Free Boundary Problems in Shock Reflection/Diffraction and Related Transonic Flow Problems

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Shock waves are steep wave fronts that are fundamental in nature, especially in high-speed fluid flows. When a shock hits an obstacle, or a flying body meets a shock, shock reflection/diffraction phenomena occur. In this paper, we show how several longstanding shock reflection/diffraction problems can be formulated as free boundary problems, discuss some recent progress in developing mathematical ideas, approaches, and techniques for solving these problems, and present some further open problems in this direction. In particular, these shock problems include von Neumann's problem for shock reflection-diffraction by two-dimensional wedges with concave corner, Lighthill's problem for shock diffraction by two-dimensional wedges with convex corner, and Prandtl-Meyer's problem for supersonic flow impinging onto solid wedges, which are also fundamental in the mathematical theory of multidimensional conservation laws.

Key words: Free boundary, shock wave, reflection, diffraction, transonic flow, von Neumann's problem, Lighthill's problem, Prandtl-Meyer configuration, transition criterion, Riemann problem, mixed elliptic-hyperbolic type, mixed equation, existence, stability, regularity, a priori estimates, iteration scheme, entropy solutions, global solutions.

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1. Introduction

Shock waves are steep fronts that propagate in the compressible fluids in which convection dominates diffusion. They are fundamental in nature, especially in high-speed fluid flows. Examples include transonic and/or supersonic shocks formed by supersonic flows impinging onto solid wedges, transonic shocks around supersonic or near sonic flying bodies, bow shocks created by solar winds in space, blast waves by explosions, and other shocks by various natural processes. When a shock hits an obstacle, or a flying body meets a shock, shock reflection/diffraction phenomena occur.

Many of such shock reflection/diffraction problems can be formulated as free boundary problems involving nonlinear partial differential equations (PDEs) of mixed elliptic-hyperbolic type. The understanding of these shock reflection/diffraction phenomena requires a complete mathematical solution of the corresponding free boundary problems for nonlinear mixed PDEs. In this paper, we show how several longstanding, fundamental multidimensional shock

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problems can be formulated as free boundary problems, discuss some recent progress in developing mathematical ideas, approaches, and techniques for solving these problems, and present some further open problems in this direction. In particular, these shock problems include von Neumann's problem for shock reflection-diffraction by two-dimensional wedges with concave corner, Lighthill's problem for shock diffraction by two-dimensional wedges with convex corner, and Prandtl-Meyer's problem for supersonic flow impinging onto solid wedges. These problems are not only longstanding open problems in fluid mechanics, but also fundamental in the mathematical theory of multidimensional conservation laws: These shock reflection/diffration configurations are the core configurations in the structure of global entropy solutions of the two-dimensional Riemann problem for hyperbolic conservation laws, while the Riemann solutions are building blocks and local structure of general solutions and determine global attractors and asymptotic states of entropy solutions, as time tends to infinity, for multidimensional hyperbolic systems of conservation laws; see [17, 33, 53, 62] and the references cited therein. In this sense, we have to understand the shock reflection/diffraction phenomena in order to understand fully global entropy solutions to multidimensional hyperbolic systems of conservation laws.

We first focus on these problems for the Euler equations for potential flow. The unsteady potential flow is governed by the conservation law of mass and Bernoulli's law:

$$\partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \nabla_{\mathbf{x}} \Phi) = 0, \tag{1.1}$$

$$\partial_t \Phi + \frac{1}{2}|\nabla_{\mathbf{x}} \Phi|^2 + h(\rho) = B \tag{1.2}$$

for the density ρ and the velocity potential Φ, where the Bernoulli constant B is determined by the incoming flow and/or boundary conditions, and $h(\rho)$ satisfies the relation

$$h'(\rho) = \frac{p'(\rho)}{\rho} = \frac{c^2(\rho)}{\rho}$$

with $c(\rho)$ being the sound speed, and p is the pressure that is a function of the density ρ. For an ideal polytropic gas, the pressure p and the sound speed c are given by $p(\rho) = \kappa\rho^\gamma$ and $c^2(\rho) = \kappa\gamma\rho^{\gamma-1}$ for constants $\gamma > 1$ and $\kappa > 0$. Without loss of generality, we may choose $\kappa = 1/\gamma$ to have

$$h(\rho) = \frac{\rho^{\gamma-1} - 1}{\gamma - 1}, \quad c^2(\rho) = \rho^{\gamma-1}. \tag{1.3}$$

This can be achieved by the following scaling: $(t, \mathbf{x}, B) \longrightarrow (\alpha^2 t, \alpha\mathbf{x}, \alpha^{-2}B)$ with $\alpha^2 = \kappa\gamma$. Taking the limit $\gamma \to 1+$, we can also consider the case of the isothermal flow ($\gamma = 1$), for which

$$i(\rho) = \ln\rho, \qquad c^2(\rho) \equiv 1.$$

In §2–§4, we first show how the shock problems can be formulated as free boundary problems for the Euler equations for potential flow and discuss recently developed mathematical ideas, approaches, and techniques for solving these free boundary problems. Then, in §5, we present mathematical formulations of these shock problems for the full Euler equations and discuss the role of the Euler equations for potential flow, (1.1)–(1.2), in these shock problems in the realm of the full Euler equations. Some further open problems in the direction are also addressed.

2. Shock Reflection-Diffraction and Free Boundary Problems

We are first concerned with von Neumann's problem for shock reflection-diffraction in [56, 57, 58]. When a vertical planar shock perpendicular to the flow direction x_1 and separating two uniform states (0) and (1), with constant velocities $(u_0, v_0) = (0, 0)$ and $(u_1, v_1) = (u_1, 0)$, and constant densities $\rho_1 > \rho_0$ (state (0) is ahead or to the right of the shock, and state (1) is behind the shock), hits a symmetric wedge:

$$W := \{(x_1, x_2) : |x_2| < x_1 \tan \theta_{\mathrm{w}}, x_1 > 0\}$$

head on at time $t = 0$, a reflection-diffraction process takes place when $t > 0$. Then a fundamental question is what type of wave patterns of reflection-diffraction configurations may be formed around the wedge. The complexity of reflection-diffraction configurations was first reported by Ernst Mach [47] in 1878, who first observed two patterns of reflection-diffraction configurations: Regular reflection (two-shock configuration; see Fig. 1 (left)) and Mach reflection (three-shock/one-vortex-sheet configuration; see Fig. 1 (center)); also see [5, 17, 28, 55]. The issues remained dormant until the 1940s when John von Neumann [56, 57, 58], as well as other mathematical/experimental scientists (*cf.* [5, 17, 28, 33, 55] and the references cited therein) began extensive research into all aspects of shock reflection-diffraction phenomena, due to its importance in applications. It has been found that the situations are much more complicated than what Mach originally observed: The Mach reflection can be further divided into more specific sub-patterns, and various other patterns of shock reflection-diffraction configurations may occur such as the double Mach reflection, von Neumann reflection, and Guderley reflection; see [5, 17, 28, 33, 55] and the references cited therein. Then the fundamental scientific issues include:

(i) Structure of the shock reflection-diffraction configurations;

(ii) Transition criteria between the different patterns of shock reflection-diffraction configurations;

(iii) Dependence of the patterns upon the physical parameters such as the wedge-angle θ_{w}, the incident-shock-wave Mach number, and the adiabatic exponent $\gamma \geq 1$.

In particular, several transition criteria between the different patterns of shock reflection-diffraction configurations have been proposed, including the sonic conjecture and the detachment conjecture by von Neumann [56, 57, 58].

Careful asymptotic analysis has been made for various reflection-diffraction configurations in Lighthill [44, 45], Keller-Blank [37], Hunter-Keller [36], Harabetian [35], Morawetz [49], and the references cited therein; also see Glimm-Majda [33]. Large or small scale numerical simulations have been also made; *cf.* [5, 33, 61] and the references cited therein. However, most of the fundamental issues for shock reflection-diffraction phenomena have not been understood,

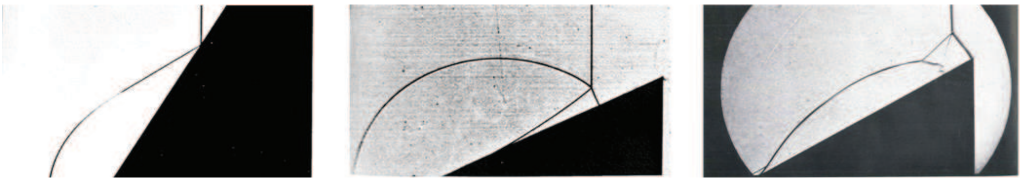

Figure 1. Regular reflection (left); Simple Mach reflection (center); Irregular Mach reflection (right). From Van Dyke [55], pp. 142–144

especially the global structure and transition between the different patterns of shock reflection-diffraction configurations. This is partially because physical and numerical experiments are hampered by various difficulties and have not yielded clear transition criteria between the different patterns. In particular, numerical dissipation or physical viscosity smear the shocks and cause boundary layers that interact with the reflection-diffraction patterns and can cause spurious Mach steams; *cf.* [61]. Furthermore, some different patterns occur when the wedge angles are only fractions of a degree apart, a resolution even by sophisticated modern experiments (*cf.* [46]) has not been able to reach. For this reason, it is almost impossible to distinguish experimentally between the sonic and detachment criteria, as pointed out in [5]. In this regard, the necessary approach to understand fully the shock reflection-diffraction phenomena, especially the transition criteria, is still via rigorous mathematical analysis. To achieve this, it is essential to formulate the shock reflection-diffraction problem as a free boundary problem and establish the global existence, regularity, and structural stability of its solution.

Mathematically, the shock reflection-diffraction problem is a multidimensional lateral Riemann problem in domain $\mathbb{R}^2 \setminus \bar{W}$.

Problem 2.1 (Lateral Riemann Problem)**.** *Piecewise constant initial data, consisting of state* (0) *on* $\{x_1 > 0\} \setminus \bar{W}$ *and state* (1) *on* $\{x_1 < 0\}$ *connected by a shock at* $x_1 = 0$*, are prescribed at* $t = 0$*. Seek a solution of the Euler system* (1.1)–(1.2) *for* $t \geq 0$ *subject to these initial data and the boundary condition* $\nabla\Phi \cdot \boldsymbol{\nu} = 0$ *on* ∂W*.*

Notice that **Problem 2.1** is invariant under the scaling:

$$(t, \mathbf{x}) \to (\alpha t, \alpha \mathbf{x}), \quad (\rho, \Phi) \to (\rho, \alpha \Phi) \qquad \text{for } \alpha \neq 0. \tag{2.1}$$

Thus, we seek self-similar solutions in the form of

$$\rho(t, \mathbf{x}) = \rho(\xi, \eta), \quad \Phi(t, \mathbf{x}) = t\phi(\xi, \eta) \qquad \text{for } (\xi, \eta) = \frac{\mathbf{x}}{t}. \tag{2.2}$$

Then the pseudo-potential function $\varphi = \phi - \frac{1}{2}(\xi^2 + \eta^2)$ satisfies the following Euler equations for self-similar solutions:

$$\operatorname{div}(\rho D\varphi) + 2\rho = 0, \tag{2.3}$$

$$\frac{\rho^{\gamma-1} - 1}{\gamma - 1} + (\frac{1}{2}|D\varphi|^2 + \varphi) = B, \tag{2.4}$$

where the divergence div and gradient D are with respect to (ξ,η). From this, we obtain the following second-order nonlinear PDE for $\varphi(\xi,\eta)$:

$$\operatorname{div}\big(\rho(|D\varphi|^2,\varphi)D\varphi\big)+2\rho(|D\varphi|^2,\varphi)=0 \tag{2.5}$$

with

$$\rho(|D\varphi|^2,\varphi)=\big(B_0-(\gamma-1)(\frac{1}{2}|D\varphi|^2+\varphi)\big)^{\frac{1}{\gamma-1}}, \tag{2.6}$$

where $B_0:=(\gamma-1)B+1$. Then we have

$$c^2(|D\varphi|^2,\varphi)=B_0-(\gamma-1)\big(\frac{1}{2}|D\varphi|^2+\varphi\big). \tag{2.7}$$

Equation (2.5) is a *nonlinear PDE of mixed elliptic-hyperbolic type*. It is elliptic if and only if

$$|D\varphi|<c(|D\varphi|^2,\varphi). \tag{2.8}$$

If ρ is a constant, then, by (2.5) and (2.6), the corresponding pseudo-potential φ is in the form of

$$\varphi(\xi,\eta)=-\frac{1}{2}(\xi^2+\eta^2)+u\xi+v\eta+k$$

for constants u, v, and k.

Then **Problem 2.1** is reformulated as a boundary value problem in unbounded domain

$$\Lambda:=\mathbb{R}^2\setminus\{(\xi,\eta)\,:\,|\eta|\le\xi\tan\theta_{\rm w},\xi>0\}$$

in the self-similar coordinates (ξ,η).

Problem 2.2 (Boundary Value Problem)**.** *Seek a solution φ of equation* (2.5) *in the self-similar domain Λ with the slip boundary condition $D\varphi\cdot\boldsymbol{\nu}|_{\partial\Lambda}=0$ on the wedge boundary $\partial\Lambda$ and the asymptotic boundary condition at infinity:*

$$\varphi\to\bar\varphi=\begin{cases}\varphi_0 & \text{for } \xi>\xi_0, |\eta|>\xi\tan\theta_{\rm w},\\ \varphi_1 & \text{for } \xi<\xi_0.\end{cases} \qquad \text{when } \xi^2+\eta^2\to\infty,$$

where

$$\varphi_0=-\frac{\xi^2+\eta^2}{2},\qquad \varphi_1=-\frac{\xi^2+\eta^2}{2}+u_1(\xi-\xi_0),$$

and $\xi_0=\rho_1\sqrt{\frac{2(c_1^2-c_0^2)}{(\gamma-1)(\rho_1^2-\rho_0^2)}}=\frac{\rho_1u_1}{\rho_1-\rho_0}$ *is the location of the incident shock in the* (ξ,η)*–coordinates.*

By symmetry, we can restrict to the upper half-plane $\{\eta>0\}\cap\Lambda$, with condition $\partial_{\boldsymbol{\nu}}\varphi=0$ on $\{\eta=0\}\cap\Lambda$.

A shock is a curve across which $D\varphi$ is discontinuous. If Ω^+ and $\Omega^-(:=\Omega\setminus\overline{\Omega^+})$ are two nonempty open subsets of $\Omega\subset\mathbb{R}^2$, and $S:=\partial\Omega^+\cap\Omega$ is a C^1-curve where $D\varphi$ has a jump, then $\varphi\in W^{1,1}_{\rm loc}\cap C^1(\Omega^\pm\cup S)\cap C^2(\Omega^\pm)$ is a global weak solution

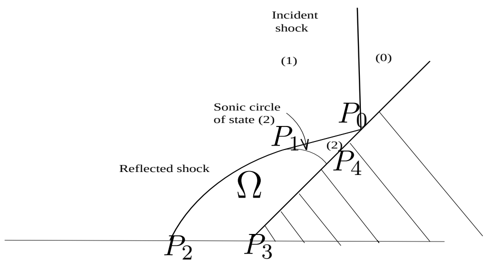

Figure 2. Supersonic regular reflection

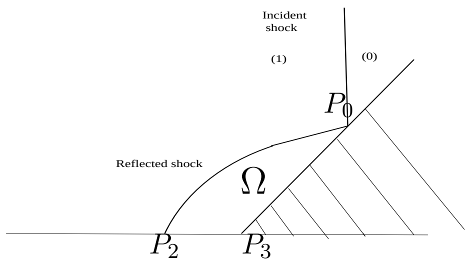

Figure 3. Subsonic regular reflection

of (2.5) in Ω if and only if φ is in $W^{1,\infty}_{loc}(\Omega)$ and satisfies equation (2.5) and the Rankine-Hugoniot condition on S:

$$[\rho(|D\varphi|^2, \varphi)D\varphi \cdot \boldsymbol{\nu}_{\rm s}]_S = 0, \tag{2.9}$$

and the physical entropy condition: *The density function $\rho(|D\varphi|^2, \varphi)$ increases across S in the relative flow direction with respect to S*, where $[F]_S$ is defined by

$$[F(\xi,\eta)]_S := F(\xi,\eta)|_{\overline{\Omega^-}} - F(\xi,\eta)|_{\overline{\Omega^+}} \qquad \text{for } (\xi,\eta) \in S,$$

and $\boldsymbol{\nu}_{\rm s}$ is a unit normal on S.

Note that the condition $\varphi \in W^{1,\infty}_{\rm loc}(\Omega)$ requires another Rankine-Hugoniot condition on S:

$$[\varphi]_S = 0. \tag{2.10}$$

If a solution has one of the regular reflection-diffraction configurations as shown in Figs. 2–3, and if φ is smooth in the subregion between the wedge and reflected shock, then it should satisfy the boundary condition $D\varphi \cdot \boldsymbol{\nu} = 0$ and the Rankine-Hugoniot conditions (2.9)–(2.10) at P_0 across the reflected shock separating it from state (1). We define the uniform state (2) with pseudo-potential $\varphi_2(\xi,\eta)$ such that

$$\varphi_2(P_0) = \varphi_1(P_0), \quad D\varphi_2(P_0) = D\varphi_1(P_0),$$

and the constant density ρ_2 of state (2) is equal to $\rho(|D\varphi|^2, \varphi)(P_0)$ defined by (2.5):

$$\rho_2 = \rho(|D\varphi|^2, \varphi)(P_0).$$

Then $D\varphi_2 \cdot \boldsymbol{\nu} = 0$ on the wedge boundary, and the Rankine-Hugoniot conditions (2.9)–(2.10) hold on the flat shock $S_1 = \{\varphi_1 = \varphi_2\}$ between states (1) and (2), which passes through P_0.

State (2) can be either subsonic or supersonic at P_0. This determines the subsonic or supersonic type of regular reflection-diffraction configurations. The supersonic regular reflection-diffraction configuration as shown in Fig. 2 consists of three uniform states (0), (1), (2), and a non-uniform state in domain Ω, where the equation is elliptic. The reflected shock $P_0P_1P_2$ has a straight part P_0P_1. The elliptic domain Ω is separated from the hyperbolic region $P_0P_1P_4$ of state (2) by a sonic arc P_1P_4. The subsonic regular reflection-diffraction configuration as shown in Fig. 3 consists of two uniform states (0) and (1), and a non-uniform state in domain Ω, where the equation is elliptic, and $\varphi_{|\Omega}(P_0) = \varphi_2(P_0)$ and $D(\varphi_{|\Omega})(P_0) = D\varphi_2(P_0)$.

Thus, a necessary condition for the existence of regular reflection-diffraction solution is the existence of the uniform state (2) determined by the conditions described above. These conditions lead to an algebraic system for the constant velocity (u_2, v_2) and density ρ_2 of state (2), which has solutions for some but not all of the wedge angles. Specifically, for fixed densities $\rho_0 < \rho_1$ of states (0) and (1), there exist a sonic-angle $\theta_{\rm w}^{\rm s}$ and a detachment-angle $\theta_{\rm w}^{\rm d}$ satisfying

$$0 < \theta_{\rm w}^{\rm d} < \theta_{\rm w}^{\rm s} < \frac{\pi}{2}$$

such that state (2) exists for all $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$ and does not exist for $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$, and the weak state (2) is supersonic at the reflection point $P_0(\theta_{\rm w})$ for $\theta_{\rm w} \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$, sonic for $\theta_{\rm w} = \theta_{\rm w}^{\rm s}$, and subsonic for $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \hat{\theta}_{\rm w}^{\rm s})$ for some $\hat{\theta}_{\rm w}^{\rm s} \in (\theta_{\rm w}^{\rm d}, \theta_{\rm w}^{\rm s}]$.

In fact, for each $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, there exists also a *strong* state (2) with $\rho_2^{\rm strong} > \rho_2^{\rm weak}$. There had been a long debate to determine which one is physical for the local theory; see [5, 28] and the references cited therein. It is expected that the strong reflection-diffraction configuration is non-physical; indeed, it is shown as in Chen-Feldman [16] that the weak reflection-diffraction configuration tends to the unique normal reflection, but the strong reflection-diffraction configuration does not, when the wedge-angle θ_w tends to $\frac{\pi}{2}$. The strength of the corresponding reflected shock in the weak reflection-diffraction configuration is relatively weak compared to the other shock given by the strong state (2), which is called a *weak shock*.

If the weak state (2) is supersonic, the propagation speeds of the solution are finite, and state (2) is completely determined by the local information: State (1), state (0), and the location of point P_0. That is, any information from the region of reflection-diffraction, especially the disturbance at corner P_3, cannot travel towards the reflection point P_0. However, if it is subsonic, the information can reach P_0 and interact with it, potentially altering a different reflection-diffraction configuration. This argument motivated the following conjecture by von Neumann in [56, 57]:

The Sonic Conjecture: *There exists a supersonic reflection-diffraction configuration when* $\theta_{\rm w} \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$ *for* $\theta_{\rm w}^{\rm s} > \theta_{\rm w}^{\rm d}$*. That is, the supersonicity of the weak state* (2) *implies the existence of a supersonic regular reflection-diffraction solution, as shown in Fig.* 2.

Another conjecture is that global regular reflection-diffraction configuration is possible whenever the local regular reflection at the reflection point is possible:

The Detachment Conjecture: *There exists a regular reflection-diffraction configuration for any wedge-angle* $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$*. That is, the existence of state* (2) *implies the existence of a regular reflection-diffraction solution, as shown in Figs.* 2–3.

It is clear that the supersonic/subsonic regular reflection-diffraction configurations are not possible without a local two-shock configuration at the reflection point on the wedge, so this is the weakest possible criterion for the existence of supersonic/subsonic regular shock reflection-diffraction configurations.

Problem 2.3 (Free Boundary Problem)**.** *For $\theta_{\rm w}\in(\theta_{\rm w}^d,\frac{\pi}{2})$, find a free boundary (curved reflected shock) P_1P_2 on Fig. 2, and P_0P_2 on Fig. 3, and a function φ defined in region Ω as shown in Figs. 2–3, such that φ satisfies*

(i) *Equation* (2.5) *in Ω;*

(ii) *$\varphi=\varphi_1$ and $\rho D\varphi\cdot\boldsymbol{\nu}_s=D\varphi_1\cdot\boldsymbol{\nu}_{\rm s}$ on the free boundary;*

(iii) *$\varphi=\varphi_2$ and $D\varphi=D\varphi_2$ on P_1P_4 in the supersonic case as shown in Fig. 2 and at P_0 in the subsonic case as shown in Fig. 3;*

(iv) *$D\varphi\cdot\boldsymbol{\nu}=0$ on $\Gamma_{\rm wedge}$,*

where $\boldsymbol{\nu}_s$ and $\boldsymbol{\nu}$ are the interior unit normals to Ω on $\Gamma_{\rm shock}$ and $\Gamma_{\rm wedge}$, respectively.

We observe that the key obstacle to the existence of regular shock reflection-diffraction configurations as conjectured by von Neumann [56, 57] is an additional possibility that, for some wedge-angle $\theta_{\rm w}^{\rm a}\in(\theta_{\rm w}^{\rm d},\frac{\pi}{2})$, shock P_0P_2 may attach to wedge-tip P_3, as observed by experimental results (*cf.* [55, Fig. 238]). To describe the conditions of such an attachment, we note that

$$\rho_1>\rho_0, \qquad u_1=(\rho_1-\rho_0)\sqrt{\frac{2(\rho_1^{\gamma-1}-\rho_0^{\gamma-1})}{\rho_1^2-\rho_0^2}}.$$

Then, for each ρ_0, there exists $\rho^{\rm c}>\rho_0$ such that

$$u_1\le c_1 \quad \text{if } \rho_1\in(\rho_0,\rho^{\rm c}]; \qquad u_1>c_1 \quad \text{if } \rho_1\in(\rho^{\rm c},\infty).$$

If $u_1\le c_1$, we can rule out the solution with a shock attached to the wedge-tip.

If $u_1>c_1$, there would be a possibility that the reflected shock could be attached to the wedge-tip as experiments show (*e.g.* [55, Fig. 238]).

Thus, in [16, 17], we have obtained the following results:

(i) *If ρ_0 and ρ_1 are such that $u_1\le c_1$, then the supersonic/subsonic regular reflection-diffraction solution exists for each wedge-angle $\theta_{\rm w}\in(\theta_{\rm w}^{\rm d},\frac{\pi}{2})$;*

(ii) *If ρ_0 and ρ_1 are such that $u_1>c_1$, then there exists $\theta_{\rm w}^{\rm a}\in[\theta_{\rm w}^{\rm d},\frac{\pi}{2})$ such that the regular reflection solution exists for each wedge-angle $\theta_{\rm w}\in(\theta_{\rm w}^{\rm a},\frac{\pi}{2})$. Moreover, if $\theta_{\rm w}^{\rm a}>\theta_{\rm w}^{\rm d}$, then, for the wedge-angle $\theta_{\rm w}=\theta_{\rm w}^{\rm a}$, there exists an attached solution, i.e., a solution of* **Problem 2.3** *with $P_2=P_3$.*

The type of regular reflection-diffraction configurations (supersonic as in Fig. 2, or subsonic as in Fig. 3) is determined by the type of state (2) at P_0. For the supersonic and sonic reflection-diffraction case, the reflected shock P_0P_2 is $C^{2,\alpha}$–smooth, and the solution φ is $C^{1,1}$ across the sonic arc for the supersonic case, which is optimal. For the subsonic reflection-diffraction case (Fig. 3), the reflected shock P_0P_2 and the solution in Ω are both $C^{1,\alpha}$ near P_0 and C^∞ away from P_0. Furthermore, the regular reflection-diffraction solution tends to the unique normal reflection, when wedge-angle θ_w tends to $\frac{\pi}{2}$.

To solve this free boundary problem (**Problem 2.3**), we define a class of admissible solutions, which are the solutions φ with weak regular reflection-diffraction configurations, such that, in the supersonic reflection case, equation (2.5) is strictly elliptic for φ in $\overline{\Omega} \setminus P_1P_4$, $\varphi_2 \leq \varphi \leq \varphi_1$ holds in Ω, and the following monotonicity properties hold:

$$\partial_\eta(\varphi_1 - \varphi) \leq 0, \quad D(\varphi_1 - \varphi) \cdot \mathbf{e} \leq 0 \qquad \text{in } \Omega$$

for $\mathbf{e} = \frac{P_0P_1}{|P_0P_1|}$. In the subsonic reflection case, admissible solutions are defined similarly, with changes corresponding to the structure of subsonic reflection-diffraction solution.

We derive uniform *a priori* estimates for admissible solutions with any wedge-angle $\theta_{\rm w} \in [\theta_{\rm w}^{\rm d} + \varepsilon, \frac{\pi}{2}]$ for each $\varepsilon > 0$, and then apply the degree theory to obtain the existence for each $\theta_{\rm w} \in [\theta_{\rm w}^{\rm d} + \varepsilon, \frac{\pi}{2}]$ in the class of admissible solutions, starting from the unique normal reflection solution for $\theta_{\rm w} = \frac{\pi}{2}$. To derive the *a priori* bounds, we first obtain the estimates related to the geometry of the shock: Show that the free boundary has a uniform positive distance from the sonic circle of state (1) and from the wedge boundary away from P_2 and P_0. This allows to estimate the ellipticity of (2.5) for φ in Ω (depending on the distance to the sonic arc P_1P_4 for the supersonic reflection-diffraction configuration and to P_0 for the subsonic reflection-diffraction configuration). Then we obtain the estimates near P_1P_4 (or P_0 for the subsonic reflection) in scaled and weighted $C^{2,\alpha}$ for φ and the free boundary, considering separately four cases depending on $\frac{D\varphi_2}{c_2}$ at P_0:

(i) Supersonic: $\frac{|D\varphi_2|}{c_2} \geq 1 + \delta$;

(ii) Supersonic (almost sonic): $1 < \frac{|D\varphi_2|}{c_2} < 1 + \delta$;

(iii) Subsonic (almost sonic): $1 - \delta \leq \frac{|D\varphi_2|}{c_2} \leq 1$;

(iv) Subsonic: $\frac{|D\varphi_2|}{c_2} \leq 1 - \delta$.

In cases (i)–(ii), equation (2.5) is degenerate elliptic in Ω near P_1P_4 on Fig. 2. In case (iii), the equation is uniformly elliptic in $\overline{\Omega}$, but the ellipticity constant is small near P_0 on Fig. 3. Thus, in cases (i)–(iii), we use the local elliptic degeneracy, which allows to find a comparison function in each case, to show the appropriately fast decay of $\varphi - \varphi_2$ near P_1P_4 in cases (i)–(ii) and near P_0 in case (iii); furthermore, combining with appropriate local non-isotropic rescaling to obtain the uniform ellipticity, we obtain the *a priori* estimates in the weighted and scaled $C^{2,\alpha}$–norms, which are different in each of cases (i)–(iii), but imply the standard $C^{1,1}$–estimates in cases (i)–(ii), and the standard $C^{2,\alpha}$–estimates in case (iii). This is an extension of the methods developed in our earlier work [16]. In the uniformly elliptic case (iv), the solution is of subsonic reflection-diffraction configuration as shown in Fig. 3, and the estimates are more technically challenging than in cases (i)–(iii), due to the lower *a priori* regularity of the free boundary and since the uniform ellipticity does not allow a comparison function that shows the decay of $\varphi - \varphi_2$ near P_0. Thus, we prove the C^α–estimates of $D(\varphi - \varphi_2)$ near P_0. With all of these, we provide a solution to the von Neumann's conjectures.

More details can be found in Chen-Feldman [17]; also see [1, 16].

3. Shock Diffraction (Lighthill's Problem) and Free Boundary Problems

We are now concerned with shock diffraction by a two-dimensional wedge with convex corner (Lighthill's problem). When a plane shock in the $(t, \mathbf{x})$–coordinates, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, with left state $(\rho, u, v) = (\rho_1, u_1, 0)$ and right state $(\rho_0, 0, 0)$ satisfying $u_1 > 0$ and $\rho_0 < \rho_1$ from the left to right along the wedge with convex corner:

$$W := \{(x_1, x_2) : x_2 < 0, \ x_1 < x_2 \tan\theta_{\rm w}\},$$

the incident shock interacts with the wedge as it passes the corner, and then the shock diffraction occurs (*cf.* [44, 45]). The mathematical study of the shock diffraction problem dates back to the 1950s by the work of Lighthill [44, 45] via asymptotic analysis; also see [7, 32, 31] via experimental analysis, as well as Courant-Friedrichs [28] and Whitham [60].

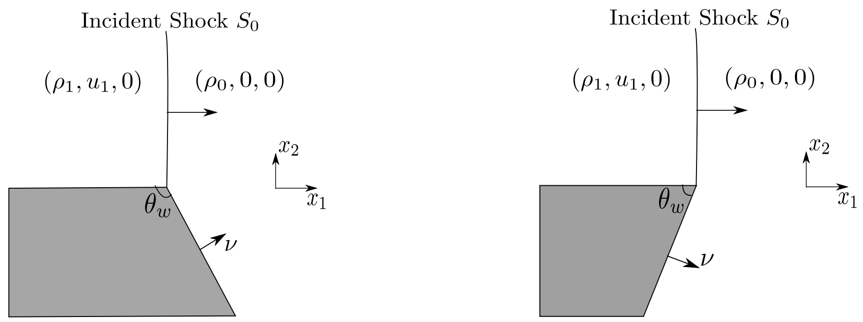

Figure 4. Lateral Riemann Problem

Similarly, this problem can be formulated as the following lateral Riemann problem for potential flow:

Problem 3.1 (Lateral Riemann Problem; see Fig. 4)**.** *Seek a solution of system* (1.1)–(1.2) *with the initial condition at* $t = 0$*:*

$$(\rho, \Phi)|_{t=0} = \begin{cases} (\rho_1, u_1 x_1) & in \ \{x_1 < 0, x_2 > 0\}, \\ (\rho_0, 0) & in \ \{\theta_{\rm w} - \pi \le \arctan\left(\frac{x_2}{x_1}\right) \le \frac{\pi}{2}\}, \end{cases} \tag{3.1}$$

and the slip boundary condition along the wedge boundary ∂W*:*

$$\nabla\Phi \cdot \boldsymbol{\nu}|_{\partial W} = 0, \tag{3.2}$$

where $\boldsymbol{\nu}$ *is the exterior unit normal to* ∂W*.*

Problem 3.1 is also invariant under the self-similar scaling (2.1). Thus, we seek self-similar solutions with form (2.2) in the self-similar domain outside the wedge:

$$\Lambda := \{\theta_w - \pi \le \arctan\big(\frac{\eta}{\xi}\big) \le \pi\}.$$

Then the shock interacts with the pseudo-sonic circle of state (1) to become a transonic shock, and **Problem 3.1** can be formulated as the following boundary value problem in the self-similar coordinates (ξ, η).

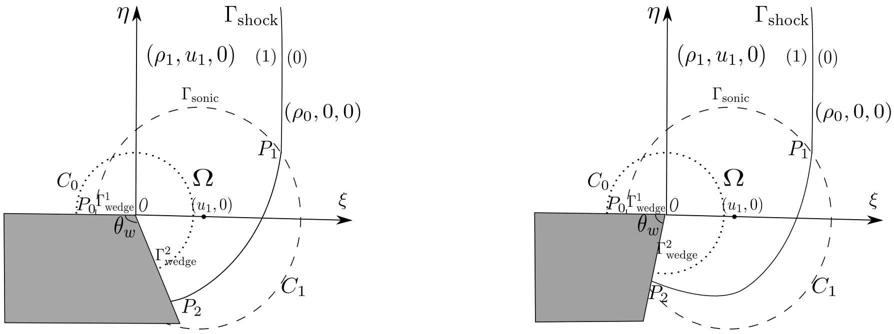

Figure 5. Boundary Value Problem; see Chen-Xiang [22]

Problem 3.2 (Boundary Value Problem; see Fig. 5)**.** *Seek a solution φ of equation* (2.5) *in the self-similar domain Λ with the slip boundary condition on the wedge boundary $\partial\Lambda$:*

$$D\varphi \cdot \boldsymbol{\nu}|_{\partial\Lambda} = 0$$

and the asymptotic boundary condition at infinity:

$$(\rho, \varphi) \to (\bar{\rho}, \bar{\varphi}) = \begin{cases} (\rho_1, \varphi_1) & \text{in } \{\xi < \xi_0,\ \eta \ge 0\}, \\ (\rho_0, \varphi_0) & \text{in } \{\xi > \xi_0,\ \eta \ge 0\} \cup \{\theta_{\rm w} - \pi \le \arctan\big(\frac{\eta}{\xi}\big) \le 0\}, \end{cases}$$

when $\xi^2 + \eta^2 \to \infty$ in the sense that $\lim_{R\to\infty} \|\varphi - \bar{\varphi}\|_{C^1(\Lambda\setminus B_R(0))} = 0$, *where φ_0, φ_1, and ξ_0 are the same as defined in* **Problem 2.2** *in the (ξ, η)–coordinates.*

Since φ does not satisfy the slip boundary condition for $\xi \ge 0$, the solution must differ from state (1) in $\{\xi < \xi_1\} \cap \Lambda$ near the wedge-corner, which forces the shock to be diffracted by the wedge. There is a critical angle θ_c so that, when $\theta_{\rm w}$ decreases to $\theta_{\rm c}$, two sonic circles C_0 and C_1 coincide at P_2 on $\Gamma_{\rm wedge}$. Then **Problem 3.2** can be formulated as the following free boundary problem:

Problem 3.3 (Free Boundary Problem)**.** *For $\theta_{\rm w} \in (\theta_{\rm c}, \pi)$, find a free boundary (curved shock) $\Gamma_{\rm shock}$ and a function φ defined in region Ω, enclosed by $\Gamma_{\rm shock}, \Gamma_{\rm sonic}$, and the wedge boundary $\Gamma_{\rm wedge} := \Gamma^1_{\rm wedge} \cup \Gamma^2_{\rm wedge}$, such that φ satisfies*

(i) *Equation* (2.5) *in Ω;*

(ii) $\varphi = \varphi_0$, $\rho D\varphi \cdot \boldsymbol{\nu}_{\rm s} = \rho_0 D\varphi_0 \cdot \boldsymbol{\nu}_{\rm s}$ *on* $\Gamma_{\rm shock}$*;*

(iii) $\varphi = \varphi_1$, $D\varphi = D\varphi_1$ *on* $\Gamma_{\rm sonic}$*;*

(iv) $D\varphi \cdot \boldsymbol{\nu} = 0$ *on* Γ_{wedge},

where $\boldsymbol{\nu}_s$ *and* $\boldsymbol{\nu}$ *are the interior unit normals to* Ω *on* Γ_{shock} *and* Γ_{wedge}, *respectively.*

In domain Ω, the solution is expected to be pseudo-subsonic and smooth, to satisfy the slip boundary condition along the wedge, and to be $C^{1,1}$–continuous across the pseudo-sonic circle to become pseudo-supersonic. Then the solution of **Problem 3.3** can be shown to be the solution of **Problem 3.1**.

The free boundary problem has been solved in [21, 22]. A crucial challenge of this problem is that the expected elliptic domain of the solution is concave so that its boundary does not satisfy the exterior ball condition, since the angle $2\pi - \theta_w$ exterior to the wedge at the origin is larger than π for the given wedge-angle $\theta_w \in (0, \pi)$, besides other mathematical difficulties including free boundary problems without uniform oblique derivative conditions. There is no general theory of elliptic PDEs on such concave domains, whose coefficients involve the gradient of the solutions. In general, the expected regularity in this domain, even for Laplace's equation, is only C^α with $\alpha < 1$; however, the coefficients in (2.5) depend on the gradient of φ so that the ellipticity of this equation depends also on the boundedness of the derivatives, which is one of the essential difficulties of this problem. To overcome the difficulty, the physical boundary conditions must be exploited to force a finer regularity of solutions at the corner to let equation (2.5) make sense. More precisely, the strategy here is that, instead of analyzing equation (2.5) directly, we study another system of equations for the physical quantities (ρ, u, v) for the existence of the velocity potential.

A tempting try would be to differentiate first equation (2.3) to obtain an equation for v, then use the irrotationality to solve u (once v has solved), and finally use (2.4) to solve the density ρ. In order to show the equivalence between these equations and the original potential flow equation (2.5), an additional one-point boundary condition is required for v. However, it is unclear for the boundary condition to be deduced for v for the problem. Moreover, along the boundaries Γ_{shock} and Γ^2_{wedge} which meet at the corner, the derivative boundary conditions of the deduced second-order elliptic equation to v are the second kind boundary conditions, *i.e.* without the viscosity, compared to [16]. This implies that the results from [42, 43] could not be directly used. To overcome this, the following directional velocity (w, z) is introduced whose relation with (u, v) is

$$(w, z) := (u \sin\theta_{\text{w}} - v\cos\theta_{\text{w}}, u\cos\theta_{\text{w}} + v\sin\theta_{\text{w}}),$$

such that the one-point boundary condition for w is not required for solving w, and then treat z as u. For (w, z), the C^α–regularity is enough.

On the other hand, for these equations, some new technical difficulties arise, for which new mathematical ideas and techniques have to be developed. First, it is a coupled system so that the coefficients of the nonlinear degenerate elliptic equation for w depend on z, which makes the uniform estimates for w near the sonic circle more challenging. Second, the obliqueness condition on the free boundary deduced from the Rankine-Hugoniot conditions depends on the smallness of z. To overcome this, a degenerate elliptic cut-off function near the pseudo–sonic circle is introduced, which is more precise in comparison with [16]. The reason why the more precise degenerate cut-off function requires to be introduced is that the uniform estimates of w are required to obtain a convergent sequence near P_1, which is crucial for the equivalence between the deduced system and the potential flow equation (2.5) with degenerate elliptic cut-off. Third, since the new feature that $\sin\theta$ may be 0 along the pseudo–sonic circle and the fact that there is no C^2–regularity at P_1 where the shock and pseudo–sonic circle meet from the optimal regularity argument by Bae-Chen-Feldman [1], more effort is needed to remove the degenerate elliptic cut-off case by case carefully, near and away from P_1 respectively. The final main difficulty is to show the equivalence of the original potential flow equation (2.5) and the deduced second-order equation for w with irrotationality and Bernoulli's law, which requires gradient estimates for w near the pseudo-sonic circle, but the estimates by scaling only provide a bound divided by the distance to this circle. This is overcome thanks to the estimates involving ϵ.

When the wedge-angle becomes smaller, several other difficulties arise. Due to the concave corner at the origin, more technical arguments are required to obtain the existence of solutions to the modified problem. Unlike in [17], since it requires to take derivatives along the shock to obtain a boundary condition for w, a new way to modify the Rankine-Hugoniot conditions is designed delicately, based on the nonlinear structure of the shock. From this modified condition, the Dirichlet condition is assigned on the shock where the modified uniform oblique condition fails. Thus, the uniform boundedness of solutions need to be controlled more carefully. Finally, the existence of shock diffraction configuration up to the critical angle $\theta_{\rm c}$ is established.

4. Prandtl-Meyer Reflection Configurations and Free Boundary Problems

We now consider with Prandtl-Meyer's problem for unsteady global solutions for supersonic flow past a solid ramp, which can be also regarded as portraying the symmetric gas flow impinging onto a solid wedge (by symmetry). When a steady supersonic flow past a solid ramp whose angle is less than the critical angle (called the detachment angle) $\theta_{\rm w}^d$, Prandtl [51] employed the shock polar analysis to show that there are two possible configurations: The weak shock reflection with supersonic or subsonic downstream flow and the strong shock reflection with subsonic downstream flow, which both satisfy the physical entropy conditions, provided that we do not give additional conditions at downstream; also see [8, 28, 48].

The fundamental question of whether one or both of the strong and the weak shocks are physically admissible has been vigorously debated over the past seventy years, but has not yet been settled in a definite manner (*cf.* [28, 29, 54]). On the basis of experimental and numerical evidence, there are strong indications that it is the weak reflection that is physically admissible. One plausible approach is to single out the strong shock reflection by the consideration of stability: The stable ones are physical. It has been shown in the steady regime that the weak reflection is not only structurally stable (*cf.* [23]), but also L^1-stable with respect to steady small perturbation of both the ramp slope and the incoming steady upstream flow (*cf.* [19]), while the strong reflection is also structurally stable for a large spectrum of physical parameters (*cf.* [12, 13]).

We are interested in the rigorous unsteady analysis of the steady supersonic weak shock solution as the long-time behavior of an unsteady flow and establishing the stability of the steady supersonic weak shock solution as the long-time asymptotics of an unsteady flow with the Prandtl-Meyer configuration for all the admissible physical parameters for potential flow. Our goal is to find a solution (ρ, Φ) to system (1.1)–(1.2) when a uniform flow in $\mathbb{R}^2_+ := \{x_1 \in \mathbb{R}, x_2 > 0\}$ with $(\rho, \nabla_{\mathbf{x}}\Phi) = (\rho_\infty, u_\infty, 0), \mathbf{x} = (x_1, x_2)$, is heading to a solid ramp at $t = 0$:

$$W := \{(x_1, x_2) : 0 < x_2 < x_1 \tan\theta_{\rm w}, x_1 > 0\}.$$

Problem 4.1 (Lateral Riemann Problem). *Seek a solution of system* (1.1)–(1.2) *with* $B = \frac{u_\infty^2}{2} + \frac{\rho_\infty^{\gamma-1}-1}{\gamma-1}$ *and the initial condition at* $t = 0$*:*

$$(\rho, \Phi)|_{t=0} = (\rho_\infty, u_\infty x_1) \qquad \textit{for } (x_1, x_2) \in \mathbb{R}^2_+ \setminus W, \tag{4.1}$$

and with the slip boundary condition along the wedge boundary ∂W*:*

$$\nabla_{\mathbf{x}}\Phi \cdot \boldsymbol{\nu}|_{\partial W \cap \{x_2 > 0\}} = 0, \tag{4.2}$$

where $\boldsymbol{\nu}$ *is the exterior unit normal to* ∂W*.*

Again, **Problem 4.1** is is invariant under the self-similar scaling (2.1). Thus, we seek self-similar solutions in form (2.2) so that the pseudo-potential function $\varphi = \phi - \frac{1}{2}(\xi^2 + \eta^2)$ satisfies the nonlinear PDE (2.5) of mixed type.

As the incoming flow has the constant velocity $(u_\infty, 0)$, the corresponding pseudo-potential φ_∞ has the expression of

$$\varphi_\infty = -\frac{1}{2}(\xi^2 + \eta^2) + u_\infty \xi. \tag{4.3}$$

Then **Problem 4.1** can be reformulated as the following boundary value problem in the domain

$$\Lambda := \mathbb{R}^2_+ \setminus \{(\xi, \eta) : \eta \le \xi \tan\theta_{\rm w}, \ \xi \ge 0\}.$$

in the self-similar coordinates (ξ, η), which corresponds to $\{(t, \mathbf{x}) : \mathbf{x} \in \mathbb{R}^2_+ \setminus W, \ t > 0\}$ in the $(t, \mathbf{x})$–coordinates:

Problem 4.2 (Boundary Value Problem). *Seek a solution* φ *of equation* (2.5) *in the self-similar domain* Λ *with the slip boundary condition:*

$$D\varphi \cdot \boldsymbol{\nu}|_{\partial\Lambda} = 0, \tag{4.4}$$

and the asymptotic boundary condition at infinity:

$$\varphi - \varphi_\infty \longrightarrow 0 \tag{4.5}$$

along each ray $R_\theta := \{\xi = \eta\cot\theta, \eta > 0\}$ *with* $\theta \in (\theta_{\rm w}, \pi)$ *as* $\eta \to \infty$ *in the sense that*

$$\lim_{r\to\infty} \|\varphi - \varphi_\infty\|_{C(R_\theta \setminus B_r(0))} = 0. \tag{4.6}$$

In particular, we seek a weak solution of **Problem 4.2** with two types of Prandtl-Meyer reflection configurations whose occurrence is determined by wedge-angle $\theta_{\rm w}$ for the two different cases: One contains a straight weak oblique shock attached to wedge-tip O and the oblique shock is connected to a normal shock through a curved shock when $\theta_{\rm w} < \theta_{\rm w}^s$, as shown in Fig. 6; the other contains a curved shock attached to the wedge-tip and connected to a normal shock when

$\theta_{\rm w}^s \le \theta_{\rm w} < \theta_{\rm w}^d$, as shown in Fig. 7, in which the curved shock $\Gamma_{\rm shock}$ is tangential to a straight weak oblique shock S_0 at the wedge-tip.

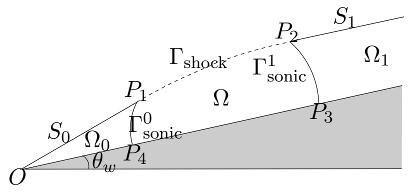

Figure 6. Admissible solutions for $\theta_{\rm w} \in (0, \theta_{\rm w}^s)$ in the self-similar coordinates (ξ, η); see Bae-Chen-Feldman [3]

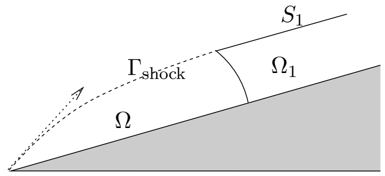

Figure 7. Admissible solutions for $\theta_{\rm w} \in [\theta_{\rm w}^s, \theta_{\rm w}^d)$ in the self-similar coordinates (ξ, η); see Bae-Chen-Feldman [3]

To seek a global entropy solution of **Problem 4.2** with the structure of Fig. 6 or Fig. 7, one needs to compute the pseudo-potential φ_0 below S_0.

Given $M_\infty > 1$, we obtain (u_0, v_0) and ρ_0 by using the shock polar curve in Fig. 8 for steady potential flow. In Fig. 8, $\theta_{\rm w}^s$ is the wedge-angle such that line $v = u \tan \theta_{\rm w}^s$ intersects with the shock polar curve at a point on the circle of radius c_∞, and $\theta_{\rm w}^d$ is the wedge-angle so that line $v = u \tan \theta_{\rm w}^d$ is tangential to the shock polar curve and there is no intersection between line $v = u \tan \theta_{\rm w}$ and the shock polar when $\theta_{\rm w} > \theta_{\rm w}^d$.

For any wedge-angle $\theta_{\rm w} \in (0, \theta_{\rm w}^s)$, line $v = u \tan \theta_{\rm w}$ and the shock polar curve intersect at a point (u_0, v_0) with $|(u_0, v_0)| > c_\infty$ and $u_0 < u_\infty$; while, for any wedge-angle $\theta_{\rm w} \in [\theta_{\rm w}^s, \theta_{\rm w}^d)$, they intersect at a point (u_0, v_0) with $u_0 > u_{\rm w}^d$ and $|(u_0, v_0)| < c_\infty$. The intersection state (u_0, v_0) is the velocity for steady potential flow behind an oblique shock S_0 attached to the wedge-tip with angle $\theta_{\rm w}$. The strength of shock S_0 is relatively weak compared to the other shock given by the other intersection point on the shock polar curve, which is a a *weak shock*, and the corresponding state (u_0, v_0) is a *weak state.*

We also note that states (u_0, v_0) smoothly depend on u_∞ and $\theta_{\rm w}$, and such states are supersonic when $\theta_{\rm w} \in (0, \theta_{\rm w}^s)$ and subsonic when $\theta_{\rm w} \in [\theta_{\rm w}^s, \theta_{\rm w}^d)$.

Once (u_0, v_0) is determined, by (2.10) and (4.3), the pseudo-potentials φ_0 and φ_1 below the weak oblique shock S_0 and the normal shock S_1 are respectively in the form of

$$\varphi_0 = -\frac{1}{2}(\xi^2 + \eta^2) + u_0 \xi + v_0 \eta, \quad \varphi_1 = -\frac{1}{2}(\xi^2 + \eta^2) + u_1 \xi + v_1 \eta + k_1 \tag{4.7}$$

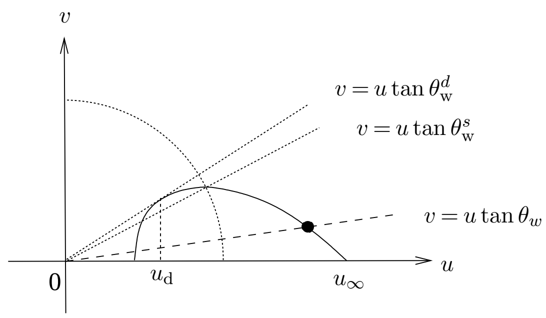

Figure 8. Shock polars in the (u, v)–plane

for constants u_0, v_0, u_1, v_1, and k_1. Then it follows from (2.6) and (4.7) that the corresponding densities ρ_0 and ρ_1 below S_0 and S_1 are constants, respectively. In particular, we have

$$\rho_0^{\gamma-1} = \rho_\infty^{\gamma-1} + \frac{\gamma-1}{2}\big(u_\infty^2 - u_0^2 - v_0^2\big). \tag{4.8}$$

Then **Problem 4.2** can be formulated into the following free boundary problem.

Problem 4.3 (Free Boundary Problem)**.** *For* $\theta_w \in (0, \theta_w^d)$, *find a free boundary (curved shock)* Γ_{shock} *and a function* φ *defined in domain* Ω, *as shown in Figs. 6–7, such that* φ *satisfies*

(i) *Equation* (2.5) *in* Ω*;*

(ii) $\varphi = \varphi_\infty$ *and* $\rho D\varphi \cdot \boldsymbol{\nu}_s = D\varphi_\infty \cdot \boldsymbol{\nu}_s$ *on* Γ_{shock}*;*

(iii) $\varphi = \hat{\varphi}$ *and* $D\varphi = D\hat{\varphi}$ *on* $\Gamma_{sonic}^0 \cup \Gamma_{sonic}^1$ *when* $\theta_w \in (0, \theta_w^s)$ *and on* $\Gamma_{sonic}^1 \cup O$ *when* $\theta_w \in [\theta_w^s, \theta_w^d)$ *for* $\hat{\varphi} := \max(\varphi_0, \varphi_1)$*;*

(iv) $D\varphi \cdot \boldsymbol{\nu} = 0$ *on* Γ_{wedge}*,*

where $\boldsymbol{\nu}_s$ *and* $\boldsymbol{\nu}$ *are the interior unit normals to* Ω *on* Γ_{shock} *and* Γ_{wedge}*, respectively.*

Let φ be a solution of **Problem 4.3** with shock Γ_{shock}. Moreover, assume that $\varphi \in C^1(\overline{\Omega})$, and Γ_{shock} is a C^1–curve up to its endpoints. To obtain a solution of **Problem 4.2** from φ, we have two cases:

For $\theta_w \in (0, \theta_w^s)$, we divide half-plane $\{\eta \geq 0\}$ into five separate regions. Let Ω_S be the unbounded domain below curve $\overline{S_0 \cup \Gamma_{shock} \cup S_1}$ and above Γ_{wedge} (see Fig. 6). In Ω_S, let Ω_0 be the bounded open domain enclosed by S_0, Γ_{sonic}^0, and

$\{\eta = 0\}$. We set $\Omega_1 := \Omega_S \setminus \overline{(\Omega_0 \cup \Omega)}$. Define a function φ_* in $\{\eta \geq 0\}$ by

$$\varphi_* = \begin{cases} \varphi_\infty & \text{in}\, \Lambda \cap \{\eta \geq 0\} \setminus \Omega_S, \\ \varphi_0 & \text{in}\, \Omega_0, \\ \varphi & \text{in}\, \Gamma^0_{\text{sonic}} \cup \Omega \cup \Gamma^1_{\text{sonic}}, \\ \varphi_1 & \text{in}\, \Omega_1. \end{cases} \tag{4.9}$$

By (2.10) and (iii) of **Problem 4.3**, φ_* is continuous in $\{\eta \geq 0\} \setminus \Omega_S$ and is C^1 in $\overline{\Omega_S}$. In particular, φ_* is C^1 across $\Gamma^0_{\text{sonic}} \cup \Gamma^1_{\text{sonic}}$. Moreover, using (i)–(iii) of **Problem 4.3**, we obtain that φ_* is a global entropy solution of equation (2.5) in $\Lambda \cap \{\eta > 0\}$, which is the Prandtl-Meyer's supersonic reflection configuration.

For $\theta_w \in [\theta^s_{\text{w}}, \theta^d_{\text{w}})$, region $\Omega_0 \cup \Gamma^0_{\text{sonic}}$ in φ_* reduces to one point O, and the corresponding φ_* is a global entropy solution of equation (2.5) in $\Lambda \cap \{\eta > 0\}$, which is the Prandtl-Meyer's subsonic reflection configuration.

The free boundary problem (**Problem 4.3**) has been solved in Bae-Chen-Feldman [2, 3].

To solve this free boundary problem, we follow the approach introduced in Chen-Feldman [17]. We first define a class of admissible solutions, which are the solutions φ with Prandtl-Meyer reflection configuration, such that, when $\theta_{\text{w}} \in (0, \theta^s_{\text{w}})$, equation (2.5) is strictly elliptic for φ in $\overline{\Omega} \setminus (\Gamma^0_{\text{sonic}} \cup \Gamma^1_{\text{sonic}})$, $\max\{\varphi_0, \varphi_1\} \leq \varphi \leq \varphi_\infty$ holds in Ω, and the following monotonicity properties hold:

$$D(\varphi_\infty - \varphi) \cdot \mathbf{e}_{S_1} \geq 0, \quad D(\varphi_\infty - \varphi) \cdot \mathbf{e}_{S_0} \leq 0 \qquad \text{in } \Omega,$$

where $\mathbf{e}_{S_0}$ and $\mathbf{e}_{S_1}$ are the unit tangential directions to lines S_0 and S_1, respectively, pointing to the positive ξ-direction. For the case $\theta_{\text{w}} \in [\theta^s_{\text{w}}, \theta^d_{\text{w}})$, admissible solutions are defined similarly, with corresponding changes to the structure of subsonic reflection solutions.

We derive uniform *a priori* estimates for admissible solutions for any wedge-angle $\theta_{\text{w}} \in [0, \theta^d_{\text{w}} - \varepsilon]$ for each $\varepsilon > 0$, and then employ the Leray-Schauder degree argument to obtain the existence for each $\theta_{\text{w}} \in [0, \theta^d_{\text{w}} - \varepsilon]$ in the class of admissible solutions, starting from the unique normal solution for $\theta_{\text{w}} = 0$.

More details can be found in [2, 3]; also see §2 above and Chen-Feldman [17].

In Chen-Feldman-Xiang [18], we have also established the strict convexity of the curved (transonic) part of the free boundary in the shock reflection-diffraction problem in §2 (also see Chen-Feldman [17]), shock diffraction in §3 (also see Chen-Xiang [21, 22]), and the Prandtl-Meyer reflection described in §4 (also see Bae-Chen-Feldman [2]). In order to prove the convexity, we employ global properties of admissible solutions, including the existence of the cone of monotonicity discussed above.

5. The Shock Reflection/Diffraction Problems and Free Boundary Problems for the Full Euler Equations

When the vortex sheets and the deviation of vorticity become significant, the full Euler equations are required. In this section, we present mathematical formulation of the shock reflection/diffraction problems for the full Euler equations and the role of the potential theory for the shock problems even in the realm of the full Euler equations. In particular, the Euler equations for potential flow, (2.3)–(2.4), are actually *exact* in an important region of the solutions to the full Euler equations.

The full Euler equations for compressible fluids in $\mathbb{R}^3_+ := \mathbb{R}_+ \times \mathbb{R}^2, t \in \mathbb{R}_+ := (0,\infty), \mathbf{x} \in \mathbb{R}^2$, are of the following form:

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0, \\ \partial_t(\rho \mathbf{v}) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = 0, \\ \partial_t(\frac{1}{2}\rho|\mathbf{v}|^2 + \rho e) + \nabla_{\mathbf{x}} \cdot \big((\frac{1}{2}\rho|\mathbf{v}|^2 + \rho e + p)\mathbf{v}\big) = 0, \end{cases} \tag{5.1}$$

where ρ is the density, $\mathbf{v} = (u, v)$ the fluid velocity, p the pressure, and e the internal energy. Two other important thermodynamic variables are the temperature θ and the energy S. The notation $\mathbf{a} \otimes \mathbf{b}$ denotes the tensor product of the vectors $\mathbf{a}$ and $\mathbf{b}$.

Choosing (ρ, S) as the independent thermodynamical variables, then the constitutive relations can be written as $(e, p, \theta) = (e(\rho, S), p(\rho, S), \theta(\rho, S))$ governed by

$$\theta dS = de + p d\tau = de - \frac{p}{\rho^2} d\rho.$$

For a polytropic gas,

$$p = (\gamma - 1)\rho e, \qquad e = c_v \theta, \qquad \gamma = 1 + \frac{R}{c_v}, \tag{5.2}$$

or equivalently,

$$p = p(\rho, S) = \kappa \rho^\gamma e^{S/c_v}, \qquad e = e(\rho, S) = \frac{\kappa}{\gamma - 1}\rho^{\gamma-1} e^{S/c_v}, \tag{5.3}$$

where $R > 0$ may be taken to be the universal gas constant divided by the effective molecular weight of the particular gas, $c_v > 0$ is the specific heat at constant volume, $\gamma > 1$ is the adiabatic exponent, and $\kappa > 0$ is any constant under scaling.

Notice that the corresponding three lateral Riemann problems in §2–§4 for system (5.1) are all invariant under the self-similar scaling: $(t, \mathbf{x}) \longrightarrow (\alpha t, \alpha \mathbf{x})$ for any $\alpha \neq 0$. Therefore, we seek self-similar solutions:

$$(\mathbf{v}, p, \rho)(t, \mathbf{x}) = (\mathbf{v}, p, \rho)(\xi, \eta), \qquad (\xi, \eta) = \frac{\mathbf{x}}{t}.$$

Then the self-similar solutions are governed by the following system:

$$\begin{cases} (\rho U)_\xi + (\rho V)_\eta + 2\rho = 0, \\ (\rho U^2 + p)_\xi + (\rho U V)_\eta + 3\rho U = 0, \\ (\rho U V)_\xi + (\rho V^2 + p)_\eta + 3\rho V = 0, \\ \big(U(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1})\big)_\xi + \big(V(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1})\big)_\eta + 2(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}) = 0, \end{cases} \tag{5.4}$$

where $q=\sqrt{U^2+V^2}$, and $(U,V)=(u-\xi, v-\eta)$ is the pseudo-velocity.

The eigenvalues of system (5.4) are

$$\lambda_0=\frac{V}{U} \text{ (repeated)}, \qquad \lambda_\pm=\frac{UV\pm c\sqrt{q^2-c^2}}{U^2-c^2},$$

where $c=\sqrt{\gamma p/\rho}$ is the sonic speed.

When the flow is pseudo-subsonic, *i.e.*, $q<c$, the eigenvalues $\lambda_\pm$ become complex and thus the system consists of two transport equations and two nonlinear equations of elliptic-hyperbolic mixed type. Therefore, system (5.4) is *hyperbolic-elliptic composite-mixed* in general.

The three lateral Riemann problems can be formulated as the corresponding boundary value problems in the unbounded domains. Then the boundary value problems can be further formulated as the three corresponding free boundary problems. The free boundary conditions are again the Rankine-Hugoniot conditions along the free boundary S:

$$[L]_S=0, \quad [\rho N]_S=0, \quad [p+\rho N^2]_S=0, \quad [\frac{\gamma p}{(\gamma-1)\rho}+\frac{N^2}{2}]_S=0, \tag{5.5}$$

where L and N are the tangential and normal components of velocity (U,V) along the free boundary, that is, $|(U,V)|^2=L^2+N^2$. The conditions along the sonic circles are the Dirichlet conditions for (U,V,p,ρ) to be continuous across the respective sonic circles.

We now discuss the role of the potential flow equation (2.5) in these free boundary problems whose boundaries also include the fixed degenerate sonic circles for the full Euler equations (5.4).

Under the Hodge-Helmoltz decomposition $(U,V)=D\varphi+W$ with $\mathrm{div}W=0$, the Euler equations (5.4) become

$$\mathrm{div}(\rho D\varphi)+2\rho=-\mathrm{div}(\rho W), \tag{5.6}$$

$$D(\frac{1}{2}|D\varphi|^2+\varphi)+\frac{1}{\rho}Dp=(D\varphi+W)\cdot DW+(D^2\varphi+I)W, \tag{5.7}$$

$$(D\varphi+W)\cdot D\omega+(1+\Delta\varphi)\omega=0, \tag{5.8}$$

$$(D\varphi+W)\cdot DS=0, \tag{5.9}$$

where $\omega=\mathrm{curl}\,W=\mathrm{curl}(U,V)$ is the vorticity of the fluid, and $S=c_v\ln(p\rho^{-\gamma})$ is the entropy.

When $\omega=0, S=const.$ and $W=0$ on a curve Γ transverse to the fluid direction, we first conclude from (5.8) that, in domain Ω_E determined by the fluid trajectories past Γ:

$$\frac{d}{dt}(\xi,\eta)=(D\varphi+W)(\xi,\eta),$$

we have

$$\omega=0, \qquad i.e. \quad \mathrm{curl}\,W=0.$$

This implies that $W=const.$ since $\mathrm{div}W=0$. Then we conclude

$$W=0 \qquad \text{in } \Omega_E,$$

since $W|_\Gamma = 0$, which yields that the right-hand side of equation (5.7) vanishes. Furthermore, from (5.9),

$$S = const. \qquad \text{in } \Omega_E,$$

which implies that

$$p = const.\, \rho^\gamma.$$

By scaling, we finally conclude that the solution of system (5.6)–(5.9) in domain Ω_E is determined by the Euler equations (2.3)–(2.4) for self-similar potential flow, or the potential flow equation (2.5) with (2.6) for self-similar solutions.

For our problems in §2–§4, we note that, in the supersonic states joint with the sonic circles (*e.g.* state (2) for **Problem 2.3**, state (1) for **Problem 3.3**, states (0) and (1) for **Problem 4.3**),

$$\omega = 0, \quad W = 0, \quad S = S_2. \tag{5.10}$$

Then, if our solution (U, V, p, ρ) is $C^{0,1}$ and the gradient of the tangential component of the velocity is continuous across the sonic arc, we still have (5.10) along $\Gamma_{\rm sonic}$ on the side of Ω. Thus, we have

Theorem 1. *Let* (U, V, p, ρ) *be a solution of one of our problems,* **Problems 2.3, 3.3**, *and* **4.3**, *such that* (U, V, p, ρ) *is* $C^{0,1}$ *in the open region formed by the reflected shock and the wedge boundary, and the gradient of the tangential component of* (U, V) *is continuous across any sonic arc. Let* Ω_E *be the subregion of* Ω *formed by the fluid trajectories past the sonic arc, then, in* Ω_E*, the potential flow equation* (2.5) *with* (2.6) *coincides with the full Euler equations* (5.6)*–*(5.9)*, that is, equation* (2.5) *with* (2.6) *is exact in the domain* Ω_E *for* **Problems 2.3, 3.3,** *and* **4.3**.

Remark 1. The regions such as Ω_E also exist in various Mach reflection-diffraction configurations. Theorem 1 applies to such regions whenever the solution (U, V, p, ρ) is $C^{0,1}$ and the gradient of the tangential component of (U, V) is continuous. In fact, Theorem 1 indicates that, for the solution φ of (2.5) with (2.6), the $C^{1,1}$–regularity of φ and the continuity of the tangential component of the velocity field $(U, V) = \nabla\varphi$ are optimal across the sonic arc Γ_{sonic}.

Remark 2. The importance of the potential flow equation (1.1) with (1.2) in the time-dependent Euler flows even through weak discontinuities was also observed by Hadamard [34] through a different argument. Moreover, for the solutions containing a weak shock, the potential flow equation (1.1)–(1.2) and the full Euler flow model (5.1) match each other well up to the third order of the shock strength. Also see Bers [6], Glimm-Majda [33], and Morawetz [49].

6. Conclusion

As we have discussed above, the three longstanding, fundamental transonic flow problems can be all formulated as free boundary problems. The understanding of these transonic flow problems requires our mathematical solution of these free boundary problems. Similar free boundary problems also arise in many other transonic flow problems, including steady transonic flow problems including transonic nozzle flow problems (*cf.* [4, 11, 15, 41]), steady transonic flows past obstacles (*cf.* [11, 12, 13, 23, 25]), supersonic bubbles in subsonic flow (*cf.* [26, 50]), local stability of Mach configurations (*cf.* [24]), as well as higher dimensional version of **Problem 2.3** (shock reflection-diffraction by a solid cone) and **Problem 4.3** (supersonic flow impinging onto a solid cone). In §2–§5, we have discussed recently developed mathematical ideas, approaches, and techniques for solving these free boundary problems. On the other hand, many free boundary problems arising from transonic flow problems are still open and demand further developments of new mathematical ideas, approaches, and techniques.

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