ON GENERALIZED SOLUTIONS OF AXISYMMETRIC TWO-PHASE INCOMPRESSIBLE VISCOUS FLOW WITH SURFACE TENSION

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Abstract. In this paper the existence of an axisymmetric with swirl varifold solution to two-phase Newtonian incompressible viscous flow problem is derived. The motivation for considering this problem is the Spin Coating process. On the free interface between the two phases we consider surface tension force. We prove that for axisymmetric, possibly with swirl, initial velocities and cylindrically symmetric initial volumes occupied by each fluid there exists a global in time axisymmetric, with swirl, solution.

1. Introduction

The spin coating process is used in industry to produce thin films. In this process a small amount of the coating material is placed on the center of a substrate. Then the substrate is rotated at high speeds and the applied coating material spreads on the disk due to centrifugal force. Finally a thin film of the coating material is produced.

Motivated by this process, in this paper we study the axisymmetric with swirl two-phase incompressible flow with surface tension force on the surface between the two fluids. To limit the length of this article we have considered a flow in $\mathbb{R}^3$. A more precise model would be a two-phase flow in the half space $\mathbb{R}^3_+ = \{x_3 > 0\}$ where the surface of the substrate is the plane $\{x_3 = 0\}$.

In the rest of this paper by axisymmetric velocities we mean axisymmetric with swirl.

Although the uniqueness of the solution is not known one expects that if the initial velocities and volumes occupied by the two fluids are axisymmetric with respect to the axis e_3, then there should exist an axisymmetric solution in all time intervals $(0, T)$.

In this paper we prove the existence of this axisymmetric solution as a varifold solution. The notion of varifold solution was introduced by Plotnikov in [9] for a two-dimensional flow of shear thickening fluids. For a varifold solution the mean curvature appearing in the surface tension force is interpreted as the first variation of a general varifold.

In [2] the existence of varifold and weak measure valued solutions for a large class of two-phase incompressible viscous flows is established.

Many authors (cf. [1, 3, 4, 8, 12–14, 17]) have worked on rather regular solutions of such free boundary problems, but by these results one has the well-posedness locally in time unless the initial state is close enough to equilibrium states. The

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approach by varifold solutions is a phase-field formulation and allows one to obtain global in time solutions with arbitrary initial states.

1.1. **Problem setting.** By $\Gamma(t)$ we denote the free boundary, by $\Omega_i(t)$ for $i = 1, 2$ respectively the volumes occupied by two phases. So we have $\mathbb{R}^3 = \Omega_1(t) \cup \Gamma(t) \cup \Omega_0(t)$. By u and p we denote respectively the velocity field and the pressure in both phases. The two phase Newtonian incompressible flow with surface tension force on the interface is the following system of equations

$$
(1.1) \quad \begin{cases}
\partial_t u + \operatorname{div}(u \otimes u) = \operatorname{div}(S(p, Du)) \text{ in } \mathbb{R}^3 \backslash \Gamma(t) \text{ for } 0 < t < T, \\
\operatorname{div}(u) = 0 \text{ in } \mathbb{R}^3 \backslash \Gamma(t) \text{ for } 0 < t < T, \\
[S(p, Du)]_{\Gamma(t)} n_\Gamma = -\kappa n_\Gamma \text{ on } \Gamma(t) \text{ for } 0 < t < T, \\
u \text{ is continuous across } \Gamma(t), \\
u(0) = u_0 \text{ in } \mathbb{R}^3, \\
\text{Velocity of } \Gamma(t) \text{ equals } u^\perp_{\Gamma(t)}, \\
\Omega_1(0) = \Omega_{1,0}.
\end{cases}
$$

Here $S(p, Du) = Du - pI$ is the stress tensor, $2Du = \nabla u + (\nabla u)^T$,

$$
\begin{aligned}
[S(p, Du)]_{\Gamma(t)} &= \text{jump of } S(p, Du) \text{ across } \Gamma(t) \\
&= \lim_{y \to x,\ y \in \Omega_2} S(Du, p) - \lim_{y \to x,\ y \in \Omega_1} S(Du, p),
\end{aligned}
$$

n_Γ is the outward with respect to $\Omega_1(t)$ normal on $\Gamma(t)$, κ is the mean curvature of the interface Γ with respect to n_Γ, u_0 is the initial velocity and $\Omega_{1,0} \subset \mathbb{R}^3$ is the initial volume occupied by the fluid with index 1 such that the initial area of the free boundary is finite, i.e. $|\Gamma(0)| = |\partial\Omega_{1,0}| < \infty$.

1.2. **Weak formulation assuming smoothness.** To describe the notion of varifold solution let us first consider the weak formulation in the case of classical solutions.

By multiplying the momentum equation in (1.1) by $\varphi \in (C_c^\infty((-\infty, T) \times \mathbb{R}^3))^3$ with $\operatorname{div}(\varphi) = 0$ and partial integrations we obtain

$$
(1.2) \quad \begin{aligned}
-\int_{\mathbb{R}^3} u_0^T \varphi(0) dx - \int_0^T \int_{\mathbb{R}^3} \big(u^T \partial_t \varphi + (u \otimes u) : \nabla\varphi\big) dx dt \\
+ \int_0^T \int_{\mathbb{R}^3} Du : D\varphi dx dt = \int_0^T \int_\Gamma (n_\Gamma \otimes n_\Gamma) : \nabla\varphi s(dx) dt
\end{aligned}
$$

with the initial value $u(0) = u_0$.

Here $\chi = 1_{\Omega_1(t)}(x)$ and by the last two lines of (1.1), χ satisfies

$$
\begin{cases}
\partial_t \chi + u \cdot \nabla\chi = 0 \text{ in } (0, T) \times \mathbb{R}^3, \\
\chi(0) = \chi_0
\end{cases}
$$

where $\chi_0 = 1_{\Omega_{1,0}}$.

1.3. **Varifold solution.** In the case of smooth Γ we can write the integral in the term on the right hand side of the equation (1.2) as follows

$$
(1.3) \quad \int_0^T \int_\Gamma (n_\Gamma \otimes n_\Gamma) : \nabla\varphi s(dx) dt = -\int_0^T \int_{\mathbb{R}^3 \times \mathbb{S}^2} (I - y \otimes y) : \nabla\varphi V(t)(d(x, y)) dt
$$

where $V(t) \in \mathcal{M}(\mathbb{R}^3 \times \mathbb{S}^2)$ is defined by

$$
V(t)(A \times B) = \int_{A \cap \Gamma(t)} \delta_{n_{\Gamma(t)}(x)}(B) s(dx) \text{ for } A \in \mathcal{B}(\mathbb{R}^3) \text{ and } B \in \mathcal{B}(\mathbb{S}^2).
$$

For each t the measure $V(t)$ describes the surface $\Gamma(t)$ together with its normal in a weak measure theoretic sense. For our purpose by saying a general 2-varifold $\mathcal{V}$ in $\mathbb{R}^3$ we understand a bounded nonnegative Radon measure on $\mathbb{R}^3\times\mathbb{S}^2$. Thus $V(t)$ is the general 2-varifold associated with the smooth surface $\Gamma(t)$. For the theory of general varifolds one may refer to [11].

We are not able to prove the existence of solutions with smooth enough interface, but we are able to prove the existence of a time dependent 2-varifold $V(t)$ describing $\Gamma(t)$. We shall show that there exists

$$V \in L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^3\times\mathbb{S}^2)) \tag{1.4}$$

such that the term on the right hand side of (1.2) is replaced by the right hand side of (1.3). The space $L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^3\times\mathbb{S}^2))$ is defined in Section 2.

For a 2-varifold $\mathcal{V}\in\mathcal{M}(\mathbb{R}^3\times\mathbb{S}^2)$ the first variation functional $\delta\mathcal{V}$ is defined by

$$\langle\delta\mathcal{V},\varphi\rangle = \int_{\mathbb{R}^3\times\mathbb{S}^2}(I-y\otimes y):\nabla\varphi(x)\mathcal{V}(d(x,y)) \text{ for } \varphi\in(C^1_0(\mathbb{R}^3))^3 \tag{1.5}$$

hence the right hand side of (1.3) is the time integral of the first variation functional.

We will show that also we have $\nabla\chi\in L^\infty_{w^*}(0,T,(\mathcal{M}(\mathbb{R}^3))^3)$ and the connection between the time dependent 2-varifold V and χ is given by the equation

$$-\int_{\mathbb{R}^3}\varphi\cdot\nabla\chi(t)(dx)dt = \int_{\mathbb{R}^3\times\mathbb{S}^2}\varphi\cdot yV(t)(d(x,y)), \qquad \forall\varphi\in(C^\infty_c(\mathbb{R}^3))^3 \text{ for a.e. } t\in(0,T). \tag{1.6}$$

1.4. **Axisymmety.** For an angle $\theta\in\mathbb{R}$ let us denote by $J(\theta)$ the rotation matrix with the angle θ in $\mathbb{R}^2$ and by $O(\theta)$ the rotation matrix with angle θ around the axis e_3 in $\mathbb{R}^3$, i.e.

$$J(\theta)=\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix} \text{ and } O(\theta)=\begin{bmatrix}J(\theta) & 0_2\\ 0_2^T & 1\end{bmatrix}$$

here 0_2 is the 0 in $\mathbb{R}^2$.

We call $h:\mathbb{R}^3\to\mathbb{R}$ axisymmetric if

$$h(x)=h(O^T(\theta)x),\ \ \forall x\in\mathbb{R}^3 \text{ and } \theta\in\mathbb{R}. \tag{1.7}$$

We call $w:\mathbb{R}^3\to\mathbb{R}^3$ axisymmetric if

$$w(x)=O(\theta)w(O^T(\theta)x),\ \ \forall x\in\mathbb{R}^3 \text{ and } \theta\in\mathbb{R}. \tag{1.8}$$

We call $\mathcal{V}\in\mathcal{M}(\mathbb{R}^3\times\mathbb{S}^2)$ axisymmetric if

$$\mathcal{V}(A\times B)=\mathcal{V}((O^T(\theta)A)\times(O^T(\theta)B)),\ \forall A\in\mathcal{B}(\mathbb{R}^3),\ B\in\mathcal{B}(\mathbb{S}^2) \text{ and } \theta\in\mathbb{R} \tag{1.9}$$

and

$$\mathcal{V}\Big(\Big\{(x,y)\in\mathbb{R}^3\times\mathbb{S}^2\ \Big|\ y\cdot(e_3\times x)\neq 0\Big\}\Big)=0. \tag{1.10}$$

1.5. **Main results.** Let us define

$$E = \text{closure of } \big\{v\in(C^\infty_c(\mathbb{R}^3))^3\ \big|\ \operatorname{div}(v)=0\big\} \text{ in } (L^2(\mathbb{R}^3))^3$$

and

$$V = \text{closure of } \big\{v\in(C^\infty_c(\mathbb{R}^3))^3\ \big|\ \operatorname{div}(v)=0\big\} \text{ in } (H^1(\mathbb{R}^3))^3.$$

In the statement of the theorem below we will encounter the spaces $L^\infty_{w^*}(0,T,(\mathcal{M}(\mathbb{R}^3))^3)$ and $L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^3\times\mathbb{S}^2))$. These are defined in Section 2.

The main results of this paper are the following two theorems.

Theorem 1. *Let $u_0 \in E$, $\chi_0 = 1_{\Omega_{1,0}} \in BV(\mathbb{R}^3)$, where*

$$BV(\mathbb{R}^3) = \Big\{ \eta \in L^1(\mathbb{R}^3) \ \Big| \ \nabla \eta \in (\mathcal{M}(\mathbb{R}^3))^3 \Big\}$$

denotes the space of functions with bounded variation in $\mathbb{R}^3$.

Then there exists a triple (u, χ, V) such that

$$u \in L^\infty(0, T, E) \cap L^2(0, T, V),$$
$$\chi \in L^\infty(0, T, L^1(\mathbb{R}^3)), \ \nabla \chi \in L^\infty_{w^*}(0, T, (\mathcal{M}(\mathbb{R}^3))^3)$$

and

$$V \in L^\infty_{w^*}(0, T, \mathcal{M}(\mathbb{R}^3 \times \mathbb{S}^2))$$

with

$$\|u\|_{L^\infty(0,T,E)} + \|u\|_{L^2(0,T,V)} + \|\chi\|_{L^\infty(0,T,L^1(\mathbb{R}^3))} + \|\nabla \chi\|_{L^\infty_{w^*}(0,T,(\mathcal{M}(\mathbb{R}^3))^3)} + \|V\|_{L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^3 \times \mathbb{S}^2))} \leq C\big(\|\chi_0\|_{BV(\mathbb{R}^3)} + \|u_0\|_E\big).$$

χ is the renormalized solution of

$$\begin{cases} \partial_t \chi + u \cdot \nabla \chi = 0 \ \textit{in} \ (0, T) \times \mathbb{R}^3, \\ \chi(0) = \chi_0 \ \textit{in} \ \mathbb{R}^3. \end{cases} \tag{1.11}$$

The connection between $\nabla \chi$ and V is given by the equation (1.6).

For all $\varphi \in (C_c^\infty((-\infty, T) \times \mathbb{R}^3))^3$ with $\operatorname{div}(\varphi) = 0$, *$u$ satisfies*

$$-\int_{\mathbb{R}^3} u_0^T \varphi(0) dx - \int_0^T \int_{\mathbb{R}^3} \big\{ u^T \partial_t \varphi + (u \otimes u) : \nabla \varphi \big\} dx dt + \int_0^T \int_{\mathbb{R}^3} Du : D\varphi dx dt = -\int_0^T \langle \delta V(t), \varphi(t) \rangle \, dt. \tag{1.12}$$

The solution triple (u, χ, V) is called a varifold solution. Fo the notion of renormalized solution to transport equation one may refer to [6].

Theorem 2. *If in the theorem 1 the initial values u_0 and χ_0 are axisymmetric then there exists a varifold solution (u, χ, V) with each component being axisymmetric, i.e. for a.e. $0 < t < T$, $\chi(t)$ satisfies* (1.7)*, $u(t)$ satisfies* (1.8) *and $V(t)$ satisfies* (1.9) *and* (1.10).

1.6. **Organization of this paper.** This paper is organized as follows, in Section 2 we have collected some definitions and facts about Banach space valued functions, in Section 3 we consider an approximate regularized problem and using the Schaefer fixed point theorem we prove the existence of a regularized solution. In Section 4 we prove the existence of a varifold solution as the limit of regularized solutions.

2. Some facts about Banach space valued functions

For the following definitions and facts a good reference is [5].

Let us denote by λ the Lebesgue measure defined on the Borel subsets of $\mathbb{R}$.

Let X be a Banach space and $I \subset \mathbb{R}$ an interval. Let us consider $f : I \to X$, then one may consider the following three kinds of measurabilities of f.

f is called λ-measurable if there exists a sequence of simple functions $s_n : I \to X$ such that $s_n(t) \to f(t)$ in X for λ-a.e. $t \in I$.

f is called weakly-λ-measurable if for any $g \in X'$ the function (as a function of t) $\langle g, f(t) \rangle_{X',X}$ is measurable.

In the case $X = Y'$ for some Banach space Y, f is called weak*-λ-measurable if for any $y \in Y$, the function (as a function of t) $\langle f(t), y \rangle_{Y',Y}$ is measurable.

The function f is called λ-essentially separably valued if there exists $E \subset I$ such that $\lambda(E) = 0$ and $f(I \backslash E)$ is a separable subset of X.

The Pettis measurability theorem states that f is λ-measurable if and only if f is λ-essentially separably valued and f is weakly-λ-measurable.

For $1 \le p \le \infty$ by $L^p(0,T,X)$ we denote the space of λ-measurable functions $f:(0,T)\to X$ such that $\|f(t)\|_X$ as a function of t is in $L^p(0,T)$.

When $X = Y'$ for some Banach space Y, for $1 \le p \le \infty$ by $L^p_{w^*}(0,T,X)$ we denote the space of w^*-λ-measurable functions $f:(0,T)\to X$ such that also $\|f(t)\|_X$ as a function of t is measurable and is in $L^p(0,T)$.

If X is a reflexive Banach space and $1 \le p < \infty$ then $(L^p(0,T,X))' = L^q(0,T,X')$ where $p^{-1}+q^{-1}=1$.

3. Approximate Regularized Problem

In subsection 3.1 we define rotation operators. In subsection 3.2 we define a regularisation operator of solenoidal time dependent vector fields Ψ_ϵ. In the subsection 3.3 we define the approximate regularised problem and in subsection 3.4 we prove the existence of a solution to the regularised problem.

3.1. **Preliminary definitions and analysis.** For a function $h:\mathbb{R}^3\to\mathbb{R}$ and $\theta\in\mathbb{R}$ we define the clockwise rotation around the axis e_3 of the function h by angle θ as $\tau_\theta(h)(x) = h(O^T(\theta)x)$ for $x\in\mathbb{R}^3$. It is easy to see that for a smooth h for $x\in\mathbb{R}^3$ we have

$$\nabla(\tau_\theta h)(x) = O(\theta)\nabla h(O^T(\theta)x). \tag{3.1}$$

For a function $w:\mathbb{R}^3\to\mathbb{R}^3$ and $\theta\in\mathbb{R}$ we define the clockwise rotation around the axis e_3 of the function w by the angle θ as follows $T_\theta(w)(x) = O(\theta)w(O^T(\theta)x)$ for $x\in\mathbb{R}^3$. It is easy to see that for a smooth w for $x\in\mathbb{R}^3$ we have

$$\nabla(T_\theta\varphi)(x) = O(\theta)\nabla\varphi(O^T(\theta)x)O^T(\theta). \tag{3.2}$$

We fix a mollifier $\psi\in C_c^\infty(B_1^3)$ such that $\int_{B_1^3}\psi(x)dx = 1$ and is radial i.e. $\psi(x)=\psi(y)$ for $|x|=|y|$. As usual notation for mollifiers $\psi_\beta(x)=\beta^{-3}\psi(\beta^{-1}x)$ for $\beta>0$.

Proposition 1. *The operators $\psi_\beta * \cdot$ and τ_θ commute. The operators $\psi_\beta * \cdot$ and T_θ commute.*

Proof. By direct computations. □

3.2. **The operator Ψ_ϵ.** In the rest of this paper, always we have $0<\epsilon<1$, $\alpha(\epsilon)=\frac{1}{2}\epsilon^2$ and $\beta(\epsilon)=\sqrt{\epsilon}$.

In this subsection we define a regularizing and compact linear operator Ψ_ϵ of solenoidal vector fields which preserves axisymmetry. We will use this operator extensively in our regularized problem.

We define

$$\Psi_\epsilon : L^2(0,T,(L^2(\mathbb{R}^3))^3) \to L^2(0,T,(L^2(\mathbb{R}^3))^3)$$

as follows

$$\Psi_\epsilon = PK_\epsilon$$

where we define the compact operator K_ϵ below and P is the Helmholtz projection operator which projects $(L^2(\mathbb{R}^3))^3$ on divergence free vector fields in this space.

We fix a mollifier $\phi\in C_c^\infty(0,1)$ such that $\int_{(0,1)}\phi(t)dt=1$ and $\phi\ge 0$. As usual $\phi_\alpha(t)=\alpha^{-1}\phi(\alpha^{-1}t)$ for $\alpha>0$. It is very crucial that the support of ϕ is on the positive numbers because this makes the value of the convolution in time $(\phi_\alpha * v)(t)$ for some function v to depend only on the vales of v in $(-\infty,0)$, i.e. historic values.

We define the cutoff function in space $\tilde{\psi}_\beta(x) = (\psi * 1_{B^3_{\beta^{-1}-1}})(x)$.

We define the operator $K_\epsilon : L^2(0,T,(L^2(\mathbb{R}^3))^3) \to (C_c^\infty(\mathbb{R}^4))^3$ as follows

$$(K_\epsilon w)(t,x) = \tilde{\psi}_{\beta(\epsilon)}(x)((\phi_{\alpha(\epsilon)}\psi_{\beta(\epsilon)}) * \bar{w})(t,x) \tag{3.3}$$

where $\bar{w}$ is equal to w for $t \in (0,T)$ and equal to 0 on $(0,T)^c$.

Proposition 2. *The operator K_ϵ commutes with T_θ. We have $K_\epsilon : L^2(0,T,(L^2(\mathbb{R}^3))^3) \to (C_c^\infty(Q_\epsilon))^3$ where $Q_\epsilon = (0,T+\alpha(\epsilon)) \times B^3_{\beta(\epsilon)^{-1}}$. For $k \in \mathbb{N} \cup \{0\}$, K_ϵ maps $L^2(0,T,(L^2(\mathbb{R}^3))^3)$ continuously into $(H_0^k(Q_\epsilon))^3$.*

Proof. One uses the commuting properties outlined in the proposition 1 and the cutoff and mollification structure of K_ϵ. □

In terms of the Fourier transform (cf. [16]) the Helmholtz projection operator $P : (L^2(\mathbb{R}^3))^3 \to (L^2(\mathbb{R}^3))^3$ might be written as $Pw = \mathcal{F}^{-1}(M\mathcal{F}(w))$ where $M(\zeta) = I - |\zeta|^{-2}\zeta\zeta^T$ and here $\mathcal{F}$ denotes the Fourier transform for functions defined on $\mathbb{R}^3$. Also for functions u defined on $\mathbb{R}^4$ we denote by $P(u)$ the function $P(u)(t,x) = P(u(t))(x)$ for $t \in \mathbb{R}$ and $x \in \mathbb{R}^3$.

Proposition 3. *The operator P commutes with T_θ.*

For all $k \in \{0\} \cup \mathbb{N}$ we have $[P(u)]_{(H^k(\mathbb{R}^3))^3} \le 2[u]_{(H^k(\mathbb{R}^3))^3}$ for all $u \in (H^k(\mathbb{R}^3))^3$ and similarly $\|P(u)\|_{(H^k(\mathbb{R}^4))^3} \le 2\|u\|_{(H^k(\mathbb{R}^4))^3}$ for all $u \in (H^k(\mathbb{R}^4))^3$.

Let $\omega(x) = \Pi_{i=1}^3(1+|x_i|)^{-\frac{1}{2}}$ for $x \in \mathbb{R}^3$ then P is a bounded linear operator mapping $(L^2(\mathbb{R}^3;\omega))^3$ to itself.

Proof. The commuting property of P with T_θ follows respectively from the fact that Fourier transform commutes with T_θ and $M(O^T(\theta)\zeta) = O^T(\theta)M(\zeta)O(\theta)$.

We have

$$|M(\zeta)|_{2,2} \le 2 \text{ for all } \zeta \in \mathbb{R}^3. \tag{3.4}$$

Let $k \in \{0\} \cup \mathbb{N}$ then from (3.4) it follows that

$$[P(u)]^2_{(H^k(\mathbb{R}^3))^3} = \int_{\mathbb{R}^3} |\zeta|^{2k}|M(\zeta)\mathcal{F}(u)|^2 d\zeta \le 4[u]^2_{(H^k(\mathbb{R}^3))^3}.$$

Similarly if we denote by $\tilde{\mathcal{F}}$ the Fourier transform of functions defined on $\mathbb{R}^4$ then by the separation of variable property of Fourier transform we have actually $P(u) = \tilde{\mathcal{F}}^{-1}(M\tilde{\mathcal{F}}(u))$ for $u \in (C_c^\infty(\mathbb{R}^4))^3$ hence again by (3.4) we obtain $\|P(u)\|_{(H^k(\mathbb{R}^4))^3} \le 2\|u\|_{(H^k(\mathbb{R}^4))^3}$.

By the definition of P we have for $j \in \{1,2,3\}$

$$(P(u))_j = u_j - R_j(\sum_{k=1}^3 R_k(u_k)) \tag{3.5}$$

where R_j is the j-th Riesz transform.

It is known that (cf. [15]) if $\eta \in A_2$ where

$$A_2 = \left\{\eta \in L^1_{loc}(\mathbb{R}^3) \;\middle|\; \eta \ge 0,\ \sup_B\left\{\frac{1}{|B|^2}\int_B \eta dx \int_B \frac{1}{\eta}dx\right\} < \infty\right\}$$

here the supremum is over all balls $B \subset \mathbb{R}^3$, then R_j is a bounded linear operator of $L^2(\mathbb{R}^3;\eta)$ to itself. Now we may check that $\omega \in A_2$ thus by (3.5) we obtain that P is a bounded linear operator mapping $(L^2(\mathbb{R}^3;\omega))^3$ to itself. □

Lemma 1. *The operators Ψ_ϵ and T_θ commute.*

For each $u \in L^2(0,T,(L^2(\mathbb{R}^3))^3)$, $\mathrm{supp}(\Psi_\epsilon(u)) \subset [0,T+\alpha(\epsilon)] \times \mathbb{R}^3$.

For each $k \in \{0\} \cup \mathbb{N}$, Ψ_ϵ is a compact map of $L^2(0,T,(L^2(\mathbb{R}^3))^3)$ to $(C_b^k(\mathbb{R}^4))^3$.

Ψ_ϵ is a bounded linear operator mapping $L^2(0,T,(L^2(\mathbb{R}^3;\omega))^3)$ to itself and its corresponding norm is uniformly bounded with respect to ϵ.

For $u \in L^2(0,T,(L^2(\mathbb{R}^3;\omega))^3)$ *we have* $\Psi_\epsilon(u) \to P(u)$ *in* $L^2(0,T,(L^2(\mathbb{R}^3;\omega))^3)$ *as* $\epsilon \to +0$.

For $u \in L^2(0,T,(H^1(\mathbb{R}^3))^3)$ *we have* $\Psi^*_\epsilon(u) \to P^*(u)$ *in* $(L^2(0,T,(H^1(\mathbb{R}^3))^3))^*$ *as* $\epsilon \to +0$.

Proof. Because P and K_ϵ commute with T_θ we obtain that Ψ_ϵ commutes with T_θ.

Fix $k \in \{0\} \cup \mathbb{N}$. Let $m \in \mathbb{N}$ large enough such that by the Sobolev embedding theorem $(H^m(\mathbb{R}^4))^3$ is continuously embedded in the space $(C^k_b(\mathbb{R}^4))^3$. The function K_ϵ maps $L^2(0,T,(L^2(\mathbb{R}^3))^3)$ in $H^{m+1}_0(Q_\epsilon)$ continuously. Thus by the compact embedding of $H^{m+1}_0(Q_\epsilon)$ in $H^m_0(Q_\epsilon)$ we have that K_ϵ is a compact operator mapping $L^2(0,T,(L^2(\mathbb{R}^3))^3)$ to $H^m_0(Q_\epsilon)$. The operator P maps $H^m(\mathbb{R}^4)$ continuously in itself. Finally by the continuous embedding of this space in $(C^k_b(\mathbb{R}^4))^3$ we prove the compactness of the operator Ψ_ϵ mapping $L^2(0,T,(L^2(\mathbb{R}^3))^3)$ to $(C^k_b(\mathbb{R}^4))^3$.

We have

$$\Psi_\epsilon(u) = \phi_{\alpha(\epsilon)} * (P(\tilde{\psi}_{\beta(\epsilon)}\psi_{\beta(\epsilon)} * \bar{u})). \tag{3.6}$$

Because $\omega \in A_2$ we have that $\psi_\beta *$ maps $L^2(\mathbb{R}^3;\omega)$ to itself with a norm uniformly bounded with respect to $0 < \beta < 1$ (cf. [15]). The operator P is also bounded mapping $(L^2(\mathbb{R}^3;\omega))^3$ to itself as discussed in proposition (3). The rest of the operators in (3.6) have also uniformly bounded in ϵ norms as operators mapping $(L^2(\mathbb{R}^3;\omega))^3$ to itself or $L^2(0,T,(L^2(\mathbb{R}^3;\omega))^3)$ to itself. And this proves the claimed boundedness and uniform bound.

The last two claims of the lemma are easy to check. □

3.3. **Approximate regularized problem.** Let us define $\chi_{0,\epsilon} = \psi_\epsilon * \chi_0$.

Proposition 4. *We have*

$$0 \le \chi_{0,\epsilon} \le 1, \tag{3.7}$$

$$\|\chi_{0,\epsilon}\|_{L^1(\mathbb{R}^3)} \le \|\chi_0\|_{L^1(\mathbb{R}^3)},\ \|\chi_{0,\epsilon}\|_{L^2(\mathbb{R}^3)} \le \|\chi_0\|_{L^2(\mathbb{R}^3)}, \tag{3.8}$$

$$\|\nabla\chi_{0,\epsilon}\|_{L^1(\mathbb{R}^3)} \le C|\nabla\chi_0|(\mathbb{R}^3) \tag{3.9}$$

and if $\Omega_{1,0}$ *is axisymmetric then* $\chi_{0,\epsilon}$ *is axisymmetric.*

Proof. Because $\chi_0(x) \in \{0,1\}$ for all $x \in \mathbb{R}^3$, $\psi \ge 0$ and $\int_{B^3_1} \psi(x)dx = 1$ we obtain (3.7). Using the Young inequality for convolution one obtains the inequalities in (3.8) and (3.9).

If $\Omega_{1,0}$ is axisymmetric then χ_0 is axisymmetric. Now because $\psi_\epsilon *$ commutes with τ_θ, $\chi_{0,\epsilon}$ is axisymmetric. □

For $\chi \in H^1(\mathbb{R}^3)$ we define $\tilde{f}^{s.t.}_\epsilon(\chi) \in ((H^1(\mathbb{R}^3))^3)^*$ by

$$\left\langle \tilde{f}^{s.t.}_\epsilon(\chi), \varphi \right\rangle = \int_{\mathbb{R}^3} \frac{\nabla\chi \otimes \nabla\chi}{(|\nabla\chi|^2 + \epsilon^2)^{\frac{1}{2}}} : \nabla\varphi dx \quad \text{for } \varphi \in (H^1(\mathbb{R}^3))^3$$

and for $\chi \in L^2(0,T,H^1(\mathbb{R}^3))$ we define $f^{s.t.}_\epsilon(\chi) \in (L^2(0,T,(L^2(\mathbb{R}^3))^3))^*$ for $\varphi \in L^2(0,T,(L^2(\mathbb{R}^3))^3)$ by

$$\left\langle f^{s.t.}_\epsilon(\chi), \varphi \right\rangle = \int_0^T \left\langle \tilde{f}^{s.t.}_\epsilon(\chi), \Psi_\epsilon(\varphi) \right\rangle dt.$$

Let us denote by J the isomorphism between the spaces $(L^2(0,T,(L^2(\mathbb{R}^3))^3))^*$ and $L^2(0,T,((L^2(\mathbb{R}^3))^3)^*)$.

Now we are in the position to state our regularized system of a transport equation

$$\begin{cases} \partial_t\chi_\epsilon + \Psi_\epsilon(u_\epsilon) \cdot \nabla\chi_\epsilon - \epsilon\triangle\chi_\epsilon = 0 \text{ in } (0,T) \times \mathbb{R}^3, \\ \chi_\epsilon(0) = \chi_{0,\epsilon} \end{cases} \tag{3.10}$$

together with the momentum equation

$$\begin{cases} \partial_t u_\epsilon + \operatorname{div}(\Psi_\epsilon(u_\epsilon) \otimes u_\epsilon) - \operatorname{div}(D(u_\epsilon)) \\ \qquad\qquad\qquad = J(f_\epsilon^{s.t.}(\chi_\epsilon)) \text{ in } (0,T) \times \mathbb{R}^3, \\ u_\epsilon(0) = u_0. \end{cases} \tag{3.11}$$

The precise sense in which the equations above should hold will be clear in the following.

3.4. **Existence of solution to the regularized problem.** In the following lemmas we prove boundedness and continuity properties of the transport equation, the force term and the momentum equation with a prescribed transport term. These results will be used in applying the Schaefer fixed point theorem.

Let us denote

$$W = \left\{ w \in (C_b^\infty(\mathbb{R}^4))^3 \;\middle|\; \operatorname{div}(w) = 0 \right\}$$

where $C_b^\infty(\mathbb{R}^4) = \cap_{k \in \{0\} \cup \mathbb{N}} C_b^k(\mathbb{R}^4)$.

Lemma 2. *Let $w \in W$ then the following equation*

$$\begin{cases} \partial_t \chi + w \cdot \nabla \chi - \epsilon \triangle \chi = 0 \text{ in } (0,T) \times \mathbb{R}^3, \\ \chi(0) = \chi_{0,\epsilon} \text{ in } \mathbb{R}^3 \end{cases} \tag{3.12}$$

has a unique classical solution $\chi \in C_b^2([0,T) \times \mathbb{R}^3)$. Let us define the nonlinear operator $G_\epsilon(w) = \chi$. We have

$$0 \le \chi \le 1 \tag{3.13}$$

and

$$\|\chi\|_{L^\infty(0,T,L^2(\mathbb{R}^3))} + \sqrt{\epsilon} \|\chi\|_{L^2(0,T,H^1(\mathbb{R}^3))} \le C \|\chi_{0,\epsilon}\|_{L^2(\mathbb{R}^3)}. \tag{3.14}$$

Considering the supremum norm for the space W, the map

$$G_\epsilon : W \to L^2(0,T,H^1(\mathbb{R}^3))$$

is (Lipschitz) continuous.

If w and $\Omega_{1,0}$ are axisymmetric then so is χ.

Proof. The inequalities (3.13) follow from maximum principle and (3.7).

By multiplying the equation (3.12) with χ, integrations by part and using that $w \in (C_b(\mathbb{R}^3))^3$ with $\operatorname{div}(w) = 0$ we obtain (3.14).

Let $w_i \in X$ for $i = 1,2$ and χ_i the corresponding solutions to (3.12). Taking the difference of the equations satisfied by χ_i and denoting $w = w_2 - w_1$ and $\chi = \chi_2 - \chi_1$ we obtain

$$\partial_t \chi + \frac{1}{2}(w_1 + w_2) \cdot \nabla \chi - \epsilon \triangle \chi = -\frac{1}{2} w \cdot \nabla(\chi_2 + \chi_1)$$

now this is an equation satisfied by χ with the initial value $\chi(0) = 0$. By similar computations as one does to obtain (3.14) and using the inequality (3.14) for χ_1 and χ_2 one obtains the inequality $\|\chi\|_{L^2(0,T,H^1(\mathbb{R}^3))} \le C_\epsilon \|\chi_{0,\epsilon}\|_{L^2(\mathbb{R}^3)} \|w\|_{C_b(\mathbb{R}^4)}$ which proves the Lipschitz continuity of G_ϵ.

If $\Omega_{1,0}$ is axisymmetric then by proposition 4, $\chi_{0,\epsilon}$ is axisymmetric. If also w is axisymmetric it is easy to see that for any $\theta \in \mathbb{R}$, the function $\tau_\theta(\chi)$ is also a solution to our equation and hence by uniqueness we should have $\chi = \tau_\theta(\chi)$ for all $\theta \in \mathbb{R}$ which proves that χ is axisymmetic. □

We call $f \in (L^2(0,T,(L^2(\mathbb{R}^3))^3))^*$ axisymmetric if $T_\theta^* f = f$ for all $\theta \in \mathbb{R}$, i.e. $\langle f, \varphi \rangle = \langle f, T_\theta \varphi \rangle$ for all $\varphi \in L^2(0,T,(L^2(\mathbb{R}^3))^3)$ and $\theta \in \mathbb{R}$.

Lemma 3. *The function $f_\epsilon^{s.t.} : L^2(0,T,H^1(\mathbb{R}^3)) \to (L^2(0,T,(L^2(\mathbb{R}^3))^3))^*$ is continuous and if χ is axisymmetric then $f_\epsilon^{s.t.}(\chi)$ is axisymmetric.*

Proof. Let $\varphi \in L^2(0,T,(L^2(\mathbb{R}^3))^3)$. Let $\chi_i \in L^2(0,T,H^1(\mathbb{R}^3))$ for $i=1,2$ then

$$\begin{aligned}(3.15)\quad &\left\langle f_\epsilon^{s.t}(\chi_2) - f_\epsilon^{s.t.}(\chi_1), \varphi\right\rangle \\ &= \int_0^T\int_{\mathbb{R}^3}\Big(\frac{\partial_i\chi_2\partial_j\chi_2}{(|\nabla\chi_2|^2+\epsilon^2)^{\frac12}} - \frac{\partial_i\chi_1\partial_j\chi_1}{(|\nabla\chi_1|^2+\epsilon^2)^{\frac12}}\Big)\partial_{x_i}(\Psi_\epsilon(\varphi))_j dxdt.\end{aligned}$$

Let us denote for $p,q\in\mathbb{R}$ and $a>0$, $A(p,q,a) = (a^2+\epsilon^2)^{-\frac12}pq$ then we have $\partial_p A(p,q,a) = (a^2+\epsilon^2)^{-\frac12}q$, $\partial_q A(p,q,a) = (a^2+\epsilon^2)^{-\frac12}p$ and $\partial_a A(p,q,a) = -(a^2+\epsilon^2)^{-\frac32}pqa$. So for $|p|,|q|\le a$ we have $|\partial_p A(p,q,a)|\le 1$, $|\partial_q A(p,q,a)|\le 1$ and $|\partial_a A(p,q,a)|\le 1$. Now let $|p_k|,|q_k|\le a_k$ for $k=1,2$ then for $0\le\tau\le 1$ we have $|(1-\tau)p_1+\tau p_2| \le (1-\tau)|p_1| + \tau|p_2| \le (1-\tau)a_1+\tau a_2$ and similarly for q_k hence we can estimate using the mean value theorem $|A(p_2,q_2,a_2)-A(p_1,q_1,a_1)| \le |p_2-p_1|+|q_2-q_1|+|a_2-a_1|$.

Considering $p_k = \partial_{x_i}\chi_k$, $q_k=\partial_{x_j}\chi_k$ and $a_k = |\nabla\chi_k|$ we have

$$\begin{aligned}(3.16)\quad &|A(\partial_{x_i}\chi_2,\partial_{x_j}\chi_2,|\nabla\chi_2|) - A(\partial_{x_i}\chi_1,\partial_{x_j}\chi_1,|\nabla\chi_1|)| \\ &\le \left\{|\partial_{x_i}\chi_2-\partial_{x_i}\chi_1| + |\partial_{x_j}\chi_2-\partial_{x_j}\chi_1| + ||\nabla\chi_2|-|\nabla\chi_1||\right\} \le 3|\nabla\chi_2-\nabla\chi_1|.\end{aligned}$$

Now by (3.15), the definition of A, (3.16) and Lemma 1 we obtain

$$\begin{aligned}(3.17)\quad |\left\langle f_\epsilon^{s.t}(\chi_2) - f_\epsilon^{s.t.}(\chi_1),\varphi\right\rangle| &\le C\int_0^T\int_{\mathbb{R}^3}|\nabla\chi_2-\nabla\chi_1||\nabla\Psi_\epsilon(\varphi)|dxdt \\ &\le C\Big\{\int_0^T\int_{\mathbb{R}^3}|\nabla\chi_2-\nabla\chi_1|^2dxdt\Big\}^{\frac12}\Big\{\int_0^T\int_{\mathbb{R}^3}|\nabla\Psi_\epsilon(\varphi)|^2dxdt\Big\}^{\frac12}.\end{aligned}$$

Denoting $v = \nabla P\tilde{\psi}_{\beta(\epsilon)}\psi_{\beta(\epsilon)} * \bar{\varphi}$ and using two times the Minkowski inequality, separately we estimate

$$\begin{aligned}(3.18)\quad &\int_0^T\int_{\mathbb{R}^3}|\nabla\Psi_\epsilon(\varphi)|^2dxdt = \int_0^T\int_{\mathbb{R}^3}|\phi_{\alpha(\epsilon)}*v|^2dxdt \\ &\le \int_0^T\|\phi_{\alpha(\epsilon)}*v\|^2_{(L^2(\mathbb{R}^3))^{3\times3}}dt \le \int_0^T(\phi_{\alpha(\epsilon)}*\|v\|_{(L^2(\mathbb{R}^3))^{3\times3}})^2dt \\ &= \int_0^T\Big(\int_0^T\phi_{\alpha(\epsilon)}(s)\|v\|_{(L^2(\mathbb{R}^3))^{3\times3}}(t-s)ds\Big)^2dt \\ &\le \Big\{\int_0^T\phi_{\alpha(\epsilon)}(s)\Big\{\int_0^T\|v\|^2_{(L^2(\mathbb{R}^3))^{3\times3}}(t-s)dt\Big\}^{\frac12}ds\Big\}^2 \\ &= \Big\{\int_0^{\alpha(\epsilon)}\phi_{\alpha(\epsilon)}(s)\Big\{\int_{-s}^{T-s}\|v\|^2_{(L^2(\mathbb{R}^3))^{3\times3}}(z)dz\Big\}^{\frac12}ds\Big\}^2 \\ &\le \Big\{\int_0^{\alpha(\epsilon)}\phi_{\alpha(\epsilon)}(s)ds\Big\}^2\int_0^T\|v\|^2_{(L^2(\mathbb{R}^3))^{3\times3}}(z)dz \\ &\le \int_0^T\|v\|^2_{(L^2(\mathbb{R}^3))^{3\times3}}(z)dz = \int_0^T[P\tilde{\psi}_{\beta(\epsilon)}\psi_{\beta(\epsilon)}*\bar{\varphi}]^2_{(H^1(\mathbb{R}^3))^3}(z)dz \\ &\le C\int_0^T[\tilde{\psi}_{\beta(\epsilon)}\psi_{\beta(\epsilon)}*\bar{\varphi}]^2_{(H^1(\mathbb{R}^3))^3}(z)dz \\ &\le C_\epsilon\int_0^T\|\varphi(t)\|^2_{(L^2(\mathbb{R}^3))^3}dt \le C_\epsilon\|\varphi\|^2_{L^2(0,T,(L^2(\mathbb{R}^3))^3)}\end{aligned}$$

and this together with (3.17) proves the continuity of $f_\epsilon^{s.t.}$.

Let $\chi \in L^2(0,T,H^1(\mathbb{R}^3))$ be axisymmetric, $\varphi \in L^2(0,T,(L^2(\mathbb{R}^3))^3)$ and $\theta \in \mathbb{R}$ then by Lemma 1, (3.1) and (3.2)

$$\begin{aligned}
\left\langle f_\epsilon^{s.t}(\chi), T_\theta\varphi\right\rangle &= \int_0^T\int_{\mathbb{R}^3} \frac{\nabla\chi^T\nabla(\Psi_\epsilon(T_\theta\varphi))\nabla\chi}{(|\nabla\chi|^2+\epsilon^2)^{\frac{1}{2}}}dxdt \\
&= \int_0^T\int_{\mathbb{R}^3} \frac{\nabla\chi^T\nabla(T_\theta(\Psi_\epsilon(\varphi)))\nabla\chi}{(|\nabla\chi|^2+\epsilon^2)^{\frac{1}{2}}}dxdt \\
&= \int_0^T\int_{\mathbb{R}^3} \frac{\nabla\chi^T(O^T(\theta)\nabla(\Psi_\epsilon(\varphi))(t,O(\theta)x)O(\theta))\nabla\chi}{(|\nabla\chi|^2+\epsilon^2)^{\frac{1}{2}}}dxdt \\
&= \int_0^T\int_{\mathbb{R}^3} \frac{(O(\theta)\nabla\chi(t,O^T(\theta)x))^T\nabla(\Psi_\epsilon(\varphi))(t,x)O(\theta)\nabla\chi(t,O^T(\theta)x)}{(|\nabla\chi(t,O^T(\theta)x)|^2+\epsilon^2)^{\frac{1}{2}}}dxdt \\
&= \int_0^T\int_{\mathbb{R}^3} \frac{(\nabla\tau_\theta\chi)^T\nabla(\Psi_\epsilon(\varphi))(\nabla\tau_\theta\chi)}{(|\nabla(\tau_\theta\chi)|^2+\epsilon^2)^{\frac{1}{2}}}dxdt \\
&= \int_0^T\int_{\mathbb{R}^3} \frac{\nabla\chi^T\nabla(\Psi_\epsilon(\varphi))\nabla\chi}{(|\nabla\chi|^2+\epsilon^2)^{\frac{1}{2}}}dxdt = \left\langle f_\epsilon^{s.t}(\chi),\varphi\right\rangle
\end{aligned}$$

which proves that $f_\epsilon^{s.t.}$ is axisymmetric. □

Lemma 4. *Let $w \in W$ and $f \in L^2(0,T,V^*)$ then there exists a unique solution $u \in C([0,T),E) \cap L^2(0,T,V)$ of the equation*

$$\begin{cases} \partial_t u + \operatorname{div}(w\otimes u) - \operatorname{div}(Du) = f \text{ in } V^* \text{ for a.e. } t \in (0,T), \\ u(0) = u_0. \end{cases}$$

Let us denote $u = A(w,f)$ then $A : W \times L^2(0,T,V^*) \to L^2(0,T,V)$ is continuous considering the $(C_b(\mathbb{R}^4))^3$ norm for W.

If w and f are axisymmetric then $u = A(w,f)$ is axisymmetric.

Proof. We divide the proof in four steps. In the first step using the Galerkin method we prove the existence of the solution, in the second step we prove the uniqueness of the solution, in the third step we prove the continuous dependence of the solution on the data and finally in the fourth step we show that if w and f are axisymmetric then so is u.

Step 1. Existence.

Let the set of the functions $v_k \in (C_c^\infty(\mathbb{R}^3))^3$ for $k \in \mathbb{N}$ with $\operatorname{div}(v_k) = 0$ form a linearly independent and complete subset of E. As usual for the Galerkin method, for each $n \in \mathbb{N}$ we first seek a solution $u_n(t,x) = \sum_{k=1}^n d_k^n(t)v_k(x)$ which satisfies the equation if the equation is tested only with the functions v_k for $k = 1,\cdots,n$ and

$$(u_n(0), v_\ell)_E = (u_0, v_\ell)_E \quad \text{for } \ell = 1,\cdots,n. \tag{3.19}$$

Using the fact that $w \in (C_b(\mathbb{R}^4))^3$, $\chi \in C_b^2([0,T)\times\mathbb{R}^3)$ and $f \in L^2(0,T,V^*)$ we obtain a well defined ordinary differential equation for d_n and obtain a unique solution $d^n \in H^1(0,T,\mathbb{R}^n)$.

Now our aim is to obtain uniform in n estimates and then to obtain a convergent subsequence of u_n. Testing the equation satisfied by u_n by itself we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3}|u_n|^2dx + \int_{\mathbb{R}^3}|Du_n|^2dx = \langle f(t), u_n\rangle.$$

By integration in time and using the fact that $f \in L^2(0,T,V^*)$ we obtain $\sup_{n\geq 1}\{\|u_n\|_{L^\infty(0,T,E)} + \|u_n\|_{L^2(0,T,V)}\} < \infty$. So there exists a subsequence n_m and

$$u \in L^\infty(0,T,E) \cap L^2(0,T,V) \tag{3.20}$$

such that

$$u_{n_m} \xrightarrow{w^*} u \text{ in } L^\infty(0,T,E) \text{ and } u_{n_m} \xrightarrow{w} u \text{ in } L^2(0,T,V).$$

Fix $\varphi \in (C_c^\infty(\mathbb{R}^4))^3$ with $\operatorname{div}(\varphi) = 0$. By appropriate approximation of φ and passing to the limit in the equation satisfied by u_{n_m} we obtain

$$\begin{aligned} -\int_{\mathbb{R}^3} u_0^T \varphi(0)dx - \int_0^T \int_{\mathbb{R}^3} u^T \partial_t \varphi dx dt \\ - \int_0^T \int_{\mathbb{R}^3} w_i(u)_j \partial_{x_i} \varphi_j dx dt + \int_0^T \int_{\mathbb{R}^3} Du : D\varphi dx dt = \int_0^T \langle f, \varphi \rangle \, dt. \end{aligned} \tag{3.21}$$

In particular considering (3.21) for $\varphi \in (C_c^\infty((0,T) \times \mathbb{R}^3))^3$ with $\operatorname{div}(\varphi) = 0$ we obtain that

$$\partial_t u \in L^2(0,T,V^*). \tag{3.22}$$

From (3.20) and (3.22) it follows that $u \in C_b([0,T],E)$ (cf. [7]). Finally from this continuity and the equation (3.21) we obtain $u(0) = u_0$.

Step 2. Uniqueness.

If u_1 and u_2 are solutions then denoting $u = u_2 - u_1$ we get

$$\partial_t u + \operatorname{div}(w \otimes u) - \operatorname{div}(Du) = 0 \text{ in } V^* \text{ for a.e. } t \in (0,T)$$

and $u(0) = 0$. Then by similar estimates as done above to show the uniform boundedness of u_n we obtain that $u = 0$.

Step 3. Continuous dependence on the data.

Let u_1 and u_2 be solutions corresponding to the pairs (w_1, f_1) and (w_2, f_2), then denoting $u = u_2 - u_1$ and taking the difference of equations satisfied by u_1 and u_2 we obtain

$$\partial_t u + \operatorname{div}((\frac{w_2 + w_1}{2}) \otimes u) - \operatorname{div}(Du) = (f_2 - f_1) - \operatorname{div}((w_2 - w_1) \otimes (\frac{u_1 + u_2}{2}))$$

in V^* for a.e. $0 < t < T$ and $u(0) = 0$. By multiplying the equation above by u and proceeding as above to get the uniform bounds on u_n we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^3} |u|^2 dx + \int_{\mathbb{R}^3} |Du|^2 dx = \langle (f_2 - f_1)(t), u \rangle + \int_{\mathbb{R}^3} (w_2 - w_1)_i \partial_{x_i} u_j (\frac{u_1 + u_2}{2})_j dx.$$

Now by estimating the right hand side from above and integration in time we obtain

$$\begin{aligned} \|u\|_{L^\infty(0,T,E)} + \|u\|_{L^2(0,T,V)} \leq C\big(\|f_2 - f_1\|_{L^2(0,T,V^*)} \\ + (\|u_1\|_{L^2(0,T,E)} + \|u_2\|_{L^2(0,T,E)})\|w_2 - w_1\|_{(C_b(\mathbb{R}^4))^3}\big). \end{aligned}$$

And by the uniform bounds on u_2 and u_1 in $L^2(0,T,E)$ we obtain the continuity of A.

Step 4. If w and f are axisymmetric then so is u.

Let w and f be axisymmetric and $\theta \in \mathbb{R}$. One may see that $T_\theta u$ is also a solution and thus by uniqueness of the solution we have $u = T_\theta u$. Because this holds for all $\theta \in \mathbb{R}$, u is axisymmetric. □

Lemma 5. *Let $w \in W$ and $\chi = G_\epsilon(w)$ then we have*

$$\int_{\mathbb{R}^3} \frac{\nabla\chi \otimes \nabla\chi}{(|\nabla\chi|^2 + \epsilon^2)^{\frac{1}{2}}} : \nabla w dx \leq -\frac{d}{dt} \int_{\mathbb{R}^3} ((|\nabla\chi|^2 + \epsilon^2)^{\frac{1}{2}} - \epsilon) dx.$$

Proof. Differentiating the equation satisfied by χ with respect the x_i, multiplying by $(|\nabla\chi|^2+\epsilon^2)^{-\frac{1}{2}}\partial_{x_i}\chi$, summing over i and integrating we obtain

$$\int_{\mathbb{R}^3}\big\{\partial_t\partial_{x_i}\chi\partial_{x_i}\chi+\partial_{x_i}w_j\partial_{x_j}\chi\partial_{x_i}\chi+w_j\partial_{x_jx_i}\chi\partial_{x_i}\chi \\ -\epsilon\triangle\partial_{x_i}\chi\partial_{x_i}\chi\big\}(|\nabla\chi|^2+\epsilon^2)^{-\frac{1}{2}}dx=0$$

now separately we evaluate each term in this equation.

We have

$$\frac{\partial_t\partial_{x_i}\chi\partial_{x_i}\chi}{(|\nabla\chi|^2+\epsilon^2)^{\frac{1}{2}}}=\frac{d}{dt}((|\nabla\chi|^2+\epsilon^2)^{\frac{1}{2}}-\epsilon), \tag{3.23}$$

$$\int_{\mathbb{R}^3}\frac{\partial_{x_i}w_j\partial_{x_j}\chi\partial_{x_i}\chi}{(|\nabla\chi|^2+\epsilon^2)^{\frac{1}{2}}}dx=\int_{\mathbb{R}^3}\frac{\nabla\chi\otimes\nabla\chi}{(|\nabla\chi|^2+\epsilon^2)^{\frac{1}{2}}}:\nabla w dx \tag{3.24}$$

and

$$\int_{\mathbb{R}^3}\frac{w_j\partial_{x_jx_i}\chi\partial_{x_i}\chi}{(|\nabla\chi|^2+\epsilon^2)^{\frac{1}{2}}}dx=\int_{\mathbb{R}^3}w_j\partial_{x_j}((|\nabla\chi|^2+\epsilon^2)^{\frac{1}{2}})dx=0 \tag{3.25}$$

so it remains to evaluate the last term involving the laplacian.

We compute

$$\begin{aligned}-\int_{\mathbb{R}^3}\epsilon\triangle\partial_{x_i}\chi\frac{\partial_{x_i}\chi}{(|\nabla\chi|^2+\epsilon^2)^{\frac{1}{2}}}dx&=\epsilon\int_{\mathbb{R}^3}\partial_{x_ix_j}\chi\partial_{x_j}(\frac{\partial_{x_i}\chi}{(|\nabla\chi|^2+\epsilon^2)^{\frac{1}{2}}})dx\\ &=\epsilon\int_{\mathbb{R}^3}\partial_{x_ix_j}\chi\partial_{x_j}(\frac{\partial_{x_i}\chi}{(|\nabla\chi|^2+\epsilon^2)^{\frac{1}{2}}})dx\\ &=\epsilon\int_{\mathbb{R}^3}\partial_{x_ix_j}\chi\{\frac{\partial_{x_ix_j}\chi}{(|\nabla\chi|^2+\epsilon^2)^{\frac{1}{2}}}-\frac{\partial_{x_i}\chi\partial_{x_k}\chi\partial_{x_kx_j}\chi}{(|\nabla\chi|^2+\epsilon^2)^{\frac{3}{2}}}\}dx\\ &=\epsilon\int_{\mathbb{R}^3}\{\frac{\mathrm{tr}(\nabla^2\chi^T\nabla^2\chi)}{(|\nabla\chi|^2+\epsilon^2)^{\frac{1}{2}}}-\frac{\nabla^T\chi\nabla^2\chi^T\nabla^2\chi\nabla\chi}{(|\nabla\chi|^2+\epsilon^2)^{\frac{3}{2}}}\}dx\\ &=\epsilon\int_{\mathbb{R}^3\cap\{\nabla\chi\neq0\}}\frac{|\nabla\chi|^2}{(|\nabla\chi|^2+\epsilon^2)^{\frac{3}{2}}}\Big\{\frac{\epsilon^2}{|\nabla\chi|^2}\mathrm{tr}(\nabla^2\chi^T\nabla^2\chi)\\ &\quad+\big\{\mathrm{tr}(\nabla^2\chi^T\nabla^2\chi)-(\frac{\nabla\chi}{|\nabla\chi|})^T\nabla^2\chi^T\nabla^2\chi(\frac{\nabla\chi}{|\nabla\chi|})\big\}\Big\}dx.\end{aligned} \tag{3.26}$$

Let $\nabla\chi\neq0$. Denoting $A=\nabla^2\chi^T\nabla^2\chi$ we know that A is a positive semi-definite matrix so we have

$$\mathrm{tr}(A)=\sum_{i=1}^3\lambda_i(A)\geq0 \tag{3.27}$$

where $\lambda_i(A)$ for $i=1,2,3$ denote the eigenvalues of A. Also $(\frac{\nabla\chi}{|\nabla\chi|})^TA(\frac{\nabla\chi}{|\nabla\chi|})\leq\max_{i=1,2,3}\lambda_i(A)$ then

$$\mathrm{tr}(A)-(\frac{\nabla\chi}{|\nabla\chi|})^TA(\frac{\nabla\chi}{|\nabla\chi|})\geq\sum_{i=1}^3\lambda_i(A)-\max_{i=1,2,3}\lambda_i(A)\geq0. \tag{3.28}$$

Now from (3.26),(3.27) and (3.28) it follows that

$$-\int_{\mathbb{R}^3}\epsilon\triangle\partial_{x_i}\chi\frac{\partial_{x_i}\chi}{(|\nabla\chi|^2+\epsilon^2)^{\frac{1}{2}}}dx\geq0. \tag{3.29}$$

Finally by (3.23),(3.24),(3.25) and (3.29) the lemma is proved. □

Lemma 6. *Fix $u \in L^2(0,T,E)$ and let $\chi = G_\epsilon(\Psi_\epsilon(u))$ then there exists $C>0$ such that for $0<s<T$ and $\eta>0$ we have*

$$\begin{aligned}(3.30)\quad \int_0^s \left\langle (J(f_\epsilon^{s.t.}(\chi)))(t), u(t)\right\rangle dt \\ \leq -\int_{\mathbb{R}^3}((|\nabla\chi(s)|^2+\epsilon^2)^{\frac{1}{2}}-\epsilon)dx + C|\nabla\chi_0|(\mathbb{R}^3) \\ + \frac{C}{\eta}\|\chi_0\|^2_{L^2(\mathbb{R}^3)} + \eta\|u\|^2_{L^\infty(0,T,E)}.\end{aligned}$$

Proof. One may observe that for $a\in\mathbb{R}$ and $\epsilon\geq 0$ the following inequality holds

$$(3.31)\qquad \frac{a^2}{(a^2+\epsilon^2)^{\frac{1}{2}}} \leq 2((a^2+\epsilon^2)^{\frac{1}{2}}-\epsilon).$$

Let $0<s<T$ then we have

$$\begin{aligned}(3.32)\quad \int_0^s \left\langle J(f_\epsilon^{s.t}(\chi))(t), u(t)\right\rangle dt &= \int_0^T \left\langle J(f_\epsilon^{s.t}(\chi))(t), 1_{(0,s)}(t)u(t)\right\rangle dt \\ = \left\langle f_\epsilon^{s.t}(\chi), 1_{(0,s)}(\cdot)u(\cdot)\right\rangle &= \int_0^T \left\langle \tilde{f}_\epsilon^{s.t}(\chi)(t), \Psi_\epsilon(1_{(0,s)}(\cdot)u(\cdot))(t)\right\rangle dt \\ &= \int_0^s \left\langle \tilde{f}_\epsilon^{s.t}(\chi)(t), \Psi_\epsilon(u(\cdot))(t)\right\rangle dt \\ &+ \int_s^{\min(s+\alpha(\epsilon),T)} \left\langle \tilde{f}_\epsilon^{s.t}(\chi)(t), \Psi_\epsilon(1_{(0,s)}(\cdot)u(\cdot))(t)\right\rangle dt.\end{aligned}$$

To estimate the first term on the right hand side of (3.32) we compute using Lemma 5

$$\begin{aligned}(3.33)\quad \left\langle \tilde{f}_\epsilon^{s.t}(\chi)(t), \Psi_\epsilon(u(\cdot))(t)\right\rangle &= \int_{\mathbb{R}^3} \frac{\nabla\chi\otimes\nabla\chi}{(|\nabla\chi|^2+\epsilon^2)^{\frac{1}{2}}} : \nabla\Psi_\epsilon(u)dx \\ &\leq -\frac{d}{dt}\int_{\mathbb{R}^3}((|\nabla\chi|^2+\epsilon^2)^{\frac{1}{2}}-\epsilon)dx.\end{aligned}$$

Now using (3.33) and (3.9) we estimate the first term on the right hand side of (3.32) as follows

$$\begin{aligned}(3.34)\quad \int_0^s \left\langle \tilde{f}_\epsilon^{s.t}(\chi)(t), \Psi_\epsilon(u(\cdot))(t)\right\rangle dt &\leq -\int_0^s \frac{d}{dt}\Big\{\int_{\mathbb{R}^3}((|\nabla\chi|^2+\epsilon^2)^{\frac{1}{2}}-\epsilon)dx\Big\}dt \\ = -\int_{\mathbb{R}^3}((|\nabla\chi(s)|^2+\epsilon^2)^{\frac{1}{2}}-\epsilon)dx &+ \int_{\mathbb{R}^3}((|\nabla\chi_{0,\epsilon}|^2+\epsilon^2)^{\frac{1}{2}}-\epsilon)dx \\ \leq -\int_{\mathbb{R}^3}((|\nabla\chi(s)|^2+\epsilon^2)^{\frac{1}{2}}-\epsilon)dx &+ \int_{\mathbb{R}^3}|\nabla\chi_{0,\epsilon}|dx \\ &\leq -\int_{\mathbb{R}^3}((|\nabla\chi(s)|^2+\epsilon^2)^{\frac{1}{2}}-\epsilon)dx + C|\nabla\chi_0|(\mathbb{R}^3).\end{aligned}$$

Now we estimate the second term on the right hand side of (3.32)

$$\begin{aligned}(3.35)\quad &\int_s^{\min(s+\alpha(\epsilon),T)} \left\langle \tilde{f}_\epsilon^{s.t}(\chi)(t), \Psi_\epsilon(1_{(0,s)}(\cdot)u(\cdot))(t)\right\rangle dt \\ &= \int_s^{\min(s+\alpha(\epsilon),T)} \int_{\mathbb{R}^3} \frac{\nabla\chi\otimes\nabla\chi}{(|\nabla\chi|^2+\epsilon^2)^{\frac{1}{2}}} : \nabla(\Psi_\epsilon(1_{(0,s)}(\cdot)u(\cdot))(t))dxdt \\ &\le C\int_s^{\min(s+\alpha(\epsilon),T)} \int_{\mathbb{R}^3} |\nabla\chi||\nabla(\Psi_\epsilon(1_{(0,s)}(\cdot)u(\cdot))(t))|dxdt \\ &\le C_1\frac{\epsilon}{\eta}\int_s^{\min(s+\alpha(\epsilon),T)} \int_{\mathbb{R}^3} |\nabla\chi|^2 dxdt \\ &\quad + \frac{\eta}{\epsilon}\int_s^{\min(s+\alpha(\epsilon),T)} \int_{\mathbb{R}^3} |\nabla(\Psi_\epsilon(1_{(0,s)}(\cdot)u(\cdot))(t))|^2 dxdt.\end{aligned}$$

Separately by the Lemma 2 and (3.8) we estimate

$$(3.36)\quad \epsilon\int_s^{\min(s+\alpha(\epsilon),T)} \int_{\mathbb{R}^3} |\nabla\chi|^2 dxdt \le 4\|\chi_{0,\epsilon}\|^2_{L^2(\mathbb{R}^3)} \le 4\|\chi_0\|^2_{L^2(\mathbb{R}^3)}.$$

There exists a constant $C>0$ such that for $\gamma\in(L^2(\mathbb{R}^3))^3$ and $0<\epsilon<1$

$$(3.37)\quad \|\nabla(\tilde{\psi}_{\beta(\epsilon)}\psi_{\beta(\epsilon)}*\gamma)\|^2_{(L^2(\mathbb{R}^3))^{3\times 3}} \le C(1+\frac{1}{\beta^2(\epsilon)})\|\gamma\|^2_{(L^2(\mathbb{R}^3))^3}.$$

Denoting $v=\nabla(P\tilde{\psi}_{\beta(\epsilon)}\psi_{\beta(\epsilon)}*1_{(0,s)}(\cdot)\bar{u}(\cdot))$ and using (3.37) we estimate

$$\begin{aligned}(3.38)\quad &\frac{1}{\epsilon}\int_s^{\min(s+\alpha(\epsilon),T)} \int_{\mathbb{R}^3} |\nabla(\Psi_\epsilon(1_{(0,s)}(\cdot)u(\cdot))(t))|^2 dxdt \\ &\le \frac{1}{\epsilon}\int_s^{\min(s+\alpha(\epsilon),T)} \|\phi_{\alpha(\epsilon)}*v\|^2_{(L^2(\mathbb{R}^3))^{3\times3}}dt \\ &\le \frac{1}{\epsilon}\int_s^{\min(s+\alpha(\epsilon),T)} (\phi_{\alpha(\epsilon)}*\|v\|_{(L^2(\mathbb{R}^3))^{3\times3}})^2 dt \\ &\le \frac{1}{\epsilon}\alpha(\epsilon)\big\|\phi_{\alpha(\epsilon)}*\|v\|_{(L^2(\mathbb{R}^3))^{3\times3}}\big\|^2_{L^\infty(s,\min(s+\alpha(\epsilon),T))} \\ &\le \frac{1}{\epsilon}\alpha(\epsilon)\|\nabla(P\tilde{\psi}_{\beta(\epsilon)}\psi_{\beta(\epsilon)}*u)\|^2_{L^\infty((s-\alpha(\epsilon))^+,s,(L^2(\mathbb{R}^3))^{3\times3})} \\ &\le \frac{C}{\epsilon}\alpha(\epsilon)(1+\frac{1}{\beta^2(\epsilon)})\|u\|^2_{L^\infty((s-\alpha(\epsilon))^+,s,(L^2(\mathbb{R}^3))^3)} \\ &\le \frac{C}{\epsilon}\alpha(\epsilon)(1+\frac{1}{\beta^2(\epsilon)})\|u\|^2_{L^\infty(0,T,E)}.\end{aligned}$$

Now by (3.35), (3.36), (3.38) and the definitions of $\alpha(\epsilon)$ and $\beta(\epsilon)$ we obtain

$$\begin{aligned}(3.39)\quad &\int_s^{\min(s+\alpha(\epsilon),T)} \left\langle \tilde{f}_\epsilon^{s.t}(\chi)(t), \Psi_\epsilon(1_{(0,s)}(\cdot)u(\cdot))(t)\right\rangle dt \\ &\le \frac{C_1}{\eta}\|\chi_0\|^2_{L^2(\mathbb{R}^3)} + C_2\eta\|u\|^2_{L^\infty(0,T,E)}.\end{aligned}$$

Finally from (3.32), (3.34) and (3.39) we obtain (3.30). □

Let us consider the function $S_\epsilon : L^2(0,T,V)\to L^2(0,T,V)$ for $v\in L^2(0,T,V)$ defined by

$$S_\epsilon(v) = A(w, J(f))$$

where

$$w=\Psi_\epsilon(v),\ \chi=G_\epsilon(w) \text{ and } f=f_\epsilon^{s.t}(\chi).$$

Lemma 7. *There exists a constant $C > 0$ independent of ϵ but depending on u_0 and χ_0 such that if for some $0 \le c \le 1$ and $u \in L^2(0, T, V)$ we have $u = cS_\epsilon(u)$ then*

$$(3.40)\quad c \sup_{0<s<T} \int_{\mathbb{R}^3} ((|\nabla\chi(s)|^2 + \epsilon^2)^{\frac{1}{2}} - \epsilon)dx + \sup_{0<s<T} \int_{\mathbb{R}^3} |u(s)|^2 dx + \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 dxdt \le C.$$

Proof. Let $u \in L^2(0, T, V)$, $0 \le c \le 1$ and $u = cS_\epsilon(u)$. Then testing the equation satisfied by $\frac{u}{c}$ by u we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |u|^2 dx + \int_{\mathbb{R}^3} |Du|^2 dx = c\left\langle (Jf_\epsilon^{s.t.}(\chi))(t), u\right\rangle$$

we then integrate in time on $(0, s)$ for some $0 < s < T$ and by Lemma 6, the inequality $c \le 1$ and small $\eta > 0$ we obtain

$$c\int_{\mathbb{R}^3} ((|\nabla\chi(s)|^2 + \epsilon^2)^{\frac{1}{2}} - \epsilon)dx + \int_{\mathbb{R}^3} |u(s)|^2 dx + \frac{1}{2}\int_0^s \int_{\mathbb{R}^3} |\nabla u|^2 dxdt$$
$$\le \int_{\mathbb{R}^3} |u_0|^2 dx + C|\nabla\chi_0|(\mathbb{R}^3) + \frac{C}{\eta}\|\chi_0\|^2_{L^2(\mathbb{R}^3)} + \eta\|u\|^2_{L^\infty(0,T,E)}.$$

And from this by choosing $\eta > 0$ small enough it follows that

$$c \sup_{0<s<T} \int_{\mathbb{R}^3} ((|\nabla\chi(s)|^2 + \epsilon^2)^{\frac{1}{2}} - \epsilon)dx + \sup_{0<s<T} \int_{\mathbb{R}^3} |u(s)|^2 dx + \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 dxdt$$
$$\le C_1\Big\{\int_{\mathbb{R}^3} |u_0|^2 dx + |\nabla\chi_0|(\mathbb{R}^3) + \|\chi_0\|^2_{L^2(\mathbb{R}^3)}\Big\}$$

which proves (3.40). □

Let us define

$$V_{a.s.} = \text{axisymmetric functions in } V.$$

It is clear that $V_{a.s.}$ is a closed subspace of V.

Theorem 3. *There exists $u_\epsilon \in L^2(0, T, V)$ solution to the system of equations* (3.10) *and* (3.11)*. If the initial values u_0 and $\Omega_{1,0}$ are axisymmetric then there exists $u_\epsilon \in L^2(0, T, V_{a.s.})$ solution to the system of equations* (3.10) *and* (3.11)*.*

Proof. By proposition 1 the map $\Psi_\epsilon : L^2(0, T, V) \to W$ is compact (we consider the $(C_b(\mathbb{R}^3))^3$ norm for W), by Lemma 2 the map $G_\epsilon : W \to L^2(0, T, H^1(\mathbb{R}^3))$ is continuous, by Lemma 3 the force term

$$f_\epsilon^{s.t} : L^2(0, T, H^1(\mathbb{R}^3)) \to (L^2(0, T, (L^2(\mathbb{R}^3))^3))^*$$

is continuous, we know that

$$J : (L^2(0, T, (L^2(\mathbb{R}^3))^3))^* \to L^2(0, T, ((L^2(\mathbb{R}^3))^3)^*)$$

is continuous and by Lemma 4 the map $A : W \times L^2(0, T, ((L^2(\mathbb{R}^3))^3)^*) \to L^2(0, T, V)$ is continuous. Thus the map S_ϵ is compact and in particular continuous. By the previous lemma we have that the set

$$X_\epsilon = \Big\{v \in L^2(0, T, V) \;\Big|\; v = cS_\epsilon(v) \text{ for some } 0 \le c \le 1\Big\}$$

is bounded and hence we might apply Schaefer's fixed point theorem to obtain a fixed point $u_\epsilon \in L^2(0, T, V)$, $u_\epsilon = S_\epsilon(u_\epsilon)$ which proves the existence of a solution.

Now let us consider the case when the initial values u_0 and $\Omega_{1,0}$ are axisymmetric. For $v \in L^2(0, T, V_{a.s.})$ by the commuting properties of Ψ_ϵ as discussed in the Lemma 1 we have that $w = \Psi_\epsilon(v)$ is axisymmetric, by the Lemma 2 we have that χ is axisymmetric, by Lemma 3 the force term $f = f_\epsilon^{s.t}(\chi)$ is axisymmetric, it is easy to see that $J(f)$ is axisymmetric and finally by Lemma 4 we have that $S_\epsilon(v)$ is axisymmetric. Thus we have $S_\epsilon : L^2(0, T, V_{a.s.}) \to L^2(0, T, V_{a.s.})$. Now because

$$X_{\epsilon,a.s.} = \Big\{ v \in L^2(0, T, V_{a.s.}) \;\Big|\; v = cS_\epsilon(v) \text{ for some } 0 \le c \le 1 \Big\} \subset X_\epsilon,$$

the set $X_{\epsilon,a.s.}$ is bounded and we might apply Schaefer's fixed point theorem to obtain a fixed point $u_\epsilon \in L^2(0, T, V_{a.s.})$, $u_\epsilon = S_\epsilon(u_\epsilon)$ which proves the existence of an axisymmetric solution. □

4. Existence of Varifold Solution

In the following for each $0 < \epsilon < 1$ we consider the pair $u_\epsilon \in L^2(0, T, V)$ and $\chi_\epsilon \in L^2(0, T, H^1(\mathbb{R}^3))$ a solution to the system of equations (3.10) and (3.11). The existence of these is proved in the theorem 3.

4.1. Uniform bounds on time derivatives.

Lemma 8. *We have* $\sup_{0<\epsilon<1} \|\partial_t \chi_\epsilon\|_{L^2(0,T,H^{-1}(\mathbb{R}^3))} < \infty$.

Proof. We have

$$\partial_t \chi_\epsilon = -\Psi_\epsilon(u_\epsilon) \cdot \nabla \chi_\epsilon + \epsilon \triangle \chi_\epsilon \tag{4.1}$$

in $H^{-1}(\mathbb{R}^3)$ for a.e. $t > 0$.

For the first term on the right hand side of (4.1) we compute for $\varphi \in H^1(\mathbb{R}^3)$ by (3.7)

$$\begin{aligned}\langle -\Psi_\epsilon(u_\epsilon) \cdot \nabla \chi_\epsilon, \varphi \rangle &= -\int_{\mathbb{R}^3} \Psi_\epsilon(u_\epsilon) \cdot \nabla \chi_\epsilon \varphi dx = \int_{\mathbb{R}^3} \Psi_\epsilon(u_\epsilon) \cdot \nabla \varphi \chi_\epsilon dx \\ &\le \int_{\mathbb{R}^3} |\Psi_\epsilon(u_\epsilon)||\nabla \varphi| dx \le \|\Psi_\epsilon(u_\epsilon)\|_{L^2(\mathbb{R}^3)} \|\varphi\|_{H^1(\mathbb{R}^3)}\end{aligned}$$

thus we have

$$\begin{aligned}\| - \Psi_\epsilon(u_\epsilon) \cdot \nabla \chi_\epsilon\|^2_{L^2(0,T,H^{-1}(\mathbb{R}^3))} &\le \int_0^T \|\Psi_\epsilon(u_\epsilon)\|^2_{(L^2(\mathbb{R}^3))^3} dt \\ &\le C \int_0^T \|u_\epsilon\|^2_{(L^2(\mathbb{R}^3))^3} dt.\end{aligned} \tag{4.2}$$

For the second term on the right hand side of (4.1) we have for $\varphi \in H^1(\mathbb{R}^3)$

$$\langle \epsilon \triangle \chi_\epsilon, \varphi \rangle = \epsilon \int_{\mathbb{R}^3} \triangle \chi_\epsilon \varphi dx = -\epsilon \int_{\mathbb{R}^3} \nabla \chi_\epsilon \cdot \nabla \varphi dx \le \epsilon \Big\{ \int_{\mathbb{R}^3} |\nabla \chi_\epsilon|^2 dx \Big\}^{\frac{1}{2}} \|\varphi\|_{H^1(\mathbb{R}^3)}$$

thus by Lemma 2 and (3.8) we have

$$\|\epsilon \triangle \chi_\epsilon\|^2_{L^2(0,T,H^{-1}(\mathbb{R}^3))} \le \epsilon^2 \int_0^T \int_{\mathbb{R}^3} |\nabla \chi_\epsilon|^2 dx dt \le C \|\chi_0\|^2_{L^2(\mathbb{R}^3)}. \tag{4.3}$$

Now by (4.1), (4.2), (4.3) and Lemma 7 the lemma is proved. □

Let

$$\tilde{V} = \text{closure of } \Big\{ v \in (C_c^\infty(\mathbb{R}^3))^3 \;\Big|\; \operatorname{div}(v) = 0 \Big\} \text{ in } (H^3(\mathbb{R}^3))^3.$$

Lemma 9. *We have* $\sup_{0<\epsilon<1} \|\partial_t u_\epsilon\|_{L^1(0,T,\tilde{V}^*)} < \infty$.

Proof. By the equation satisfied by u_ϵ we have

$$\partial_t u_\epsilon = -\operatorname{div}(\Psi_\epsilon(u_\epsilon) \otimes u_\epsilon) + \operatorname{div}(D(u_\epsilon)) + J(f_\epsilon^{s.t.}(\chi_\epsilon)) \tag{4.4}$$

in V^* for a.e. $0 < t < T$.

By Sobolev inequality in $\mathbb{R}^3$, for $u \in H^1(\mathbb{R}^3)$ we have $\|u\|_{L^6(\mathbb{R}^3)} \leq C\|\nabla u\|_{(L^2(\mathbb{R}^3))^3}$ from here it follows that $\|u\|_{L^4(\mathbb{R}^3)} \leq C_1\|u\|_{H^1(\mathbb{R}^3)}$. Now for the first term on the right hand side of (4.4) we have for $\varphi \in \tilde{V}$

$$\begin{aligned}\langle -\operatorname{div}(\Psi_\epsilon(u_\epsilon) \otimes u_\epsilon), \varphi\rangle_{V^*,V} &= \int_{\mathbb{R}^3} (\Psi_\epsilon(u_\epsilon))_i (u_\epsilon)_j \partial_{x_i}\varphi_j dx \\ &\leq C_1 \int_{\mathbb{R}^3} |\Psi_\epsilon(u_\epsilon)||u_\epsilon||\nabla\varphi| dx \\ &\leq C_1\Big\{\int_{\mathbb{R}^3} |\Psi_\epsilon(u_\epsilon)|^4 dx\Big\}^{\frac{1}{4}}\Big\{\int_{\mathbb{R}^3} |u_\epsilon|^4 dx\Big\}^{\frac{1}{4}}\Big\{\int_{\mathbb{R}^3} |\nabla\varphi|^2 dx\Big\}^{\frac{1}{2}} \\ &\leq C_2\|\Psi_\epsilon(u_\epsilon)\|_{(H^1(\mathbb{R}^3))^3}\|u_\epsilon\|_{(H^1(\mathbb{R}^3))^3}\|\varphi\|_{\tilde{V}}.\end{aligned}$$

One may check that there exists a constant $C > 0$ such that for $v \in L^2(0, T, (H^1(\mathbb{R}^3))^3)$ we have

$$\int_0^T \|\Psi_\epsilon(v)\|^2_{(H^1(\mathbb{R}^3))^3} dt \leq C \int_0^T \|v\|^2_{(H^1(\mathbb{R}^3))^3} dt$$

thus

$$\begin{aligned}\| -\operatorname{div}(\Psi_\epsilon(u_\epsilon) \otimes u_\epsilon)\|_{L^1(0,T,\tilde{V}^*)} &\leq C_2\Big\{\int_0^T \|\Psi_\epsilon(u_\epsilon)\|^2_{(H^1(\mathbb{R}^3))^3} dt + \int_0^T \|u_\epsilon\|^2_{(H^1(\mathbb{R}^3))^3} dt\Big\} \\ &\leq C_3 \int_0^T \|u_\epsilon\|^2_{(H^1(\mathbb{R}^3))^3} dt.\end{aligned} \tag{4.5}$$

By the Sobolev inequality there exists a constant $C > 0$ such that

$$\|v\|_{C_b^1(\mathbb{R}^3)} \leq C\|v\|_{H^3(\mathbb{R}^3)} \text{ for all } v \in H^3(\mathbb{R}^3). \tag{4.6}$$

For the second term on the right hand side of (4.4) we have for $\varphi \in \tilde{V}$

$$\langle \operatorname{div}(D(u_\epsilon)), \varphi\rangle_{V^*,V} = -\int_{\mathbb{R}^3} Du_\epsilon : D\varphi dx \leq C\Big\{\int_{\mathbb{R}^3} |\nabla u_\epsilon|^2 dx\Big\}^{\frac{1}{2}}\|\varphi\|_{\tilde{V}}$$

thus

$$\begin{aligned}\|\operatorname{div}(Du_\epsilon)\|_{L^1(0,T,\tilde{V}^*)} &\leq C\int_0^T \Big\{\int_{\mathbb{R}^3} |\nabla u_\epsilon|^2 dx\Big\}^{\frac{1}{2}} dt \\ &\leq CT^{\frac{1}{2}}\Big\{\int_0^T \|u_\epsilon\|^2_{(H^1(\mathbb{R}^3))^3} dt\Big\}^{\frac{1}{2}}.\end{aligned} \tag{4.7}$$

For the third term on the right hand side of (4.4) we have for $\varphi \in (C_c^\infty(\mathbb{R}^4))^3$

$$\int_0^T \left\langle J(f_\epsilon^{s.t.}(\chi_\epsilon))(t), \varphi\right\rangle dt = \left\langle f_\epsilon^{s.t.}(\chi_\epsilon), \varphi\right\rangle = \int_0^T \left\langle \tilde{f}_\epsilon^{s.t.}(\chi_\epsilon(t)), \Psi_\epsilon(\varphi)\right\rangle dt$$
$$= \int_0^T \int_{\mathbb{R}^3} \frac{\nabla\chi_\epsilon \otimes \nabla\chi_\epsilon}{(|\nabla\chi_\epsilon|^2+\epsilon^2)^{\frac{1}{2}}} : \nabla\Psi_\epsilon(\varphi) dx dt$$
$$\le C_1 \int_0^T \int_{\mathbb{R}^3} \frac{|\nabla\chi_\epsilon|^2}{(|\nabla\chi_\epsilon|^2+\epsilon^2)^{\frac{1}{2}}} |\nabla\Psi_\epsilon(\varphi)| dx dt$$
$$\le C_2 \int_0^T \int_{\mathbb{R}^3} ((|\nabla\chi_\epsilon|^2+\epsilon^2)^{\frac{1}{2}} - \epsilon)|\nabla\Psi_\epsilon(\varphi)| dx dt$$
$$\le C_2 \Big\{ \sup_{0<t<T} \int_{\mathbb{R}^3} ((|\nabla\chi_\epsilon|^2+\epsilon^2)^{\frac{1}{2}} - \epsilon) dx \Big\} \int_0^T \|\nabla\Psi_\epsilon(\varphi)\|_{(C(\mathbb{R}^3))^{3\times 3}} dt.$$

denoting $v = \nabla P(\tilde{\psi}_{\beta(\epsilon)}\psi_{\beta(\epsilon)} * \bar{\varphi})$ using the Minkowski inequality once and then the Sobolev inequality (4.6), separately we estimate

$$\int_0^T \|\nabla\Psi_\epsilon(\varphi)\|_{(C(\mathbb{R}^3))^{3\times 3}} dt \le \int_0^T \phi_{\alpha(\epsilon)} * \|v\|_{(C(\mathbb{R}^3))^{3\times 3}} dt$$
$$\le \int_0^T \|v\|_{(C(\mathbb{R}^3))^{3\times 3}} dt \le C_1 \int_0^T \|P(\tilde{\psi}_{\beta(\epsilon)}\psi_{\beta(\epsilon)} * \varphi)\|_{(H^3(\mathbb{R}^3))^3} dt$$
$$\le C_2 \int_0^T \|\tilde{\psi}_{\beta(\epsilon)}\psi_{\beta(\epsilon)} * \varphi\|_{(H^3(\mathbb{R}^3))^3} dt \le C_3 \|\varphi\|_{L^1(0,T,(H^3(\mathbb{R}^3))^3)}$$

so we have

$$\int_0^T \left\langle J(f_\epsilon^{s.t.}(\chi_\epsilon))(t), \varphi\right\rangle dt \le C\Big\{ \sup_{0<t<T} \int_{\mathbb{R}^3} ((|\nabla\chi_\epsilon|^2+\epsilon^2)^{\frac{1}{2}} - \epsilon) dx \Big\} \|\varphi\|_{L^1(0,T,\tilde{V})}.$$

As mentioned in Section 2, because V^* is a Hilbert space it is in particular a reflexive Banach space and hence $L^\infty(0,T,\tilde{V}^*) = (L^1(0,T,\tilde{V}))^*$ so we have

$$\|J(f_\epsilon^{s.t.}(\chi_\epsilon))\|_{L^1(0,T,\tilde{V}^*)} \le T\|J(f_\epsilon^{s.t.}(\chi_\epsilon))\|_{L^\infty(0,T,\tilde{V}^*)}$$
$$\le C_1 T \|J(f_\epsilon^{s.t.}(\chi_\epsilon))\|_{(L^1(0,T,\tilde{V}))^*} \le C_2 T \sup_{0<t<T} \int_{\mathbb{R}^3} ((|\nabla\chi_\epsilon|^2+\epsilon^2)^{\frac{1}{2}} - \epsilon) dx. \tag{4.8}$$

Now by (4.4), (4.5), (4.7), (4.8) and Lemma 7 the lemma is proved. □

4.2. Let us denote by $\mathcal{M}(\mathbb{R}^d)$ the space of bounded Radon-Borel measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

Lemma 10. *Let for some* $\Lambda > 0$

$$\|\nu_k\|_{L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d))} \le \Lambda, \ \ \forall k \in \mathbb{N} \tag{4.9}$$

then there exists a subsequence k_n *and* $\nu \in L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d))$ *such that*

$$\int_0^T \int_{\mathbb{R}^d} \varphi \nu_{k_n}(t)(dx) dt \to \int_0^T \int_{\mathbb{R}^d} \varphi(t,x)\nu(t)(dx) dt, \ \forall \varphi \in L^1(0,T,C_0(\mathbb{R}^d)) \tag{4.10}$$

and

$$\|\nu\|_{L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d))} \le \liminf_{k\to\infty} \|\nu_k\|_{L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d))}.$$

Proof. **Step 1.** If $m \in \mathbb{N}$ such that $m > \frac{d}{2}$ then it follows from Sobolev embedding theorem that we have the continuous, dense and injective embedding

$$H^m(\mathbb{R}^d) \hookrightarrow C_0(\mathbb{R}^d) \tag{4.11}$$

and by the Ritz representation theorem it follows the continuous, dense and injective embedding

$$\mathcal{M}(\mathbb{R}^d) = (C_0(\mathbb{R}^d))' \hookrightarrow H^{-m}(\mathbb{R}^d). \tag{4.12}$$

Step 2. In this step our aim is to show the continuous embedding

$$L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d)) \hookrightarrow L^\infty_{w^*}(0,T,H^{-m}(\mathbb{R}^d)) \tag{4.13}$$

holds.

Let $\gamma \in L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d))$. By (4.12) we have that $\gamma : (0,T) \to H^{-3}(\mathbb{R}^d)$.

To show that γ with values in $H^{-3}(\mathbb{R}^d)$ is w^* λ-measurable let $\phi \in H^m(\mathbb{R}^d)$ then

$$\langle \gamma(t), \phi\rangle_{H^{-m}(\mathbb{R}^d),H^m(\mathbb{R}^d)} = \langle \gamma(t), \phi\rangle_{\mathcal{M}(\mathbb{R}^d),C_0(\mathbb{R}^d)} \tag{4.14}$$

and because γ with values in $\mathcal{M}(\mathbb{R}^d)$ is w^* λ-measurable the function on the right hand side of (4.14) is measurable and hence so is the function on the left hand side, which proves the desired property.

Now from the separability of $H^m(\mathbb{R}^d)$ it follows that $\|\gamma(t)\|_{H^{-m}(\mathbb{R}^d)}$ as a function of t is measurable.

Finally by (4.12) we have that for a.e. $0 < t < T$ we have

$$\|\gamma(t)\|_{H^{-m}(\mathbb{R}^d)} \leq C\|\gamma(t)\|_{\mathcal{M}(\mathbb{R}^d)} \leq C\|\gamma\|_{L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d))}$$

which shows that $\|\gamma\|_{L^\infty_{w^*}(0,T,H^{-m}(\mathbb{R}^d))} \leq C\|\gamma\|_{L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d))}$ and this proves (4.13).

Step 3. In this step our aim is to show that

$$L^\infty_{w^*}(0,T,H^{-m}(\mathbb{R}^d)) = L^\infty(0,T,H^{-m}(\mathbb{R}^d)). \tag{4.15}$$

It is obvious that by the identity map $L^\infty(0,T,H^{-m}(\mathbb{R}^d)) \hookrightarrow L^\infty_{w^*}(0,T,H^{-m}(\mathbb{R}^d))$ continuously as λ-measurability is stronger than w^* λ-measurability. So it remains to show that by the identity map we have the continuous embedding

$$L^\infty_{w^*}(0,T,H^{-m}(\mathbb{R}^d)) \hookrightarrow L^\infty(0,T,H^{-m}(\mathbb{R}^d)). \tag{4.16}$$

Now let $\gamma \in L^\infty_{w^*}(0,T,H^{-m}(\mathbb{R}^d))$ and we should show that γ is λ-measurable.

Because $H^{-m}(\mathbb{R}^d)$ is separable we have that γ is λ-essentially separably valued.

Because $H^m(\mathbb{R}^d)$ is a Hilbert space in particular it is a reflexive space and hence by w^* λ-measurability of γ, γ is weakly λ-measurable.

Hence by Pettis measurability theorem γ is λ-measurable and this proves (4.16) which in turn proves (4.15).

Step 4. In this step our aim is to obtain a w^* convergent subsequence of ν_k in $L^\infty(0,T,H^{-m}(\mathbb{R}^d))$.

There exists a subsequence k_n such that the limit $\lim_{n\to\infty} \|\nu_{k_n}\|_{L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d))}$ exists and

$$\lim_{n\to\infty} \|\nu_{k_n}\|_{L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d))} = \liminf_{k\to\infty} \|\nu_k\|_{L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d))}. \tag{4.17}$$

In the following for ease of notation let us denote $\Lambda' = \liminf_{k\to\infty} \|\nu_k\|_{L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d))}$.

Because $H^{-m}(\mathbb{R}^d)$ is a Hilbert space it has the Radon-Nykodym property and thus we have $L^\infty(0,T,H^{-m}(\mathbb{R}^d)) = (L^1(0,T,H^m(\mathbb{R}^d)))'$.

By (4.9) and (4.13) we have that ν_{k_n} is uniformly bounded in $L^\infty_{w^*}(0,T,H^{-m}(\mathbb{R}^d))$. By the previous step we obtain that ν_k is uniformly bounded in $L^\infty(0,T,H^{-m}(\mathbb{R}^d))$. Now by Anaoglu theorem there exists a $\nu \in L^\infty(0,T,H^{-m}(\mathbb{R}^d))$ and a subsequence k_{n_ℓ} such that $\nu_{k_{n_\ell}}$, w^* converges to ν in $L^\infty(0,T,H^{-m}(\mathbb{R}^d))$, i.e.

$$\int_0^T \left\langle \nu_{k_{n_\ell}}(t), \phi(t)\right\rangle_{H^{-m}(\mathbb{R}^d),H^m(\mathbb{R}^d)} dt \to \int_0^T \langle \nu(t), \phi(t)\rangle_{H^{-m}(\mathbb{R}^d),H^m(\mathbb{R}^d)} dt \tag{4.18}$$

for all $\phi \in L^1(0,T,H^m(\mathbb{R}^d))$ as $\ell \to \infty$.

Step 5. In this step our aim is to show that $\nu \in L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d))$ with

$$\|\nu\|_{L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d))} \le \Lambda'. \tag{4.19}$$

Let $\phi \in L^1(0,T,H^m(\mathbb{R}^d))$ then by the uniform bound (4.9) we have that

$$\int_0^T \Big\langle \nu_{k_{n_\ell}}(t), \phi(t) \Big\rangle_{H^{-m}(\mathbb{R}^d),H^m(\mathbb{R}^d)} dt = \int_0^T \Big\langle \nu_{k_{n_\ell}}(t), \phi(t) \Big\rangle_{\mathcal{M}(\mathbb{R}^d),C_0(\mathbb{R}^d)} dt$$
$$\le \|\nu_{k_{n_\ell}}\|_{L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d))} \|\phi\|_{L^1(0,T,C_0(\mathbb{R}^d))}$$

by (4.17) and (4.18) passing to the limit in the inequality above we obtain

$$\int_0^T \langle \nu(t), \phi(t) \rangle_{H^{-m}(\mathbb{R}^d),H^m(\mathbb{R}^d)} dt \le \Lambda' \|\phi\|_{L^1(0,T,C_0(\mathbb{R}^d))}$$

and by Lebesgue differentiation theorem we obtain that

$$\langle \nu(t), \varphi \rangle_{H^{-m}(\mathbb{R}^d),H^m(\mathbb{R}^d)} \le \Lambda' \|\varphi\|_{C_0(\mathbb{R}^d)} \text{ for a.e. } t \in (0,T) \text{ and } \varphi \in H^m(\mathbb{R}^d). \tag{4.20}$$

From (4.20) and the density of the embedding (4.11) it follows that $\nu(t) \in \mathcal{M}(\mathbb{R}^d)$ for a.e. $t \in (0,T)$ and

$$\|\nu(t)\|_{\mathcal{M}(\mathbb{R}^d)} \le \Lambda' \text{ for a.e. } t \in (0,T). \tag{4.21}$$

Now let us show that ν with a.e. values in $\mathcal{M}(\mathbb{R}^d)$ is w^* λ-measurable. Let $\varphi \in C_0(\mathbb{R}^d)$ and $\varphi_n \in H^m(\mathbb{R}^d)$ such that $\varphi_n \to \varphi$ in $C_0(\mathbb{R}^d)$. Then we have

$$\langle \nu(t), \varphi \rangle_{\mathcal{M}(\mathbb{R}^d),C_0(\mathbb{R}^d)} = \lim_{n\to\infty} \langle \nu(t), \varphi_n \rangle_{H^{-m}(\mathbb{R}^d),H^m(\mathbb{R}^d)}$$

and because $\nu \in L^\infty(0,T,H^{-m}(\mathbb{R}^d))$ and $\varphi_n \in H^m(\mathbb{R}^d)$ each of the functions (as function of t) on the right hand side is measurable and thus left hand side as the limit of a sequence is measurable.

From separability of $C_0(\mathbb{R}^d)$ it follows that $\|\nu(t)\|_{\mathcal{M}(\mathbb{R}^d)}$ as a function of t is measurable. These measurability results together with (4.21) prove (4.19).

Step 6. In this step our aim is to prove the continuous and dense embedding

$$L^1(0,T,H^m(\mathbb{R}^d)) \hookrightarrow L^1(0,T,C_0(\mathbb{R}^d)). \tag{4.22}$$

Let $\phi \in L^1(0,T,H^m(\mathbb{R}^d))$ then by the embedding (4.11) we have that $\phi : (0,T) \to C_0(\mathbb{R}^d)$. By the λ-measurability of ϕ with values in $H^m(\mathbb{R}^d)$ there exists a sequence of simple functions s_n with values in $H^m(\mathbb{R}^d)$ such that $\|s_n(t) - \phi(t)\|_{H^m(\mathbb{R}^d)} \to 0$ for a.e. $t \in (0,T)$ hence by (4.11), also $\|s_n(t) - \phi(t)\|_{C_0(\mathbb{R}^d)} \to 0$ for a.e. $t \in (0,T)$. Thus ϕ is also λ-measurable with values in $C_0(\mathbb{R}^d)$.

By the embedding (4.11) it is easy to see that the embedding (4.22) is continuous.

Now let us show the density of the embedding (4.22).

Let $\phi \in L^1(0,T,C_0(\mathbb{R}^d))$. Let us show that there exists a sequence of simple functions $s_k \in L^1(0,T,H^m(\mathbb{R}^d))$ such that $s_k \to \phi$ in $L^1(0,T,C_0(\mathbb{R}^d))$.

By the separability of $H^m(\mathbb{R}^d)$ and its density by (4.11) in $C_0(\mathbb{R}^d)$ there exists a countable sequence $g_n \in H^m(\mathbb{R}^d)$ which is dense in $C_0(\mathbb{R}^d)$.

Let us define for $k,n \in \mathbb{N}$

$$E_{k,n} = \big\{ t \in (0,T) \;\big|\; \|\phi(t) - g_n\|_{C_0(\mathbb{R}^d)} < k^{-1} \big\}.$$

By the λ measurability of ϕ we have that $\|\phi(t) - g_n\|_{C_0(\mathbb{R}^d)}$ as a function of t is measurable, thus $E_{k,n}$ is Borel measurable. By the density of the sequence g_n in $C_0(\mathbb{R}^d)$ we have $(0,T) = \cup_{n\in\mathbb{N}} E_{k,n}$. Let us define $G_{k,1} = E_{k,1}$ and for $n \in \mathbb{N}\backslash\{1\}$, $G_{k,n} = E_{k,n} \backslash (\cup_{i=1}^{n-1} E_{k,i})$. Then for $n \in \mathbb{N}$, $G_{k,n}$ are disjoint and cover $(0,T)$. Let us define the simple function $p_{k,n} = \sum_{i=1}^n 1_{G_{k,i}} g_i$. We have the estimate

$$\|\phi - p_{k,n}\|_{L^1(0,T,C_0(\mathbb{R}^d))} \le \frac{T}{k} + \int_{(0,T)\backslash \cup_{i=1}^n G_{k,i}} \|\phi(t)\|_{C_0(\mathbb{R}^d)} dt. \tag{4.23}$$

Let $A_{k,n} = (0,T)\setminus \cup_{i=1}^n G_{k,i}$ then $A_{k,n+1} \subset A_{k,n}$ and $\emptyset = \cap_{n\in\mathbb{N}} A_{k,n}$. Because $\|\phi(t)\|_{C_0(\mathbb{R}^d)} \in L^1(0,T)$ by the absolute continuity of Leabesgue integral we have $\lim_{n\to\infty}\int_{A_{k,n}} \|\phi(t)\|_{C_0(\mathbb{R}^d)} dt = 0$. From this convergence and (4.23), taking for each k the $n = n_k$ sufficiently large we obtain $p_{k,n_k} \to \phi$ in $L^1(0,T,C_0(\mathbb{R}^d))$.

Step 7. In this step our aim is to prove (4.10). Let $\phi \in L^1(0,T,C_0(\mathbb{R}^d))$ then by the previous step there exists a sequence $\phi_q \in L^1(0,T,H^m(\mathbb{R}^d))$ such that $\phi_q \to \phi$ in $L^1(0,T,C_0(\mathbb{R}^d))$. By (4.9) we have

$$\Big|\int_0^T\int_{\mathbb{R}^d} \phi\nu_{k_{n_\ell}}(t)(dx)dt - \int_0^T\int_{\mathbb{R}^d}\phi_q\nu_{k_{n_\ell}}(t)(dx)dt\Big| \le \Lambda\|\phi-\phi_q\|_{L^1(0,T,C_0(\mathbb{R}^d))}$$

from which it follows that the left hand side converges to 0 as $q\to\infty$ uniformly with respect to ℓ. We write

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d} \phi(t)(x)\nu_{k_{n_\ell}}(t)(dx)dt - \int_0^T\int_{\mathbb{R}^d}\phi(t)(x)\nu(t)(dx)dt \\ = \int_0^T\int_{\mathbb{R}^d} \phi\nu_{k_{n_\ell}}(t)(dx)dt - \int_0^T\int_{\mathbb{R}^d}\phi_q\nu_{k_{n_\ell}}(t)(dx)dt \\ + \int_0^T\int_{\mathbb{R}^d} \phi_q\nu_{k_{n_\ell}}(t)(dx)dt - \int_0^T\int_{\mathbb{R}^d}\phi_q\nu(t)(dx)dt \\ + \int_0^T\int_{\mathbb{R}^d} \phi_q\nu(t)(dx)dt - \int_0^T\int_{\mathbb{R}^d}\phi\nu(t)(dx)dt\end{aligned}$$

using the uniform convergence described above first by choosing q large enough we can make the first and third terms on the right hand side small then fixing q and choosing ℓ large enough using (4.18) we make the second term small. ☐

Lemma 11. *Let there exists $\Lambda > 0$ and for each $r \in \mathbb{N}$, $\Lambda_r > 0$ such that*

$$\|\nu_k\|_{L^\infty_{w^*}(0,T,\mathcal{M}(\overline{B_r}))} \le \frac{\Lambda_r}{k} + \Lambda, \quad \forall k \in \mathbb{N} \tag{4.24}$$

then there exists a subsequence k_n and $\nu \in L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d))$ such that for all compact $K \subset \mathbb{R}^d$

$$\int_0^T\int_{\mathbb{R}^d} \phi\nu_{k_n}(t)(dx)dt \to \int_0^T\int_{\mathbb{R}^d}\phi(t,x)\nu(t)(dx)dt, \quad \forall\phi\in L^1(0,T,C_c(K)) \tag{4.25}$$

and $\|\nu\|_{L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d))} \le \Lambda$.

If for an open set $U \subset \mathbb{R}^d$ and each $k \in \mathbb{N}$ we have $\nu_k(t)(U) = 0$ for a.e. $t \in (0,T)$ then $\nu(t)(U) = 0$ for a.e. $t \in (0,T)$.

Proof. **Step 1.** Defining β_k.

Let for $k \in \mathbb{N}$, $r_k \in \mathbb{N}$ be a non-decreasing sequence such that $r_k \to \infty$ and

$$\frac{\Lambda_{r_k}}{k} \le \Lambda_1 \text{ and } \frac{\Lambda_{r_k}}{k} \to 0 \text{ as } k\to\infty.$$

Let us define $\beta_k : (0,T) \to \mathcal{M}(\mathbb{R}^d)$ by

$$\beta_k = \nu_k|_{\overline{B_{r_k}}}.$$

Step 2. $\beta_k \in L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d))$ and uniformly bounded.

Let us show that β_k is w^* λ-measurable. Let $\phi \in C_0(\mathbb{R}^d)$ then $\phi \in C(\overline{B_{r_k}})$ and because $\nu_k \in L^\infty_{w^*}(0,T,\mathcal{M}(\overline{B_{r_k}}))$ we have

$$\langle\beta_k(t),\phi\rangle = \Big\langle \nu_k(t)|_{\overline{B_{r_k}}}, \phi(\cdot)\Big\rangle$$

is λ-measurable. Thus β_k is w^* λ-measurable. Now from the separability of $C_0(\mathbb{R}^d)$ it follows that $\|\beta_k(t)\|_{\mathcal{M}(\mathbb{R}^d)}$ is measurable.

Because of the bound (4.24) we have

$$\|\beta_k\|_{L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d))} \le \frac{\Lambda_{r_k}}{k} + \Lambda \le \Lambda_1 + \Lambda. \tag{4.26}$$

Step 3. Apply the previous lemma to the sequence β_k.

By the uniform bound (4.26) and the previous lemma there exists a subsequence k_n and $\nu \in L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d))$ such that

$$\int_0^T \int_{\mathbb{R}^d} \phi \beta_{k_n}(t)(dx)dt \to \int_0^T \int_{\mathbb{R}^d} \phi \nu(t)(dx)dt,\ \forall \phi \in L^1(0,T,C_0(\mathbb{R}^d)) \tag{4.27}$$

and

$$\|\nu\|_{L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d))} \le \liminf_{k\to\infty} \|\beta_k\|_{L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^d))} \le \Lambda.$$

Step 4. Proving (4.25).

Let $K \subset \mathbb{R}^d$ be a compact set and $\phi \in L^1(0,T,C_c(K))$ then for n large enough such that $K \subset \overline{B_{r_{k_n}}}$ by (4.27) we have

$$\int_0^T \int_{\mathbb{R}^d} \phi \nu_{k_n}(t)(dx)dt = \int_0^T \int_{\mathbb{R}^d} \phi \beta_{k_n}(t)(dx)dt \to \int_0^T \int_{\mathbb{R}^d} \phi \nu(t)(dx)dt.$$

Step 5. Proof of the last claim.

Let K a compact subset of U. For $\phi \in L^1(0,T,C_c(K))$ we have for each $k \in \mathbb{N}$, $\int_0^T \int_{\mathbb{R}^d} \phi \nu_{k_n}(t)(dx)dt = 0$ hence by (4.25) we have $\int_0^T \int_{\mathbb{R}^d} \phi \nu(t)(dx)dt = 0$.

Thus for all compact subsets $K \subset U$ we have

$$\int_0^T \int_{\mathbb{R}^d} \phi \nu(t)(dx)dt = 0 \text{ for all } \phi \in L^1(0,T,C_c(K)).$$

Let the sequence $\psi_n \in C_c(U)$ be dense in $C_0(\overline{U})$. Choosing $\phi = 1_I(t)\psi_n(x)$ for some interval $I \subset (0,T)$ we have $\int_I \int_{\mathbb{R}^d} \psi_n \nu(t)(dx)dt = 0$. Now because this holds for all intervals $I \subset (0,T)$ we obtain that there exists $E_n \in \mathcal{B}((0,T))$ with $\lambda(E_n) = 0$ such that

$$\int_{\mathbb{R}^d} \psi_n(x)\nu(t)(dx) = 0 \text{ for } t \in (0,T)\backslash E_n.$$

Let $E = \cup_{n\in\mathbb{N}} E_n$ then $\lambda(E) = 0$ and

$$\int_{\mathbb{R}^d} \psi_n(x)\nu(t)(dx) = 0 \text{ for } t \in (0,T)\backslash E \text{ and } n \in \mathbb{N}$$

hence for $t \in (0,T)\backslash E$ we have $\nu(t)(U) = 0$. □

4.3. Existence. In the proof of theorem 1 we will need the compactness result of Aubin. For ease of reading we bring here the statement of this result as it is in [10] for the special cases that we will need.

Theorem 4 (Aubin's compactness result)**.** *Let X_1, X_2 and X_3 be normed linear spaces and $T > 0$. Let $f : X_1 \to X_2$ be linear and compact and $g : X_2 \to X_3$ be linear, bounded and injective. If for $n \in \mathbb{N}$, $v_n \in L^2(0,T,X_1)$ and $v \in L^2(0,T,X_1)$ such that $v_n \to v$ weakly in $L^2(0,T,X_1)$ and*

$$\frac{d}{dt} g(f(v_n)) \text{ uniformly bounded in } L^1(0,T,X_3)$$

then we have $f(v_n) \to f(v)$ in $L^2(0,T,X_2)$.

Proof of theorem 1. **Step 1.** Passing to the limit in the transport equation and obtaining (1.11).

By (3.14) and (3.8) we have

$$\|\chi_\epsilon\|_{L^2(0,T,L^2(\mathbb{R}^3))} \le C\|\chi_0\|_{L^2(\mathbb{R}^3)}. \tag{4.28}$$

Now by (4.28) and Lemma 7 there exists a sequence ϵ_k and

$$\chi \in L^2(0,T,L^2(\mathbb{R}^3)),\ u \in L^2(0,T,V)$$

such that

$$\chi_{\epsilon_k} \to \chi \text{ weakly in } L^2(0,T,L^2(\mathbb{R}^3)) \tag{4.29}$$

and

$$u_{\epsilon_k} \to u \text{ weakly in } L^2(0,T,V). \tag{4.30}$$

Let $\zeta \in C^1(\mathbb{R}^3)$, $\zeta > 0$ and $\zeta(x) \to 0$ as $|x| \to \infty$.

Let $f : L^2(\mathbb{R}^3;\zeta) \to H^{-1}(\mathbb{R}^3)$ be defined for $\chi \in L^2(\mathbb{R}^3;\zeta)$ by $\langle f(\chi), \varphi\rangle_{H^{-1}(\mathbb{R}^3),H^1(\mathbb{R}^3)} = (\chi,\varphi)_{L^2(\mathbb{R}^3;\zeta)}$ for $\varphi \in H^1(\mathbb{R}^3)$.

We claim that

$$f(\chi_{\epsilon_k}) \to f(\chi) \text{ in } L^2(0,T,H^{-1}(\mathbb{R}^3)). \tag{4.31}$$

To apply Aubin's theorem let us choose $X_1 = L^2(\mathbb{R}^3;\zeta)$, $X_2 = X_3 = H^{-1}(\mathbb{R}^3)$.

It is easy to see that

$$f = f_1^* \circ f_2^{-1}$$

where $f_2 : (L^2(\mathbb{R}^3;\zeta))^* \to L^2(\mathbb{R}^3;\zeta)$ is the Riesz representation function which has a continuous inverse and $f_1 : H^1(\mathbb{R}^3) \to L^2(\mathbb{R}^3;\zeta)$ is the natural embedding function. It is easy to see that $H^1(\mathbb{R}^3)$ is compactly embedded in $L^2(\mathbb{R}^3;\zeta)$, i.e. f_1 is compact and thus f_1^* is compact. Therefore f is a compact operator.

Also because $X_2 = X_3$, X_2 is injectively embedded by the identity map in X_3.

By (4.28), χ_{ϵ_k} is uniformly bounded in $L^2(0,T,L^2(\mathbb{R}^3;\zeta))$ and by Lemma 8, $\partial_t\chi_\epsilon$ is uniformly bounded in $L^2(0,T,H^{-1}(\mathbb{R}^3))$ and therefore in $L^1(0,T,H^{-1}(\mathbb{R}^3))$. Thus by the Aubin theorem there exists a subsequence of ϵ_k which we again denote by ϵ_k and $\gamma \in L^2(0,T,H^{-1}(\mathbb{R}^3))$ such that

$$f(\chi_{\epsilon_k}) \to \gamma \text{ in } L^2(0,T,H^{-1}(\mathbb{R}^3)). \tag{4.32}$$

Now we have for $\varphi \in L^2(0,T,H^1(\mathbb{R}^3))$

$$\begin{aligned}\langle \gamma,\varphi\rangle = \lim_{k\to\infty}\int_0^T \langle f(\chi_{\epsilon_k}),\varphi(t)\rangle_{(H^1(\mathbb{R}^3))^*,H^1(\mathbb{R}^3)}\,dt &= \lim_{k\to\infty}\int_0^T\int_{\mathbb{R}^3}\chi_{\epsilon_k}\varphi(t)\zeta dxdt \\ = \int_0^T\int_{\mathbb{R}^3}\chi\varphi(t)\zeta dxdt &= \int_0^T \langle f(\chi(t)),\varphi(t)\rangle_{H^{-1}(\mathbb{R}^3),H^1(\mathbb{R}^3)}\,dt\end{aligned}$$

hence $\gamma = f(\chi)$ and (4.31) follows from (4.32).

We claim that

$$\Psi_{\epsilon_k}(u_{\epsilon_k}) \to u \text{ weakly in } L^2(0,T,(H^1(\mathbb{R}^3))^3). \tag{4.33}$$

To prove (4.33) let $v \in L^2(0,T,(H^1(\mathbb{R}^3))^3)$ then

$$(v,\Psi_{\epsilon_k}(u_{\epsilon_k}))_{L^2(0,T,(H^1(\mathbb{R}^3))^3)} = (\Psi^*_{\epsilon_k}(v),u_{\epsilon_k})_{L^2(0,T,(H^1(\mathbb{R}^3))^3)}$$

now as mentioned in Lemma 1 because $\Psi^*_{\epsilon_k}(v)$ converges strongly in $(L^2(0,T,(H^1(\mathbb{R}^3))^3))^*$ to P^*v we obtain

$$\begin{aligned}(\Psi^*_{\epsilon_k}(v),u_{\epsilon_k})_{L^2(0,T,(H^1(\mathbb{R}^3))^3)} &\to (P^*v,u)_{L^2(0,T,(H^1(\mathbb{R}^3))^3)} \\ &= (v,Pu)_{L^2(0,T,(H^1(\mathbb{R}^3))^3)} = (v,u)_{L^2(0,T,(H^1(\mathbb{R}^3))^3)}\end{aligned}$$

which proves (4.33).

By (3.12) for $\varphi \in C_c^\infty((-\infty,T)\times\mathbb{R}^3)$ we have

$$(4.34)\quad -\int_{\mathbb{R}^3}\chi_{0,\epsilon}\varphi(0,x)dx - \int_0^\infty\int_{\mathbb{R}^3}\chi_\epsilon\partial_t\varphi dxdt \\ - \int_0^\infty\int_{\mathbb{R}^3}\chi_\epsilon\Psi_\epsilon(u_\epsilon)\cdot\nabla\varphi dxdt + \epsilon\int_0^\infty\int_{\mathbb{R}^3}\nabla\chi_\epsilon\cdot\nabla\varphi dxdt = 0.$$

Our aim is now for the sequence ϵ_k to pass to the limit in (4.34). We pass to the limit in the first term in (4.34) using the convergence $\chi_{0,\epsilon_k}\to\chi_0$ in $L^2(\mathbb{R}^3)$. In the second term in (4.34) we pass to the limit using the weak convergence (4.29). To pass to the limit in the third term in (4.34), by (4.31) and (4.33) we have

$$\int_0^T\int_{\mathbb{R}^3}\chi_{\epsilon_k}\Psi_{\epsilon_k}(u_{\epsilon_k})\cdot\nabla\varphi dxdt = \int_0^T\int_{\mathbb{R}^3}\chi_{\epsilon_k}\Psi_{\epsilon_k}(u_{\epsilon_k})\cdot(\zeta^{-1}\nabla\varphi)\zeta dxdt \\ = \int_0^T\left\langle f(\chi_{\epsilon_k}),\Psi_{\epsilon_k}(u_{\epsilon_k})\cdot(\zeta^{-1}\nabla\varphi)\right\rangle_{H^{-1}(\mathbb{R}^3),H^1(\mathbb{R}^3)}dt \\ \to \int_0^T\left\langle f(\chi),u\cdot(\zeta^{-1}\nabla\varphi)\right\rangle_{H^{1-}(\mathbb{R}^3),H^1(\mathbb{R}^3)}dt \\ = \int_0^T\int_{\mathbb{R}^3}\chi u\cdot(\zeta^{-1}\nabla\varphi)\zeta dxdt = \int_0^T\int_{\mathbb{R}^3}\chi u\cdot\nabla\varphi dxdt.$$

For the fourth term in (4.34) by (3.14) and (3.8) we have

$$\big|\epsilon\int_0^T\int_{\mathbb{R}^3}\nabla\chi_\epsilon\cdot\nabla\varphi dxdt\big| \le \sqrt{\epsilon}\big\{\epsilon\int_0^T\int_{\mathbb{R}^3}|\nabla\chi_\epsilon|^2dxdt\big\}^{\frac12}\big\{\int_0^T\int_{\mathbb{R}^3}|\nabla\varphi|^2dxdt\big\}^{\frac12} \\ \le C\sqrt{\epsilon}\|\chi_{0,\epsilon}\|_{L^2(\mathbb{R}^3)}\big\{\int_0^T\int_{\mathbb{R}^3}|\nabla\varphi|^2dxdt\big\}^{\frac12} \\ \le C\sqrt{\epsilon}\|\chi_0\|_{L^2(\mathbb{R}^3)}\big\{\int_0^T\int_{\mathbb{R}^3}|\nabla\varphi|^2dxdt\big\}^{\frac12}\to 0.$$

Thus by passing to the limit for $\epsilon=\epsilon_k$ in (4.34) we obtain for all $\varphi\in C_c^\infty((-\infty,T)\times\mathbb{R}^3)$

$$-\int_{\mathbb{R}^3}\chi_0\varphi(0,x)dx - \int_0^T\int_{\mathbb{R}^3}\chi\partial_t\varphi dxdt - \int_0^T\int_{\mathbb{R}^3}\chi u\cdot\nabla\varphi dxdt = 0$$

thus χ is the renormalized solution of (1.11).

Step 2. Obtaining the terms on the left hand side of (1.12).

Because u_ϵ satisfies (3.11) we have for all $\varphi\in(C_c^\infty((-\infty,T)\times\mathbb{R}^3))^3$ with $\mathrm{div}(\varphi)=0$

$$(4.35)\quad -\int_{\mathbb{R}^3}u_0^T\varphi(0)dx - \int_0^T\int_{\mathbb{R}^3}u_\epsilon^T\partial_t\varphi dxdt \\ - \int_0^T\int_{\mathbb{R}^3}(\Psi_\epsilon(u_\epsilon)\otimes u_\epsilon):\nabla\varphi dxdt + \int_0^T\int_{\mathbb{R}^3}Du_\epsilon : D\varphi dxdt \\ = \int_0^T\int_{\mathbb{R}^3}\frac{\nabla\chi_\epsilon\otimes\nabla\chi_\epsilon}{(|\nabla\chi_\epsilon|^2+\epsilon^2)^{\frac12}}:\nabla\Psi_\epsilon(\varphi)dxdt.$$

For the second term on the left hand side of (4.35) by the weak convergence (4.30) we have

$$\int_0^T\int_{\mathbb{R}^3}u_{\epsilon_k}^T\partial_t\varphi dxdt \to \int_0^T\int_{\mathbb{R}^3}u^T\partial_t\varphi dxdt.$$

To pass to the limit in the third term on the left hand side of (4.35) let us notice that by Lemma 7 we have $\sup_{0<\epsilon<1}\|u_\epsilon\|_{L^2(0,T,V)} < \infty$ and also by Lemma 9 we have $\sup_{0<\epsilon<1}\|\partial_t u_\epsilon\|_{L^1(0,T,\tilde{V}^*)} < \infty$.

Let ω be as in Proposition 3. Let us define

$$E_\omega = \text{closure of } \Big\{ v \in (C_c^\infty(\mathbb{R}^3))^3 \ \Big| \ \operatorname{div}(v) = 0 \Big\} \text{ in } (L^2(\mathbb{R}^3;\omega))^3.$$

To apply the Aubin theorem let us choose $X_1 = V$, $X_2 = E_\omega$ and $X_3 = \tilde{V}^*$.

It is easy to check that the natural embedding of V in E_ω is compact, let f be this compact embedding.

We have that $\tilde{V}$ is densely and continuously embedded in E and in turn this is densely and continuously embedded in E_ω. Finally E_ω as a Hilbert space is isometrically isomorphic to its dual E_ω^*. Hence $\tilde{V}$ is densely and continuously embedded in E_ω^*. Now it follows that E_ω is injectively and continuously embedded in $\tilde{V}^*$ and let g be this embedding.

Hence we may apply the theorem of Aubin to obtain that for a subsequence that we denote again by ϵ_k, $u_{\epsilon_k} \to u$ in $L^2(0,T,E_\omega)$.

By Lemma 1 we compute

$$\begin{aligned}
&\|\Psi_{\epsilon_k}(u_{\epsilon_k}) - u\|_{L^2(0,T,(L^2(\mathbb{R}^3;\omega))^3)} \\
&\quad \le \|\Psi_{\epsilon_k}(u_{\epsilon_k} - u)\|_{L^2(0,T,(L^2(\mathbb{R}^3;\omega))^3)} + \|\Psi_{\epsilon_k}(u) - u\|_{L^2(0,T,(L^2(\mathbb{R}^3;\omega))^3)} \\
&\qquad \le C\|u_{\epsilon_k} - u\|_{L^2(0,T,(L^2(\mathbb{R}^3;\omega))^3)} + \|\Psi_{\epsilon_k}(u) - u\|_{L^2(0,T,(L^2(\mathbb{R}^3;\omega))^3)}
\end{aligned}$$

thus

$$\Psi_{\epsilon_k}(u_{\epsilon_k}) \to u \text{ in } L^2(0,T,E_\omega). \tag{4.36}$$

Finally using the fact that φ has compact support from (4.30) and (4.36) we obtain

$$-\int_0^T \int_{\mathbb{R}^3} (\Psi_{\epsilon_k}(u_{\epsilon_k}) \otimes u_{\epsilon_k}) : \nabla\varphi dxdt \to -\int_0^T \int_{\mathbb{R}^3} (u \otimes u) : \nabla\varphi dxdt.$$

For the fourth term on the left hand side of (4.35) by (4.30) we have

$$\int_0^T \int_{\mathbb{R}^3} Du_{\epsilon_k} : D\varphi dxdt \to \int_0^T \int_{\mathbb{R}^3} Du : D\varphi dxdt.$$

Step 3. Existence of the varifold and obtaining the term on the right hand side of (1.12).

Let $r \in \mathbb{N}$. We compute for $0 < t < T$

$$\begin{aligned}
\int_{B_r} |\nabla\chi_\epsilon| dx \le \int_{B_r} (|\nabla\chi_\epsilon|^2 + \epsilon^2)^{\frac{1}{2}} dx \le \epsilon|B_r| + \int_{B_r} \big\{ (|\nabla\chi_\epsilon|^2 + \epsilon^2)^{\frac{1}{2}} - \epsilon \big\} dx \\
\le \epsilon|B_r| + \int_{\mathbb{R}^3} \big\{ (|\nabla\chi_\epsilon|^2 + \epsilon^2)^{\frac{1}{2}} - \epsilon \big\} dx
\end{aligned}$$

hence by Lemma 7 there exists $\Lambda > 0$ such that

$$\|\nabla\chi_\epsilon\|_{L^\infty(0,T,(L^1(B_r))^3)} \le \epsilon|B_r| + \Lambda. \tag{4.37}$$

By the embedding $L^1(B_r) \hookrightarrow \mathcal{M}(\overline{B_r})$ we have

$$\|\nabla\chi_\epsilon\|_{L^\infty(0,T,(\mathcal{M}(\overline{B_r}))^3)} \le \epsilon|B_r| + \Lambda. \tag{4.38}$$

Then by Lemma 11 there exists a subsequence of ϵ_k which we again denote by ϵ_k and $\mu \in L^\infty_{w^*}(0,T,(\mathcal{M}(\mathbb{R}^3))^3)$ such that

$$\int_0^T \int_{\mathbb{R}^3} \varphi \cdot \nabla\chi_{\epsilon_k} dxdt \to \int_0^T \int_{\mathbb{R}^3} \varphi \cdot \mu(t)(dx)dt, \ \ \forall\varphi \in (C_c(\mathbb{R}^4))^3 \tag{4.39}$$

and

$$\|\mu\|_{L^\infty_{w^*}(0,T,(\mathcal{M}(\mathbb{R}^3))^3)} \leq C\Lambda.$$

Also for $\varphi \in (C^1_c(\mathbb{R}^4))^3$

$$\int_0^T \int_{\mathbb{R}^3} \varphi \cdot \nabla \chi_{\epsilon_k} dxdt = -\int_0^T \int_{\mathbb{R}^3} \operatorname{div}(\varphi)\chi_{\epsilon_k} dxdt \to -\int_0^T \int_{\mathbb{R}^3} \operatorname{div}(\varphi)\chi dxdt \tag{4.40}$$

as $k \to \infty$.

Thus by (4.39) and (4.40) we have for $\varphi \in (C^1_c(\mathbb{R}^4))^3$

$$\int_0^T \int_{\mathbb{R}^3} \varphi \cdot \mu(t)(dx)dt = -\int_0^T \int_{\mathbb{R}^3} \operatorname{div}(\varphi)\chi dxdt$$

hence as distributions

$$\mu(t) = \nabla\chi(t,\cdot) \text{ for a.e. } 0 < t < T.$$

Let us define for $0 < t < T$ and $0 < \epsilon < 1$ the linear functional $V_\epsilon(t)$ on $C_c(\mathbb{R}^6)$ by

$$\langle V_\epsilon(t), \varphi\rangle = \int_{\mathbb{R}^3\cap\{\nabla\chi_\epsilon\neq 0\}} \varphi(x, -\frac{\nabla\chi_\epsilon}{|\nabla\chi_\epsilon|})|\nabla\chi_\epsilon|dx \text{ for } \varphi \in C_c(\mathbb{R}^6).$$

As a positive linear functional on $C_c(\mathbb{R}^6)$ by Riesz representation theorem $V_\epsilon(t)$ corresponds to a unique Radon measure on $\mathbb{R}^6$ which we again denote by $V_\epsilon(t)$.

For a.e. $0 < t < T$ and $r \in \mathbb{N}$ we have by (4.37)

$$\|V_\epsilon(t)\|_{\mathcal{M}(\overline{B^6_r})} \leq \|V_\epsilon(t)\|_{\mathcal{M}(\overline{B^3_r}\times\mathbb{R}^3)} \leq \|\nabla\chi_\epsilon(t)\|_{L^1(B^3_r)} \leq \epsilon|B^3_r| + \Lambda. \tag{4.41}$$

Now let us show that $V_\epsilon(t)$ with values in $\mathcal{M}(\overline{B^6_r})$ is w^* λ-measureable. Let $\varphi \in C(\overline{B^6_r})$ then by the dominated convergence theorem we have

$$\begin{aligned}\langle V_\epsilon(t), \varphi\rangle_{\mathcal{M}(\overline{B^6_r}),C(\overline{B^6_r})} &= \int_{\mathbb{R}^3\cap\{\nabla\chi_\epsilon\neq 0\}} \varphi(x, -\frac{\nabla\chi_\epsilon}{|\nabla\chi_\epsilon|})|\nabla\chi_\epsilon|1_{(x,-\frac{\nabla\chi_\epsilon}{|\nabla\chi_\epsilon|})\in\overline{B^6_r}}dx \\ &= \lim_{\ell\to\infty}\int_{\mathbb{R}^3} \varphi(x, -\frac{\nabla\chi_\epsilon}{|\nabla\chi_\epsilon|+\frac{1}{\ell}})|\nabla\chi_\epsilon|1_{(x,-\frac{\nabla\chi_\epsilon}{|\nabla\chi_\epsilon|+\frac{1}{\ell}})\in\overline{B^6_r}}dx\end{aligned}$$

hence the right hand side as the limit of measurable functions is measurable.

By the separability of $C(\overline{B^6_r})$ it follows that $\|V_\epsilon(t)\|_{\mathcal{M}(\overline{B^6_r})}$ is measurable.

By these measurabilities and (4.41) we obtain $V_\epsilon \in L^\infty_{w^*}(0,T,\mathcal{M}(\overline{B^6_r}))$ with

$$\|V_\epsilon\|_{L^\infty_{w^*}(0,T,\mathcal{M}(\overline{B^6_r}))} \leq \epsilon|B^3_r| + \Lambda.$$

By Lemma 11 there exists a subsequence of ϵ_k which we denote again by ϵ_k and $V \in L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^3\times\mathbb{S}^2))$ such that

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^3\times\mathbb{S}^2} \varphi V_{\epsilon_k}(t)(d(x,y))dt \\ &\qquad\to \int_0^T \int_{\mathbb{R}^3\times\mathbb{S}^2} \varphi V(t)(d(x,y))dt,\ \ \forall\varphi \in C_c(\mathbb{R}^7)\end{aligned} \tag{4.42}$$

and

$$\|V\|_{L^\infty_{w^*}(0,T,\mathcal{M}(\mathbb{R}^3\times\mathbb{S}^2))} \leq \Lambda.$$

Let us prove that

$$-\int_0^T \int_{\mathbb{R}^3} \varphi \cdot \mu(t)(dx)dt = \int_0^T \int_{\mathbb{R}^3\times\mathbb{S}^2} \varphi \cdot yV(t)(d(x,y))dt,\ \forall\varphi \in (C_c(\mathbb{R}^4))^3. \tag{4.43}$$

Let $\varphi \in (C_c(\mathbb{R}^4))^3$. For $(t,x) \in \mathbb{R}^4$ and $y \in \mathbb{S}^2$ let us define $\tilde{\varphi}(t,x,y) = \varphi(t,x)\cdot y$, then one may extend $\tilde{\varphi}$ to a function in $C_c(\mathbb{R}^7)$ and by (4.42) we obtain

$$\begin{aligned}
-\int_0^T\int_{\mathbb{R}^3}\varphi\cdot\mu(t)(dx)dt &= -\lim_{k\to\infty}\int_0^T\int_{\mathbb{R}^3}\varphi\cdot\nabla\chi_{\epsilon_k}dxdt\\
&= \lim_{k\to\infty}\int_0^T\int_{\mathbb{R}^3\cap\{\nabla\chi_{\epsilon_k}\neq 0\}}\tilde{\varphi}(t,x,-\frac{\nabla\chi_{\epsilon_k}}{|\nabla\chi_{\epsilon_k}|})|\nabla\chi_{\epsilon_k}|dxdt\\
&= \lim_{k\to\infty}\int_0^T\int_{\mathbb{R}^3\times\mathbb{S}^2}\tilde{\varphi}(t,x,y)V_{\epsilon_k}(t)(d(x,y))dt\\
&= \int_0^T\int_{\mathbb{R}^3\times\mathbb{S}^2}\tilde{\varphi}(t,x,y)V(t)(d(x,y))dt = \int_0^T\int_{\mathbb{R}^3\times\mathbb{S}^2}\varphi\cdot yV(t)(d(x,y))dt
\end{aligned}$$

thus we have proved (4.43). Now (1.6) follows from (4.43).

Let us prove that

$$\begin{aligned}
(4.44)\quad &\int_0^T\int_{\mathbb{R}^3\times\mathbb{S}^2}\varphi V(t)(d(x,y))dt\\
&= \lim_{k\to\infty}\int_0^T\int_{\mathbb{R}^3\times\mathbb{S}^2}\varphi\frac{|\nabla\chi_{\epsilon_k}|}{(|\nabla\chi_{\epsilon_k}|^2+\epsilon_k^2)^{\frac{1}{2}}}V_{\epsilon_k}(t)(d(x,y))dt,\ \forall\varphi\in C_c(\mathbb{R}^7).
\end{aligned}$$

Let $\varphi \in C_c(\mathbb{R}^7)$. We compute

$$\begin{aligned}
(4.45)\quad &\Big|\int_0^T\int_{\mathbb{R}^3\times\mathbb{S}^2}\varphi V(t)(d(x,y))dt\\
&\quad -\int_0^T\int_{\mathbb{R}^3\times\mathbb{S}^2}\varphi\frac{|\nabla\chi_{\epsilon_k}|}{(|\nabla\chi_{\epsilon_k}|^2+\epsilon_k^2)^{\frac{1}{2}}}V_{\epsilon_k}(t)(d(x,y))dt\Big|\\
&\le \Big|\int_0^T\int_{\mathbb{R}^3\times\mathbb{S}^2}\varphi V(t)(d(x,y))dt - \int_0^T\int_{\mathbb{R}^3\times\mathbb{S}^2}\varphi V_{\epsilon_k}(t)(d(x,y))dt\Big|\\
&\quad + \int_0^T\int_{\mathbb{R}^3\times\mathbb{S}^2}|\varphi|\big(1-\frac{|\nabla\chi_{\epsilon_k}|}{(|\nabla\chi_{\epsilon_k}|^2+\epsilon_k^2)^{\frac{1}{2}}}\big)V_{\epsilon_k}(t)(d(x,y))dt.
\end{aligned}$$

By (4.42) the first term on the right hand side of (4.45) converges to 0.

To show that the second term on the right hand side of (4.45) also converges to 0 we estimate

$$\begin{aligned}
\frac{(|\nabla\chi_{\epsilon_k}|^2+\epsilon_k^2)^{\frac{1}{2}}-|\nabla\chi_{\epsilon_k}|}{(|\nabla\chi_{\epsilon_k}|^2+\epsilon_k^2)^{\frac{1}{2}}}|\nabla\chi_{\epsilon_k}| &= \frac{\epsilon_k^2}{(|\nabla\chi_{\epsilon_k}|^2+\epsilon_k^2)^{\frac{1}{2}}}\frac{|\nabla\chi_{\epsilon_k}|}{(|\nabla\chi_{\epsilon_k}|^2+\epsilon_k^2)^{\frac{1}{2}}+|\nabla\chi_{\epsilon_k}|}\\
&\le \frac{\epsilon_k^2}{(|\nabla\chi_{\epsilon_k}|^2+\epsilon_k^2)^{\frac{1}{2}}} \le \epsilon_k
\end{aligned}$$

and using the fact that φ has compact support and is bounded we compute

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^3\times\mathbb{S}^2}|\varphi|\big(1-\frac{|\nabla\chi_{\epsilon_k}|}{(|\nabla\chi_{\epsilon_k}|^2+\epsilon_k^2)^{\frac{1}{2}}}\big)V_{\epsilon_k}(t)(d(x,y))dt\\
&= \int_0^T\int_{\mathbb{R}^3\cap\{\nabla\chi_{\epsilon_k}\neq 0\}}|\varphi(t,x,-\frac{\nabla\chi_{\epsilon_k}}{|\nabla\chi_{\epsilon_k}|})|\big(1-\frac{|\nabla\chi_{\epsilon_k}|}{(|\nabla\chi_{\epsilon_k}|^2+\epsilon_k^2)^{\frac{1}{2}}}\big)|\nabla\chi_{\epsilon_k}|dxdt\\
&\le \epsilon_k\int_0^T\int_{\mathbb{R}^3\cap\{\nabla\chi_{\epsilon_k}\neq 0\}}|\varphi(t,x,-\frac{\nabla\chi_{\epsilon_k}}{|\nabla\chi_{\epsilon_k}|})|dxdt \le C\epsilon_k \to 0 \text{ as } k\to\infty.
\end{aligned}$$

Now we proved that both terms on the right hand side of (4.45) converge to 0 hence the left hand side converges to 0 and this proves (4.44).

Let us prove that for all $\varphi \in (C_c^\infty(\mathbb{R}^4))^3$ with $\mathrm{div}(\varphi) = 0$

$$\lim_{k\to\infty}\int_0^T\int_{\mathbb{R}^3}\frac{\nabla\chi_{\epsilon_k}\otimes\nabla\chi_{\epsilon_k}}{(|\nabla\chi_{\epsilon_k}|^2+\epsilon_k^2)^{\frac12}}:\nabla\Psi_{\epsilon_k}(\varphi)dxdt = -\int_0^T\langle\delta V(t),\varphi(t)\rangle\, dt. \tag{4.46}$$

Let $\varphi \in (C_c^\infty(\mathbb{R}^4))^3$ with $\mathrm{div}(\varphi) = 0$. We write

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^3}&\frac{\nabla\chi_{\epsilon_k}\otimes\nabla\chi_{\epsilon_k}}{(|\nabla\chi_{\epsilon_k}|^2+\epsilon^2)^{\frac12}}:\nabla\Psi_{\epsilon_k}(\varphi)dxdt\\ &=\int_0^T\int_{\mathbb{R}^3}\frac{\nabla\chi_{\epsilon_k}\otimes\nabla\chi_{\epsilon_k}}{(|\nabla\chi_{\epsilon_k}|^2+\epsilon^2)^{\frac12}}:(\nabla\Psi_{\epsilon_k}(\varphi)-\nabla\varphi)dxdt\\ &\qquad+\int_0^T\int_{\mathbb{R}^3}\frac{\nabla\chi_{\epsilon_k}\otimes\nabla\chi_{\epsilon_k}}{(|\nabla\chi_{\epsilon_k}|^2+\epsilon^2)^{\frac12}}:\nabla\varphi dxdt.\end{aligned}\tag{4.47}$$

For the first term on the right hand side of the equation above because $\nabla\Psi_{\epsilon_k}(\varphi)$ converges in $(C(\mathbb{R}^4))^{3\times3}$ to $\nabla\varphi$ we obtain that

$$\begin{aligned}\Big|\int_0^T\int_{\mathbb{R}^3}&\frac{\nabla\chi_{\epsilon_k}\otimes\nabla\chi_{\epsilon_k}}{(|\nabla\chi_{\epsilon_k}|^2+\epsilon^2)^{\frac12}}:(\nabla\Psi_{\epsilon_k}(\varphi)-\nabla\varphi)dxdt\Big|\\ &\le C\int_0^T\int_{\mathbb{R}^3}((|\nabla\chi_{\epsilon_k}|^2+\epsilon_k^2)^{\frac12}-\epsilon_k)|\nabla\Psi_{\epsilon_k}(\varphi)-\nabla\varphi|dxdt\\ \le CT\Big\{\sup_{0<t<T}&\int_{\mathbb{R}^3}((|\nabla\chi_{\epsilon_k}|^2+\epsilon_k^2)^{\frac12}-\epsilon_k)dx\Big\}\||\nabla\Psi_{\epsilon_k}(\varphi)-\nabla\varphi|\|_{C_b([0,T]\times\mathbb{R}^3)}\\ &\le C_1\||\nabla\Psi_{\epsilon_k}(\varphi)-\nabla\varphi|\|_{C_b([0,T]\times\mathbb{R}^3)}\to 0\end{aligned}\tag{4.48}$$

in the last inequality we used Lemma 7.

For the second term on the right hand side of the equation above using (4.44) we have

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^3}&\frac{\nabla\chi_{\epsilon_k}\otimes\nabla\chi_{\epsilon_k}}{(|\nabla\chi_{\epsilon_k}|^2+\epsilon_k^2)^{\frac12}}:\nabla\varphi dxdt\\ &=\int_0^T\int_{\mathbb{R}^3}((-\frac{\nabla\chi_{\epsilon_k}}{|\nabla\chi_{\epsilon_k}|})\otimes(-\frac{\nabla\chi_{\epsilon_k}}{|\nabla\chi_{\epsilon_k}|}):\nabla\varphi\frac{|\nabla\chi_{\epsilon_k}|^2}{(|\nabla\chi_{\epsilon_k}|^2+\epsilon_k^2)^{\frac12}}dxdt\\ =-\int_0^T&\int_{\mathbb{R}^3}(I-(-\frac{\nabla\chi_{\epsilon_k}}{|\nabla\chi_{\epsilon_k}|})\otimes(-\frac{\nabla\chi_{\epsilon_k}}{|\nabla\chi_{\epsilon_k}|}):\nabla\varphi\frac{|\nabla\chi_{\epsilon_k}|}{(|\nabla\chi_{\epsilon_k}|^2+\epsilon_k^2)^{\frac12}}|\nabla\chi_{\epsilon_k}|dxdt\\ &=-\int_0^T\int_{\mathbb{R}^3\times\mathbb{S}^2}(I-y\otimes y):\nabla\varphi\frac{|\nabla\chi_{\epsilon_k}|}{(|\nabla\chi_{\epsilon_k}|^2+\epsilon_k^2)^{\frac12}}V_{\epsilon_k}(t)(d(x,y))dt\\ &\to-\int_0^T\int_{\mathbb{R}^3\times\mathbb{S}^2}(I-y\otimes y):\nabla\varphi(t,x)V(t)(d(x,y))dt\\ &=-\int_0^T\langle\delta V(t),\varphi(t)\rangle\, dt.\end{aligned}\tag{4.49}$$

Now by (4.47), (4.48) and (4.49) we prove (4.46).

And this completes the existence of a varifold solution. □

Proof of theorem 2. In the case of axisymmetric initial values and boundary condition by the theorem 3 for each $0 < \epsilon < 1$ there exists $u_\epsilon \in L^2(0, T, V_{a.s.})$ and axisymmetric $\chi_\epsilon \in L^2(0, T, H^1(\mathbb{R}^3))$, such that together these are a solution to the system of equations (3.10) and (3.11).

In the following when writing u_ϵ or χ_ϵ we mean these axisymmetric solutions.

We proceed as in the proof of theorem 1 and prove the existence of a Varifold solution triple (u, χ, V) to our equations. As described in the proof of theorem 1 these are appropriate limits of the regularized solutions corresponding to a sequence ϵ_k.

In the following using the axisymmetry of the regularized solutions we prove the axisymmetry properties of the varifold solution triple.

For $\theta \in \mathbb{R}$ by the axisymmetry of χ_{ϵ_k} we have $\tau_\theta \chi_{\epsilon_k} = \chi_{\epsilon_k}$. Thus for $\varphi \in C_c(\mathbb{R}^4)$ by the weak convergence (4.29) we have

$$\int_0^T \int_{\mathbb{R}^3} \tau_\theta \chi \varphi dxdt = \int_0^T \int_{\mathbb{R}^3} \chi \tau_{-\theta} \varphi dxdt = \lim_{k\to\infty} \int_0^T \int_{\mathbb{R}^3} \chi_{\epsilon_k} \tau_{-\theta} \varphi dxdt$$
$$= \lim_{k\to\infty} \int_0^T \int_{\mathbb{R}^3} \tau_\theta \chi_{\epsilon_k} \varphi dxdt = \lim_{k\to\infty} \int_0^T \int_{\mathbb{R}^3} \chi_{\epsilon_k} \varphi dxdt = \int_0^T \int_{\mathbb{R}^3} \chi \varphi dxdt$$

and from here by the arbitrariness of φ we obtain $\tau_\theta \chi = \chi$ for a.e. $0 < t < T$ and a.e. $x \in \mathbb{R}^3$ which proves the axisymmetry of χ.

For $\theta \in \mathbb{R}$ by the axisymmetry of u_{ϵ_k} we have $T_\theta u_{\epsilon_k} = u_{\epsilon_k}$. Thus for $\varphi \in (C_c(\mathbb{R}^4))^3$ by the weak convergence (4.30) we have

$$\int_0^T \int_{\mathbb{R}^3} (T_\theta u)^T \varphi dxdt = \int_0^T \int_{\mathbb{R}^3} u^T (T_{-\theta}\varphi) dxdt = \lim_{k\to\infty} \int_0^T \int_{\mathbb{R}^3} u_{\epsilon_k}^T (T_{-\theta}\varphi) dxdt$$
$$= \lim_{k\to\infty} \int_0^T \int_{\mathbb{R}^3} (T_\theta u_{\epsilon_k})^T \varphi dxdt = \lim_{k\to\infty} \int_0^T \int_{\mathbb{R}^3} u_{\epsilon_k}^T \varphi dxdt = \int_0^T \int_{\mathbb{R}^3} u^T \varphi dxdt$$

and from here by the arbitrariness of φ we obtain $T_\theta u = u$ for a.e. $0 < t < T$ and a.e. $x \in \mathbb{R}^3$ which proves the axisymmetry of u.

Now let us prove the axisymmetry properties of V. Let $\theta \in \mathbb{R}$ and $\varphi \in C_c(\mathbb{R}^7)$ then using (3.1) we compute

$$\int_0^T \int_{\mathbb{R}^3\times\mathbb{S}^2} \varphi(t,x,y)(V(t)\circ O^T(\theta))(d(x,y))dt$$
$$= \int_0^T \int_{\mathbb{R}^3\times\mathbb{S}^2} \varphi(t, O(\theta)x, O(\theta)y) V(t)(d(x,y))dt$$
$$= \lim_{k\to\infty} \int_0^T \int_{\mathbb{R}^3\times\mathbb{S}^2} \varphi(t, O(\theta)x, O(\theta)y) V_{\epsilon_k}(t)(d(x,y))dt$$
$$= \lim_{k\to\infty} \int_0^T \int_{\mathbb{R}^3\cap\{|\nabla\chi_{\epsilon_k}|\neq 0\}} \varphi(t, O(\theta)x, O(\theta)(-\frac{\nabla\chi_{\epsilon_k}}{|\nabla\chi_{\epsilon_k}|})) |\nabla\chi_{\epsilon_k}(x)| dxdt$$
$$= \lim_{k\to\infty} \int_0^T \int_{\mathbb{R}^3\cap\{x\in\mathbb{R}^3 | |\nabla\chi_{\epsilon_k}(t,O^T(\theta)x)|\neq 0\}}$$
$$\varphi(t, x, O(\theta)(-\frac{\nabla\chi_{\epsilon_k}(O^T(\theta)x)}{|\nabla\chi_{\epsilon_k}(O^T(\theta)x)|})) |\nabla\chi_{\epsilon_k}(O^T(\theta)x)| dxdt$$
$$= \lim_{k\to\infty} \int_0^T \int_{\mathbb{R}^3\cap\{x\in\mathbb{R}^3 | |\nabla\chi_{\epsilon_k}(t,x)|\neq 0\}} \varphi(t, x, -\frac{\nabla\chi_{\epsilon_k}(x)}{|\nabla\chi_{\epsilon_k}(x)|})) |\nabla\chi_{\epsilon_k}(x)| dxdt$$
$$= \lim_{k\to\infty} \int_0^T \int_{\mathbb{R}^3\times\mathbb{S}^2} \varphi(t,x,y) V_{\epsilon_k}(t)(d(x,y))dt$$
$$= \int_0^T \int_{\mathbb{R}^3\times\mathbb{S}^2} \varphi(t,x,y) V(t)(d(x,y))dt$$

and by the arbitrariness of φ we obtain that for a.e. $0 < t < T$, $V(t)$ satisfies (1.9).

It is easy to compute and see that

$$\frac{d}{d\theta} O(\theta) = O(\theta + \frac{\pi}{2})\Pi.$$

By the axisymmetry of χ_ϵ we can compute

$$0 = \frac{d}{d\theta}\chi_\epsilon(t,x) = \frac{d}{d\theta}\chi_\epsilon(t, O^T(\theta)x) = \nabla\chi_\epsilon(t, O^T(\theta)x)^T O(\theta + \frac{\pi}{2})\Pi x$$

and in particular for $\theta = 0$ we obtain

$$0 = \nabla\chi_\epsilon(t,x)^T O(\frac{\pi}{2})\Pi x \text{ for } 0 < t < T \text{ and } x \in \mathbb{R}^3. \tag{4.50}$$

Let us define

$$Q = \left\{(x,y) \in \mathbb{R}^6 \;\middle|\; y^T O(\frac{\pi}{2})\Pi x = 0\right\}$$

then Q is closed and is not equal to $\mathbb{R}^6$.

Let $\varphi \in C_c(\mathbb{R} \times Q^c)$ then

$$\varphi = h(t,x,y) y^T O(\frac{\pi}{2})\Pi x$$

where

$$h = \frac{\varphi}{y^T O(\frac{\pi}{2})\Pi x} \in C_c(\mathbb{R}^7).$$

Now we may compute using (4.50)

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^3\times\mathbb{S}^2} \varphi V(t)(d(x,n))dt &= \lim_{k\to\infty} \int_0^T \int_{\mathbb{R}^3\times\mathbb{S}^2} \varphi V_{\epsilon_k}(t)(d(x,n))dt \\ &= \lim_{k\to\infty} \int_0^T \int_{\mathbb{R}^3\cap\{|\nabla\chi_{\epsilon_k}|\neq 0\}} \varphi(t,x,-\frac{\nabla\chi_{\epsilon_k}}{|\nabla\chi_{\epsilon_k}|})|\nabla\chi_{\epsilon_k}|dxdt \\ = \lim_{k\to\infty} \int_0^T \int_{\mathbb{R}^3\cap\{|\nabla\chi_{\epsilon_k}|\neq 0\}} & h(t,x,-\frac{\nabla\chi_{\epsilon_k}}{|\nabla\chi_{\epsilon_k}|})(-\frac{\nabla\chi_{\epsilon_k}}{|\nabla\chi_{\epsilon_k}|})^T O(\frac{\pi}{2})\Pi x |\nabla\chi_{\epsilon_k}|dxdt \\ &= -\lim_{k\to\infty} \int_0^T \int_{\mathbb{R}^3\cap\{|\nabla\chi_{\epsilon_k}|\neq 0\}} h(t,x,-\frac{\nabla\chi_{\epsilon_k}}{|\nabla\chi_{\epsilon_k}|})\nabla\chi_{\epsilon_k}^T O(\frac{\pi}{2})\Pi x dxdt = 0. \end{aligned} \tag{4.51}$$

Now by (4.51) and the arbitrariness of $\varphi \in C_c(\mathbb{R} \times Q^c)$ we obtain that for a.e. $0 < t < T$, $V(t)$ satisfies (1.10). □

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