that

$$\int_{Q_t} R\,(u-\sigma) \le c \int_{Q_t} |u|\,|\sigma_1 - \gamma\mu_1|\,(|u| + |\sigma|) + c \int_{Q_t} |\sigma - \gamma\mu|\,(|u| + |\sigma|)$$
$$\le \frac{1}{4} \int_{Q_t} |\mu|^2 + c \int_0^t \bigl(1 + \|\sigma_1(s) - \gamma\mu_1(s)\|_\infty\bigr)\bigl(\|u(s)\|_2^2 + \|\sigma(s)\|_2^2\bigr)\,ds\,.$$

In order to estimate the last term of (7.34), we recall that u_1 and u_2 are bounded and that $\mathcal{W}'$ is Lipschitz continuous on every bounded interval. Hence, we have

$$\int_{Q_t} \bigl(\mathcal{W}'(u_1) - \mathcal{W}'(u_2)\bigr)\mu \le c \int_{Q_t} |u|\,|\mu| \le \frac{1}{4} \int_{Q_t} |\mu|^2 + c \int_{Q_t} |u|^2\,.$$

At this point, we collect (7.34) and the above inequalities and apply the Gronwall lemma by noting that the function $s \mapsto \|\sigma_1(s) - \gamma\mu_1(s)\|_\infty$ belongs to $L^2(0,T)$ (since both σ_1 and μ_1 belong to $L^2(0,T;W) \subset L^2(0,T;L^\infty(\Omega))$). This immediately leads to $u = \mu = \sigma = 0$ and the proof is complete.

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References

[1] V. Barbu, “Nonlinear semigroups and differential equations in Banach spaces”, Noordhoff International Publishing, Leyden, 1976.

[2] H. Brezis, “Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert”, North-Holland Math. Stud. **5**, North-Holland, Amsterdam, 1973.

[3] J.W. Cahn and J.E. Hilliard, *Free energy of a nonuniform system I. Interfacial free energy*, J. Chem. Phys., **2** (1958), pp. 258–267.

[4] V. Cristini, H.B. Frieboes, X. Li, J.S. Lowengrub, P. Macklin, S. Sanga, S.M. Wise, and X. Zheng, *Nonlinear modeling and simulation of tumor growth*, in Selected Topics in Cancer Modeling: Genesis, Evolution, Immune Competition, and Therapy, N. Bellomo, M. Chaplain, and E. De Angelis, eds., Modeling and Simulation in Science, Engineering and Technology, Birkhäuser, 2008, ch. 6, pp. 113–181.

[5] V. Cristini, X. Li, J.S. Lowengrub, and S.M. Wise, *Nonlinear simulations of solid tumor growth using a mixture model: invasion and branching*, J. Math. Biol., **58** (2009), pp. 723–763.

[6] V. Cristini and J.S. Lowengrub, "Multiscale Modeling of Cancer: An Integrated Experimental and Mathematical Modeling Approach", Cambridge University Press, Cambridge, 2010.

[7] V. Cristini, J.S. Lowengrub, and Q. Nie, *Nonlinear simulation of tumor growth*, J. Math. Biol., **46** (2003), pp. 191–224.

[8] R. Denk, M. Hieber, and J. Prüss, *Optimal L^p-L^q-estimates for parabolic boundary value problems with inhomogeneous data*, Math. Z., **257** (2007), pp. 193–224.

[9] C.M. Elliott and A.M. Stuart, *Viscous Cahn-Hilliard equation. II. Analysis*, J. Differential Equations, **128** (1996), pp. 387–414.

[10] C.M. Elliott and S. Zheng, *On the Cahn-Hilliard equation*, Arch. Rational Mech. Anal., **96** (1986), pp. 339–357.

[11] S. Frigeri, M. Grasselli, and E. Rocca, *Well-posedness, regularity and global attractor for a diffuse-interface tumor-growth model*, in preparation.

[12] A. Hawkins-Daarud, K.G. van der Zee, and J.T. Oden,, *Numerical simulation of a thermodynamically consistent four-species tumor growth model*, Int. J. Numer. Methods Biomed. Eng., **28** (2012), pp. 3–24.

[13] D. Hilhorst, J. Kampmann, T.N. Nguyen, and K.G. van der Zee, *Formal asymptotic limit of a diffuse-interface tumor-growth model*, in preparation.

[14] J.-L. Lions, "Quelques méthodes de résolution des problèmes aux limites non linéaires", Dunod; Gauthier-Villars, Paris, 1969.

[15] J.S. Lowengrub, H.B. Frieboes, F. Jin, Y.-L. Chuang, X. Li, P. Macklin, S.M. Wise, and V. Cristini, *Nonlinear modeling of cancer: bridging the gap between cells and tumors*, Nonlinearity, **23** (2010), pp. R1–R91.

[16] J.T. Oden, A. Hawkins, and S. Prudhomme, *General diffuse-interface theories and an approach to predictive tumor growth modeling*, Math. Models Methods Appl. Sci., **20** (2010), pp. 477–517.

[17] R.E. Showalter, "Monotone operators in Banach spaces and nonlinear partial differential equations", Math. Surveys Monogr. **49**, American Mathematical Society, Providence, RI, 1997.

[18] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. (4), **146** (1987), pp. 65-96.

[19] S. Zheng, "Nonlinear evolution equations", Chapman Hall/CRC Monogr. Surv. Pure Appl. Math. **133**, Chapman & Hall/CRC, Boca Raton, FL, 2004.