

The structure of completely positive matrices according to their CP-rank and CP-plus-rank

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Abstract

We study the topological properties of the cp-rank operator $\text{cp}(A)$ and the related cp-plus-rank operator $\text{cp}^+(A)$ (which is introduced in this paper) in the set \mathcal{S}^n of symmetric $n \times n$ -matrices. For the set of completely positive matrices, \mathcal{CP}^n , we show that for any fixed p the set of matrices A satisfying $\text{cp}(A) = \text{cp}^+(A) = p$ is open in $\mathcal{S}^n \setminus \text{bd}(\mathcal{CP}^n)$. By making use of the Perron-Frobenius vector we also prove that the set \mathcal{A}^n of matrices with $\text{cp}(A) = \text{cp}^+(A)$ is dense in \mathcal{S}^n . By applying the theory of semi-algebraic sets we are able to show that membership in \mathcal{A}^n is even a generic property. We furthermore answer several questions on the existence of matrices satisfying $\text{cp}(A) = \text{cp}^+(A)$ or $\text{cp}(A) \neq \text{cp}^+(A)$, and comment on genericity of having infinitely many minimal cp-decompositions.

1 Introduction

We define a symmetric matrix A to be completely positive if there exists nonnegative vectors $\mathbf{b}_1, \dots, \mathbf{b}_p$ such that $A = \sum_{i=1}^p \mathbf{b}_i \mathbf{b}_i^T$. The set of completely positive matrices forms a proper cone, i.e. a cone which is closed, convex, pointed and full-dimensional. This cone plays an important role in the field of copositive optimisation (see, e.g., [Dür10, Bom12, Bur12]).

In this paper we investigate the cp- and cp-plus-ranks of matrices, which are closely related to complete positivity. These are defined below, where we let \mathcal{S}^n be the set of symmetric $n \times n$ matrices, \mathbb{N} be the set of nonnegative integers, \mathbb{R}_+^n be the set of nonnegative real n -vectors and \mathbb{R}_{++}^n be the set of strictly positive real n -vectors:

Definition 1.1. For $A \in \mathcal{S}^n$ we define its cp-rank and its cp-plus-rank respectively as:

$$\begin{aligned} \text{cp}(A) &:= \min \left\{ p \in \mathbb{N} \mid \exists \mathbf{b}_1, \dots, \mathbf{b}_p \in \mathbb{R}_+^n \text{ s.t. } A = \sum_{i=1}^p \mathbf{b}_i \mathbf{b}_i^T \right\}, \\ \text{cp}^+(A) &:= \min \left\{ p \in \mathbb{N} \mid \exists \mathbf{b}_1, \dots, \mathbf{b}_p \in \mathbb{R}_{++}^n \text{ s.t. } A = \sum_{i=1}^p \mathbf{b}_i \mathbf{b}_i^T \right\}. \end{aligned}$$

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Note that a matrix $A \in \mathcal{S}^n$ is completely positive if and only if $\text{cp}(A) < \infty$.

One motivation for the study of the cp-plus-rank is given by the following theorem, where \mathcal{CP}^n denotes the cone of completely positive matrices of order n , $\text{int}(\mathcal{CP}^n)$ denotes its interior, and $\text{rank}(A)$ denotes the standard linear rank of the matrix A :

Theorem 1.2. *For $A \in \mathcal{S}^n$ we have*

$$A \in \text{int}(\mathcal{CP}^n) \iff \text{cp}^+(A) < \infty \text{ and } \text{rank}(A) = n.$$

Proof. This comes from [Dic10, Theorem 3.8], after noting that for any matrix $B \in \mathbb{R}^{m \times n}$ we have $\text{rank}(B^\top B) = \text{rank}(B)$. \square

Another point of interest is when $\text{cp}(M) = \text{cp}^+(M)$. We then have the following two properties, where for $\varepsilon > 0$ and $M \in \mathcal{S}^n$ we define $N_\varepsilon(M) = \{X \in \mathcal{S}^n \mid \|M - X\| \leq \varepsilon\}$, and for a matrix $A = (a_{ij}) \in \mathcal{S}^n$, by $\|A\|$ we mean the Frobenius norm, *i.e.*, $\|A\| = \sqrt{\sum_{i,j} a_{ij}^2}$:

Theorem 1.3. *Consider $M \in \mathcal{S}^n$ such that $2 \leq \text{cp}(M) = \text{cp}^+(M) < \infty$. Then M has infinitely many minimal cp-decompositions, where a minimal cp-decomposition is a set $\{\mathbf{b}_1, \dots, \mathbf{b}_p\} \subseteq \mathbb{R}_+^n$ such that $p = \text{cp}(M)$ and $M = \sum_{i=1}^p \mathbf{b}_i \mathbf{b}_i^\top$.*

Proof. This will follow directly from Lemma 2.10. \square

Theorem 1.4. *Consider $M \in \text{int}(\mathcal{CP}^n)$ such that $\text{cp}(M) = \text{cp}^+(M)$. Then there exists $\varepsilon > 0$ such that for all $X \in N_\varepsilon(M)$ we have $\text{cp}(X) = \text{cp}^+(X) = \text{cp}(M)$.*

Proof. This will be shown in Theorem 2.7. \square

The aim of the present paper is to study the topological properties of the functions $\text{cp}(M)$ and $\text{cp}^+(M)$.

In Section 2 we will look at some basic preliminary results on these ranks. In Section 3 we show how orthogonal matrices can be used in considering them. In Section 4 properties of the rank functions are analysed by using Perron-Frobenius vectors. In Section 5 we are interested in properties of the maximum cp- and cp-plus-ranks. Finally, in Section 6 of this paper we shall show that membership in the set $\{M \in \mathcal{S}^n \mid \text{cp}(M) = \text{cp}^+(M)\}$ is a generic property. This yields

Theorem 1.5. *The following properties are generic within the completely positive cone:*

1. *Having infinitely many minimal cp-decompositions,*
2. *The cp- and cp-plus- ranks being equal and locally constant.*

Proof. This will be shown in Corollary 6.8. \square

Notation

In this paper we shall always consider n to be an integer which is strictly greater than one. In addition to the notation introduced earlier in this section, we shall let \mathbb{R}^n denote the set of real n -vectors; \mathcal{S}_+^n the set of positive semidefinite matrices of order n ; \mathcal{N}^n the set of nonnegative symmetric matrices of order n ; and $\text{bd}(\mathcal{CP}^n)$ the boundary of the set of completely positive matrices. For a vector $\mathbf{a} \in \mathbb{R}^n$, whenever we mention a norm we mean the Euclidean norm, *i.e.*, $\|\mathbf{a}\| = \sqrt{\sum_i a_i^2}$.

2 Preliminary results

In this section we shall consider some basic results connected to the cp- and cp-plus-ranks. We start with the following three trivial results.

Lemma 2.1. *For all $M \in \mathcal{S}^n$, we have $\text{cp}^+(M) \geq \text{cp}(M) \geq \text{rank}(M)$.*

Lemma 2.2. *If $M \in \mathcal{S}^n \setminus \{0\}$ such that $\text{cp}^+(M)$ is finite, then $M \in \text{int}(\mathcal{N}^n) \cap \mathcal{CP}^n$.*

Lemma 2.3. *For all $A, B \in \mathcal{S}^n$ and $\alpha, \beta > 0$ we have*

$$\text{cp}(\alpha A + \beta B) \leq \text{cp}(A) + \text{cp}(B).$$

We shall now consider how the cp- and cp-plus-ranks vary in a neighbourhood of a matrix $M \in \mathcal{S}^n$.

Theorem 2.4. *Let $M \in \mathcal{S}^n$. Then there exists $\varepsilon > 0$ such that $\text{cp}(P) \geq \text{cp}(M)$ for all $P \in N_\varepsilon(M)$.*

Proof. This was shown in [SMBJS13, Proposition 2.4]. □

A similar result also holds for the cp-plus-rank, although with the inequality reversed.

Theorem 2.5. *Let $M \in \mathcal{S}^n \setminus \text{bd}(\mathcal{CP}^n)$. Then there exists $\varepsilon > 0$ such that $\text{cp}^+(P) \leq \text{cp}^+(M)$ for all $P \in N_\varepsilon(M)$.*

Proof. If $M \notin \mathcal{CP}^n$ then there exists $\varepsilon > 0$ such that for all $P \in N_\varepsilon(M)$ we have $P \notin \mathcal{CP}^n$, and thus $\text{cp}^+(P) = \infty = \text{cp}^+(M)$.

If $M \in \text{int}(\mathcal{CP}^n)$ then the result comes directly from considering the proof of [DS08, Theorem 2.3]. □

Remark 2.6. The result of the previous theorem does not in general hold when $M \in \text{bd}(\mathcal{CP}^n)$. For example, if $M \in \text{bd}(\mathcal{CP}^n)$ such that $\text{cp}^+(M) \neq \infty$, then for all $\varepsilon > 0$ there exists $P \in N_\varepsilon(M) \setminus \mathcal{CP}^n$ and thus $\text{cp}^+(P) = \infty > \text{cp}^+(M)$.

Combining Lemma 2.1 and Theorems 2.4 and 2.5, we get the following result.

Theorem 2.7. *Let $M \in \mathcal{S}^n \setminus \text{bd}(\mathcal{CP}^n)$ such that $\text{cp}(M) = \text{cp}^+(M) = p$. Then there exists $\varepsilon > 0$ such that $\text{cp}^+(P) = \text{cp}(P) = p$ for all $P \in N_\varepsilon(M)$.*

Corollary 2.8. *The following sets are open for all $p \in \mathbb{N}$:*

$$\begin{aligned} & \{M \in \mathcal{S}^n \setminus \text{bd}(\mathcal{CP}^n) \mid \text{cp}(M) = \text{cp}^+(M) = p\}, \\ & \{M \in \mathcal{S}^n \setminus \text{bd}(\mathcal{CP}^n) \mid \text{cp}(M) = \text{cp}^+(M)\}. \end{aligned}$$

We finish this section by considering some equivalent definitions of the cp- and cp-plus-ranks, which will be used regularly throughout the paper.

We begin with the following trivial result:

Lemma 2.9. *For all $A \in \mathcal{S}^n \setminus \{0\}$ we have*

$$\begin{aligned} \text{cp}(A) &= \min \{p \in \mathbb{N} \mid \exists B \in \mathbb{R}_+^{p \times n} \text{ s.t. } A = B^\top B\}, \\ \text{cp}^+(A) &= \min \{p \in \mathbb{N} \mid \exists B \in \mathbb{R}_{++}^{p \times n} \text{ s.t. } A = B^\top B\}. \end{aligned}$$

Proof. This comes from noting that if we have a matrix $B \in \mathbb{R}^{p \times n}$ whose rows are given by $\mathbf{b}_1^\top, \dots, \mathbf{b}_p^\top$ then $B^\top B = \sum_{i=1}^p \mathbf{b}_i \mathbf{b}_i^\top$. \square

We now consider another equivalent definition of the cp-plus-rank which is less trivial. These results come from [Dic13, Lemma 7.13] and its proof.

Lemma 2.10. *Consider $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, and for all $\theta \in \mathbb{R}$ let $\mathbf{c}_\theta = \mathbf{a} \sin \theta + \mathbf{b} \cos \theta$ and $\mathbf{d}_\theta = \mathbf{a} \cos \theta - \mathbf{b} \sin \theta$. Then we have:*

1. $\mathbf{a} \mathbf{a}^\top + \mathbf{b} \mathbf{b}^\top = \mathbf{c}_\theta \mathbf{c}_\theta^\top + \mathbf{d}_\theta \mathbf{d}_\theta^\top$ for all $\theta \in \mathbb{R}$, and
2. if $\mathbf{a} \in \mathbb{R}_{++}^n$ and $\mathbf{b} \in \mathbb{R}_+^n$ then there exists $\Theta > 0$ such that $\mathbf{c}_\theta, \mathbf{d}_\theta \in \mathbb{R}_{++}^n$ for all $\theta \in (0, \Theta]$.

Corollary 2.11. *For $A \in \mathcal{S}^n \setminus \{0\}$, we have*

$$\text{cp}^+(A) = \min \left\{ p \in \mathbb{N} \mid \exists \mathbf{b}_1, \dots, \mathbf{b}_p \in \mathbb{R}_+^n \text{ s.t. } \mathbf{b}_1 \in \mathbb{R}_{++}^n \text{ and } A = \sum_{i=1}^p \mathbf{b}_i \mathbf{b}_i^\top \right\}.$$

This result leads to an inequality for the function cp-plus-rank similar to Lemma 2.3, but note that here we have a mixture of cp- and cp-plus-ranks.

Corollary 2.12. *For all $A, B \in \mathcal{S}^n \setminus \{0\}$ and $\alpha, \beta > 0$ we have*

$$\text{cp}^+(\alpha A + \beta B) \leq \text{cp}^+(A) + \text{cp}(B).$$

3 Orthogonal matrices

The concept of cp-plus-rank connects to orthogonal matrices through the following lemma.

Lemma 3.1. *Let $A, B \in \mathbb{R}^{p \times n}$. Then $A^\top A = B^\top B$ if and only if there exists an orthogonal matrix $Q \in \mathbb{R}^{p \times p}$ such that $A = QB$.*

Proof. The reverse implication (which we will need below) is trivial. The forward implication is a well known result in linear algebra, and a sketch of the proof is presented in [Xu04, Lemma 1]. \square

In the paper [SSMS13] the authors considered matrices $B \in \mathbb{R}_+^{p \times n}$ and defined such matrices to be *nearly positive* if there exist orthogonal matrices $\{Q_i \mid i \in \mathbb{N}\}$ such that $Q_i B > 0$ for all i and $\lim_{i \rightarrow \infty} Q_i = I$ (where I is the identity matrix). Using the lemma above we then get the following sufficient condition for when the cp-rank of a matrix is equal to its cp-plus rank.

Corollary 3.2. *Let $X \in \mathcal{CP}^n$ with $\text{cp}(X) = p$, and let $B \in \mathbb{R}_+^{p \times n}$ such that $X = B^\top B$. If B is a nearly positive matrix, then $\text{cp}^+(X) = p$.*

In [SSMS13, Example 7.4] it was shown that the reverse implication to this does not hold. In that example, for $n \geq 4$, they considered a family of $M = A^\top A \in \text{int}(\mathcal{CP}^n)$, with $A \in \mathbb{R}_+^{n \times n}$ not being a nearly positive matrix, but $\text{cp}^+(M) = \text{cp}(M) = \text{rank}(M) = n$.

In [SSMS13] the authors looked at many interesting results on nearly positive matrices, including the following:

Theorem 3.3. *Let $X \in \mathcal{CP}^n \cap \text{int}(\mathcal{N}^n)$ and let $B \in \mathbb{R}_+^{p \times n}$ such that $X = B^\top B$. If either $n \leq 3$ or $p \leq 2$ (or both) then B is nearly positive.*

Translating this result for the cp-plus-rank we get the following corollary.

Corollary 3.4. *Let $X \in \mathcal{CP}^n \cap \text{int}(\mathcal{N}^n)$. If $n \leq 3$ or $\text{cp}(X) \leq 2$ (or both) then $\text{cp}(X) = \text{cp}^+(X)$.*

4 Perron-Frobenius Vectors

In this section we shall analyse the cp- and cp-plus-ranks using the theory of Perron-Frobenius vectors. We begin by recalling some basic definitions and results on Perron-Frobenius vectors, applied to matrices in $\mathcal{N}^n \setminus \{0\}$.

Theorem 4.1. *Let $M \in \mathcal{N}^n \setminus \{0\}$. Then there exists $\lambda \in \mathbb{R}_{++}$ such that:*

1. λ is an eigenvalue of M ,
2. the absolute values of all eigenvalues of M are less than or equal to λ ,
3. there is an eigenvector $\mathbf{x} \in \mathbb{R}_+^n$, with $\|\mathbf{x}\| = 1$, corresponding to the eigenvalue λ . We refer to this as a Perron-Frobenius (P-F) vector of M .

Furthermore, if $M \in \text{int}(\mathcal{N}^n)$, then for λ and \mathbf{x} given above we have:

4. the absolute values of all eigenvalues of M , excluding λ , are strictly less than λ ,
5. $\mathbf{x} \in \mathbb{R}_{++}^n$, and \mathbf{x} is the unique eigenvector of M corresponding to λ , up to multiplication by a scalar (i.e. the eigenvalue λ has multiplicity one). We shall denote this eigenvector by \mathbf{x}_M .

Remark 4.2. Note that any matrix $M \in \text{int}(\mathcal{CP}^n)$ satisfies $M \in \text{int}(\mathcal{N}^n)$. Also note that if in the theorem above we have $M \notin \text{int}(\mathcal{N}^n)$, then we do not necessarily have a unique P-F vector. For example, consider M being equal to the identity matrix.

We now recall the following well known lemma on eigenvectors and eigenvalues.

Lemma 4.3. *Consider a matrix $A \in \mathcal{S}^n$ with eigenvectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$, whose corresponding eigenvalues are $\lambda_1, \lambda_2 \in \mathbb{R}$. If $\lambda_1 \neq \lambda_2$ then $\mathbf{x}_1^\top \mathbf{x}_2 = 0$.*

Proof. This comes from noting that $\lambda_1 \mathbf{x}_1^\top \mathbf{x}_2 = \mathbf{x}_1^\top A \mathbf{x}_2 = \lambda_2 \mathbf{x}_1^\top \mathbf{x}_2$. □

From this we then get the following result on P-F vectors.

Lemma 4.4. *Consider $M \in \mathcal{N}^n \setminus \{0\}$ and let $\mathbf{x} \in \mathbb{R}_{++}^n$ be an eigenvector of M such that $\|\mathbf{x}\| = 1$. Then \mathbf{x} is a P-F vector of M .*

Proof. Assume for the sake of contradiction that the eigenvector \mathbf{x} with corresponding eigenvalue μ is not a P-F vector. Then there exists a P-F vector $\mathbf{y} \in \mathbb{R}_+^n \setminus \{0\}$ with eigenvalue $\lambda > \mu$. By Lemma 4.3 it would follow $\mathbf{y}^\top \mathbf{x} = 0$, a contradiction to $0 \neq \mathbf{y} \in \mathbb{R}_+^n$, $\mathbf{x} \in \mathbb{R}_{++}^n$. □

Another well known lemma on eigenvectors is the following.

Lemma 4.5. *Consider $P, Q \in \mathcal{S}^n$ and $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$ such that $P = Q + \mu \mathbf{x} \mathbf{x}^\top$ for some $\mu \in \mathbb{R}$. Then \mathbf{x} is an eigenvector of P if and only if it is an eigenvector of Q .*

Proof. Without loss of generality let $\|\mathbf{x}\| = 1$. Then we have

$$Q\mathbf{x} = \lambda\mathbf{x} \quad \Leftrightarrow \quad P\mathbf{x} = (\lambda + \mu)\mathbf{x}. \quad \square$$

Now combining Lemmas 4.4 and 4.5 we get the following result:

Lemma 4.6. Consider $P, Q \in \mathcal{N}^n \setminus \{0\}$ and $\mathbf{x} \in \mathbb{R}_{++}^n$ such that $P = Q + \mu \mathbf{x} \mathbf{x}^\top$ for some $\mu \in \mathbb{R}$. Then \mathbf{x} is a P-F vector of P if and only if it is a P-F vector of Q .

We will now look at what P-F vectors can tell us about the cp- and cp-plus-ranks. In order to do this for $M \in \text{int}(\mathcal{N}^n)$ and $\mu \in \mathbb{R}$ we let

$$M(\mu) := M + \mu \mathbf{x}_M \mathbf{x}_M^\top.$$

Note from Theorem 4.1 that this is well defined. Furthermore, from the definition, we have that $\|M - M(\mu)\| = |\mu|$ and thus $M(\mu) \in N_{|\mu|}(M)$. We also note the following basic result.

Lemma 4.7. Let $M, P \in \text{int}(\mathcal{N}^n)$ and $\mu \in \mathbb{R}$ such that $P = M(\mu)$. Then we have $M = P(-\mu)$.

Proof. This comes directly from Theorem 4.1 and Lemma 4.6. \square

We are now ready to present the main results of this section.

Theorem 4.8. For all $M \in \text{int}(\mathcal{N}^n)$ and all $\mu > 0$ we have $\text{cp}(M) \geq \text{cp}^+(M(\mu))$.

Proof. This proof is an adaptation of one from [SMBJS13].

In this proof we will in fact prove the more general result that considers $M \in \mathcal{N}^n \setminus \{0\}$ with a P-F vector $\mathbf{x} \in \mathbb{R}_{++}^n$. Under these circumstances, we have

$$\text{cp}(M) \geq \text{cp}^+(M + \mu \mathbf{x} \mathbf{x}^\top) \quad \text{for all } \mu > 0.$$

Indeed, if $M \notin \mathcal{CP}^n$ then we have $\text{cp}(M) = \infty$ and the result is trivial. From now on we assume $M \in \mathcal{CP}^n \setminus \{0\}$ and consider an arbitrary $\mu > 0$.

Letting $p = \text{cp}(M) \in (0, \infty)$, there exists $V \in \mathbb{R}_+^{p \times n}$ such that $M = V^\top V$. All rows of V are nonzero, otherwise $\text{cp}(M) < p$. Therefore, letting $\mathbf{y} = V \mathbf{x}$, we have $\mathbf{y} \in \mathbb{R}_{++}^p$.

As \mathbf{x} is a P-F vector of M , there exists $\lambda > 0$ such that $\lambda \mathbf{x} = M \mathbf{x} = V^\top V \mathbf{x} = V^\top \mathbf{y}$. We thus have that $\mathbf{y}^\top \mathbf{y} = \mathbf{x}^\top V^\top V \mathbf{x} = \lambda \mathbf{x}^\top \mathbf{x} = \lambda$.

The proof is now completed by letting $\nu = \sqrt{1 + (\mu/\lambda)} - 1 > 0$, noting that we have $(V + \nu \mathbf{y} \mathbf{x}^\top) \in \mathbb{R}_{++}^{p \times n}$ and considering the following:

$$\begin{aligned} (V + \nu \mathbf{y} \mathbf{x}^\top)^\top (V + \nu \mathbf{y} \mathbf{x}^\top) &= V^\top V + \nu(V^\top \mathbf{y} \mathbf{x}^\top + \mathbf{x} \mathbf{y}^\top V) + \nu^2 \mathbf{x} \mathbf{y}^\top \mathbf{y} \mathbf{x}^\top \\ &= M + \nu \lambda (2 + \nu) \mathbf{x} \mathbf{x}^\top \\ &= M + \mu \mathbf{x} \mathbf{x}^\top. \end{aligned} \quad \square$$

Theorem 4.9. Consider $M \in \text{int}(\mathcal{CP}^n)$ with $p := \text{cp}(M) \leq \text{cp}^+(M) =: q$. Then there exists $\hat{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \hat{\varepsilon}]$ we have

$$\text{cp}(M(\varepsilon)) = \text{cp}^+(M(\varepsilon)) = p \quad \text{and} \quad \text{cp}(M(-\varepsilon)) = \text{cp}^+(M(-\varepsilon)) = q.$$

Proof. From Theorem 2.4, there exists $\varepsilon_+ > 0$ such that for all $\varepsilon \in (0, \varepsilon_+]$ we have $p \leq \text{cp}(M(\varepsilon))$, and from Theorem 4.8 we have $\text{cp}^+(M(\varepsilon)) \leq p$. We now note from Lemma 2.1 that $\text{cp}(M(\varepsilon)) \leq \text{cp}^+(M(\varepsilon))$, and combining these three inequalities together we get $\text{cp}(M(\varepsilon)) = \text{cp}^+(M(\varepsilon)) = p$ for all $\varepsilon \in (0, \varepsilon_+]$.

Similarly, from Theorem 2.5, there exists $\varepsilon_- > 0$ such that for all $\varepsilon \in (0, \varepsilon_-]$ we have $q \geq \text{cp}^+(M(-\varepsilon))$. For such ε , as the cp-plus-rank is finite, we have $M(-\varepsilon) \in \text{int}(\mathcal{N}^n)$, and thus from Lemma 4.7 and Theorem 4.8 we have $\text{cp}(M(-\varepsilon)) \geq q$. We now note from Lemma 2.1 that $\text{cp}^+(M(-\varepsilon)) \geq \text{cp}(M(-\varepsilon))$, and combining these three inequalities together we get $\text{cp}(M(-\varepsilon)) = \text{cp}^+(M(-\varepsilon)) = q$ for all $\varepsilon \in (0, \varepsilon_-]$.

Now letting $\hat{\varepsilon} = \min\{\varepsilon_-, \varepsilon_+\}$, this completes the proof. \square

5 Maximum cp- and cp-plus-ranks

Let us define the numbers

$$\begin{aligned} p_n &:= \max\{\text{cp}(M) \mid M \in \mathcal{CP}^n\}, \\ p_n^+ &:= \max\{\text{cp}^+(M) \mid \text{cp}^+(M) < \infty\}. \end{aligned}$$

In the following theorem we collect some known results on these numbers, along with a couple of new ones.

Theorem 5.1. *We have that*

$$p_n = \max\{\text{cp}(M) \mid M \in \text{bd}(\mathcal{CP}^n)\}, \quad (1)$$

$$= \max\{\text{cp}(M) \mid M \in \text{int}(\mathcal{CP}^n)\} \quad (2)$$

$$= \max\{\text{cp}^+(M) \mid M \in \text{int}(\mathcal{CP}^n)\} \quad (3)$$

$$p_n \leq p_n^+ \leq p_n + 1 \quad (4)$$

$$p_n = n \quad \text{for all } n = 2, 3, 4, \quad (5)$$

$$p_n \geq \lfloor n^2/4 \rfloor > n \quad \text{for all } n \geq 5, \quad (6)$$

$$p_n \geq \frac{1}{2}n(n+1) - 4 - \sqrt{2}n^{\frac{3}{2}} + \frac{3}{2}n > \lfloor n^2/4 \rfloor \quad \text{for all } n \geq 15, \quad (7)$$

$$p_n \leq \frac{1}{2}n(n+1) - 4 \quad \text{for all } n \geq 5, \quad (8)$$

$$\begin{aligned} \text{for all } k \in \{1, \dots, n-1\} \exists M \in \mathcal{CP}^n \text{ s.t. } \text{cp}(M) = \text{cp}^+(M) = k \\ \text{and we have } M \in \text{bd}(\mathcal{CP}^n), \end{aligned} \quad (9)$$

$$\text{for all } k \in \{n, \dots, p_n\} \exists M \in \text{int}(\mathcal{CP}^n) \text{ s.t. } \text{cp}(M) = \text{cp}^+(M) = k, \quad (10)$$

$$\text{for all } k \in \{n+1, \dots, p_n\} \exists M \in \text{int}(\mathcal{CP}^n) \text{ s.t. } k-1 = \text{cp}(M) \neq \text{cp}^+(M) = k. \quad (11)$$

Proof. (1) and (2) were proven in [SMBJS13], and (3) follows directly from (2) and Theorem 4.9. The leftmost inequality in (4) follows from (3). To prove the rightmost inequality in (4), we consider an arbitrary $M \in \mathcal{S}^n \setminus \{0\}$ such that $\text{cp}^+(M) < \infty$. From the definitions, there exists $\mathbf{v} \in \mathbb{R}_{++}^n$ such that $M - \mathbf{v}\mathbf{v}^\top \in \mathcal{CP}^n \setminus \{0\}$ and from Corollary 2.12 we have $\text{cp}^+(M) \leq \text{cp}^+(\mathbf{v}\mathbf{v}^\top) + \text{cp}(M - \mathbf{v}\mathbf{v}^\top) \leq 1 + p_n$.

While (5) and (6) are well known since long, see for example [BSM03], the bounds in (7) and (8) were established quite recently, namely in [SMBB⁺13] and in [BSU14b]. For $n = 5$ we have $p_n = \lfloor n^2/4 \rfloor$ [SMBJS13]. It was conjectured in [DJL94] that this equality holds for all $n \geq 5$, however counter examples to this conjecture for $n = 7, \dots, 11$ were recently presented in [BSU14a]. For $n \geq 15$ this conjecture is refuted by (7), and for $n = 12, 13, 14$ tighter lower bounds also refute it [BSU14b].

We shall now prove (9), (10) and (11). From Theorem 1.2 and Lemma 2.1, if $\text{cp}(M) < n$ then $M \in \text{bd}(\mathcal{CP}^n)$. From (2), (5), (6) and Theorem 4.9, there exists $M \in \text{int}(\mathcal{CP}^n)$ such that $\text{cp}(M) = \text{cp}^+(M) = p_n \geq n$, and thus statement (10) holds for $k = p_n$. From Theorem 1.2 and using that $\text{rank}(M) = \text{rank}(B)$ holds for $M = \mathbf{B}^\top \mathbf{B}$, there exists $\mathbf{b}_1, \dots, \mathbf{b}_{p_n} \in \mathbb{R}_{++}^n$ such that $\text{span}\{\mathbf{b}_1, \dots, \mathbf{b}_{p_n}\} = \mathbb{R}^n$ and $M = \sum_{i=1}^{p_n} \mathbf{b}_i \mathbf{b}_i^\top$. For all $k \in \{1, \dots, p_n\}$, $\theta \in [0, 1]$ we let $M_k(\theta) := \sum_{i=1}^{k-1} \mathbf{b}_i \mathbf{b}_i^\top + \theta \mathbf{b}_k \mathbf{b}_k^\top$. From Theorem 1.2 we have

$$\begin{aligned} M_k(\theta) \in \text{int}(\mathcal{CP}^n) & \quad \text{for all } k \in \{n, \dots, p_n\}, \theta \in (0, 1], \\ M_k(\theta) \in \text{bd}(\mathcal{CP}^n) & \quad \text{for all } k \in \{1, \dots, n-1\}, \theta \in [0, 1]. \end{aligned}$$

Furthermore, for all $k \in \{1, \dots, p_n\}$, $\theta \in [0, 1]$ we have $M = M_k(\theta) + (1-\theta)\mathbf{b}_k\mathbf{b}_k^\top + \sum_{i=k+1}^{p_n} \mathbf{b}_i\mathbf{b}_i^\top$, and thus by Lemma 2.3 we have

$$\begin{aligned} p_n = \text{cp}(M) &\leq \text{cp}(M_k(\theta)) + \text{cp}\left((1-\theta)\mathbf{b}_k\mathbf{b}_k^\top + \sum_{i=k+1}^{p_n} \mathbf{b}_i\mathbf{b}_i^\top\right) \\ &\leq \text{cp}(M_k(\theta)) + 1 + p_n - k. \end{aligned}$$

It is also trivial to see from the definitions that $\text{cp}(M_k(\theta)) \leq \text{cp}^+(M_k(\theta)) \leq k$. Combining these inequalities together, we get

$$k - 1 \leq \text{cp}(M_k(\theta)) \leq \text{cp}^+(M_k(\theta)) \leq k \quad \text{for all } k \in \{1, \dots, p_n\}, \theta \in [0, 1].$$

For all $k \in \{1, \dots, p_n - 1\}$ we have $M_{k+1}(0) = M_k(1)$ and thus using the above we get $\text{cp}(M_k(1)) = \text{cp}^+(M_k(1)) = k$, which completes the proof for statements (9) and (10). Similar arguments can also be found in [SMBB⁺13, Prop.4.1, Thm.4.1].

For an arbitrary $k \in \{n+1, \dots, p_n\}$, we now let $\vartheta_k = \sup_{\theta \in [0, 1]} \{\theta \mid \text{cp}(M_k(\theta)) = k-1\}$ and note by Corollary 2.8 that $0 < \vartheta_k < 1$. For all $\theta \in (\vartheta_k, 1]$ we have $k = \text{cp}(M_k(\theta)) = \text{cp}^+(M_k(\theta))$. Therefore, by Theorem 2.5, we have $k \leq \text{cp}^+(M_k(\vartheta_k))$, and thus $\text{cp}^+(M_k(\vartheta_k)) = k$. Additionally, for all $\varepsilon > 0$ there exists $\theta \in [\vartheta_k - \varepsilon, \vartheta_k]$ such that $k - 1 = \text{cp}(M_k(\theta))$. Therefore, by Theorem 2.4, we have $k - 1 \geq \text{cp}(M_k(\vartheta_k))$, and thus $\text{cp}(M_k(\vartheta_k)) = k - 1$, which completes the proof. \square

From the following lemma we get $p_n = p_n^+$ for $n = 2, 3, 4$. It is an open question whether this equality continues to hold for $n \geq 5$.

Lemma 5.2. *For $n = 2, 3, 4$ let $M \in \mathcal{CP}^n \cap \text{int}(\mathcal{N}^n)$. Then $\text{cp}^+(M) \leq p_n = n$.*

Proof. From Corollary 3.4, for $n = 2, 3$ we already have $\text{cp}(M) = \text{cp}^+(M)$. However, the following proof will be for a general $n = 2, 3, 4$, as nothing is lost in doing this.

We begin by recalling that for $n = 2, 3, 4$ we have $\mathcal{CP}^n = \mathcal{S}_+^n \cap \mathcal{N}^n$ and thus $\mathcal{CP}^n \cap \text{int}(\mathcal{N}^n) = \mathcal{S}_+^n \cap \text{int}(\mathcal{N}^n)$, see [MM62].

Let $M \in \mathcal{CP}^n \cap \text{int}(\mathcal{N}^n)$, with P-F vector \mathbf{x} . For $\varepsilon > 0$ small enough we have $P = (M - \varepsilon\mathbf{x}\mathbf{x}^\top) \in \text{int}(\mathcal{N}^n)$ and thus from Lemma 4.6, \mathbf{x} is also the P-F vector of P . When going from M to P , the only eigenvalue that we are affecting is the eigenvalue corresponding to \mathbf{x} , which remains strictly positive. Therefore we have $P \in \mathcal{S}_+^n$. This implies that $P \in \mathcal{CP}^n$, and thus $\text{cp}(P) \leq p_n$. Finally, from Theorem 4.8, we have $\text{cp}^+(M) \leq \text{cp}(P) \leq p_n$, completing the proof. \square

For $n \geq 5$ this lemma no longer holds, consider for example the following:

Example. In [DS08, Example 2.2], the authors showed that the following matrix is on the boundary of the completely positive cone:

$$B = \begin{pmatrix} 8 & 5 & 1 & 1 & 5 \\ 5 & 8 & 5 & 1 & 1 \\ 1 & 5 & 8 & 5 & 1 \\ 1 & 1 & 5 & 8 & 5 \\ 5 & 1 & 1 & 5 & 8 \end{pmatrix}.$$

We have $\text{rank}(B) = 5$, but $B \notin \text{int}(\mathcal{CP}^5)$, and thus from Theorem 1.2 we have $\text{cp}^+(B) = \infty$.

6 Genericity of the property $\text{cp}^+(M) = \text{cp}(M)$

6.1 Genericity vs. open and dense

In this section we consider the topological properties of the following set:

$$\mathcal{A}^n := \{M \in \mathcal{S}^n \mid \text{cp}(M) = \text{cp}^+(M)\}. \quad (12)$$

As usual, we say that a set $\mathcal{A} \subseteq \mathcal{S}^n$ is dense if for all $X \in \mathcal{S}^n$ and $\varepsilon > 0$ we have $N_\varepsilon(X) \cap \mathcal{A} \neq \emptyset$. From the results so far it follows that the set \mathcal{A}^n contains an open and dense subset of \mathcal{S}^n :

Theorem 6.1. *The set $\mathcal{A}^n := \mathcal{A}^n \setminus \text{bd}(\mathcal{CP}^n)$ is open and dense in \mathcal{S}^n .*

Proof. For an arbitrary $M \in \mathcal{S}^n$, we consider the following cases, which will complete the proof:

1. $M \notin \mathcal{CP}^n$: We have $M \in \mathcal{A}^n$, and as the set of completely positive matrices is closed, there exists $\varepsilon > 0$ such that $N_\varepsilon(M) \subseteq \mathcal{S}^n \setminus \mathcal{CP}^n \subseteq \mathcal{A}^n$.
2. $M \in \text{bd}(\mathcal{CP}^n)$: We have $M \notin \mathcal{A}^n$, and for all $\varepsilon > 0$ there exists $M_\varepsilon \in N_\varepsilon(M)$ such that $M_\varepsilon \in \mathcal{S}^n \setminus \mathcal{CP}^n \subseteq \mathcal{A}^n$.
3. $M \in \text{int}(\mathcal{CP}^n) \cap \mathcal{A}^n$: From Theorem 2.7, there exists $\varepsilon > 0$ such that $N_\varepsilon(M) \subseteq \mathcal{A}^n$.
4. $M \in \text{int}(\mathcal{CP}^n) \setminus \mathcal{A}^n$: From Theorem 4.9, for all $\varepsilon > 0$ we have $N_\varepsilon(M) \cap \mathcal{A}^n \neq \emptyset$. □

By this theorem we know that the set \mathcal{A}^n contains an open and dense set. But it is well known that for a set $\mathcal{A} \subset \mathcal{S}^n$, being dense and open does not necessarily imply that the Lebesgue measure of $\mathcal{S}^n \setminus \mathcal{A}$, denoted $\mu_L(\mathcal{S}^n \setminus \mathcal{A})$, is equal to zero. Indeed, the set of rational numbers, $\mathbb{Q} \subseteq \mathbb{R}$, is a well-known dense set with $\mu_L(\mathbb{Q}) = 0$ [Bea04, p.133]. Considering approximations of measurable sets [Bea04, p.139], for all $\varepsilon > 0$ there exists an open set \mathcal{A}_ε such that $\mathbb{Q} \subseteq \mathcal{A}_\varepsilon \subseteq \mathbb{R}$ and $\mu_L(\mathcal{A}_\varepsilon) \leq \mu_L(\mathbb{Q}) + \varepsilon = \varepsilon$. We then have that \mathcal{A}_ε is an open and dense set in \mathbb{R} with $\mu_L(\mathbb{R} \setminus \mathcal{A}_\varepsilon) = \infty \neq 0$.

In what follows we wish to strengthen the statement of Theorem 6.1, and we will show that the membership in \mathcal{A}^n is a generic property.

Recall that in topology for a subset $\mathcal{A} \subseteq \mathbb{R}^N$, we say that membership in \mathcal{A} is generic in \mathbb{R}^N if \mathcal{A} contains a set \mathcal{A}_0 such that the following two statements hold:

1. the set \mathcal{A}_0 is open, and
2. the Lebesgue measure of $\mathbb{R}^N \setminus \mathcal{A}_0$ is equal to zero.

Statement (1) means that membership in \mathcal{A}_0 is stable for small variations. Statement (2) means that ‘almost all’ elements of \mathbb{R}^N are in \mathcal{A}_0 (and thus also in \mathcal{A}).

In the next subsection we prove that indeed membership in \mathcal{A}^n is a generic property.

6.2 Semi-algebraic sets

In order to show that membership in the set \mathcal{A}^n is generic we make use of the theory of semi-algebraic sets and only need the density part of Theorem 6.1.

We prove that \mathcal{A}^n is a semi-algebraic set and apply the fact that for a semi-algebraic set, being dense is a sufficient condition for membership in the set being generic. We note that similar arguments have been used recently to obtain genericity results in cone programming [BDL11].

The results on semi-algebraic sets will be stated for the space \mathbb{R}^N . The results can then be trivially applied to the space $\mathcal{S}^n \equiv \mathbb{R}^{(n+1)n/2}$.

We begin by recalling some preliminary definitions and results on semi-algebraic sets (see [BR90, GWdL76]).

Definition 6.2. A set $\mathcal{A} \subset \mathbb{R}^N$ is called semi-algebraic if it is given by a finite union of sets of the form

$$\{\mathbf{x} \in \mathbb{R}^N \mid p_i(\mathbf{x}) = 0 \text{ for all } i = 1, \dots, k, \quad q_j(\mathbf{x}) > 0 \text{ for all } j = 1, \dots, s\}$$

with $k, s \in \mathbb{N}$ and polynomial functions $p_i, q_j \in \mathbb{R}[\mathbf{x}]$.

Remark 6.3. Since $\{\mathbf{x} \mid p(\mathbf{x}) \geq 0\} = \{\mathbf{x} \mid p(\mathbf{x}) = 0\} \cup \{\mathbf{x} \mid p(\mathbf{x}) > 0\}$, also sets defined by polynomial inequalities $p(\mathbf{x}) \geq 0$ are semi-algebraic.

The following theorem states some well-known facts on semi-algebraic sets.

Theorem 6.4. For $N, M \in \mathbb{N}$, consider semi-algebraic sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^N$ and a polynomial function $h : \mathbb{R}^N \rightarrow \mathbb{R}^M$ (e.g. a projection operator). Then the following sets are also semi-algebraic:

$$\mathcal{A} \cup \mathcal{B}, \quad \mathcal{A} \cap \mathcal{B}, \quad \mathcal{A} \setminus \mathcal{B}, \quad h(\mathcal{A}).$$

Proof. For a proof we refer to [BR90, Section 2.1–2.3]. □

We shall now show that the set \mathcal{A}^n from (12) is a semi-algebraic set.

Lemma 6.5. The set \mathcal{A}^n from (12) is semi-algebraic.

Proof. Recalling from the definition that the union of finitely many semi-algebraic sets is also semi-algebraic and recalling from Theorem 5.1 that we have $p_n < \frac{1}{2}n(n+1)$, it is sufficient to show that the following set is semi-algebraic for all $p \in \mathbb{N} \cup \{\infty\}$:

$$\mathcal{A}_p^n := \{A \in \mathcal{S}^n \mid \text{cp}(A) = \text{cp}^+(A) = p\}.$$

For all $p \in \mathbb{N}$ we have that the following sets are trivially semi-algebraic:

$$\begin{aligned} \mathcal{E} &:= \{(X, V) \in \mathcal{S}^n \times \mathbb{R}^{p \times n} \mid v_{ij} \geq 0 \text{ for all } i, j, \quad X = V^T V\}, \\ \mathcal{F} &:= \{(X, V) \in \mathcal{S}^n \times \mathbb{R}^{p \times n} \mid v_{ij} > 0 \text{ for all } i, j, \quad X = V^T V\}. \end{aligned}$$

From Theorem 6.4, the projections $\text{proj}_X(\mathcal{E})$ and $\text{proj}_X(\mathcal{F})$ are also semi-algebraic for all $p \in \mathbb{N}$; but obviously

$$\text{proj}_X(\mathcal{E}) = \{X \in \mathcal{S}^n \mid \text{cp}(X) \leq p\} \quad \text{and} \quad \text{proj}_X(\mathcal{F}) = \{X \in \mathcal{S}^n \mid \text{cp}^+(X) \leq p\}.$$

Therefore again considering Theorem 5.1 and Theorem 6.4, the following sets are semi-algebraic for all $p \in \mathbb{N}$:

$$\begin{aligned} \{X \in \mathcal{S}^n \mid \text{cp}(X) = \infty\} &= \mathcal{S}^n \setminus \{X \in \mathcal{S}^n \mid \text{cp}(X) \leq p_n\}, \\ \{X \in \mathcal{S}^n \mid \text{cp}^+(X) = \infty\} &= \mathcal{S}^n \setminus \{X \in \mathcal{S}^n \mid \text{cp}^+(X) \leq p_n^+\}, \quad \text{as well as} \\ \{X \in \mathcal{S}^n \mid \text{cp}(X) = p\} &= \{X \in \mathcal{S}^n \mid \text{cp}(X) \leq p\} \setminus \{X \in \mathcal{S}^n \mid \text{cp}(X) \leq p-1\}, \\ \{X \in \mathcal{S}^n \mid \text{cp}^+(X) = p\} &= \{X \in \mathcal{S}^n \mid \text{cp}^+(X) \leq p\} \setminus \{X \in \mathcal{S}^n \mid \text{cp}^+(X) \leq p-1\}. \end{aligned}$$

Since

$$\mathcal{A}_p^n = \{X \in \mathcal{S}^n \mid \text{cp}(X) = p\} \cap \{X \in \mathcal{S}^n \mid \text{cp}^+(X) = p\},$$

this finally implies that also \mathcal{A}_p^n is semi-algebraic for all $p \in \mathbb{N} \cup \{\infty\}$. \square

We can also combine Theorem 6.4 with other well-known results to obtain the following which may be of general interest in algebraic geometry:

Theorem 6.6. *Let $\mathcal{A} \subseteq \mathbb{R}^N$ be a semi-algebraic set. Then the membership in \mathcal{A} is generic if and only if \mathcal{A} is dense in \mathbb{R}^N .*

Proof. The forward implication is trivial. To prove the reverse implication we make use of the following facts on semi-algebraic sets.

From [GwDL76, 2.7], we have that any semi-algebraic set $\mathcal{A} \subset \mathbb{R}^N$ admits a (stratification) partition $\mathcal{A} = \bigcup_{i=0}^d \mathcal{S}_i$ with some $d \in \mathbb{N}$ such that

1. $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$ for $i \neq j$ and
2. the sets \mathcal{S}_i are smooth manifolds of \mathbb{R}^N of dimension i (or are empty).

It is a well-known result, see for example [GP74], that the manifolds of dimension N in \mathbb{R}^N are precisely the open sets in \mathbb{R}^N . Furthermore, manifolds of dimension $k < N$ in \mathbb{R}^N have Lebesgue measure zero (*cf.*, *e.g.*, [GP74, p.45]).

We first consider the set $\mathcal{M} = \mathbb{R}^N \setminus \mathcal{A}$, and note from Theorem 6.4 that this set is semi-algebraic. So \mathcal{M} allows a stratification $\mathcal{M} = \bigcup_{i=0}^d \mathcal{S}_i$ with some $d \in \mathbb{N}$. As \mathcal{A} is dense, \mathcal{M} cannot contain any open sets. This implies that for all $0 \leq i \leq d$ we have $\dim \mathcal{S}_i < N$ and thus $\mu_L(\mathcal{S}_i) = 0$, implying:

$$0 \leq \mu_L(\mathcal{M}) = \mu_L\left(\bigcup_{i=0}^d \mathcal{S}_i\right) \leq \sum_{i=0}^d \mu_L(\mathcal{S}_i) = 0.$$

Now we take the semi-algebraic set \mathcal{A} and a stratification $\mathcal{A} = \bigcup_{i=0}^q \tilde{\mathcal{S}}_i$ with some $q \in \mathbb{N}$.

We claim that the manifold $\tilde{\mathcal{S}}_q$ with (highest) dimension q must be of dimension $q = N$. So by the remark above, $\tilde{\mathcal{S}}_q$ must be an open set. Indeed, the condition $\dim \tilde{\mathcal{S}}_q < N$ would also imply $\mu_L(\mathcal{A}) = 0$ and then

$$\mu_L(\mathbb{R}^N) = \mu_L((\mathbb{R}^N \setminus \mathcal{A}) \cup \mathcal{A}) \leq \mu_L(\mathbb{R}^N \setminus \mathcal{A}) + \mu_L(\mathcal{A}) = 0,$$

a contradiction. Altogether we have shown that the set \mathcal{A} contains the open set $\mathcal{A}_0 := \tilde{\mathcal{S}}_N$ with complement

$$\mathbb{R}^N \setminus \mathcal{A}_0 = (\mathcal{A} \setminus \tilde{\mathcal{S}}_N) \cup (\mathbb{R}^N \setminus \mathcal{A}) = \left(\bigcup_{i=0}^{N-1} \tilde{\mathcal{S}}_i\right) \cup (\mathbb{R}^N \setminus \mathcal{A})$$

of Lebesgue measure

$$\mu_L(\mathbb{R}^N \setminus \mathcal{A}_0) \leq \sum_{i=0}^{N-1} \mu_L(\tilde{\mathcal{S}}_i) + \mu_L(\mathbb{R}^N \setminus \mathcal{A}) = 0.$$

So membership in the set \mathcal{A} is a generic property. \square

We are now ready to present the main result of this section.

Theorem 6.7. *Membership in the set \mathcal{A}^n from (12) is generic in \mathcal{S}^n .*

Proof. By Theorem 6.1 the set \mathcal{A}^n is dense in \mathcal{S}^n . The result then follows by Lemma 6.5 and Theorem 6.6. \square

Corollary 6.8. *The following properties are generic within the completely positive cone:*

1. *Having infinitely many minimal cp-decompositions,*
2. *The cp- and cp-plus-ranks being equal and locally constant.*

Proof. From Theorems 1.2 to 1.4 and Lemma 2.1 it is sufficient to show that membership of the open set $\mathcal{A}^n \cap \text{int}(\mathcal{CP}^n)$ is generic in \mathcal{CP}^n . Since \mathcal{CP}^n is convex we have $\mu_L(\text{bd}(\mathcal{CP}^n)) = 0$ (see e.g. [Lan86]), and from Theorem 6.7 we have $\mu_L(\mathcal{S}^n \setminus \mathcal{A}^n) = 0$. The proof is then completed by noting the following:

$$\begin{aligned} \mu_L(\mathcal{CP}^n \setminus (\mathcal{A}^n \cap \text{int}(\mathcal{CP}^n))) &= \mu_L((\mathcal{CP}^n \setminus \mathcal{A}^n) \cup \text{bd}(\mathcal{CP}^n)) \\ &\leq \mu_L(\mathcal{S}^n \setminus \mathcal{A}^n) + \mu_L(\text{bd}(\mathcal{CP}^n)) = 0. \end{aligned} \quad \square$$

7 Concluding Remarks

In this paper we studied the distribution of completely positive matrices according to their cp- and cp-plus-ranks. One interesting result found was that whereas it was previously known that in a sufficiently small neighbourhood of a matrix $M \in \mathcal{S}^n$ the cp-rank cannot go down, we have shown that in a sufficiently small neighbourhood of a matrix $M \in \mathcal{S}^n \setminus \text{bd}(\mathcal{CP}^n)$ the cp-plus-rank can not go up. As the cp-plus-rank of a matrix is an upper bound on its cp-rank, this means that for a matrix $M \in \mathcal{S}^n \setminus \text{bd}(\mathcal{CP}^n)$ with its cp-rank equal to its cp-plus rank, in a sufficiently small neighbourhood of the matrix, neither the cp-rank nor the cp-plus-rank will change.

Motivated by this result we considered the open sets

$$\{M \in \mathcal{S}^n \setminus \text{bd}(\mathcal{CP}^n) \mid \text{cp}(M) = \text{cp}^+(M) = p\},$$

which were shown to be nonempty for all $p \in \{n, \dots, p_n, \infty\}$. An interesting open question is whether these are also connected sets.

We have also established that the sets

$$\{M \in \text{int}(\mathcal{CP}^n) \mid k - 1 = \text{cp}(M) < \text{cp}^+(M) = k\}$$

are nonempty for all $k \in \{n + 1, \dots, p_n\}$.

Considering the set $\mathcal{A}^n = \{M \in \mathcal{S}^n \mid \text{cp}(M) = \text{cp}^+(M)\}$, we have shown that this is dense in \mathcal{S}^n and open in $\mathcal{S}^n \setminus \text{bd}(\mathcal{CP}^n)$. By applying the theory of semi-algebraic sets we in addition established that membership in \mathcal{A}^n is a generic property in \mathcal{S}^n .

Some interesting questions are still open: For example, is the set $\mathcal{A}^n \setminus \{0\}$ open in \mathcal{S}^n ? Note that around the zero matrix the set of matrices M satisfying $\text{cp}(M) = \text{cp}^+(M)$ is not open. Indeed, take any matrix $B \in \mathcal{CP}^n$ with $\text{cp}(B) \neq \text{cp}^+(B)$. For all $\lambda > 0$ we have $\text{cp}(\lambda B) = \text{cp}(B) \neq \text{cp}^+(B) = \text{cp}^+(\lambda B)$, but for $A = 0$ we have $A = \lim_{\lambda \searrow 0} \lambda B$ and $\text{cp}(A) = \text{cp}^+(A) = 0$.

On the other hand in contrast to the behaviour on \mathcal{S}^n the set of matrices \mathcal{A}^n is not dense on $\text{bd}(\mathcal{CP}^n)$. Indeed there exist matrices $A \in \text{bd}(\mathcal{CP}^n)$ and $\varepsilon > 0$, such that for all $M \in \text{bd}(\mathcal{CP}^n) \cap N_\varepsilon(A)$ we have $\text{cp}(M) \neq \text{cp}^+(M) = \infty$. Take for example the identity matrix $I \in \text{bd}(\mathcal{CP}^n)$. Since I has full rank n , by Theorem 1.2 we must have $\text{cp}^+(M) = \infty$ (otherwise I would be in the interior of \mathcal{CP}^n) and this argument holds for all $M \in \text{bd}(\mathcal{CP}^n) \cap N_\varepsilon(I)$ for some $\varepsilon > 0$.

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References

- [BDL11] Jérôme Bolte, Aris Daniilidis, and Adrian S. Lewis. Generic optimality conditions for semialgebraic convex programs. *Mathematics of Operations Research*, 36(1):55–70, 2011.
- [Bea04] Richard Beals. *Analysis*. Cambridge University Press, 2004.
- [Bom12] Immanuel M. Bomze. Copositive optimization – recent developments and applications. *European J. Oper. Res.*, 216:509–520, 2012.
- [BR90] Riccardo Benedetti and Jean-Jacques Risler. *Real algebraic and semi-algebraic sets*. Hermann, Éditeurs des Sciences et des Arts, Paris, 1990.
- [BSM03] Abraham Berman and Naomi Shaked-Monderer. *Completely Positive Matrices*. World Scientific, 2003.
- [BSU14a] Immanuel M. Bomze, Werner Schachinger, and Reinhard Ullrich. From seven to eleven: Completely positive matrices with high cp-rank. *Linear Algebra Appl.*, 459:208–221, 2014.

- [BSU14b] Immanuel M. Bomze, Werner Schachinger, and Reinhard Ullrich. New lower bounds and asymptotics for the cp-rank. *SIAM Journal on Matrix Analysis and Applications*, to appear. Also available as: INI14048–POP, Isaac Newton Institute, Cambridge UK, 2014.
- [Bur12] Samuel Burer. Copositive programming. In Miguel F. Anjos and Jean Bernard Lasserre, editors, *Handbook of Semidefinite, Cone and Polynomial Optimization: Theory, Algorithms, Software and Applications*, International Series in Operations Research and Management Science, pages 201–218. Springer, New York, 2012.
- [Dic10] Peter J.C. Dickinson. An improved characterisation of the interior of the completely positive cone. *Electronic Journal of Linear Algebra*, 20:723–729, 2010.
- [Dic13] Peter J.C. Dickinson. *The Copositive Cone, the Completely Positive Cone and their Generalisations*. PhD thesis, University of Groningen, 2013.
- [DJL94] John H. Drew, Charles R. Johnson, and Raphael Loewy. Completely positive matrices associated with M-matrices. *Linear and Multilinear Algebra*, 37:303–310, 1994.
- [DS08] Mirjam Dür and Georg Still. Interior points of the completely positive cone. *Electronic Journal of Linear Algebra*, 17:48–53, 2008.
- [Dür10] Mirjam Dür. Copositive programming — a survey. In Moritz Diehl, Francois Glineur, Elias Jarlebring, and Wim Michiels, editors, *Recent Advances in Optimization and its Applications in Engineering*, pages 3–20. Springer, Berlin Heidelberg New York, 2010.
- [GP74] Victor Guillemin and Alan Pollack. *Differential Topology*. Prentice-Hall, Englewood Cliffs, N.J., 1974.
- [GWL76] Christopher G. Gibson, Klaus Wirthmüller, Andrew A. du Plessis, and Eduard J. N. Looijenga. *Topological Stability of Smooth Mappings*. Springer, 1976.
- [Lan86] Robert Lang. A note on the measurability of convex sets. *Archiv der Mathematik*, 47(1):90–92, 1986.
- [MM62] John E. Maxfield and Henryk Minc. On the matrix equation $X'X = A$. *Proceedings of the Edinburgh Mathematical Society (Series 2)*, 13(02):125–129, 1962.
- [SMBB⁺13] Naomi Shaked-Monderer, Abraham Berman, Immanuel M. Bomze, Florian Jarre, and Werner Schachinger. New results on the cp rank and related properties of co(mpletely)positive matrices. *Linear Multilinear Algebra*, to appear. Also available at: arxiv.org/abs/1305.0737, 2013.
- [SMBJS13] Naomi Shaked-Monderer, Immanuel M. Bomze, Florian Jarre, and Werner Schachinger. On the cp-rank and minimal cp factorizations of a completely positive matrix. *SIAM Journal on Matrix Analysis and Applications*, 34(2):355–368, 2013.

- [SSMS13] Bryan Shader, Naomi Shaked-Monderer, and Daniel B. Szyld. Nearly positive matrices. *Preprint*, 2013.
- [Xu04] Changqing Xu. Completely positive matrices. *Linear Algebra Appl.*, 379:319–327, 2004.