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To Fully Net or Not to Net: Adverse Effects of Partial Multilateral Netting

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To Fully Net or Not to Net:
Adverse Effects of Partial Multilateral Netting*

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Abstract

We study a financial network where forced liquidations of an illiquid asset have a negative impact on its price, thus reinforcing network contagion. We prove uniqueness of the clearing asset price and liability payments under no, partial, and full multilateral netting of interbank liabilities. We show that partial versus full multilateral netting increases bank shortfall, and reduces clearing asset price and aggregate bank surplus. We also show that partial multilateral netting can be worse than no netting at all.

Keywords: Over the Counter Markets, Financial Network, Partial Multilateral Netting.

JEL classification: C44, C54, C62, G01, G18, G32.

1 Introduction

We study a financial network in which banks can be thought of as dealers in an over the counter (OTC) market and their costumers. Banks hold interbank liabilities, cash, and shares of an illiquid asset. The settlement of interbank liabilities may force banks to liquidate some shares of the illiquid asset. This has a negative impact on the price of the illiquid asset. Marking to market of banks' balance sheets reinforces network contagion: lower asset prices may force other banks to default on their interbank liability payments. This results in an entanglement of price mediated contagion and network mediated contagion.

We model the price impact by a given inverse demand function, similarly as in [11]. In equilibrium, this leads to a clearing price and liability payments, given as solution of a fixed point equation. Existence of the fixed point follows by Tarski's fixed-point theorem, as shown

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in [11]. Uniqueness has remained an open problem. As first main result of this paper, we prove uniqueness under some mild and natural technical assumptions. Key assumption is that the cash proceeds from asset liquidations are strictly increasing in the number of shares liquidated. We also provide an algorithm for computing the fixed point that terminates in at most m steps, where m denotes the number of banks in the network.

We then study the effect of full and partial multilateral netting on the clearing price, bank shortfall and aggregate surplus in equilibrium. We find that aggregate surplus depends on the multilateral netting policy only through the clearing price of the illiquid asset. As second main result of this paper, we show that partial versus full multilateral netting has adverse effects on the clearing price, bank shortfall and aggregate surplus. Full multilateral netting maximizes the clearing price and thus the aggregate surplus and minimizes bank shortfall. Moreover, we illustrate by example that the effects of partial versus full multilateral netting can be strictly adverse, and that no netting can be better than partial multilateral netting.

Our paper is part of the literature that seeks to explore the risks in financial networks in presence of a central clearing counterparty (CCP), see e.g., [15, 19, 6, 4, 7, 13, 14, 18]. The influential paper [15] shows that the efficiency of a CCP critically depends on the tradeoff between bilateral netting across derivative classes and multilateral netting via the CCP. The tradeoff is assessed based on the average interbank liability and the main insight is that partial netting of derivative classes may remove bilateral netting opportunities and thus increase average liability. In this paper, we consider another type of adverse effects of partial netting than [15]. Our setup is deliberately chosen such that under any partial multilateral netting arrangement the total liability of all banks decreases. Our results are driven by the tradeoff between this decrease in the total liability and the increase of the number of counterparties to which a bank is exposed. Full netting limits contagion to a single round, as opposed to the case of uncleared networks in which contagion can go through several rounds. Under certain conditions, partial netting leads to even more rounds of contagion in the network due to the increase in the number of counterparties and thus to strict adverse effects.

Our paper is also related to the literature where various mechanisms may reinforce network contagion, e.g, [20, 1, 17, 2, 3, 5], and the literature on price mediated contagion in networks of common asset holdings [8, 9].

The reminder of the paper is structured as follows. In Section 2 we introduce the financial network and inverse demand function that models the negative price impact of forced liquidations on the illiquid asset. In Section 3 we prove existence and uniqueness of the clearing equilibrium and provide a finite algorithm for the fixed point. In Section 4 we study partial and full multilateral netting of interbank liabilities. Section 5 concludes. The proofs of all lemmas and theorems in the main text are given in Appendix A.

2 Financial network

We consider the payment network model of [11] which extends the model of [16] where we account for the price impact of the liquidation of external assets. The financial network consists of m interlinked financial institutions ("banks") $i \in [m] = \{1, \ldots, m\}$. These can be thought of as dealers in an OTC market and their customers. Since customers are included, we assume there is no outside creditor to the financial network. Dealer-to-customers and dealer-to-dealer contracts have the same bilateral and multilateral netting opportunities. This is consistent with the post-reform environment, in which, with some exceptions, customer-to-dealer trades are required to be centrally cleared [14].

There is one period $t = 0, 1$. At $t = 0$, bank i holds $\gamma_i \geq 0$ units of a liquid asset (cash), and $y_i \geq 0$ units of an illiquid asset. Cash has constant value one. The illiquid asset has a positive fundamental value P at $t = 1$. The total illiquid asset holdings of the banks is denoted by $y_{tot} := \sum_{i\in[m]} y_i$.

Nominal interbank liabilities. Interbank liabilities realize at $t = 1$. They are represented by a matrix of nominal liabilities (L_{ij}), where $L_{ij} \geq 0$ denotes the cash-amount that bank i owes bank j at $t = 1$. The total nominal liabilities of bank i sum up to

$$L_i = \sum_{j\in[m]} L_{ij}.$$

Bank i in turn claims a total nominal cash amount of $\sum_{j\in[m]} L_{ji}$ from the other banks. The nominal balance sheet of bank i is then given by:

- Assets: $\gamma_i + \sum_{j\in[m]} L_{ji} + y_i P$,
- Liabilities: $L_i +$ nominal net worth.

The nominal cash balance is $\gamma_i + \sum_{j\in[m]} L_{ji} - L_i$.

Price impact of liquidations. If bank i's nominal cash balance is negative, then it has a liquidity shortfall and sells some of its shares of the illiquid asset. This has a negative price impact on the illiquid asset, which we model by an inverse demand function similarly as in [11]. We assume there is an outside market for the illiquid asset that can absorb the total illiquid asset holdings of the banks at a distressed price. It is beyond the scope of this paper to endogenize both the demand function for the illiquid asset and the financial network payments. Instead, we consider a given inverse demand function satisfying some mild technical assumptions and we analyze the interplay between the forced liquidations and the payment equilibrium in the network of interbank liabilities.

The inverse demand function $f(x, P)$ gives the equilibrium price for the illiquid asset when x units of the asset are sold. We assume that $f(x, P)$ satisfies

(i) $f(0, P) = P$,

(ii) $x \longmapsto f(x, P)$ is non-increasing;

(iii) $x \longmapsto xf(x, P)$ is strictly increasing for $x \in [0, y_{tot}]$.

The first property states that in absence of liquidations the price is given exogenously by P. The second property states that the price is non-increasing with the excess supply x. The third property specifies that the cash proceeds from liquidations do increase with the liquidated quantity x. This third assumption, although very natural[1], turns out to be also be necessary for the uniqueness of an equilibrium: [10] show that the [11] model features multiple equilibria.

We denote by $P_{min} = f(y_{tot}, P)$ the price when the total illiquid asset holdings of the banks y_{tot} are sold. We then have

$$f(x, P) \geq P_{min} > 0, \quad \text{for all } x \in [0, y_{tot}].$$

If the revenue from selling y_i units of the illiquid asset does not cover the negative cash-balance, then bank i defaults on its interbank liabilities. Interbank claims are of equal seniority, so that counterparty bank j will in turn receive a proportion

$$\Pi_{ij} = \begin{cases} L_{ij}/L_i & \text{if } L_i > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

of the cash-value of bank i's total assets. This means that the assets are distributed among the creditors according to the proportionality rule, see e.g. [16].

Negative price externalities resulting from liquidity shortages are intertwined with negative network externalities resulting from non-payment of liabilities. In the next section, we characterize jointly the clearing price and the clearing total liability vector in equilibrium.

3 Existence and uniqueness of equilibrium

In equilibrium, the previous characterization of actual cash flows and price impact lead to a clearing price P^* and total liability vector $\mathbf{L}^* = (L_1^*, \ldots, L_m^*)$, which can be determined as a fixed point, $\Phi(P^*, \mathbf{L}^*) = (P^*, \mathbf{L}^*)$, of the non-linear map Φ on $[P_{min}, P] \times [0, (L_1, \ldots, L_m)]$, given by

$$\begin{cases} \Phi_0(p, \ell) &= f\left(\sum_{i \in [m]} \frac{(L_i - \gamma_i - \sum_{j \in [m]} \ell_j \Pi_{ji})^+}{p} \wedge y_i, P \right) \\ \Phi_i(p, \ell) &= L_i \wedge \left(y_i \cdot p + \gamma_i + \sum_{j \in [m]} \ell_j \Pi_{ji} \right), \ i \in [m], \end{cases} \tag{2}$$

similar as in [16, 11].

We have the following lemma.

Lemma 1. *The mapping Φ is monotone, continuous and bounded.*

[1] Removing this assumption would mean that the cash proceeds from liquidations may decrease with the number of liquidated shares, which means that the marginal price of the asset becomes zero or negative beyond a certain point.

As shown in [11, 16], Lemma 1 and Tarski's fixed-point theorem [21] implies the existence of a clearing price and total liability vector. However, uniqueness in the setup with price impact has remained an open problem. Our first result now solves this: uniqueness holds under very mild assumptions.

Theorem 2. *If all banks hold external assets, i.e., $\gamma_i > 0$ or $y_i > 0$ for all $i \in [m]$, then the mapping Φ has a unique fixed point.*

This result extends the uniqueness result of [16] who assumed no price impact on the illiquid asset under liquidation.

Net worth. Given the unique clearing price P^* and total liability vector L^*, by Theorem 2, we define the equilibrium net worth of bank i as

$$C_i = y_i P^* + \gamma_i + \sum_{j\in[m]} L_j^* \Pi_{ji} - L_i.$$

The shortfall is given by

$$C_i^- = L_i - L_i^*,$$

and for the surplus we can easily check the following identity

$$C_i^+ = y_i P^* + \gamma_i + \sum_{j\in[m]} L_j^* \Pi_{ji} - L_i^*. \tag{3}$$

The surplus of bank i depends on the network structure both through the clearing price and liability payments. In contrast, we have the following *aggregate surplus* identity

$$\sum_{i\in[m]} C_i^+ = \sum_{i\in[m]} y_i P^* + \sum_{i\in[m]} \gamma_i, \tag{4}$$

where we used (3) and the fact that $\sum_{i\in[m]} \Pi_{ji} = 1$ for all $j \in [m]$. Hence the aggregate surplus of the banks depends on the clearing price only. In particular, if there are no price effects in the clearing equilibrium, then the aggregate surplus does not depend on the network structure.

Using this important insight, in Section 4 we will focus on the impact of multilateral netting on bank shortfall and on the clearing illiquid asset price.

Before that, we give the following iterative procedure to identify the clearing liability payment vector and asset price in at most m steps. This algorithm is similar to the one given by [20] for the construction of the largest clearing vector and involves solving at each step a system of linear equations.

Algorithm 3 (Constructing the clearing vector)**.** Under the assumptions of Theorem 2, the unique fixed point of the mapping Φ can be found by the following algorithm, in at most m steps. We start with $k = 0$, $p(0) = P$ and $\ell_i(0) = L_i$ for every bank $i \in [m]$. Repeat:

(i) Set $k \to k + 1$;

(ii) For any bank $i \in [m]$, define the total cash balance by

$$c_i(k) := y_i \cdot p(k-1) + \gamma_i + \sum_{j\in[m]} \ell_j(k-1)\Pi_{ji} - L_i;$$

(iii) Denote the set of illiquid banks by

$$\mathcal{D}(k) := \{i \in [m] \mid c_i(k) < 0\},$$

and the set of liquid banks by

$$\mathcal{L}(k) := \{i \in [m] \mid c_i(k) \geq 0\};$$

(We set $\mathcal{D}(0) = \emptyset$ and $\mathcal{L}(0) = [m]$.)

(iv) If $\mathcal{D}(k) = \mathcal{D}(k-1)$ terminate the algorithm;

(v) Set $\ell_i(k) = L_i$ for all banks $i \in \mathcal{L}(k)$, $\ell_i(k) = x_i$ for all $i \in \mathcal{D}(k)$ and $p(k) = q$ where $(q, (x_i)_{i\in\mathcal{D}(k)})$ is determined as the maximal solution on $[P_{min}, p(k-1)] \times \prod_{i\in\mathcal{D}(k)}[0, \ell_i(k-1)]$ of the following system of equations:

$$x_i = y_i \cdot q + \gamma_i + \sum_{j\in\mathcal{L}(k)} L_j\Pi_{ji} + \sum_{j\in\mathcal{D}(k)} x_j\Pi_{ji}, \quad i \in \mathcal{D}(k), \tag{5}$$

$$q = f\left(\sum_{i\in\mathcal{D}(k)} y_i + \sum_{i\in\mathcal{L}(k)} \frac{(L_i - \gamma_i - \sum_{j\in\mathcal{L}(k)} L_j\Pi_{ji} - \sum_{j\in\mathcal{D}(k)} x_j\Pi_{ji})^+}{q} \wedge y_i, P\right). \tag{6}$$

Lemma 4. *Algorithm 3 converges in at most m steps to the fixed point of map* (2).

4 Effects of multilateral netting

We now consider the effects of multilateral netting of the interbank liabilities, in full or in part, on clearing price of the illiquid asset, bank shortfall and aggregate surplus. Formally, we extend the financial network by an auxiliary node $i = 0$. The node 0 can be interpreted as a central clearing counterparty, or more abstractly as a facility whose role is to provide multi-lateral netting services. In practice, such facilities exist for tri-party netting agreements, which are performed by custodian banks in multiple deal multiple broker netting scenarios.[2]

Any nominal interbank liability is multilaterally netted in full or in part through node 0. Partial multilateral netting accounts for the case when only a part of the contracts, for example the standardized contracts, are eligible for multilateral netting. It can also account for the case when some transactions, e.g. customer-dealer, are not eligible for central clearing.

We denote by $\alpha_{ij} \in [0, 1]$ the fraction of nominal interbank liability L_{ij} that is intermediated through node 0. We deliberately assume that $\alpha_{ij} \leq 1$. That is, we can interpret L_{ij} as gross

[2]See `www.isitc.org/market_practice`.

nominal liability of bank i towards bank j, before bilateral netting. The case $\alpha_{ij} > 1$ would correspond to the situation when partial clearing of the liabilities of bank i towards bank j increases its nominal liabilities due to the breaking of bilateral netting opportunities. The tradeoff between bilateral netting and multilateral netting and that partial clearing may lead to an increase in the average liability in the network is well understood for a variety of distributions of the liabilities [15, 12]. With $\alpha_{ij} \leq 1$, our results are driven by network effects and not by the break-up of bilateral netting opportunities. The total liabilities of the banks decrease in nominal terms, but the network is "rewired" as described in the following.

We denote $\boldsymbol{\alpha} = (\alpha_{ij})_{i,j\in[m]} \in [0,1]^{m\times m}$, and write $\mathbf{0}$ for the all-zero matrix, and $\mathbf{1}$ for the all-one matrix. We denote the net exposure of node i to node 0 under partial clearing by

$$\Lambda_i(\boldsymbol{\alpha}) = \sum_{j\in[m]} \alpha_{ji} L_{ji} - \sum_{j\in[m]} \alpha_{ij} L_{ij}. \tag{7}$$

The net exposure of bank i to the other banks,

$$\Lambda_i(\mathbf{1}) = \sum_{j\in[m]} L_{ji} - \sum_{j\in[m]} L_{ij}, \tag{8}$$

is equal to the exposure of bank i to node 0 under full multilateral netting, i.e., $\boldsymbol{\alpha} = \mathbf{1}$.

The total nominal liabilities of node i (towards the node 0 and the other banks) sum up to

$$\widehat{L}_i(\boldsymbol{\alpha}) = \begin{cases} \sum_{j\in[m]}(1-\alpha_{ij})L_{ij} + \Lambda_i^-(\boldsymbol{\alpha}), & i \in [m]; \\ \sum_{j\in[m]} \Lambda_j^+(\boldsymbol{\alpha}), & i = 0. \end{cases}$$

We define the relative liability weights for bank $i \in [m]$ by

$$\bar{\Pi}_{ij}(\boldsymbol{\alpha}) = \begin{cases} \frac{(1-\alpha_{ij})L_{ij}}{\widehat{L}_i(\boldsymbol{\alpha})}, & j \neq 0; \\ \frac{\Lambda_i^-(\boldsymbol{\alpha})}{\widehat{L}_i(\boldsymbol{\alpha})}, & j = 0; \end{cases} \tag{9}$$

and for node 0 by

$$\bar{\Pi}_{0j}(\boldsymbol{\alpha}) = \frac{\Lambda_j^+(\boldsymbol{\alpha})}{\widehat{L}_0(\boldsymbol{\alpha})}.$$

Using the above equations, we can interpret the partial clearing as a "rewiring" of the financial network. The "rewired" network introduces for each bank, in case of partial clearing, liabilities to more banks than the initial network. The resulting relative interbank liability weights are given as

$$\widehat{\Pi}_{ij}(\boldsymbol{\alpha}) = \bar{\Pi}_{ij}(\boldsymbol{\alpha}) + \bar{\Pi}_{i0}(\boldsymbol{\alpha}) \times \bar{\Pi}_{0j}(\boldsymbol{\alpha}), \quad i,j \in [m].$$

Indeed, it is straightforward to verify that

$$\sum_{i\in[m]} \widehat{\Pi}_{ij}(\boldsymbol{\alpha}) = 1 \quad \text{and} \quad \widehat{\Pi}_{ij}(\mathbf{0}) = \Pi_{ij}.$$

In equilibrium, this leads to a clearing price $\widehat{P}^* = \widehat{P}^*(\boldsymbol{\alpha})$ and total liability vector $\widehat{\mathbf{L}}^* = (\widehat{L}_1^*, \ldots, \widehat{L}_m^*) = \widehat{\mathbf{L}}^*(\boldsymbol{\alpha})$, which can be determined as a fixed point, $\widehat{\Phi}(\widehat{P}^*, \widehat{\mathbf{L}}^*) = (\widehat{P}^*, \widehat{\mathbf{L}}^*)$, of the non-linear map $\widehat{\Phi}$ on $[P_{min}, P] \times [0, (\widehat{L}_1, \ldots, \widehat{L}_m)]$, given by

$$\begin{cases} \widehat{\Phi}_0(p,\ell) &= f\left(\sum_{i\in[m]} \frac{(\widehat{L}_i - \gamma_i - \sum_{j\in[m]} \ell_j \widehat{\Pi}_{ji})^+}{p} \wedge y_i, P \right) \\ \widehat{\Phi}_i(p,\ell) &= \widehat{L}_i \wedge \left(y_i \cdot p + \gamma_i + \sum_{j\in[m]} \ell_j \widehat{\Pi}_{ji} \right), \ i \in [m], \end{cases} \tag{10}$$

similar to (2). Theorem 2 implies that the mapping $\widehat{\Phi}$ has a unique fixed point under the stated assumptions. As before, we define the equilibrium net worth of each bank $i \in [m]$ by

$$\widehat{C}_i(\boldsymbol{\alpha}) := y_i \widehat{P}^*(\boldsymbol{\alpha}) + \gamma_i + \sum_{j\in[m]} \widehat{L}_j^*(\boldsymbol{\alpha}) \widehat{\Pi}_{ji}(\boldsymbol{\alpha}) - \widehat{L}_i(\boldsymbol{\alpha}).$$

The shortfall is given by

$$\widehat{C}_i^-(\boldsymbol{\alpha}) = \widehat{L}_i(\boldsymbol{\alpha}) - \widehat{L}_i^*(\boldsymbol{\alpha}).$$

Similarly to (4), the aggregate surplus satisfies

$$\sum_{i\in[m]} \widehat{C}_i^+(\boldsymbol{\alpha}) = \sum_{i\in[m]} y_i \widehat{P}^*(\alpha) + \sum_{i\in[m]} \gamma_i.$$

Using (10) and the definition of the net exposure (8), we find that the shortfall vector $(\widehat{C}_1^-, \ldots, \widehat{C}_m^-)$ solves the fixed point equation

$$\begin{aligned} \widehat{C}_i^-(\boldsymbol{\alpha}) &= \left(\widehat{L}_i(\boldsymbol{\alpha}) - \gamma_i - y_i \widehat{P}^* - \sum_{j\in[m]} \widehat{L}_j^*(\boldsymbol{\alpha}) \widehat{\Pi}_{ji} \right)^+ \\ &= \left(\sum_{j\in[m]} \widehat{L}_j(\boldsymbol{\alpha}) \widehat{\Pi}_{ji} - \Lambda_i(\mathbf{1}) - \gamma_i - y_i \widehat{P}^* - \sum_{j\in[m]} \widehat{L}_j^*(\boldsymbol{\alpha}) \widehat{\Pi}_{ji} \right)^+ \\ &= \left(\sum_{j\in[m]} \left(\widehat{L}_j(\boldsymbol{\alpha}) - \widehat{L}_j^*(\boldsymbol{\alpha}) \right) \widehat{\Pi}_{ji} - \Lambda_i(\mathbf{1}) - \gamma_i - y_i \widehat{P}^* \right)^+ \\ &= \left(\sum_{j\in[m]} \widehat{C}_j^-(\boldsymbol{\alpha}) \widehat{\Pi}_{ji}(\boldsymbol{\alpha}) - \gamma_i - y_i \widehat{P}^* - \Lambda_i(\mathbf{1}) \right)^+. \end{aligned} \tag{11}$$

From (11) we see that the shortfall of bank $j \neq i$ is borne by bank i according to the proportionality rule $\widehat{\Pi}_{ji}(\boldsymbol{\alpha})$. Similarly, we find that the clearing price $\widehat{P}^* = \widehat{P}^*(\boldsymbol{\alpha})$ satisfies

$$\begin{aligned}\widehat{P}^* &= f\left(\sum_{i\in[m]} \frac{\left(\widehat{L}_i - \gamma_i - \sum_{j\in[m]} \widehat{L}_j^*\widehat{\Pi}_{ji}\right)^+}{\widehat{P}^*} \wedge y_i, P\right) \\ &= f\left(\sum_{i\in[m]} \frac{\left(\sum_{j\in[m]} \widehat{C}_j^-(\boldsymbol{\alpha})\widehat{\Pi}_{ji}(\boldsymbol{\alpha}) - \gamma_i - \Lambda_i(\mathbf{1})\right)^+}{\widehat{P}^*} \wedge y_i, P\right).\end{aligned} \tag{12}$$

The fixed point equation $\widehat{\Phi}(\widehat{P}^*, \widehat{\mathbf{L}}^*) = (\widehat{P}^*, \widehat{\mathbf{L}}^*)$ is thus equivalent to (11)–(12).

We now analyze the effect of partial versus full multilateral netting on banks' shortfall, clearing price, and aggregate surplus.

Theorem 5. *For any $\boldsymbol{\alpha} \in [0,1]^{m\times m}$, partial versus full multilateral netting increases the shortfall of bank i,*

$$\widehat{C}_i^-(\boldsymbol{\alpha}) \geq \widehat{C}_i^-(\mathbf{1}),$$

reduces the clearing price of the illiquid asset,

$$\widehat{P}^*(\boldsymbol{\alpha}) \leq \widehat{P}^*(\mathbf{1}),$$

and reduces the aggregate surplus,

$$\sum_{i\in[m]} \widehat{C}_i^+(\boldsymbol{\alpha}) \leq \sum_{i\in[m]} \widehat{C}_i^+(\mathbf{1}).$$

The following example illustrates the adverse effects of partial versus full multilateral netting listed in Theorem 5, and it shows that partial multilateral netting may also be worse than no netting.

Example 6. We consider the financial network consisting of $m = 5$ banks as shown in Figure 1, where $L_{21} = 2x, L_{13} = L_{14} = x$ and $L_{45} = X \gg x$. Assume that $\gamma_2 \geq 2x$ so that bank 2 can pay its liability in full. We also set $\gamma_1 = 0$ and $y_1 > 0$ and $\gamma_4 > 0$, with $x \gg \gamma_4$ and $y_4 = 0$. Only bank 1 holds the liquid asset, so the price of this asset depends only on the quantity liquidated by bank 1.

Under no netting ($\boldsymbol{\alpha} = \mathbf{0}$) and under full netting ($\boldsymbol{\alpha} = \mathbf{1}$), we have $\widehat{C}_1^-(\mathbf{0}) = \widehat{C}_1^-(\mathbf{1}) = 0$ and there are no liquidations by bank 1. As a consequence, $\widehat{P}^*(\mathbf{0}) = \widehat{P}^*(\mathbf{1}) = P$. Also, we have $\widehat{C}_4^-(\mathbf{0}) = \widehat{C}_4^-(\mathbf{1}) = X - x - \gamma_4$. Banks 2, 3 and 5 do not have any shortfall.

Consider now the case when all liabilities are cleared, except for L_{13}. That is, we set $\boldsymbol{\alpha} = \mathbf{1} - \boldsymbol{e}_{13}$, where $\boldsymbol{e}_{ij}$ denotes the matrix which is all zero except for the element (i, j) which is one. We have $\widehat{C}_4^- = X - x - \gamma_4$. This shortfall of bank 4 is borne by both banks 1 and 5 in the respective proportions, $\widehat{\Pi}_{41} = \frac{x}{X+x}$ and $\widehat{\Pi}_{45} = \frac{X}{X+x}$. The total shortfall imposed on bank 1 is thus given by $\frac{x}{X+x} \times (X - x - \gamma_4)$, which is approximately x when $X \gg x$. Therefore, bank

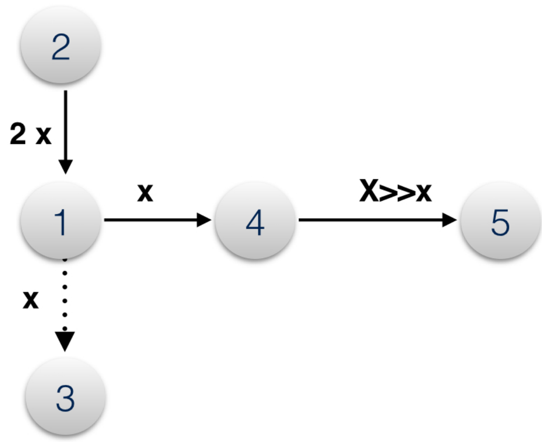

Figure 1: Partial multilateral netting may be worse. The dashed line indicates uncleared liabilities.

1 has a liquidity shortfall that drives the price of the illiquid asset down. More precisely, the clearing price $\widehat{P}^*$ satisfies

$$\widehat{P}^*(\alpha) = f\left(\frac{\left(\frac{x}{X+x} \times (X - x - \gamma_4) - \gamma_1\right)^+}{\widehat{P}^*} \wedge y_1, P\right) < P = \widehat{P}^*(\mathbf{0}),$$

as soon as $\frac{x}{X+x} \times (X - x - \gamma_4) > \gamma_1$. We also have that for $\widehat{C}_1^-(\alpha) \leq \widehat{C}_1^-(\mathbf{0})$ and, for all $i \neq 1$, $\widehat{C}_i^-(\alpha) = \widehat{C}_i^-(\mathbf{0})$. We thus conclude that the partial netting may increase the shortfall of all banks and consequently decreases the asset price.

Remark 7. Recall that the partial netting can also be interpreted as excluding the customer-dealer transactions from multi-lateral clearing. In the example above, we can think of node 3 as the customers of node 1. We conclude that excluding customer-dealer transactions from multi-lateral netting may have negative effects on shortfall and the asset price.

5 Conclusion

We have shown a uniqueness result for the payment and price equilibrium in a network of liabilities. Our results hold under mild and natural assumptions on the price impact function: monotonicity of the price impact function and strict monotonicity of the proceeds of liquidation in the liquidated quantity.

We have also considered the transformation of the network of liabilities by using multilateral netting and analyzed its impact on the equilibrium payment vector and the asset price. Under full multilateral netting, shortfall and price impact of forced liquidations are minimized, while not including certain liabilities in the netting agreement may lead to more forced liquidations, higher price impact and shortfall. The adverse effects of partial versus full clearing are not due to removing bilateral netting opportunities as in previous papers, such as [15, 12]. Our results are driven entirely by network effects.

We have not considered a regulated clearing counter party (CCP) for the multilateral netting. A regulated CCP would also have capital requirements. The capitalization and design of a CCP is of critical importance and is treated in [4]. However, multilateral netting is the main aspect of central clearing, and our results expose the risks of partial clearing on asset prices and shortfall. Our example would still hold in case of an undercapitalized CCP, and partial clearing may still be worse than no clearing.

A Proofs

In this section we present the proofs of all lemmas and theorems in the main text.

Proof of Lemma 1

First, note that $\Phi_0(p, \ell)$ is a non-decreasing continuous function of p and ℓ. Also for $i \in [m]$, we have that $\Phi_i(p, \ell)$ is a non-decreasing continuous function of p and ℓ, as it is the composition of the non-decreasing continuous maps $\ell \to yp + \gamma + \Pi^T \ell$ and $\ell \to \ell \wedge L$. Last, note that, $\Phi(P_{min}, \mathbf{0}) \geq (P_{min}, \mathbf{0})$ and $\Phi(P, \mathbf{L}) \leq (P, \mathbf{L})$. This implies that the map Φ is bounded, which concludes the proof.

Proof of Theorem 2

First note that by assumption, $y_i p + \gamma_i > 0$ for all $p > 0$ and for all banks $i \in [m]$. By [16, Theorem 1], when the price of the illiquid asset is fixed at $p > 0$, there exists a unique clearing vector of the interbank payments. Let us denote it by $\ell^*(p)$ and we have that $p \to \ell^*(p)$ is continuous and non-decreasing by Lemma 1.

Let us denote by

$$\zeta(p) := \sum_{i \in [m]} \frac{(L_i - \gamma_i - \sum_{j \in [m]} \ell_j^*(p) \Pi_{ji})^+}{p} \wedge y_i \in [0, y_{tot}],$$

the total liquidated value at price p. To prove our theorem we need to show that the function $f(\zeta(\cdot), P)$ has a unique fixed point.

Now note that the function $\zeta(.)$ is continuous non-increasing and the inverse demand function $f(., P)$ is continuous non-decreasing. Thus, $f(\zeta(\cdot), P)$ is a non-decreasing continuous function. Furthermore, we have that $f(\zeta(P), P) \leq P$ and that $f(\zeta(P_{min}), P) \geq P_{min}$, since P_{min} is the priced reached when all available quantity if the asset is liquidated. Therefore, by Tarski's fixed point theorem, there exists a fixed point of the function $f(\zeta(\cdot), P)$.

Suppose now by way of contradiction that there exist two fixed points p_1 and p_2, with $p_1 < p_2$ such that $f(\zeta(p_1), P) = p_1$, and $f(\zeta(p_2), P) = p_2$.

Since $\zeta(.)$ is non-increasing, we have $\zeta(p_1) \geq \zeta(p_2)$. Moreover, if $\zeta(p_1) = \zeta(p_2)$ then

$$f(\zeta(p_1), P) = f(\zeta(p_2), P),$$

which contradicts $p_1 < p_2$ and $\zeta(p_1) = \zeta(p_2)$. Thus, $\zeta(p_1) > \zeta(p_2)$. By property (iii) of the demand function, we have that $\zeta(p_1) f(\zeta(p_1), P) > \zeta(p_2) f(\zeta(p_2), P)$, and thus

$$\zeta(p_1) p_1 > \zeta(p_2) p_2. \tag{13}$$

Denote by $\mathcal{D}(p_1)$ and $\mathcal{D}(p_2)$ the set of defaulted banks when the price of the illiquid asset is p_1 and respectively p_2. We have

$$\begin{aligned}
\mathcal{D}(p_1) &:= \{i \mid L_i - \gamma_i - \sum_{j \in [m]} \ell_j^*(p_1) \Pi_{ji} > y_i p_1\}, \text{ and} \\
\mathcal{D}(p_2) &:= \{i \mid L_i - \gamma_i - \sum_{j \in [m]} \ell_j^*(p_2) \Pi_{ji} > y_i p_2\}.
\end{aligned}$$

Clearly $\mathcal{D}(p_2) \subseteq \mathcal{D}(p_1)$ since $p_1 < p_2$. For $k \in \{1, 2\}$, we have that

$$\begin{aligned}
\zeta(p_k) p_k &= p_k \sum_{i \in \mathcal{D}(p_k)} y_i + \sum_{i \in [m] \setminus \mathcal{D}(p_k)} (L_i - \gamma_i - \sum_{j \in [m]} \ell_j^*(p_k) \Pi_{ji})^+ \\
&= p_k \sum_{i \in \mathcal{D}(p_k)} y_i + \sum_{i \in [m] \setminus \mathcal{D}(p_k)} L_i - \gamma_i - \sum_{j \in [m]} \ell_j^*(p_k) \Pi_{ji} + (L_i - \gamma_i - \sum_{j \in [m]} \ell_j^*(p_k) \Pi_{ji})^-.
\end{aligned}$$

Thus, we infer

$$\begin{aligned}
\zeta(p_2) p_2 - \zeta(p_1) p_1 = (p_2 - p_1) \sum_{i \in \mathcal{D}(p_2)} y_i - p_1 \sum_{i \in \mathcal{D}(p_1) \setminus \mathcal{D}(p_2)} y_i \\
&- \sum_{i \in [m] \setminus \mathcal{D}(p_2)} \sum_{j \in [m]} \big(\ell_j^*(p_2) - \ell_j^*(p_1)\big) \Pi_{ji} + \sum_{i \in \mathcal{D}(p_1) \setminus \mathcal{D}(p_2)} \big(L_i - \gamma_i - \sum_{j \in [m]} \ell_j^*(p_1) \Pi_{ji}\big) \\
&+ \sum_{i \in [m] \setminus \mathcal{D}(p_2)} (L_i - \gamma_i - \sum_{j \in [m]} \ell_j^*(p_2) \Pi_{ji})^- - (L_i - \gamma_i - \sum_{j \in [m]} \ell_j^*(p_1) \Pi_{ji})^- \\
&+ \sum_{i \in \mathcal{D}(p_1) \setminus \mathcal{D}(p_2)} (L_i - \gamma_i - \sum_{j \in [m]} \ell_j^*(p_1) \Pi_{ji})^-.
\end{aligned}$$

Furthermore, since $\ell^*(p)$ is non-decreasing, we have

$$(L_i - \gamma_i - \sum_{j \in [m]} \ell_j^*(p_2) \Pi_{ji})^- \geq (L_i - \gamma_i - \sum_{j \in [m]} \ell_j^*(p_1) \Pi_{ji})^-.$$

We conclude

$$\begin{aligned}\zeta(p_2)p_2 - \zeta(p_1)p_1 \;&\geq (p_2 - p_1)\sum_{i\in\mathcal{D}(p_2)} y_i - p_1 \sum_{i\in\mathcal{D}(p_1)\setminus\mathcal{D}(p_2)} y_i \\ &\quad - \sum_{i\in[m]\setminus\mathcal{D}(p_2)} \sum_{j\in[m]} \big(\ell_j^*(p_2) - \ell_j^*(p_1)\big)\Pi_{ji} \\ &\quad + \sum_{i\in\mathcal{D}(p_1)\setminus\mathcal{D}(p_2)} (L_i - \gamma_i - \sum_{j\in[m]} \ell_j^*(p_1)\Pi_{ji}). \end{aligned} \tag{14}$$

Using the fact that banks that are not in default pay their liabilities in full, while banks that are in default pay the total of their assets, it follows that

$$\begin{aligned}\sum_{i,j\in[m]} \big(\ell_j^*(p_2) - \ell_j^*(p_1)\big)\Pi_{ji} &= \sum_{i\in[m]} \big(\ell_i^*(p_2) - \ell_i^*(p_1)\big) \\ &= \sum_{i\in\mathcal{D}(p_1)\setminus\mathcal{D}(p_2)} \Big(L_i - (\gamma_i + y_i p_1 + \sum_{j\in[m]} \ell_j^*(p_1)\Pi_{ji})\Big) \\ &\quad + \sum_{i\in\mathcal{D}(p_2)} \big(y_i(p_2 - p_1) + \sum_{j\in[m]} \big(\ell_j^*(p_2) - \ell_j^*(p_1)\big)\Pi_{ji}\big). \end{aligned}$$

Rearranging terms, we obtain

$$\begin{aligned}(p_2 - p_1)\sum_{i\in\mathcal{D}(p_2)} y_i - p_1 \sum_{i\in\mathcal{D}(p_1)\setminus\mathcal{D}(p_2)} y_i &= \sum_{i\in[m]\setminus\mathcal{D}(p_2)} \sum_{j\in[m]} \big(\ell_j^*(p_2) - \ell_j^*(p_1)\big)\Pi_{ji} \\ &\quad - \sum_{i\in\mathcal{D}(p_1)\setminus\mathcal{D}(p_2)} \Big(L_i - \gamma_i - \sum_{j\in[m]} \ell_j^*(p_1)\Pi_{ji}\Big). \end{aligned} \tag{15}$$

We now plug (15) in (14) to obtain that $\zeta(p_2)p_2 - \zeta(p_1)p_1 \geq 0$, which is in contradiction to (13). This finishes the proof of Theorem 2.

Proof of Lemma 4

Fix the notations of Algorithm 3 and let $k \geq 1$. Consider the map

$$(q, (x_j)_{j\in\mathcal{D}(k)}) \to (\Phi_i(q, (x_j)_{j\in\mathcal{D}(k)}), (L_j)_{j\in\mathcal{L}(k)}))_{i\in\{0\}\cup\mathcal{D}(k)} \tag{16}$$

defined on $[P_{min}, p(k-1)] \times \prod_{i\in\mathcal{D}(k)}[0, \ell_i(k-1)]$, with Φ defined in (2).

Since for all $i \in \mathcal{D}(k)$ we have that

$$y_i \cdot p(k-1) + \gamma_i + \sum_{j\in[m]} \ell_j(k-1)\Pi_{ji} < L_i,$$

then on the above domain, $\Phi_0(q, (x_j)_{j\in\mathcal{D}(k)}), (L_j)_{j\in\mathcal{L}(k)})$ writes as the right-hand side of (6) and $\Phi_i(q, (x_j)_{j\in\mathcal{D}(k)}), (L_j)_{j\in\mathcal{L}(k)})$ writes as the right-hand side of (5) for all $i \in \mathcal{D}(k)$.

The existence of a maximal solution in step k, (v) of Algorithm 3 is guaranteed then by Tarski's fixed-point theorem [21] for the map (16). Indeed, as in Lemma 1, this map is increasing on $[P_{min}, p(k-1)] \times \prod_{i\in\mathcal{D}(k)}[0, \ell_i(k-1)]$. Moreover, by induction on step k of the algorithm, this map takes values in $[P_{min}, p(k-1)] \times \prod_{i\in\mathcal{D}(k)}[0, \ell_i(k-1)]$.

The algorithm will stop in at most m steps since at each step we have $|\mathcal{D}(k)\setminus\mathcal{D}(k-1)| \geq 1$. We let k^* the stopping time. We have

$$\begin{aligned}
p(k^*-1) &= \Phi_0(p(k^*-1), (\ell_j(k^*-1))_{j\in\mathcal{D}(k^*-1)}, (L_j)_{j\in\mathcal{L}(k^*-1)}),\\
\ell_i(k^*-1) &= \Phi_i(p(k^*-1), (\ell_j(k^*-1))_{j\in\mathcal{D}(k^*-1)}, (L_j)_{j\in\mathcal{L}(k^*-1)}) \text{ for all } i \in \mathcal{D}(k^*-1),\\
L_i &\leq y_i \cdot p(k^*-1) + \gamma_i + \sum_{j\in[m]} \ell_j(k^*-1)\Pi_{ji} \text{ for all } i \in \mathcal{L}(k^*-1),
\end{aligned}$$

and thus $(p(k^*-1, (\ell_i(k^*-1))_{i\in\mathcal{D}(k-1^*)}, (L_i)_{i\in\mathcal{L}(k^*-1)})$ is a fixed point of the map (2). By Theorem 2, this is the unique fixed point.

Proof of Theorem 5

We prove that $\widehat{C}_i^-(\boldsymbol{\alpha}) \geq \widehat{C}_i^-(\mathbf{1})$ and $\widehat{P}^*(\boldsymbol{\alpha}) \leq \widehat{P}^*(\mathbf{1})$. Note that $\widehat{\Pi}_{ij}(\mathbf{1}) = \frac{\Lambda_i^+}{\sum_{j\in[m]}\Lambda_j^+}$. We thus have, for $i \in [m]$,

$$\begin{aligned}
\widehat{C}_i^-(\mathbf{1}) &= \left(\sum_{j\in[m]} \widehat{C}_j^-(\mathbf{1}) \frac{\Lambda_i^+(\mathbf{1})}{\sum_{j\in[m]}\Lambda_j^+(\mathbf{1})} - \gamma_i - y_i\widehat{P}^*(\mathbf{1}) - \Lambda_i(\mathbf{1})\right)^+\\
&= \left(\Lambda_i^+(\mathbf{1}) \frac{\sum_{j\in[m]}\widehat{C}_j^-(\mathbf{1})}{\sum_{j\in[m]}\Lambda_j^-(\mathbf{1})} - \gamma_i - y_i\widehat{P}^*(\mathbf{1}) - \Lambda_i(\mathbf{1})\right)^+\\
&= \left(-\Lambda_i^+(\mathbf{1})\left(1 - \frac{\sum_{j\in[m]}\widehat{C}_j^-(\mathbf{1})}{\sum_{j\in[m]}\Lambda_j^-(\mathbf{1})}\right) - \gamma_i - y_i\widehat{P}^*(\mathbf{1}) + \Lambda_i^-(\mathbf{1})\right)^+.
\end{aligned}$$

Since $\widehat{C}_j^-(\mathbf{1}) \leq \Lambda_j^-(\mathbf{1})$ for all $j \in [m]$ and $\gamma_i + y_i\widehat{P}^*(\mathbf{1}) \geq 0$ we have $\widehat{C}_i^- = 0$ for $\Lambda_i > 0$. We thus have

$$\widehat{C}_i^-(\mathbf{1}) = \left(\gamma_i + y_i\widehat{P}^*(\mathbf{1}) + \Lambda_i(\mathbf{1})\right)^-. \tag{17}$$

Similarly $\widehat{P}^*(\mathbf{1})$ satisfies

$$\widehat{P}^*(\mathbf{1}) = f\left(\sum_{i\in[m]} \frac{(\gamma_i + \Lambda_i(\mathbf{1}))^-}{\widehat{P}^*(\mathbf{1})} \wedge y_i, P\right). \tag{18}$$

We thus have, for all $i \in [m]$ and $\boldsymbol{\alpha}$,

$$\begin{aligned}\widehat{C}_i^-(\boldsymbol{\alpha}) &= \left(\sum_{j\in[m]} \widehat{C}_j^-(\boldsymbol{\alpha})\widehat{\Pi}_{ji}(\boldsymbol{\alpha}) - \gamma_i - y_i\widehat{P}^*(\boldsymbol{\alpha}) - \Lambda_i(\mathbf{1})\right)^+ , \\ &\geq \Big(\gamma_i + y_i\widehat{P}^*(\boldsymbol{\alpha}) + \Lambda_i(\mathbf{1})\Big)^- ,\end{aligned}$$

and (since f is non increasing)

$$\begin{aligned}\widehat{P}^*(\boldsymbol{\alpha}) &= f\left(\sum_{i\in[m]} \frac{(\sum_{j\in[m]} \widehat{C}_j^-(\boldsymbol{\alpha})\widehat{\Pi}_{ji}(\boldsymbol{\alpha}) - \gamma_i - \Lambda_i(\mathbf{1}))^+}{\widehat{P}^*(\boldsymbol{\alpha})} \wedge y_i, P\right) \\ &\leq f\left(\sum_{i\in[m]} \frac{(\gamma_i + \Lambda_i(\mathbf{1}))^-}{\widehat{P}^*(\boldsymbol{\alpha})} \wedge y_i, P\right).\end{aligned}$$

We thus conclude that $\widehat{C}_i^-(\boldsymbol{\alpha}) \geq \widehat{C}_i^-(\mathbf{1})$ and $\widehat{P}^*(\boldsymbol{\alpha}) \leq \widehat{P}^*(\mathbf{1})$.

Moreover, the aggregate surplus satisfies:

$$\sum_{i\in[m]} \widehat{C}_i^+(\mathbf{1}) - \sum_{i\in[m]} \widehat{C}_i^+(\boldsymbol{\alpha}) = \Big(\widehat{P}^*(\mathbf{1}) - \widehat{P}^*(\boldsymbol{\alpha})\Big) \sum_{i\in[m]} y_i \geq 0.$$

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