

# Killip-Simon problem and Jacobi flow on GSMP matrices

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## Abstract

One of the first and therefore most important theorems in perturbation theory claims that for an arbitrary self-adjoint operator  $A$  there exists a perturbation  $B$  of Hilbert-Schmidt class with arbitrary small operator norm, which destroys completely the absolutely continuous (a.c.) spectrum of the initial operator  $A$  (von Neumann). However, if  $A$  is the discrete free 1-D Schrödinger operator and  $B$  is an arbitrary Jacobi matrix (of Hilbert-Schmidt class) the a.c. spectrum remains perfectly the same, that is, the interval  $[-2, 2]$ . Moreover, Killip and Simon described explicitly the spectral properties for such  $A + B$ . Jointly with Damanik they generalized this result to the case of perturbations of periodic Jacobi matrices in the non-degenerated case. Recall that the spectrum of a periodic Jacobi matrix is a system of intervals of a very specific nature. Christiansen, Simon and Zinchenko posed in a review dedicated to F. Gesztesy (2013) the following question: “is there an extension of the Damanik-Killip-Simon theorem to the general finite system of intervals case?” In this paper we solve this problem completely. Our method deals with the Jacobi flow on GSMP matrices. GSMP matrices are probably a new object in the spectral theory. They form a certain Generalization of matrices related to the Strong Moment Problem, the latter ones are a very close relative of Jacobi and CMV matrices. The Jacobi flow on them is also a probably new member of the rich family of integrable systems. Finally, related to Jacobi matrices of Killip-Simon class, analytic vector bundles and their curvature play a certain role in our construction and, at least on the level of ideology, this role is quite essential.

## 1 Introduction

### 1.1 Main result

(1) *Von Neumann Theorem* [42] states that for an arbitrary self-adjoint operator  $A$ , having a nontrivial absolutely continuous (a.c.) component of the spectrum, there exists a self-adjoint perturbation  $\delta A$  of Hilbert-Schmidt class such that  $A + \delta A$  has a pure point spectrum. Moreover,  $\delta A$  may have an arbitrary small operator norm.

Therefore, the following result is already quite non-trivial.

(2) *Deift-Killip Theorem* [11]. For a discrete one-dimensional Schrödinger operator with square summable potential, the absolutely continuous part of the spectrum is  $[-2, 2]$ .

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Thus, under a special perturbations of Hilbert-Schmidt class (the square summable potential) the absolutely continuous spectrum of the free, discrete 1-D Schrödinger operator is perfectly preserved. It is totally surprising that one can find a *complete explicit characterization* of the spectral data if the perturbation is an arbitrary Jacobi matrix of Hilbert-Schmidt class.

(3) *Killip-Simon Theorem* [20]. Let  $d\sigma$  be a probability measure on  $\mathbb{R}$  with bounded but infinite support. As it is well known the orthonormal polynomials  $P_n(x)$  with respect to this measure obey a three-term recurrence relation

$$xP_n(x) = a(n)P_{n-1}(x) + b(n)P_n(x) + a(n+1)P_{n+1}(x), \quad a(n) > 0.$$

The following are equivalent:

(op)  $\sum_{n \geq 1} |a(n) - 1|^2 < \infty$  and  $\sum_{n \geq 0} |b(n)|^2 < \infty$ .

(sp) The measure  $d\sigma$  is supported on  $[-2, 2] \cup X$ , and moreover

$$\int_{-2}^2 |\log \sigma'(x)| \sqrt{4 - x^2} dx + \sum_{x_k \in X} \sqrt{x_k^2 - 4}^3 < \infty. \quad (1.1)$$

**Remark 1.1.** Of course the (op)-condition means that the Jacobi matrix

$$J_+ = \begin{bmatrix} b(0) & a(1) & & & \\ a(1) & b(1) & a(2) & & \\ & \ddots & \ddots & \ddots & \\ & & & & \ddots \end{bmatrix}$$

represents a Hilbert-Schmidt class perturbation of the matrix  $\overset{\circ}{J}_+$  with the constant coefficients  $\overset{\circ}{a}(n) = 1$  and  $\overset{\circ}{b}(n) = 0$ . In this case we consider  $J_+$  as an operator acting in the standard space of one-sided sequences  $l_+^2$ . In its turn, the (sp)-condition means that the related *spectral measure*  $d\sigma$  has an absolutely continuous component supported on  $[-2, 2]$ . Moreover, the spectral density  $\sigma'(x)$  with respect to the Lebesgue measure satisfies an explicitly given integral condition, which in particular means that  $\sigma'(x) \neq 0$  a.e. on this interval. Besides that, the measure may have at most countably many mass points (the set  $X$ ) outside of the given interval. Again, the corresponding set  $X$  satisfies an explicitly given condition, which in particular means that the only possible accumulation points of this set are the endpoints  $\pm 2$ . Finally, note that there is no restriction on the *singular component* of the measure  $d\sigma$  on the interval  $[-2, 2]$ .

Later, the authors jointly with David Damanik generalized their result on the case of perturbations of *periodic* Jacobi matrices. To state this theorem we need a couple of definitions.

We define a distance between two one-sided sequences  $b = \{b(n)\}_{n \geq 0}$  and  $\tilde{b} = \{\tilde{b}(n)\}_{n \geq 0}$  from  $l_+^\infty$  by

$$\text{dist}^2(b, \tilde{b}) = \text{dist}_\eta^2(b, \tilde{b}) := \sum_{n \geq 0} |b(n) - \tilde{b}(n)|^2 \eta^{2n}, \quad \eta \in (0, 1). \quad (1.2)$$

The distance  $\text{dist}(J_+, \tilde{J}_+)$  between two Jacobi matrices is defined via the distances between the generating coefficient sequences.

Let  $J(E)$  be the isospectral set of periodic two-sided Jacobi matrices with a given spectral set  $E \subset \mathbb{R}$ . The distance between  $J_+$  and  $J(E)$  is defined in a standard way

$$\text{dist}(J_+, J(E)) = \inf\{\text{dist}(J_+, \overset{\circ}{J}_+) : \overset{\circ}{J} \in J(E)\},$$

where  $\overset{\circ}{J}_+$  is the restriction of a two-sided matrix  $\overset{\circ}{J}$  on the positive half-axis.

(4) *Damanik-Killip-Simon Theorem* (DKST) [9]. Assume that  $J_+$  is a Jacobi matrix and let  $d\sigma$  be the associated spectral measure. The following are equivalent:

(opp) Let  $S_+$  denote the shift operator in  $l^2_+$ . Then

$$\sum_{n \geq 0} \text{dist}^2((S_+^*)^n J_+ S_+^n, J(E)) < \infty. \quad (1.3)$$

(spp) The measure  $d\sigma$  is supported on  $E \cup X$ , and moreover

$$\int_E |\log \sigma'(x)| \sqrt{\text{dist}(x, \mathbb{R} \setminus E)} dx + \sum_{x_k \in X} \sqrt{\text{dist}(x_k, E)}^3 < \infty. \quad (1.4)$$

**Remark 1.2.** Note that (1.3) means that the shifts of the given Jacobi matrix  $J_+$  approach to the isospectral set  $J(E)$ , but possibly not to a specific element  $\overset{\circ}{J}$  of this set. In the same time (1.4) looks as a straightforward counterpart of the condition (1.1).

**Remark 1.3.** Let us point out that the spectral set of any periodic two-sided Jacobi matrix  $\overset{\circ}{J}$  is a system of interval of a very special nature: the system of intervals  $E = [\mathbf{b}_0, \mathbf{a}_0] \setminus \cup_{j=1}^g (\mathbf{a}_j, \mathbf{b}_j)$  represents the spectrum of a periodic Jacobi matrix if and only if  $E = T_m^{-1}([-2, 2])$ , where  $T_m(z)$  is a polynomial with only real critical points, that is,  $T'_m(c) = 0$  for  $c \in \mathbb{R}$ , and its critical values  $T_m(c)$  obey the conditions  $|T_m(c)| \geq 2$ . Actually, the Damanik-Killip-Simon Theorem was proved under an additional regularity condition  $|T_m(c)| > 2$  for all critical points  $c$ . In this case the degree  $m = g + 1$ .

The paper [7] reviews recent progress in the understanding of the class of so-called *finite gap* Jacobi matrices and their perturbations. In the end of the article the authors posed the following question: “Is there an extension of the Damanik-Killip-Simon theorem to the general finite system of intervals  $E$  case?” In the present paper *we solve completely this problem*, see Theorem 1.5 below. Naturally, this question was posed as soon as the original Killip-Simon theorem was published or even presented or proved. From this point of view [7] is just an explicit recent reference.

Finite gap Jacobi matrices were discovered in the context of approximation theory [2, 3], [5, Chapter X]. They became especially famous because of their relation with the theory of integrable systems, for historical comments we would refer to [24] with many references therein. But the true meaning of this class was significantly clarified recently

by C. Remling: for a system of intervals  $E$  the finite gap class  $J(E)$  consists of all limit points of Jacobi matrices with an essential spectrum on  $E$ , having this  $E$  as the support of their a.c. spectrum.

(5) *Remling Theorem* [32]. Let  $E$  be a system of intervals. Let  $J_+$  be a Jacobi matrix with the generating coefficient sequences  $\{a(n), b(n)\}$  such that its spectrum  $\sigma(J_+) = E \cup X$ , where  $X$  is a set of points, which accumulate only to the endpoints of the intervals, and  $\sigma'(x) \neq 0$  for a.e.  $x \in E$ . If

$$\overset{\circ}{a}(n) = \lim_{m_k \rightarrow +\infty} a(n + m_k), \quad \overset{\circ}{b}(n) = \lim_{m_k \rightarrow +\infty} b(n + m_k),$$

for all  $n \in \mathbb{Z}$ , then the corresponding two-sided Jacobi matrix  $\overset{\circ}{J}$  belongs to  $J(E)$ .

Note that the system of shifts  $\{(S_+^*)^n J_+ S_+^n\}_{n \geq 0}$  forms a precompact set in the compact-open topology (generated by the distance (1.2)).

For  $E = [\mathbf{b}_0, \mathbf{a}_0] \setminus \cup_{j=1}^g (\mathbf{a}_j, \mathbf{b}_j)$  the class  $J(E)$  represents a  $g$ -dimensional torus, which can be parametrized explicitly.

(6) *Baker-Akhiezer parametrization for the class  $J(E)$* , see e.g. [40, Theorem 9.4]. For  $\alpha \in \mathbb{R}^g / \mathbb{Z}^g$  let

$$\mathcal{A}(\alpha) = \bar{a}^2 \frac{\theta(\alpha + \mu + \bar{\alpha})\theta(\alpha - \mu + \bar{\alpha})}{\theta(\alpha + \bar{\alpha})^2}, \quad \mathcal{B}(\alpha) = \bar{b} + \partial_\xi \ln \frac{\theta(\alpha - \mu + \bar{\alpha})}{\theta(\alpha + \bar{\alpha})}, \quad (1.5)$$

where

$$\theta(z) = \theta(z, \Omega) = \sum_{n \in \mathbb{Z}^g} e^{\pi i \langle \Omega n, n \rangle + 2\pi i \langle z, n \rangle}, \quad z \in \mathbb{C}^g,$$

with the following system of parameters depending on  $E$ :

- $\Omega$  is a symmetric  $g \times g$  matrix with a positive imaginary part,  $\text{Im } \Omega > 0$ ;
- $\bar{\alpha} \in \mathbb{C}^g$  is an appropriate shift;
- $\mu \in \mathbb{R}^g / \mathbb{Z}^g$  and  $\xi \in \mathbb{R}^g$  are certain fixed directions of discrete and continuous translations on the torus  $\mathbb{R}^g / \mathbb{Z}^g$ , respectively;
- $\bar{a} > 0$  and  $\bar{b} \in \mathbb{R}$  are normalization constants.

Then  $\overset{\circ}{J} \in J(E)$  if and only if

$$\overset{\circ}{a}(n)^2 = \mathcal{A}(\alpha - \mu n), \quad \overset{\circ}{b}(n) = \mathcal{B}(\alpha - \mu n), \quad (1.6)$$

for some  $\alpha \in \mathbb{R}^g / \mathbb{Z}^g$ . In this case we write  $\overset{\circ}{J} = J(\alpha)$ . Thus,

$$J(E) = \{J(\alpha) : \alpha \in \mathbb{R}^g / \mathbb{Z}^g\}. \quad (1.7)$$

**Definition 1.4.** For an arbitrary finite system of intervals  $E$ , we say that a Jacobi matrix  $J_+$  belongs to the Killip-Simon class  $\text{KS}(E)$  if for some  $X$  the corresponding spectral measure  $d\sigma$  is supported on  $E \cup X$  and obeys (1.4).

**Theorem 1.5.**  $J_+$  belongs to  $\text{KS}(E)$  if and only if there exist  $\epsilon_\alpha(n) \in l_+^2(\mathbb{R}^g)$  and  $\epsilon_a(n) \in l_+^2$ ,  $\epsilon_b(n) \in l_+^2$  such that (cf. (1.6))

$$a(n)^2 = \mathcal{A}\left(\sum_{k=0}^n \epsilon_\alpha(k) - \mu n\right) + \epsilon_a(n), \quad b(n) = \mathcal{B}\left(\sum_{k=0}^n \epsilon_\alpha(k) - \mu n\right) + \epsilon_b(n), \quad (1.8)$$

where  $\mathcal{A}(\alpha)$  and  $\mathcal{B}(\alpha)$  are defined in (1.5).

**Remark 1.6.** In the one interval case the functions  $\mathcal{A}$  and  $\mathcal{B}$  are constants, e.g. if  $E = [-2, 2]$ , then  $\mathcal{A} = 1$  and  $\mathcal{B} = 0$  and we obtain the original Killip-Simon Theorem.

**Remark 1.7.** It is easy to see that a Jacobi matrix of the form (1.8) satisfies (1.3), see Lemma 7.2. Moreover, from our explicit formulas one can give immediately a suitable approximant for  $(S_+^*)^n J_+ S_+^n$ , this is  $J(\alpha_n) \in J(E)$ ,  $\alpha_n = \sum_{k=0}^n \epsilon_\alpha(k) - \mu n$ ; or conclude that, if the series  $\beta := \sum_{k=0}^\infty \epsilon_\alpha(k)$  conditionally converges, then the coefficients of  $J_+$  approach, in fact, to the coefficients of the fixed element  $J(\beta) \in J(E)$ ,

$$a^2(n) - \mathcal{A}(\beta - \mu n) \rightarrow 0 \text{ and } b(n) - \mathcal{B}(\beta - \mu n) \rightarrow 0, \text{ where } n \rightarrow \infty.$$

The representation (1.8) contains a certain ambiguity, for the reason see Remark 7.1.

## 1.2 Basic ideas of the method and the structure of the paper

The proof of DKST was based on two things:

- (i) Magic formula for periodic Jacobi matrices
- (ii) Matrix version of the Killip-Simon theorem

The first one is the following identity. Let  $S$  be the shift in the space of two sided sequences  $l^2$ . If  $E = [\mathbf{b}_0, \mathbf{a}_0] \setminus \cup_{j=1}^g (\mathbf{a}_j, \mathbf{b}_j) = T_{g+1}^{-1}([-2, 2])$ , then

$$T_{g+1}(\overset{\circ}{J}) = S^{g+1} + S^{-(g+1)}$$

for all  $\overset{\circ}{J} \in J(E)$ . The last matrix can be understood as the  $(g+1) \times (g+1)$ -block Jacobi matrix with the constant block coefficients  $\overset{\circ}{A}(n) = I_{g+1}$  and  $\overset{\circ}{B}(n) = \mathbf{0}_{g+1}$ .

Now, for  $J_+$  the matrix  $T_{g+1}(J_+)$  is a  $(2g+3)$ -diagonal matrix, or, also a one-sided  $(g+1) \times (g+1)$  Jacobi block-matrix, see survey [10],

$$T_{g+1}(J_+) = \begin{bmatrix} B(0) & A(1) & & & \\ A(1) & B(1) & A(2) & & \\ & & \ddots & \ddots & \ddots \\ & & & & \ddots \end{bmatrix}.$$

Such matrix has a spectral  $(g+1) \times (g+1)$  matrix-measure, say  $d\Xi$ . According to [9] the matrix analog of (1.1) is of the form

$$\int_{-2}^2 |\log \det \Xi'(y)| \sqrt{4-y^2} dy + \sum_{y_k \in Y} \sqrt{y_k^2 - 4}^3 < \infty, \quad (1.9)$$

as before  $[-2, 2] \cup Y$  is the support of  $d\Xi$ . On the one hand this condition can be rewritten by means of the spectral measure  $d\sigma$  of the initial Jacobi matrix  $J_+$  into the form (1.4),  $y = T_{g+1}(x)$ . On the other hand, due to the matrix version of the Killip-Simon theorem, (1.9) is equivalent to  $T_{g+1}(J_+) - (S_+^{g+1} + (S_+^*)^{g+1})$  belongs to the Hilbert-Schmidt class. This is a certain bunch of conditions on the coefficients of  $J_+$ , but we should recognize that extracting from this simple-looking condition the final one (1.3), is a very non-trivial task.

Our *first basic observation* is the following.

**Lemma 1.8.** *For a system of intervals  $E$  there exists a unique rational function  $\Delta(z)$  such that*

$$E = [\mathbf{b}_0, \mathbf{a}_0] \setminus \bigcup_{j=1}^g (\mathbf{a}_j, \mathbf{b}_j) = \Delta^{-1}([-2, 2]),$$

and  $\text{Im } \Delta(z) > 0$  for  $\text{Im } z > 0$ .

*Proof.* Let  $\Psi(z)$  be the Ahlfors function in the domain  $\bar{\mathbb{C}} \setminus E$ . Among all analytic functions in this domain, which vanish at infinity and are bounded by one in absolute value, this function has the biggest possible value  $\text{Cap}_a(E) = |z\Psi(z)|_{z=\infty}$  (the so-called analytic capacity) [1]. As it is well known [30]

$$\frac{1 - \Psi(z)}{1 + \Psi(z)} = \sqrt{\prod_{j=0}^g \frac{z - \mathbf{a}_j}{z - \mathbf{b}_j}}. \quad (1.10)$$

Then

$$\Delta(z) = \frac{1}{\Psi(z)} + \Psi(z) = \lambda_0 z + \mathbf{c}_0 + \sum_{j=1}^g \frac{\lambda_j}{\mathbf{c}_j - z}, \quad (1.11)$$

where  $\lambda_j > 0$ ,  $j \geq 0$ , and  $\Psi(\mathbf{c}_j) = 0$ ,  $\mathbf{c}_j \in (\mathbf{a}_j, \mathbf{b}_j)$ ,  $j \geq 1$ . □

Note that in this proof we represented  $\Delta(z)$  as a superposition of a function  $\Psi : \bar{\mathbb{C}} \setminus E \rightarrow \mathbb{D}$  with the Zhukovskii map. Essentially, (1.11) is our *generalized magic formula*, though it holds of course not for Jacobi matrices.

*Jacobi, CMV and SMP matrices.* Jacobi matrices probably the oldest object in the spectral theory of self-adjoint operators generated by the moment problem [4]

$$s_k = \int x^k d\sigma. \quad (1.12)$$

In this problem we are looking for a measure  $d\sigma$  supported on the real axis, which provides the representation (1.12) for the given moments  $\{s_k\}_{k \geq 0}$ . In this sense CMV matrices are related to the *trigonometric moment problem*, which corresponds to the same question with respect to a measure supported on the unit circle. Note that this problem is also classical [4], but corresponding CMV matrices are a comparably fresh object in the spectral theory [33, 34]. The *strong moment problem* corresponds to

measures on the real axis in the case that the moments are given for *all integers*  $k$ . An extensive bibliography of works on the strong moment problem can be found in the survey [19], concerning its matrix generalization see [37, 38].

As usual the solution of the problem deals with the orthogonalization of the generating system of functions, that is, the system

$$1, \frac{-1}{x}, x, \frac{(-1)^2}{x^2}, x^2, \dots$$

in the given case. The multiplication operator by the independent variable in  $L^2_{d\sigma}$  with respect to the related *orthonormal basis we call SMP matrix* (this is exactly the way of the appearance of Jacobi and CMV matrices in connection with the power and trigonometric moment problem, respectively). In another terminology they are called Laurent-Jacobi matrices [6, 12, 18]. Very similar to the CMV-case, this is a five-diagonal matrix of a special structure, say  $A_+ = A_+(d\sigma)$ . We assume that the measure is compactly supported and the origin does not belong to the support of this measure. In this case our  $A_+$  is bounded, moreover  $A_+^{-1}$  is also a bounded operator of a similar five-diagonal structure (just shifted by one element!)

Note that, by a linear change of variable, we can always normalize an arbitrary *two intervals* system to the form  $\mathbf{c}_1 = 0$ , see (1.11), that is,

$$E = [\mathbf{b}_0, \mathbf{a}_0] \setminus (\mathbf{a}_1, \mathbf{b}_1) = \Delta^{-1}([-2, 2]), \quad \Delta(z) = \lambda_0 + \mathbf{c}_0 - \frac{\lambda_1}{z}. \quad (1.13)$$

Without going into detail, dealing with the structure of SMP matrices, we can formulate our *second basic observation*.

**Proposition 1.9.** [13] *Let  $A(E)$  be the set of all two sided SMP matrices of period two with their spectrum on  $E$  (1.13). Then  $\overset{\circ}{A} \in A(E)$  if and only if*

$$\Delta(\overset{\circ}{A}) = \lambda_0 \overset{\circ}{A} + \mathbf{c}_0 - \lambda_1 (\overset{\circ}{A})^{-1} = S^2 + S^{-2}. \quad (1.14)$$

**Remark 1.10.** It is highly important in (1.14) to be hold that both  $\overset{\circ}{A}$  and  $(\overset{\circ}{A})^{-1}$  are five-diagonal matrices.

Naturally, (1.13)-(1.14) have to be generalized to the multi-interval case. This leads to the concept of GSMP matrices (G for generalized), see the next subsection. However, even after such a generalization the result on spectral properties of ("some") GSMP matrices of Killip-Simon class would be interesting probably only to a small circle of specialists, working with the strong moment problem. The point is that GSMP matrices are used here as a certain intermediate (but very important) object. In a sense, this is the best possible choice of a *system of coordinates*. We can try to clarify the last sentence. The standard point of view on  $J(E)$  is to associate it with the hyperelliptic Riemann surface  $\mathfrak{R}_E = \{(z, w) : w^2 = \prod_{j=0}^g (z - \mathbf{a}_j)(z - \mathbf{b}_j)\}$ . Then  $J(E)$  corresponds to the "real part" of the Jacobian variety  $\text{Jac}(\mathfrak{R}_E)$  of this surface see e.g. [24, 25]. Periodic GSMP matrices, satisfying

$$\Delta(\overset{\circ}{A}) = S^{g+1} + S^{-(g+1)} \quad (1.15)$$

for  $\Delta(z)$  given in (1.11), are most likely the best possible choice for a coordinate system on the affine part of  $\text{Jac}(\mathfrak{R}_E)$ , at least in the application to spectral theory.

Thus, the point is to *go back to Jacobi matrices*. Let  $d\sigma$  be compactly supported and 0 does not belong to its support. We can define the map

$$\mathcal{F}_+ : \text{SMP} \rightarrow \text{Jacobi}$$

just setting  $J_+(\sigma)$  in correspondence with the given  $A_+(\sigma)$ . If so, we can define (in a naive way) a discrete dynamical system (*Jacobi flow on SMP matrices*) by the map  $\mathcal{J}_+$ , which corresponds to the following commutative diagram:

$$\begin{array}{ccc} \text{SMP} & \xrightarrow{\mathcal{J}_+} & \text{SMP} \\ \mathcal{F}_+ \downarrow & & \mathcal{F}_+ \downarrow \\ \text{Jacobi} & \xrightarrow{\mathcal{S}_+} & \text{Jacobi} \end{array} \quad (1.16)$$

where  $\mathcal{S}_+ J_+ = S_+^* J_+ S_+$ .

The *third basic observation* deals with the idea of getting properties of the class  $\text{KS}(E)$  from the corresponding properties of the class of SMP (or, generally, GSMP) matrices using the above introduced dynamical system  $A_+(n) = \mathcal{J}_+^n(A_+)$ .

This definition (1.16) is naive for the following reason. In the transformation  $J_+(n) = \mathcal{S}_+^n J_+$  the eigenvalues in the gaps start to move. E.g., in a generic case for an initial  $\overset{\circ}{J}_+$ , which corresponds to one of our fundamental operators  $\overset{\circ}{J} \in J(E)$ , the eigenvalues will cover densely the spectral gaps  $(\mathbf{a}_j, \mathbf{b}_j)$ . Thus, corresponding to such measures  $A_+(n)$  just can not be properly defined. The easiest way to explain that nevertheless our program is doable is the following: pass to two-sided Jacobi matrices and enjoy unitarity of the shift  $S$  in  $l^2$ ! (One can actually work with one-sided matrices but still use methods related to two dimensional cyclic subspaces, which are naturally required if one works with two-sided matrices). In fact, an arbitrary one-sided Jacobi matrix  $J_+$ , with its essential spectrum on  $E$ , can be extended by a Jacobi matrix  $J_- = P_- \overset{\circ}{J} P_-$ ,  $\overset{\circ}{J} \in J(E)$ , such that  $(\mathbf{c}_j - J)^{-1}$  exists for the resulting two sided matrix  $J$  and all  $\mathbf{c}_j$ , see Lemma 5.1.

In the next subsection we give formal definitions for GSMP matrices and the Jacobi flow on them, but probably we can already outline the structure of the current paper:

Section 2. We recall the functional model for finite gap Jacobi matrices. In this model each operator is marked by a Hardy space  $H^2(\alpha)$  of character-automorphic functions in the domain  $\bar{\mathbb{C}} \setminus E$ , where  $\alpha$  is a character of the fundamental group of this domain (2.1), so, as before,  $\alpha \in \mathbb{R}^g / \mathbb{Z}^g$  cf. (1.7). Here  $J(\alpha)$  is the multiplication operator by the independent variable with respect to the basis  $\{e_n^\alpha\}_{n \in \mathbb{Z}}$  (2.3), and  $\{e_n^\alpha\}_{n \geq 0}$  is an intrinsic basis in  $H^2(\alpha)$ . The point is that in this domain the *inner* function  $\Psi(z)$  and the fixed ordering  $\mathbf{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_g\}$  of its zeros generate another natural basis  $\{f_n^\alpha\}_{n \geq 0}$  in  $H^2(\alpha)$  (2.7). Thus, we obtain a new family of operators (the multiplication



operator in the new bases)

$$A(E, \mathbf{C}) = \{A(\alpha, \mathbf{C}) : \alpha \in \mathbb{R}^g / \mathbb{Z}^g\}.$$

This is the collection of all periodic GSMP matrices associated with the given spectral set  $E$  and a fixed ordering  $\mathbf{C}$  of zeros of the Ahlfors function  $\Psi(z)$ . The fact that  $\Psi(z)$  is *single valued* (the character corresponding to this function is trivial) is responsible for the periodicity of an arbitrary  $A(\alpha, \mathbf{C})$ .

Another characteristic feature of  $\Psi(z)$  is its certain conformal invariance. Indeed, if  $w = w_j = \frac{1}{c_j - z}$ , then  $\Psi_j(w) := \Psi(z)$  is the Ahlfors function in the  $w$ -plane. The given ordering  $\mathbf{C}$  generates the specific ordering

$$\mathbf{C}_j = \left\{ \frac{1}{\mathbf{c}_{j+1} - \mathbf{c}_j}, \dots, \frac{1}{\mathbf{c}_g - \mathbf{c}_j}, 0, \frac{1}{\mathbf{c}_1 - \mathbf{c}_j}, \dots, \frac{1}{\mathbf{c}_{j-1} - \mathbf{c}_j} \right\}$$

and the multiplication by  $w$  is again a *periodic GSMP matrix* (up to an appropriate shift). That is,

$$S^{-j}(\mathbf{c}_j - A(\alpha, \mathbf{C}))^{-1}S^j \in A(E_j, \mathbf{C}_j), \quad (1.17)$$

where  $E_j = \{y = \frac{1}{c_j - x} : x \in E\}$ . Note that  $0 = w(\infty)$ . Let us point out that the spectral condition (1.4) possesses the same conformal invariance property. Thus, passing from the  $e$ -basis to the  $f$ -basis in  $H^2(\alpha)$ , we payed a certain prize:  $J(\alpha)$  is three diagonal and  $A(\alpha, \mathbf{C})$  is a  $(2g + 3)$ -diagonal matrix. In the same time we essentially win, since  $(\mathbf{c}_j - J(\alpha))^{-1}$  has infinitely many non-trivial diagonals, but due to (1.17) all matrices  $(\mathbf{c}_j - A(\alpha, \mathbf{C}))^{-1}$  are still  $(2g + 3)$  diagonal. For them (1.11) (in the chosen basis) is nothing but the magic formula (1.15).

The Jacobi flow on  $A(E, \mathbf{C})$  can be defined in a very natural way. Since  $S^{-1}J(\alpha)S = J(\alpha - \mu)$ , we set  $\mathcal{J}A(\alpha, \mathbf{C}) = A(\alpha - \mu, \mathbf{C})$ . As we see, this is just one, probably new, object in the family of integrable systems.

As a result, thanks to this section we are well prepared to understand and describe the structure of GSMP matrices,  $A \in \text{GSMP}(\mathbf{C})$ , and the Jacobi flow on them,  $A(n) = \mathcal{J}^n A$ , in the general case. This is done in the Sections 3 and 4, respectively.

Section 5. Using the block-matrix version of the Killip-Simon theorem, it is a fairly simple task to write the necessary and sufficient condition for  $A \in \text{GSMP}(\mathbf{C})$  with the spectral data (1.4) in the form

$$\Delta(A) - (S^{-(g+1)} + S^{g+1}) \text{ is in the Hilbert-Schmidt class.} \quad (1.18)$$

Note that the relation between corresponding spectral densities of  $V(A)$  and  $A$  has a quite elegant form (5.5). Further, (1.18) is equivalent to

$$H_{\pm}(A) < \infty \quad (1.19)$$

for the *Killip-Simon functional of the problem*, which is basically the  $l_{\pm}^2$ -part of the trace of  $(\Delta(A) - (S^{-(g+1)} + S^{g+1}))^2$ , for the precise expression see (5.9). In the spirit of our

third basic observation, we compute the "derivative" of this functional in the direction of the Jacobi flow, that is, the value

$$\delta_{\mathcal{J}}H_+(A) := H_+(A) - H_+(\mathcal{J}A),$$

see Lemma 5.5. *This derivative represents a finite sum of squares!* Now, we can rewrite (1.19) as the "integral"  $\sum_{n \geq 0} \delta_{\mathcal{J}}H_+(\mathcal{J}^n A) < \infty$  and, thus, get certain  $l^2$ -properties. Note that they are already more related to the Jacobi matrix  $J = \mathcal{F}A$  than to the given GSMP matrix  $A$  itself.

Section 6. Nevertheless, all these conditions were given by means the coefficients of  $V(A)$ , not by the ones of  $A$  (or the system of iterates  $A(n)$ , to be more precise). This is probably the hardest technical part of the work. To indicate the difficulty, we would mention the following. In [26] we found higher-order generalizations of Killip-Simon sum rules (relations between coefficients of  $J_+$  and the spectral measure  $d\sigma$ ), for a *single interval spectrum*. But only for a very special family (related to Chebyshev polynomials of an arbitrary degree  $n$ ), which was initially found in [22], we were able to convert the result of the form (1.18) to explicit relations on the coefficients of the given  $J_+$ . Otherwise, each particular case becomes a reason for an interesting research, see e.g. [21, 16, 36]. Moreover, a nice looking general conjecture was recently disproved by M. Lukic [23]. By the way, for a highly interesting new development in this area see [15]. In this section we prove Theorem 1.21. Practically, this is already a parametric representation for coefficients of Jacobi matrices of  $\text{KS}(E)$ .

Section 7. In this section we finalize the parametric representation for Killip-Simon Jacobi matrices associated to an arbitrary system of intervals  $E$ , that is, we prove the main Theorem 1.5. In the end of this section we demonstrate implicitly our *last basic for this paper observation* that the spectral theory in the spirit of [8] could be more powerful than the classical orthogonal polynomials approach [4, 35], see Subsection 7.2 and especially Remark 7.4. Explicitly this was demonstrated in [28, 41, 27], as well as in Section 2 of the current paper. At the moment we are not able to present a theory of spaces of vector bundles, which corresponds as model spaces to Jacobi matrices of Killip-Simon class (in full generality) even in a finite gap case.

### 1.3 GSMP matrices and Jacobi flow on them in solving the Killip-Simon problem

In this subsection we give formal definitions for the named objects so that in the end of it we are able to state Theorem 1.21. This is the main ingredient in our proof of Theorem 1.5.

Let  $\{e_n\}$  be the standard basis in  $l^2$ . Depending on the context,  $l_+^2$  is the set of square-summable one-sided sequences or the subspace of  $l^2$  spanned by  $\{e_n\}_{n \geq 0}$ . In the last case  $l_-^2 := l^2 \ominus l_+^2$  and  $P_+ : l^2 \rightarrow l_+^2$  is the orthogonal projector. Also  $\{\delta_k\}_{k=0}^g$  denotes the standard basis in the Euclidian space  $\mathbb{C}^{g+1}$ .

By  $T^*$  we denote the conjugated operator to an operator  $T$ , or the conjugated matrix if  $T$  is a matrix. In particular, for a vector-column  $\vec{p} \in \mathbb{C}^{g+1}$ ,  $(\vec{p})^*$  is a  $(g+1)$ -dimensional vector-row. Consequently, the scalar product in  $\mathbb{C}^{g+1}$  can be given in the

following form  $\langle \vec{p}, \vec{q} \rangle = (\vec{q})^* \vec{p}$ . The notation  $T^-$  denotes the upper triangular part of a matrix  $T$  (including the main diagonal), respectively  $T^+ := T - T^-$  is its lower triangular part (excluding the main diagonal).

GSMP matrices form a certain special subclass of real symmetric  $(2g + 3)$ -diagonal matrices,  $g \geq 1$ . First of all, the class depends on an ordered collection of distinct points  $\mathbf{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_g\}$ . That is, if needed we will specify the notation  $\text{GSMP}(\mathbf{C})$ . We will define two-sided GSMP matrices, but their restrictions on the positive half-axis will be highly important.

**Definition 1.11.** We say that  $A$  is *GSMP-structured* if it is a  $(g + 1)$ -block Jacobi matrix

$$A = \begin{bmatrix} \ddots & & & & & & \\ & \ddots & & & & & \\ & & A^*(\vec{p}_{-1}) & B(\vec{\mathbf{p}}_{-1}) & A(\vec{p}_0) & & \\ & & & A^*(\vec{p}_0) & B(\vec{\mathbf{p}}_0) & A(\vec{p}_1) & \\ & & & & \ddots & \ddots & \ddots \end{bmatrix} \quad (1.20)$$

such that

$$\vec{\mathbf{p}} = (\vec{p}, \vec{q}) \in \mathbb{R}^{2g+2}, \quad A(\vec{p}) = \delta_g \vec{p}^*, \quad B(\vec{\mathbf{p}}) = (\vec{q} \vec{p}^*)^- + (\vec{p} \vec{q}^*)^+ + \tilde{\mathbf{C}}, \quad (1.21)$$

and

$$\tilde{\mathbf{C}} = \begin{bmatrix} \mathbf{c}_1 & & & & \\ & \ddots & & & \\ & & \mathbf{c}_g & & \\ & & & & 0 \end{bmatrix}, \quad \vec{p}_j = \begin{bmatrix} p_0^{(j)} \\ \vdots \\ p_g^{(j)} \end{bmatrix}, \quad \vec{q}_j = \begin{bmatrix} q_0^{(j)} \\ \vdots \\ q_g^{(j)} \end{bmatrix}, \quad p_g^{(j)} > 0. \quad (1.22)$$

We call  $\{\vec{\mathbf{p}}_j\}_{j \in \mathbb{Z}}$  the generating coefficient sequences (for the given  $A$ ).

**Remark 1.12.** Concerning the last condition in (1.22): actually, it is important that  $p_g^{(j)} \neq 0$ . The choice  $p_g^{(j)} > 0$  is a matter of normalization. Further, throughout this paper we will assume in this definition that the much stronger condition

$$\inf_{j \in \mathbb{Z}} p_g^{(j)} > 0 \quad (1.23)$$

holds. Note that these coefficients  $\{p_g^{(j)}\}_{j \in \mathbb{Z}}$  form the non-trivial part of the last upper non-vanishing  $(g + 1)$ -th diagonal of a GSMP-structured matrix  $A$ .

**Definition 1.13.** Let  $S$  be the shift operator  $S e_n = e_{n+1}$ . A GSMP-structured matrix  $A$  belongs to the GSMP class if the matrices  $\{\mathbf{c}_k - A\}_{k=1}^g$  are invertible, and moreover  $S^{-k}(\mathbf{c}_k - A)^{-1} S^k$  are GSMP-structured, see (1.20)-(1.22). To abbreviate we write  $A \in \text{GSMP}(\mathbf{C})$ .

**Remark 1.14.** As it follows from the definition the entries of the last upper non-trivial  $(g + 1)$ -th diagonal of the matrix  $S^{-k}(\mathbf{c}_k - A)^{-1} S^k$  should satisfy a counterpart of the condition (1.23). This set of conditions can be written explicitly by means of the coefficients of the initial GSMP-structured matrix  $A$ , see (3.9). Moreover, this set

of conditions on the forming sequences  $\{\vec{p}_j\}_{j \in \mathbb{Z}}$  can be considered as a *constructive definition* of GSMP matrices, see Theorem 3.3. That is,  $A \in \text{GSMP}(\mathbf{C})$  if it is GSMP-structured and (3.9) holds for the generating sequences.

Let  $J$  be a Jacobi matrix with coefficients  $\{a(n), b(n)\}$ :

$$Je_n = a(n)e_{n-1} + b(n)e_n + a(n+1)e_{n+1}, \quad a(n) > 0, \quad n \in \mathbb{Z}. \quad (1.24)$$

The two-dimensional space spanned by  $e_{-1}$  and  $e_0$  forms a cyclic subspace for  $J$ . Also,  $J$  can be represented as a two-dimensional perturbation of the orthogonal sum with respect to the decomposition  $l^2 = l_-^2 \oplus l_+^2$

$$J = \begin{bmatrix} J_- & 0 \\ 0 & J_+ \end{bmatrix} + a(0)(e_0 \langle \cdot, e_{-1} \rangle + e_{-1} \langle \cdot, e_0 \rangle). \quad (1.25)$$

We have a similar decomposition for  $A \in \text{GSMP}(\mathbf{C})$

$$A = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix} + \|\vec{p}_0\|(\tilde{e}_0 \langle \cdot, \tilde{e}_{-1} \rangle + \tilde{e}_{-1} \langle \cdot, \tilde{e}_0 \rangle), \quad \tilde{e}_{-1} = e_{-1}, \quad \tilde{e}_0 := \frac{1}{\|\vec{p}_0\|} P_+ A e_{-1}. \quad (1.26)$$

**Definition 1.15.** For  $A \in \text{GSMP}$  the Jacobi matrix  $J = \mathcal{F}A$  is uniquely defined by the conditions

$$r_{\pm}(z) := \langle (J_{\pm} - z)^{-1} e_{-\frac{1 \pm 1}{2}}, e_{-\frac{1 \pm 1}{2}} \rangle = \langle (A_{\pm} - z)^{-1} \tilde{e}_{-\frac{1 \pm 1}{2}}, \tilde{e}_{-\frac{1 \pm 1}{2}} \rangle, \quad a(0) = \|\vec{p}_0\|. \quad (1.27)$$

Note that the image of  $\mathcal{F}$  consists of Jacobi matrices  $J$  with  $\mathbf{c}_j \in \mathbb{C} \setminus \sigma(J)$ . For an explicit construction see Subsection 7.2, particularly (7.9).

**Definition 1.16.** Let  $\mathcal{S}J := S^{-1}JS$ . The Jacobi flow on GSMP matrices is generated by the transformation  $\mathcal{J}$ , which makes the following diagram commutative

$$\begin{array}{ccc} \text{GSMP} & \xrightarrow{\mathcal{J}} & \text{GSMP} \\ \mathcal{F} \downarrow & & \mathcal{F} \downarrow \\ \text{Jacobi} & \xrightarrow{\mathcal{S}} & \text{Jacobi} \end{array} \quad (1.28)$$

The corresponding discrete dynamical system (Jacobi flow) is of the form

$$A(n+1) = \mathcal{J}A(n), \quad A(0) = A.$$

Essentially, it can be reduced to an open (input-output) dynamical system (4.14). The coefficients of the Jacobi matrix  $J = \mathcal{F}A$  are easily represented by means of the Jacobi flow acting on the initial  $A$ . Namely,

**Corollary 1.17.** Let  $J = \mathcal{F}A$  and  $A(n) = \mathcal{J}^{on}A$ . In the above notations (1.24)

$$a(n) = \|\vec{p}_0(n)\|, \quad b(n-1) = q_g^{(-1)}(n)p_g^{(-1)}(n). \quad (1.29)$$

Now we can define the Killip-Simon class of GSMP matrices. Let  $E$  be a system of  $g + 1$  disjoint intervals,  $E = [\mathbf{b}_0, \mathbf{a}_0] \setminus \cup_{j=1}^g (\mathbf{a}_j, \mathbf{b}_j)$ . Let  $\Delta(z) = \Delta_E(z)$  be the unique function, which was given in (1.11).

**Proposition 1.18.**  $\overset{\circ}{A} \in \text{GSMP}(\mathbf{C})$ , generated by coefficients  $\vec{\mathbf{p}} = (\vec{p}, \vec{q})$ , belongs to the isospectral set of periodic matrices  $A(E, \mathbf{C})$  if and only if it obeys the magic formula (1.15). Moreover, the isospectral surface  $\mathcal{IS}_E$  is given by, see (2.24), (2.25) and (3.2),

$$p_g = \frac{1}{\lambda_0}, \quad q_g = -\mathbf{c}_0 - \lambda_0 \sum_{j=1}^{g-1} p_j q_j, \quad \Lambda_k(\vec{\mathbf{p}}) = \lambda_k, \quad k = 1, \dots, g. \quad (1.30)$$

**Definition 1.19.** Let  $A \in \text{GSMP}(\mathbf{C})$ . Let  $\sigma_{\pm}$  be the related spectral measures, that is,

$$r_{\pm}(z) = \int \frac{d\sigma_{\pm}(x)}{x - z},$$

where  $r_{\pm}(z)$  are given in (1.27). We say that  $A$  belongs to the Killip-Simon class  $\text{KSA}(E, \mathbf{C})$  if the measures  $\sigma_{\pm}$  are supported on  $E \cup X_{\pm}$ , and both satisfy (1.4).

The following theorem is just a consequence of the matrix version of Killip-Simon theorem.

**Theorem 1.20.**  $A \in \text{GSMP}(\mathbf{C})$  belongs to the Killip-Simon class  $\text{KSA}(E, \mathbf{C})$  if the difference  $\Delta_E(A) - (S^{-(g+1)} + S^{g+1})$  belongs to the Hilbert-Schmidt class.

However, the next statement is already highly non-trivial. Practically, it gives a parametrization of the coefficients of *Jacobi matrices* of Killip-Simon class with the essential spectrum on  $E$ .

**Theorem 1.21.** For  $A \in \text{GSMP}(\mathbf{C})$ , let  $A(n+1) = \mathcal{J}A(n)$ ,  $A(0) = A$ . Let  $\{\vec{\mathbf{p}}_j(n)\}_{j \in \mathbb{Z}}$  be the forming  $A(n)$  coefficient sequences. The given  $A$  belongs to  $\text{KSA}(E, \mathbf{C})$  if and only if (see (1.30))

$$\{p_j^{(-1)}(n) - p_j^{(0)}(n)\}_{n \geq 0} \in l_+^2, \quad \{q_j^{(-1)}(n) - q_j^{(0)}(n)\}_{n \geq 0} \in l_+^2, \quad (1.31)$$

$$\{\lambda_0 p_g^{(0)}(n) - 1\}_{n \geq 0} \in l_+^2, \quad \{\lambda_0 \langle \vec{p}_0(n), \vec{q}_0(n) \rangle + \mathbf{c}_0\}_{n \geq 0} \in l_+^2, \quad (1.32)$$

$$\{\Lambda_k(\vec{\mathbf{p}}_0(n)) - \lambda_k\}_{n \geq 0} \in l_+^2 \quad (1.33)$$

hold for all  $j = 0, \dots, g - 1$  and all  $k = 1, \dots, g$ .

To summarise, in this paper solving Killip-Simon problem

- we introduce GSMP matrices as possibly the best coordinate system on the Jacobians of hyperelliptic Riemann surfaces associated to finite band operators;
- we introduce and study the Jacobi flow on GSMP matrices as a one more important object in a rich family of integrable systems;
- our study is based essentially on the Damanik-Killip-Simon theorem on Hilbert-Schmidt perturbations of Jacobi block-matrices with constant coefficients;
- we follow the ideology of application of analytic vector bundles in spectral theory, explicitly in Section 2 and implicitly in Section 7.

## 2 Functional models for $J(E)$ and $A(E, \mathbf{C})$ . Jacobi flow on periodic GSMP matrices

### 2.1 Hardy spaces and class $J(E)$

In what follows, we will use functional models for the class of reflectionless matrices  $J(E)$  in the form as considered in [39]. To this end, we need to recall certain special functions related to function theory in the common resolvent domain  $\Omega = \overline{\mathbb{C}} \setminus E$  for  $J \in J(E)$ . Note that in this case,  $E$  can be a set of an essentially more complicated structure [17, 29, 43], than a system of intervals.

Let  $\mathbb{D}/\Gamma \simeq \overline{\mathbb{C}} \setminus E$  be a uniformization of the domain  $\Omega$ . It means that there exists a Fuchsian group  $\Gamma$  and a meromorphic function  $\mathfrak{z} : \mathbb{D} \rightarrow \overline{\mathbb{C}} \setminus E$ ,  $\mathfrak{z} \circ \gamma = \mathfrak{z}$  for all  $\gamma \in \Gamma$ , such that

$$\forall z \in \overline{\mathbb{C}} \setminus E \exists \zeta \in \mathbb{D}: \mathfrak{z}(\zeta) = z \text{ and } \mathfrak{z}(\zeta_1) = \mathfrak{z}(\zeta_2) \Rightarrow \zeta_1 = \gamma(\zeta_2).$$

We assume that  $\mathfrak{z}$  meets the normalization  $\mathfrak{z}(0) = \infty$ ,  $(\zeta \mathfrak{z})(0) > 0$ .

Let  $\Gamma^*$  be the group of characters of the discrete group  $\Gamma$ ,

$$\Gamma^* = \{\alpha \mid \alpha : \Gamma \rightarrow \mathbb{R}/\mathbb{Z} \text{ such that } \alpha(\gamma_1 \gamma_2) = \alpha(\gamma_1) + \alpha(\gamma_2)\}$$

Since  $\Gamma$  is formed by  $g$  independent generators, say  $\{\overset{\circ}{\gamma}_j\}_{j=1}^g$ , the group  $\Gamma^*$  is equivalent to  $\mathbb{R}^g/\mathbb{Z}^g$ ,

$$\alpha \simeq \{\alpha(\overset{\circ}{\gamma}_1), \dots, \alpha(\overset{\circ}{\gamma}_g)\} \in \mathbb{R}^g/\mathbb{Z}^g. \quad (2.1)$$

**Definition 2.1.** For  $\alpha \in \Gamma^*$  we define the Hardy space of character automorphic functions as

$$H^2(\alpha) = H_\Omega^2(\alpha) = \{f \in H^2 : f \circ \gamma = e^{2\pi i \alpha(\gamma)} f, \gamma \in \Gamma\},$$

where  $H^2$  denotes the standard Hardy class in  $\mathbb{D}$ .

Fix  $z_0 \in \Omega$  and let  $\text{orb}(\zeta_0) = \mathfrak{z}^{-1}(z_0) = \{\gamma(\zeta_0)\}_{\gamma \in \Gamma}$ . The Blaschke product  $b_{z_0}$  with zeros at  $\mathfrak{z}^{-1}(z_0)$  is called the Green function of the group  $\Gamma$  (cf. [39]). It is related to the standard Green function  $G(z, z_0)$  in the domain  $\Omega$  by  $\log \frac{1}{|b_{z_0}(\zeta)|} = G(\mathfrak{z}(\zeta), z_0)$ . The function  $b_{z_0}$  is character automorphic, that is,  $b_{z_0} \circ \gamma = e^{2\pi i \mu_{z_0}} b_{z_0}$ , where  $\mu_{z_0} \in \Gamma^*$ . For  $b_{z_0}$  we fix the normalization  $b_{z_0}(0) > 0$  if  $z_0 \neq \infty$  and  $(\mathfrak{z}b)(0) > 0$  for the Blaschke product  $b$  related to infinity.

We define  $k_{\zeta_0}^\alpha(\zeta) = k^\alpha(\zeta, \zeta_0)$  as the reproducing kernel of the space  $H^2(\alpha)$ , that is,

$$\langle f, k_{\zeta_0}^\alpha \rangle = f(\zeta_0) \quad \forall f \in H^2(\alpha).$$

**Remark 2.2.** Let us point out that in our case this reproducing kernels possess a representation by means of  $\theta$  functions associated with the given Riemann surface [14]. As already mentioned,  $k^\alpha$  has sense in a much more general situation, say, domains of Widom type. Although, generally speaking, they can not be represented via  $\theta$  functions, they still play a role of special functions in the related problems.

Let  $k^\alpha(\zeta) = k_0^\alpha(\zeta)$ ,  $b(\zeta) = b_{\mathfrak{z}(0)}(z)$ , and  $\mu = \mu_{\mathfrak{z}(0)}$ . We have an evident decomposition

$$H^2(\alpha) = \{e^\alpha\} \oplus bH^2(\alpha - \mu), \quad e^\alpha = \frac{k^\alpha(\zeta)}{\sqrt{k^\alpha(0)}}. \quad (2.2)$$

This decomposition plays an essential role in the proof of the following theorem.

**Theorem 2.3.** *The system of functions*

$$e_n^\alpha(\zeta) = b^n(\zeta) \frac{k^{\alpha-n\mu}(\zeta)}{\sqrt{k^{\alpha-n\mu}(0)}} \quad (2.3)$$

(i) *forms an orthonormal basis in  $H^2(\alpha)$  for  $n \in \mathbb{N}$  and*

(ii) *forms an orthonormal basis in  $L^2(\alpha)$  for  $n \in \mathbb{Z}$ ,*

where

$$L^2(\alpha) = \{f \in L^2: f \circ \gamma = e^{2\pi i \alpha(\gamma)} f, \gamma \in \Gamma\}.$$

*Proof.* Item (i) follows from the above paragraphs and a proof for (ii) in a much more general case can be found in [39, Theorem E].  $\square$

The following theorem describes all elements of  $J(E)$  for a given finite-gap set  $E$ .

**Theorem 2.4.** *The multiplication operator by  $\mathfrak{z}$  in  $L^2(\alpha)$  with respect to the basis  $\{e_n^\alpha\}$  from Theorem 2.3 is the following Jacobi matrix  $J = J(\alpha)$ :*

$$\mathfrak{z}e_n^\alpha = a(n; \alpha)e_{n-1}^\alpha + b(n; \alpha)e_n^\alpha + a(n+1; \alpha)e_{n+1}^\alpha,$$

where

$$a(n; \alpha) = \mathcal{A}(\alpha - n\mu), \quad \mathcal{A}(\alpha) = (\mathfrak{z}b)(0) \sqrt{\frac{k^\alpha(0)}{k^{\alpha+\mu}(0)}}$$

and

$$b(n; \alpha) = \mathcal{B}(\alpha - n\mu), \quad \mathcal{B}(\alpha) = \frac{\mathfrak{z}b(0)}{b'(0)} + \left\{ \frac{(k^\alpha)'(0)}{k^\alpha(0)} - \frac{(k^{\alpha+\mu})'(0)}{k^{\alpha+\mu}(0)} \right\} + \frac{(\mathfrak{z}b)'(0)}{b'(0)}.$$

*This Jacobi matrix  $J(\alpha)$  belongs to  $J(E)$ . Thus, we have a map from  $\Gamma^*$  to  $J(E)$ . Moreover, this map is one-to-one.*

**Remark 2.5.** Using the representation of the reproducing kernels via  $\theta$  functions, see Remark 2.2, one gets  $\mathcal{A}(\alpha)$  and  $\mathcal{B}(\alpha)$  in the form (1.5).

**Remark 2.6.** The following important relation is an immediate consequence of the above functional model

$$S^{-1}J(\alpha)S = J(\alpha - \mu), \quad Se_n := e_{n+1}. \quad (2.4)$$

In particular,  $J(\alpha)$  is periodic if and only if  $N\mu = \mathbf{0}_{\Gamma^*}$  for a certain positive integer  $N$ .

## 2.2 Class $A(E, \mathbf{C})$ and Jacobi flow

Now we turn to the functional model for  $A(E, \mathbf{C})$ . The rational function  $\Delta(z)$  and the single valued function  $\Psi(z)$ ,  $z \in \bar{\mathbb{C}} \setminus E$ , were defined in (1.10)-(1.11). Let us list characteristic properties of  $\Psi(z)$ :

- (i)  $|\Psi| < 1$  in  $\Omega$  and  $|\Psi| = 1$  on  $E$ ,
- (ii)  $\Psi(\infty) = \Psi(\mathbf{c}_j) = 0$ ,  $1 \leq j \leq g$ , otherwise  $\Psi(z) \neq 0$ .

All this implies that  $\Psi(z)$  is given by  $\Psi(\mathfrak{z}(\zeta)) = b(\zeta) \prod_{j=1}^g b_{\mathbf{c}_j}(\zeta)$ . In particular,  $\mu + \sum_{j=1}^g \mu_{\mathbf{c}_j} = \mathbf{0}_{\Gamma^*}$ .

Let us fix  $\zeta_j \in \mathbb{D}$  such that  $\mathfrak{z}(\zeta_j) = \mathbf{c}_j$  and  $\overset{\circ}{\gamma}_j(\zeta_j) = \bar{\zeta}_j$  for the generator  $\overset{\circ}{\gamma}_j$  of the group  $\Gamma$ . In order to construct a functional model for operators from  $A(E, \mathbf{C})$ , we start with the following counterpart of the orthogonal decomposition (2.2):

$$H^2(\alpha) = \{k_{\zeta_1}^\alpha, \dots, k_{\zeta_g}^\alpha, k^\alpha\} \oplus \Psi H^2(\alpha) = \{f_0^\alpha\} \oplus \dots \oplus \{f_g^\alpha\} \oplus \Psi H^2(\alpha), \quad (2.5)$$

where

$$f_0^\alpha = \frac{e^{-\pi i \alpha(\overset{\circ}{\gamma}_1)} k_{\zeta_1}^\alpha}{\sqrt{k_{\zeta_1}^\alpha(\zeta_1)}}, \quad f_1^\alpha = \frac{e^{-\pi i(\alpha - \mu_{\mathbf{c}_1})(\overset{\circ}{\gamma}_2)} b_{\mathbf{c}_1} k_{\zeta_2}^{\alpha - \mu_{\mathbf{c}_1}}}{\sqrt{k_{\zeta_2}^{\alpha - \mu_{\mathbf{c}_1}}(\zeta_2)}}, \dots, \quad f_g^\alpha = \frac{\prod_{j=1}^g b_{\mathbf{c}_j} k^{\alpha + \mu}}{\sqrt{k^{\alpha + \mu}(0)}}. \quad (2.6)$$

**Theorem 2.7.** *The system of functions*

$$f_n^\alpha = \Psi^m f_j^\alpha, \quad n = (g+1)m + j, \quad j \in [0, \dots, g] \quad (2.7)$$

(i) *forms an orthonormal basis in  $H^2(\alpha)$  for  $n \in \mathbb{N}$  and*

(ii) *forms an orthonormal basis in  $L^2(\alpha)$  for  $n \in \mathbb{Z}$ .*

*Proof.* Item (i) follows from (2.5) and for (ii) we have to use the description of the orthogonal complement  $L^2(\alpha) \ominus H^2(\alpha)$ , see [39].  $\square$

Similarly as we had before, this allows us to parametrize all elements of  $A(E, \mathbf{C})$  for a given  $E$  by the characters of  $\Gamma^*$ .

**Theorem 2.8.** *In the above notations the multiplication operator by  $\mathfrak{z}$  with respect to the basis  $\{f_n^\alpha\}$  is a GSMP matrix  $A(\alpha; \mathbf{C}) \in A(E, \mathbf{C})$ . Moreover, this map  $\Gamma^* \rightarrow A(E; \mathbf{C})$  is one-to-one up to the identification  $(p_j, q_j) \mapsto (-p_j, -q_j)$  in  $A(E; \mathbf{C})$ ,  $0 \leq j \leq g-1$ .*

*Proof.* The structure of the matrix is fixed by the choice of the orthonormal basis. We only need to check that, under the normalization (2.6),  $p_j(\alpha)$  and  $q_j(\alpha)$  are real. For  $\beta_j = \alpha - \sum_{k=1}^j \mu_{\mathbf{c}_k}$ , we have

$$p_j(\alpha) = \langle \mathfrak{z} f_j^\alpha, f_{-1}^\alpha \rangle = (b\mathfrak{z})(0) \prod_{k=1}^{j-1} b_{\mathbf{c}_k}(0) e^{-\pi \beta_j(\overset{\circ}{\gamma}_j)} \frac{k^{\beta_j}(0, \zeta_j)}{\sqrt{k_{\zeta_j}^{\beta_j}(\zeta_j) k^{\alpha + \mu}(0)}}.$$



Since  $\overline{k^\beta(\bar{\zeta})} = k^\beta(\zeta)$  for all  $\beta \in \Gamma^*$ , we get

$$k^\beta(\zeta_j) = \overline{k^\beta(\bar{\zeta}_j)} = \overline{k^\beta(\overset{\circ}{\gamma}_j(\zeta_j))} = e^{-2\pi i\beta(\overset{\circ}{\gamma}_j)} \overline{k^\beta(\bar{\zeta}_j)}.$$

Therefore,  $e^{-\pi i\beta(\overset{\circ}{\gamma}_j)} \overline{k^\beta(\bar{\zeta}_j)} = e^{-\pi i\beta(\overset{\circ}{\gamma}_j)} k^\beta(0, \zeta_j)$  is real. Note that the square root of  $e^{-2\pi i\beta(\overset{\circ}{\gamma}_j)}$  is defined up to the multiplier  $\pm 1$ . Similarly, we prove that  $q_j(\alpha)$  are real based on  $p_g(\alpha)q_j(\alpha) = \langle f_g^\alpha, f_j^\alpha \rangle$ .

If a periodic  $\overset{\circ}{A} \in A(E, \mathbf{C})$  is given, we introduce its resolvent function  $r_+(z)$ , see Theorem 2.13 below, and define  $\alpha$  exactly as in the Jacobi matrix case, see e.g. [39, (2.3.2)-(2.3.3) and Theorem A].  $\square$

**Definition 2.9.** We define the Jacobi flow on  $A(E; \mathbf{C})$  as the dynamical system generated by the following map (see (2.4), (1.28)):

$$\mathcal{J}A(\alpha) = A(\alpha - \mu), \quad \alpha \in \Gamma^*.$$

We can describe this operation in a very explicit form.

**Lemma 2.10.** *Let*

$$\mathbf{o}(\phi) = \begin{bmatrix} \sin \phi & \cos \phi \\ \cos \phi & -\sin \phi \end{bmatrix}. \quad (2.8)$$

*Let  $\mathcal{O}(\alpha)$  be the unitary, periodic  $(g+1) \times (g+1)$ -block diagonal matrix given by*

$$\mathcal{O}(\alpha) [e_{(g+1)m} \quad \cdots \quad e_{(g+1)m+g}] = [e_{(g+1)m} \quad \cdots \quad e_{(g+1)m+g}] \mathcal{O}(\alpha), \quad (2.9)$$

*where*

$$\mathcal{O}(\alpha) = \begin{bmatrix} I_{g-2} & 0 \\ 0 & \mathbf{o}(\phi(\alpha)) \end{bmatrix}, \quad [\sin(\phi(\alpha)) \quad \cos(\phi(\alpha))] = \frac{[p_{g-1}(\alpha) \quad p_g(\alpha)]}{\sqrt{p_{g-1}^2(\alpha) + p_g^2(\alpha)}}. \quad (2.10)$$

*Then*

$$\mathcal{O}A(\alpha; \mathbf{C}) := S^{-1} \mathcal{O}(\alpha)^* A(\alpha; \mathbf{C}) \mathcal{O}(\alpha) S = A(\alpha + \mu_{\mathbf{c}_g}; \mathbf{c}_g, \mathbf{c}_1, \dots, \mathbf{c}_{g-1}). \quad (2.11)$$

*Proof.* Actually, in this operation we just switched the order of two reproducing kernels related to  $\mathbf{c}_g$  and  $\infty$ . This is a rotation in the two dimensional space. Then, up to the shift, we derived a GSMP basis of the form (2.6), but with the new ordering  $(\mathbf{c}_g, \mathbf{c}_1, \dots, \mathbf{c}_{g-1})$  and the new character  $\alpha + \mu_{\mathbf{c}_g}$ .  $\square$

**Theorem 2.11.** *In the above notations*

$$\mathcal{J}A(\alpha; \mathbf{C}) = \mathcal{O}^{\circ g} A(\alpha; \mathbf{C}) \quad (2.12)$$

*Proof.* We use (2.11), having in mind that  $\sum_{j=1}^g \mu_{\mathbf{c}_j} = -\mu$  and that after all permutations we obtain the original ordering  $\mathbf{C}$ .  $\square$

The next lemma allows us to estimate components of the vector  $f_j^\alpha$ ,  $j = 0, \dots, g$ , in its decomposition with respect to the basis  $\{e_n^\alpha\}_{n \geq 0}$ .

**Lemma 2.12.** *Let  $f_j^\alpha = \sum_{k=0}^\infty F_k^j(\alpha) e_k^\alpha$ . Then*

$$|F_k^j(\alpha)| \leq C(E)\eta^k, \quad j = 0, \dots, g, \quad (2.13)$$

where  $1 > \eta > \max\{|b(\zeta_1)|, \dots, |b(\zeta_g)|\}$ .

*Proof.* First of all, we note that  $\underline{C}(E) \leq \|k_{\zeta_n}^\alpha\| \leq \overline{C}(E)$  uniformly on  $\alpha \in \Gamma^*$ . Also  $|b_{\mathbf{c}_n}(\zeta_j)| \geq \underline{c}(E)$  and by definition (2.3),  $|e_k^\alpha(\zeta_n)| \leq \overline{c}(E)\eta^k$ . Since

$$\begin{aligned} \langle e_k^\alpha, \prod_{n=1}^{j-1} b_{\mathbf{c}_n} k_{\zeta_j}^{\beta_j} \rangle &= \left\langle \frac{e_k^\alpha}{\prod_{n=1}^{j-1} b_{\mathbf{c}_n}} - \sum_{n=1}^{j-1} \frac{k_{\zeta_n}^{\beta_j + \mu_{\mathbf{c}_n}} e_k^\alpha(\zeta_n)}{b_{\mathbf{c}_n} k_{\zeta_n}^{\beta_j + \mu_{\mathbf{c}_n}}(\zeta_n) \prod_{l=1, l \neq n}^{j-1} b_{\mathbf{c}_l}(\zeta_n)}, k_{\zeta_j}^{\beta_j} \right\rangle \\ &= \frac{e_k^\alpha(\zeta_j)}{\prod_{n=1}^{j-1} b_{\mathbf{c}_n}(\zeta_j)} - \sum_{n=1}^{j-1} \frac{k_{\zeta_n}^{\beta_j + \mu_{\mathbf{c}_n}}(\zeta_j) e_k^\alpha(\zeta_n)}{b_{\mathbf{c}_n}(\zeta_j) k_{\zeta_n}^{\beta_j + \mu_{\mathbf{c}_n}}(\zeta_n) \prod_{l=1, l \neq n}^{j-1} b_{\mathbf{c}_l}(\zeta_n)} \end{aligned}$$

and this is  $e^{-\pi i \beta_j(\tilde{\gamma}_j)} \|k_{\zeta_j}^{\beta_j}\| \overline{F_k^j(\alpha)}$ , we get (2.13).  $\square$

### 2.3 Transfer matrix

In this subsection we discuss briefly the direct spectral problem of the class  $A(E, \mathbf{C})$ . For  $\vec{v} = \vec{p}, \vec{q}$ , we use the following notations

$$(u_k \vec{v})^* = [v_0 \quad \dots \quad v_{g-k}], \quad (d_k \vec{v})^* = [v_k \quad \dots \quad v_g].$$

Let  $\{\delta_j\}_{j=0}^g$  be the standard basis in  $\mathbb{C}^{g+1}$  and let  $M_j$ 's be upper triangular matrices such that

$$B(\vec{p}) - \vec{p}(\vec{q})^* = M(\vec{p}) := M_0 = \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix} + (-\vec{p}q_g + \vec{q}p_g)\delta_g^* \quad (2.14)$$

and

$$M_j = \begin{bmatrix} M_{j+1} & 0 \\ 0 & \mathbf{c}_{g+1-j} \end{bmatrix} + (-u_j \vec{p}q_{g-j} + u_j \vec{q}p_{g-j})\delta_{g-j}^*, \quad j \geq 1. \quad (2.15)$$

**Theorem 2.13.** *Let*

$$\begin{bmatrix} R_{00}(z) & R_{0g}(z) \\ R_{g0}(z) & R_{gg}(z) \end{bmatrix} = \begin{bmatrix} \langle (B_0 - z)^{-1} \vec{p}, \vec{p} \rangle & \langle (B_0 - z)^{-1} \delta_g, \vec{p} \rangle \\ \langle (B_0 - z)^{-1} \vec{p}, \delta_g \rangle & \langle (B_0 - z)^{-1} \delta_g, \delta_g \rangle \end{bmatrix} \quad (2.16)$$

and  $r_+(z) = \|\vec{p}\|^2 \langle (A_+ - z)^{-1} \tilde{e}_0, \tilde{e}_0 \rangle$ . Then the shift by one block for a one-sided GSMP matrix  $A_+ \mapsto A_+^{(1)}$ , see (2.18), by means of the spectral function has the following form

$$r_+(z) = \frac{\mathfrak{A}_{11}(z)r_+^{(1)}(z) + \mathfrak{A}_{12}(z)}{\mathfrak{A}_{21}(z)r_+^{(1)}(z) + \mathfrak{A}_{22}(z)},$$

where

$$\mathfrak{A}(z) := \begin{bmatrix} \mathfrak{A}_{11} & \mathfrak{A}_{12} \\ \mathfrak{A}_{21} & \mathfrak{A}_{22} \end{bmatrix} (z) = \frac{1}{R_{0g}(z)} \begin{bmatrix} R_{00}R_{gg} - R_{0g}^2 & -R_{00} \\ R_{gg} & -1 \end{bmatrix} (z). \quad (2.17)$$

*Proof.* We represent  $A_+$  as a two dimensional perturbation of the block diagonal matrix

$$A_+ = \begin{bmatrix} B(\vec{\mathbf{p}}) & 0 \\ 0 & A_+^{(1)} \end{bmatrix} + \|\vec{p}^{(1)}\| \langle e_g \langle \cdot, \tilde{e}_0^{(1)} \rangle + \tilde{e}_0^{(1)} \langle \cdot, e_g \rangle \rangle \quad (2.18)$$

and apply the resolvent perturbation formula.  $\square$

Note that in the definition (2.17) we use the normalization  $\det \mathfrak{A}(z) = 1$ .

**Definition 2.14.** Let  $\mathbf{p}^* = [p \quad q]$ . The matrix function

$$\mathfrak{a}(z, \mathbf{c}; \mathbf{p}) = I - \frac{1}{\mathbf{c} - z} \mathbf{p} \mathbf{p}^* \mathbf{j}, \quad \mathbf{j} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (2.19)$$

represents the so-called Blaschke-Potapov factor of the third kind with a real pole  $\mathbf{c}$  [31]. A specific factor related to infinity we introduce in the form

$$\mathfrak{a}(z; \mathbf{p}) = \mathfrak{a}(z, \infty; \mathbf{p}) = \begin{bmatrix} 0 & -p \\ \frac{1}{p} & \frac{z-pq}{p} \end{bmatrix}. \quad (2.20)$$

**Theorem 2.15.** Let  $\mathbf{p}_j^* = [p_j \quad q_j]$ . The matrix function  $\mathfrak{A}(z)$ , given in (2.17), possesses the following multiplicative representation

$$\mathfrak{A}(z) = \mathfrak{a}(z, \mathbf{c}_1; \mathbf{p}_0) \mathfrak{a}(z, \mathbf{c}_2; \mathbf{p}_1) \dots \mathfrak{a}(z, \mathbf{c}_g; \mathbf{p}_{g-1}) \mathfrak{a}(z; \mathbf{p}_g). \quad (2.21)$$

*Proof.* We use the representation (2.14) and definitions (2.16), (2.20) to get  $\mathfrak{A}(z) = \mathfrak{A}_0(z) \mathfrak{a}(z; \mathbf{p}_g)$ , where

$$\mathfrak{A}_0(z) = I - \begin{bmatrix} \langle (M_1 - z)^{-1} u_1 \vec{p}, u_1 \vec{p} \rangle & \langle (M_1 - z)^{-1} u_1 \vec{q}, u_1 \vec{p} \rangle \\ \langle (M_1 - z)^{-1} u_1 \vec{p}, u_1 \vec{q} \rangle & \langle (M_1 - z)^{-1} u_1 \vec{q}, u_1 \vec{q} \rangle \end{bmatrix} \mathbf{j}. \quad (2.22)$$

Then, we use one after another (2.15) and definitions (2.19) to get

$$\mathfrak{A}_{j-1}(z) = \mathfrak{A}_j(z) \mathfrak{a}(z, \mathbf{c}_{g+1-j}; \mathbf{p}_{g-j}),$$

where

$$\mathfrak{A}_{j-1}(z) = I - \begin{bmatrix} \langle (M_j - z)^{-1} u_j \vec{p}, u_j \vec{q} \rangle & \langle (M_j - z)^{-1} u_j \vec{q}, u_j \vec{p} \rangle \\ \langle (M_j - z)^{-1} u_j \vec{p}, u_j \vec{q} \rangle & \langle (M_j - z)^{-1} u_j \vec{q}, u_j \vec{q} \rangle \end{bmatrix} \mathbf{j}.$$

That is, we obtain (2.21).  $\square$

**Definition 2.16.** Let  $\mathring{A} \in A(E, \mathbf{C})$ . Then the product (2.21) is called the *transfer matrix* associated with the given  $\mathring{A}$ .

The role of the transfer matrix is described in the following theorem.

**Theorem 2.17.** Let  $\mathring{A} \in A(E, \mathbf{C})$  with the transfer matrix  $\mathfrak{A}(z)$ , given in (2.21), and let  $\Delta(z) := \text{tr } \mathfrak{A}(z)$ . Then the spectrum  $E$  of  $\mathring{A}$  is given by

$$E = \Delta^{-1}([-2, 2]) = \{x : \Delta(x) \in [-2, 2]\}. \quad (2.23)$$

Moreover,  $\Delta(z)$  is of the form (1.11), where

$$\lambda_0 p_g = 1, \quad \lambda_0 \sum_{j=0}^g p_j q_j + \mathbf{c}_0 = 0, \quad (2.24)$$

and  $\lambda_k = \Lambda_k(\vec{\mathbf{p}}) := -\text{Res}_{\mathbf{c}_k} \text{tr } \mathfrak{A}(z)$ , i.e.,

$$\lambda_k = -\text{tr} \left\{ \prod_{j=0}^{k-2} \mathfrak{a}(\mathbf{c}_k, \mathbf{c}_{j+1}; \mathbf{p}_j) \mathbf{p}_{k-1} \mathbf{p}_{k-1}^* \prod_{j=k}^{g-1} \mathfrak{a}(\mathbf{c}_k, \mathbf{c}_{j+1}; \mathbf{p}_j) \mathfrak{a}(\mathbf{c}_k; \mathbf{p}_g) \right\}. \quad (2.25)$$

*Proof.* A proof of (2.23) is the same as for the transfer matrix in the Jacobi matrices case. The relations (2.24) and (2.25) follow immediately from (2.21).  $\square$

*Proof of Proposition 1.18.* First of all, we have a parametrization of  $A(E, \mathbf{C})$  by the characters  $\Gamma^*$ . It is evident that, in the basis (2.7), multiplication by  $\Psi$  is the shift  $S^{g+1}$ ,  $\Psi f_n^\alpha = f_{n+(g+1)}^\alpha$ . Thus, the magic formula for GSMP matrices corresponds to the definition (1.11). The relations (2.24), (2.25) imply the form of the isospectral surface, that is, (1.30).  $\square$

Later, in Section 6, we will use another representation for  $q_g$ .

**Lemma 2.18.**  $q_g$  allows the following alternative representation

$$q_g + \mathbf{c}_0 = \sum_{k=1}^g \text{tr} \left\{ \prod_{j=0}^{k-2} \mathfrak{a}(\mathbf{c}_k, \mathbf{c}_{j+1}; \mathbf{p}_j) \mathbf{p}_{k-1} \mathbf{p}_{k-1}^* \prod_{j=k}^{g-1} \mathfrak{a}(\mathbf{c}_k, \mathbf{c}_{j+1}; \mathbf{p}_j) \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{p_g} \end{bmatrix} \right\}. \quad (2.26)$$

*Proof.* From the second relation in (2.24) and (2.22) one has

$$q_g + \mathbf{c}_0 = \frac{1}{2\pi i} \oint_{|z|=R} \text{tr } \mathfrak{A}_0(z) \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{p_g} \end{bmatrix} dz = \sum_{k=1}^g \text{Res}_{\mathbf{c}_k} \text{tr } \mathfrak{A}_0(z) \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{p_g} \end{bmatrix},$$

which is (2.26).  $\square$

### 3 GSMP matrices, general case.

We hope after Theorem 2.17, and especially (2.25), it would be easy to perceive the following notations.

**Notation 3.1.** For  $k = 1, \dots, g$  the following functions (polynomials) are given by

$$\begin{aligned} \Lambda_{j,k}^\# &= \Lambda_k^\#(\mathbf{p}_0^{(j+1)}, \dots, \mathbf{p}_{k-1}^{(j+1)}; \mathbf{p}_{k-1}^{(j)}, \dots, \mathbf{p}_g^{(j)}) \\ &= -\text{tr} \left\{ \prod_{m=0}^{k-2} \mathbf{a}(\mathbf{c}_k, \mathbf{c}_{m+1}; \mathbf{p}_m^{(j+1)}) \mathbf{p}_{k-1}^{(j+1)} (\mathbf{p}_{k-1}^{(j)})^* \mathbf{j} \prod_{m=k}^{g-1} \mathbf{a}(\mathbf{c}_k, \mathbf{c}_{m+1}; \mathbf{p}_m^{(j)}) \mathbf{a}(\mathbf{c}_k; \mathbf{p}_g^{(j)}) \right\} \end{aligned} \quad (3.1)$$

If  $\mathbf{p}_m^{(j)} = \mathbf{p}_m^{(j+1)} = \mathbf{p}_m$  for all  $m \in [0, g]$  this notation is simplified to

$$\Lambda_k(\vec{\mathbf{p}}) = \Lambda_k^\#(\mathbf{p}_0, \dots, \mathbf{p}_{k-1}; \mathbf{p}_{k-1}, \dots, \mathbf{p}_g). \quad (3.2)$$

**Lemma 3.2.** Let  $A$  belong to the GSMP class. Then the vector  $f(\mathbf{c}_k) := (\mathbf{c}_k - A)^{-1} e_{k-1}$ ,  $k = 1, \dots, g$ , is of the form

$$f_j(\mathbf{c}_k) = 0, \quad j \notin \{-1, 0, 1\}, \quad f_j = f_j(\mathbf{c}_k) = \{(f_j(\mathbf{c}_k))_n\}_{n=0}^g \in \mathbb{C}^{g+1}$$

and  $(f_{-1})_0 = \dots = (f_{-1})_{k-2} = 0$ ,  $(f_1)_k = \dots = (f_1)_g = 0$ . The non-zero entries are given by:

$$\Lambda_{-1,k}^\#(f_{-1})_{k-1} = 1, \quad \Lambda_{0,k}^\#(f_1)_{k-1} = 1, \quad (3.3)$$

$$(f_{-1})_l = -\frac{(\mathbf{p}_{k-1}^{(-1)})^* \mathbf{j}}{\mathbf{c}_{l+1} - \mathbf{c}_k} \prod_{j=k}^{l-1} \mathbf{a}(\mathbf{c}_k, \mathbf{c}_{j+1}, \mathbf{p}_j^{(-1)}) \frac{\mathbf{p}_l^{(-1)}}{\Lambda_{-1,k}^\#}, \quad (3.4)$$

$$(f_1)_m = -\frac{(\mathbf{p}_m^{(1)})^* \mathbf{j}}{\mathbf{c}_{m+1} - \mathbf{c}_k} \prod_{j=m+1}^{k-2} \mathbf{a}(\mathbf{c}_k, \mathbf{c}_{j+1}, \mathbf{p}_j^{(1)}) \frac{\mathbf{p}_{k-1}^{(1)}}{\Lambda_{0,k}^\#}. \quad (3.5)$$

In particular,

$$\langle \Delta(A) e_{-1}, e_{-1} \rangle = \lambda_0 p_g^{(-1)} q_g^{(-1)} + \mathbf{c}_0 - \sum_{k=1}^g \lambda_k \frac{\pi_{k-1}^{(-1)} \tilde{\rho}_{k-1}^{(0)}}{p_g^{(-1)} \Lambda_{-1,k}^\#}, \quad (3.6)$$

where

$$\begin{bmatrix} \tilde{\pi}_{k-1}^{(0)} \\ \tilde{\rho}_{k-1}^{(0)} \end{bmatrix} = \prod_{j=0}^{k-2} \mathbf{a}(\mathbf{c}_k, \mathbf{c}_{j+1}; \mathbf{p}_j^{(0)}) \mathbf{p}_{k-1}^{(0)}, \quad \begin{bmatrix} \pi_{k-1}^{(-1)} & \rho_{k-1}^{(-1)} \end{bmatrix} = -(\mathbf{p}_{k-1}^{(-1)})^* \mathbf{j} \prod_{j=k}^{g-1} \mathbf{a}(\mathbf{c}_k, \mathbf{c}_{j+1}; \mathbf{p}_j^{(-1)}) \mathbf{j} \quad (3.7)$$

*Proof.* This is a purely linear algebra. Using (2.14), we solve the system

$$\begin{aligned} -A(\vec{p}_{-1}) f_{-1} &= 0, \\ (\mathbf{c}_k - B(\vec{p}_{-1})) f_{-1} - A_0 f_0 &= 0, \\ -A^*(\vec{p}_0) f_{-1} + (\mathbf{c}_k - B(\vec{p}_0)) f_0 - A(\vec{p}_1) f_1 &= \delta_{k-1}, \\ -A^*(\vec{p}_1) f_0 + (\mathbf{c}_k - B(\vec{p}_1)) f_1 &= 0, \\ -A^*(\vec{p}_2) f_1 &= 0. \end{aligned} \quad (3.8)$$

It is worth to recall that  $B(\vec{\mathbf{p}})$  is an upper or lower triangular matrix up to a one-dimensional perturbation and its main diagonal in this case is  $\tilde{\mathbf{C}}$ , see definition (1.21). That is, all inverse matrices can be found exactly like in the previous section.  $\square$



Also,

$$\begin{bmatrix} \tilde{q}_g^{(-1)} \tilde{p}_g^{(-1)} & \tilde{p}_0^{(0)} \\ \tilde{p}_0^{(0)} & \tilde{q}_0^{(0)} \tilde{p}_0^{(0)} + \mathbf{c}_g \end{bmatrix} = \mathbf{o}(\phi_{-1})^* \begin{bmatrix} q_{g-1}^{(-1)} p_{g-1}^{(-1)} + \mathbf{c}_g & q_{g-1}^{(-1)} p_g^{(-1)} \\ q_{g-1}^{(-1)} p_g^{(-1)} & q_g^{(-1)} p_g^{(-1)} \end{bmatrix} \mathbf{o}(\phi_{-1}).$$

Thus, the  $q$ -entries have the form

$$\tilde{q}_0^{(0)} \tilde{p}_g^{(0)} = -\sin \phi_{-1} \sqrt{(p_{g-1}^{(0)})^2 + (p_g^{(0)})^2}, \quad \tilde{q}_j^{(0)} \tilde{p}_g^0 = q_{j-1}^{(0)} \sqrt{(p_{g-1}^{(0)})^2 + (p_g^{(0)})^2}. \quad (4.4)$$

Our next definition is a counterpart of (2.12).

**Definition 4.2.** We define the Jacobi flow transform

$$\mathcal{J} : \text{GSMP}(\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_g) \rightarrow \text{GSMP}(\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_g)$$

by

$$\mathcal{J}A = S^{-(g+1)} \mathcal{O}^{\circ g} A S^{g+1} = \mathcal{O}^{\circ g} (S^{-(g+1)} A S^{g+1}). \quad (4.5)$$

Let us note that

$$S^{-(g+1)} \mathcal{O}(A) S^{g+1} = \mathcal{O}(S^{-(g+1)} A S^{g+1}). \quad (4.6)$$

This has an important consequence.

**Corollary 4.3.**

$$\mathcal{O}(\mathcal{J}^{\circ n} A) = \mathcal{J}^{\circ n}(\mathcal{O}A). \quad (4.7)$$

*Proof.* Due to (4.5) and (4.6) we get

$$\mathcal{J}(\mathcal{O}A) = \mathcal{O}^{\circ g} (S^{-(g+1)} \mathcal{O}A S^{g+1}) = \mathcal{O}^{\circ(g+1)} (S^{-(g+1)} A S^{g+1}) = \mathcal{O}(\mathcal{J}A). \quad \square$$

Let us turn to explicit formulas for the given transform. First of all, we note that

$$\mathcal{J}A = S^{-1} U_A^* A U_A S, \quad (4.8)$$

where  $U_A$  is a  $(g+1) \times (g+1)$ -block diagonal matrix

$$U_A = U = \begin{bmatrix} \ddots & & & & & \\ & U(\vec{p}_{-1}) & & & & \\ & & U(\vec{p}_0) & & & \\ & & & \ddots & & \\ & & & & & \ddots \end{bmatrix}.$$

The block matrices  $U = U(\vec{p})$  are given by the products of orthogonal matrices, i.e.,

$$U(\vec{p}) = \begin{bmatrix} I_{g-2} & 0 \\ 0 & \mathbf{o}(\phi_g) \end{bmatrix} \cdots \begin{bmatrix} \mathbf{o}(\phi_1) & 0 \\ 0 & I_{g-2} \end{bmatrix}, \quad [\sin \phi_k \quad \cos \phi_k] = \frac{[p_{k-1} \quad \|d_k \vec{p}\|]}{\|d_{k-1} \vec{p}\|}. \quad (4.9)$$

**Lemma 4.4.** *In the notation above, we have for  $1 \leq k \leq g$*

$$U(\vec{p})\delta_0 = \frac{1}{\|\vec{p}\|}\vec{p}, \quad U(\vec{p})\delta_k = \frac{1}{\|d_{k-1}\vec{p}\|\|d_k\vec{p}\|} \begin{bmatrix} 0 \\ \|d_k\vec{p}\|^2 \\ -p_{k-1}d_k\vec{p} \end{bmatrix}. \quad (4.10)$$

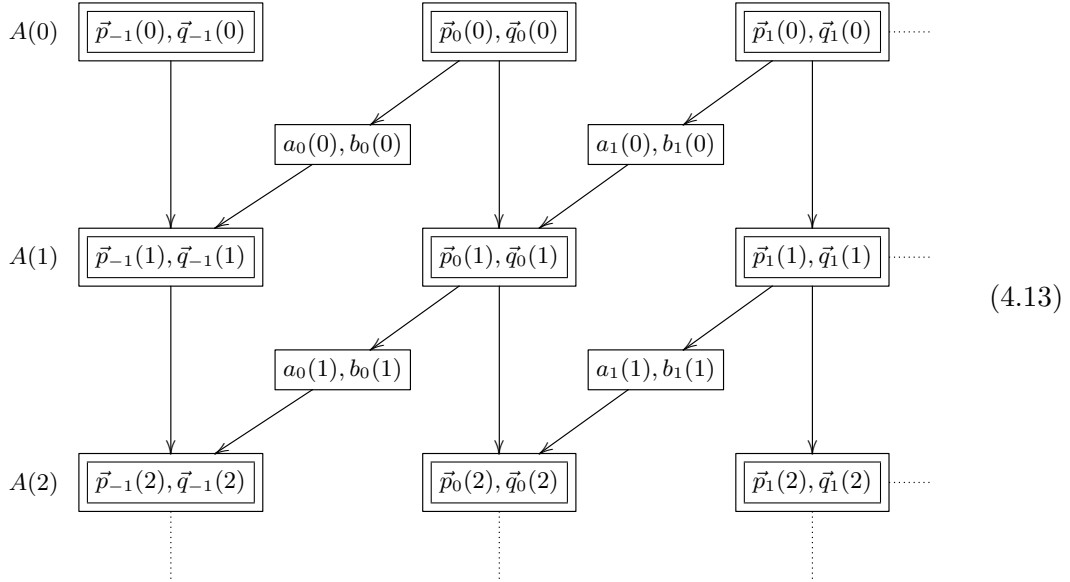
*Proof.* It follows from (4.9).  $\square$

**Theorem 4.5.** *Let  $A(1) = \mathcal{J}A$  and let  $\{p_k^{(j)}(1), q_k^{(j)}(1)\}$  be generating coefficient sequences of  $A(1)$ . Then*

$$\begin{bmatrix} q_0^{(j)}(1) \\ \vdots \\ q_{g-1}^{(j)}(1) \end{bmatrix} = \|\vec{p}_j\| \begin{bmatrix} \vdots \\ -\frac{p_k^{(j)}}{\|d_k\vec{p}_j\|\|d_{k+1}\vec{p}_j\|} \\ \vdots \end{bmatrix}, \quad \begin{bmatrix} * \\ p_0^{(j)}(1) \\ \vdots \\ p_{g-1}^{(j)}(1) \end{bmatrix} = U^*(\vec{p}_j)B(\vec{\mathbf{p}}_j) \frac{\vec{p}_j}{\|\vec{p}_j\|}, \quad (4.11)$$

$$p_g^{(j)}(1) = \frac{\|\vec{p}_{j+1}\|}{\|\vec{p}_j\|} p_g^{(j)}, \quad q_g^{(j)}(1) = \frac{\|\vec{p}_j\|}{p_g^{(j)}\|\vec{p}_{j+1}\|} \frac{\langle B(\vec{\mathbf{p}}_{j+1})\vec{p}_{j+1}, \vec{p}_{j+1} \rangle}{\|\vec{p}_{j+1}\|^2}. \quad (4.12)$$

*Proof.* We obtain (4.11) and (4.12) from (4.8) by Lemma 4.4.  $\square$



**Remark 4.6.** In view of Theorem 4.5 the Jacobi flow on GSMP matrices can be related to an open (input-output) dynamical system, see (4.13). Let us fix a block-position  $j = 0$ , but vary  $n$  in  $A(n+1) = \mathcal{J}A(n)$ . Then the coefficients related to the next block  $j = 1$  are involved only in (4.12) and in a very specific way. If we define the *two dimensional input* by

$$a^{\text{in}}(n) = \|\vec{p}_1(n)\|, \quad b^{\text{in}}(n) = \frac{\langle B(\vec{\mathbf{p}}_1(n))\vec{p}_1(n), \vec{p}_1(n) \rangle}{\|\vec{p}_1(n)\|^2}$$



and consider the parameters  $\{\vec{p}_0(n), \vec{q}_0(n)\}$  as the *internal state* of the system, then the open dynamical system is defined by

$$\begin{bmatrix} b^{\text{out}}(n-1) & \vec{p}_0(n+1)^* \\ \vec{p}_0(n+1) & B(\vec{\mathbf{p}}_0(n+1)) \end{bmatrix} = \begin{bmatrix} U^*(\vec{p}_0(n)) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} B(\vec{\mathbf{p}}_0(n)) & \delta_g a^{\text{in}}(n) \\ a^{\text{in}}(n) \delta_g^* & b^{\text{in}}(n) \end{bmatrix} \begin{bmatrix} U(\vec{p}_0(n)) & 0 \\ 0 & 1 \end{bmatrix}, \quad (4.14)$$

and  $a^{\text{out}}(n) = \|\vec{p}_0(n+1)\|$ . Note that the *output*  $\{a^{\text{out}}(n), b^{\text{out}}(n-1)\}$  are the Jacobi parameters of  $J = \mathcal{F}A$  (cf. (1.29)) and the input is related to  $\mathcal{F}S^{-(g+1)}AS^{g+1}$ . That is, this system represents the *GSMP transform on Jacobi matrices*.

## 5 KS-functional

### 5.1 Killip-Simon spectral conditions for one- and two-sided Jacobi matrices

First of all we mention the following

**Lemma 5.1.** *Assume that  $J_+$  is a one-sided Jacobi matrix with essential spectrum on  $E$ . Then it can be extended by a matrix  $J_- = P_- \overset{\circ}{J} P_-$ ,  $\overset{\circ}{J} \in J(E)$ , such that each  $\mathbf{c}_j$  belongs to the resolvent set (domain) of the resulting matrix (1.25).*

*Proof.* Let

$$R(z) := \mathcal{E}^*(J - z)^{-1} \mathcal{E} = \int \frac{d\Sigma}{x - z} = \begin{bmatrix} r_-(z)^{-1} & a(0) \\ a(0) & r_+(z)^{-1} \end{bmatrix}^{-1}, \quad (5.1)$$

where  $\mathcal{E} : \mathbb{C}^2 \rightarrow l^2$  such that  $\mathcal{E} \begin{bmatrix} c_- \\ c_+ \end{bmatrix} = c_- e_{-1} + c_+ e_0$ . In particular, for the diagonal entries of  $R(z)$  we have

$$-\frac{1}{R_{-1,-1}(z)} = -\frac{1}{r_-(z)} + a(0)^2 r_+(z), \quad -\frac{1}{R_{0,0}(z)} = -\frac{1}{r_+(z)} + a(0)^2 r_-(z). \quad (5.2)$$

If  $r_+(\mathbf{c}_j)$  is zero or infinity, we chose  $\overset{\circ}{J}$  such that  $r_-(\mathbf{c}_j)$  is regular, that is,  $r_-(\mathbf{c}_j) \neq 0$ ,  $r_-(\mathbf{c}_j) \neq \infty$ . And vice versa, if  $r_+(\mathbf{c}_j)$  is regular, we set  $r_-(\mathbf{c}_j) = 0$ . In both cases  $R_{-1,-1}(\mathbf{c}_j) \neq \infty$  and  $R_{0,0}(\mathbf{c}_j) \neq \infty$ . Therefore the whole matrix  $(\mathbf{c}_j - J)$  is invertible.  $\square$

The spectral Killip-Simon condition can be formulated either in terms of measures  $\sigma_{\pm}$ , see Definition 1.19, or by means of the matrix measure  $\Sigma$ .

**Lemma 5.2.** *The measures  $\sigma_{\pm}$  both satisfy the Killip-Simon condition if and only if the matrix measure  $d\Sigma$  is supported on  $E \cup Y$  and obeys*

$$\int_E |\log \det \Sigma'(x)| \sqrt{\text{dist}(x, \mathbb{R} \setminus E)} dx + \sum_{y_k \in Y} \sqrt{\text{dist}(y_k, E)^3} < \infty. \quad (5.3)$$

*Proof.* We note two properties of an arbitrary function  $F(z)$ , which is analytic in the upper half-plane and has positive imaginary part. If such a function has a meromorphic extension in an interval  $(\mathbf{a}_j, \mathbf{b}_j) \subset \mathbb{R}$  when its zeros and poles interlay. Secondary,  $F(z)$  is of bounded characteristic in the upper half-plane, and therefore

$$\int_{\mathbb{R}} \frac{|\log F(x)|}{1+x^2} dx < \infty \quad (5.4)$$

From the first property we get that all poles of the first and second functions in (5.2) satisfy the Killip-Simon condition in  $\mathbb{R} \setminus E$ . Applying this fact once again we obtain that poles of  $R_{-1,-1}$  and  $R_{0,0}$ , that is the set  $Y$ , satisfy this condition. Similar observations show the opposite directions.

With respect to the a.c. part of the measure we have

$$\Sigma'(x) = \begin{bmatrix} r_-(x)^{-1} & a(0) \\ a(0) & r_+(x)^{-1} \end{bmatrix}^{-1} \begin{bmatrix} \sigma'_-(x) & 0 \\ 0 & \sigma'_+(x) \end{bmatrix} \begin{bmatrix} r_-(x)^{-1} & a(0) \\ a(0) & r_+(x)^{-1} \end{bmatrix}^{-1}.$$

Therefore

$$\det \Sigma'(x) = \left| \frac{r_-(x)}{-r_+^{-1}(x) + a(0)^2 r_-(x)} \right|^2 \sigma'_-(x) \sigma'_+(x).$$

Applying (5.4) to  $r_-(z)$  and  $-r_+^{-1}(z) + a(0)^2 r_-(z)$ , we obtain an equivalence of the conditions for  $\det \Sigma'(x)$  and  $\sigma'_\pm(x)$ .  $\square$

## 5.2 Scalar and block-matrix spectral Killip-Simon conditions

**Theorem 5.3.** *Let  $A \in \text{GSMP}(\mathbb{C})$ . Its spectral measure satisfies (5.3) if and only if the block Jacobi matrix  $\Delta(A)$  belongs to the Killip-Simon class.*

A proof follows from the lemma given below. We prove the corresponding lemma for a scalar measure  $\sigma$ , assuming that  $\sigma(\mathbf{c}_j) = 0$ . A proof for a  $2 \times 2$  matrix measure  $\Sigma$  is essentially the same but requires more space in a presentation. Note even if we start with an initial one-sided matrix  $J_+$  such that  $\sigma_+(\mathbf{c}_j) > 0$  for some  $j$ , due to the Lemma 5.1, we always can get a two-sided  $J$  such that  $\Sigma(\mathbf{c}_j) = 0$ . Also, it is more uniform to let  $\Delta(z) = \sum_{j=1}^g \frac{\lambda_j}{\mathbf{c}_j - z}$ . To pass to our case, where  $\Delta(z)$  is of the form (1.11), it is enough to send one of this  $\mathbf{c}_j$  to infinity by a suitable linear fractional transform.

So, let  $d\sigma$  be a scalar measure with an essential support on  $E = \Delta^{-1}([-2, 2])$  such that  $\sigma(\mathbf{c}_j) = 0$ . We define the matrix measure  $d\Xi$  by

$$\int \frac{d\Xi(y)}{y-z} := \int \frac{1}{V(x)-z} W^*(x) d\sigma(x) W(x),$$

where

$$W(x) = \begin{bmatrix} 1 & & \\ \mathbf{c}_1 - x & \dots & \\ & & \mathbf{c}_g - x \end{bmatrix}.$$

In other words,  $d\Xi$  is the matrix measure of the multiplication by  $\Delta(x)$  in  $L_{d\sigma}^2$  with respect to a suitable cyclic subspace. Note that one can normalize this measure by a triangular (constant) matrix  $L$  such that  $L^* \int d\Xi(y) L = I$ , that is, to choose an appropriate orthonormal basis in the fixed cyclic subspace.

**Lemma 5.4.** *Let  $\Xi'(y)$  be the density of the a.c. part of the measure  $d\Xi$  on  $[-2, 2]$  and  $\sigma'(x)$  be the density of  $d\sigma$ , respectively. Then*

$$\det \Xi'(y) = \frac{\prod_{\Delta(x)=y} \sigma'(x)}{\prod_{k=1}^g \lambda_k}. \quad (5.5)$$

*Proof.* Let  $\{x_1, \dots, x_g\} = \Delta^{-1}(y)$ ,  $y \in [-2, 2]$ . Then

$$\begin{aligned} \Xi'(y) &= \sum_{\Delta(x)=y} W^*(x) \frac{\sigma'(x)}{\Delta'(x)} W(x) \\ &= \begin{bmatrix} \frac{1}{\mathbf{c}_1 - x_1} & \cdots & \frac{1}{\mathbf{c}_g - x_1} \\ \vdots & \cdots & \vdots \\ \frac{1}{\mathbf{c}_1 - x_g} & \cdots & \frac{1}{\mathbf{c}_g - x_g} \end{bmatrix}^* \begin{bmatrix} \frac{\sigma'(x_1)}{\Delta'(x_1)} & & \\ & \ddots & \\ & & \frac{\sigma'(x_g)}{\Delta'(x_g)} \end{bmatrix} \begin{bmatrix} \frac{1}{\mathbf{c}_1 - x_1} & \cdots & \frac{1}{\mathbf{c}_g - x_1} \\ \vdots & \cdots & \vdots \\ \frac{1}{\mathbf{c}_1 - x_g} & \cdots & \frac{1}{\mathbf{c}_g - x_g} \end{bmatrix}. \end{aligned}$$

As it is well known

$$\det \begin{bmatrix} \frac{1}{\mathbf{c}_1 - x_1} & \cdots & \frac{1}{\mathbf{c}_g - x_1} \\ \vdots & \cdots & \vdots \\ \frac{1}{\mathbf{c}_1 - x_g} & \cdots & \frac{1}{\mathbf{c}_g - x_g} \end{bmatrix} = (-1)^{\frac{g(g-1)}{2}} \frac{\prod_{k < j} (x_k - x_j) \prod_{k < j} (\mathbf{c}_k - \mathbf{c}_j)}{\prod_{j,k} (\mathbf{c}_j - x_k)}. \quad (5.6)$$

On the other hand,

$$y - \Delta(x) = y \frac{\prod (x - x_j)}{\prod (x - \mathbf{c}_j)}.$$

Therefore,

$$-\Delta'(x_k) = y \frac{\prod_{k \neq j} (x_k - x_j)}{\prod_j (x_k - \mathbf{c}_j)} \quad \text{and} \quad -\lambda_k = y \frac{\prod_j (\mathbf{c}_k - x_j)}{\prod_{k \neq j} (\mathbf{c}_k - \mathbf{c}_j)}.$$

That is,

$$\Delta'(x_k) = \lambda_k \frac{\prod_{k \neq j} (x_k - x_j) \prod_{k \neq j} (\mathbf{c}_k - \mathbf{c}_j)}{\prod_j (x_k - \mathbf{c}_j) \prod_j (\mathbf{c}_k - x_j)}.$$

Thus,

$$\prod \Delta'(x_k) = \frac{\prod_{k < j} (x_k - x_j)^2 (\mathbf{c}_k - \mathbf{c}_j)^2}{\prod_{k,j} (\mathbf{c}_k - x_j)^2} \prod_k \lambda_k. \quad (5.7)$$

Combining (5.6) and (5.7), we obtain (5.5).  $\square$

*Proof of Theorem 5.3.* Clearly, the eigenvalue spectral condition on  $A$  corresponds to the eigenvalue spectral condition for  $\Delta(A)$  of the Killip-Simon class matrices with asymptotically constant matrix-block coefficients. By (5.5), we get the corresponding condition on the a.c. spectrum of  $\Delta(A)$ .  $\square$



## 6 Proof of Theorem 1.21

**Lemma 6.1.** *Let  $A \in \text{KSA}(E, \mathbf{C})$  and  $A(n+1) = \mathcal{J}A(n)$ ,  $A(0) = A$ . Then (1.31) and the first relation in (1.32) are satisfied.*

First we prove the following sublemma.

**Lemma 6.2.** *Assume that for sequences  $\psi_n$  and  $\tilde{\psi}_n$  there are sequences  $\tau_n$  and  $\tilde{\tau}_n$  such that*

$$\begin{bmatrix} \tau_n & 0 \\ 0 & 1 \end{bmatrix} \mathbf{o}(\psi_n) - \mathbf{o}(\tilde{\psi}_n) \begin{bmatrix} 1 & 0 \\ 0 & \tilde{\tau}_n \end{bmatrix} \in l_+^2, \quad (6.1)$$

that is, all entries of the above matrix form  $l_+^2$ -sequences. Assume in addition that there is  $\eta > 0$  such that for all  $n$  we have a priori estimations

$$\cos \psi_n \geq \eta, \quad \cos \tilde{\psi}_n \geq \eta, \quad \frac{1}{\eta} \geq \tau_n \geq \eta, \quad \frac{1}{\eta} \geq \tilde{\tau}_n \geq \eta. \quad (6.2)$$

Then  $\{e^{i\psi_n} - e^{i\tilde{\psi}_n}\}_{n \geq 0} \in l_+^2$ .

*Proof.* Directly from (6.1) we have

$$\{\cos \psi_n - \cos \tilde{\psi}_n\}_{n \geq 0} \in l_+^2 \quad \text{and} \quad \{\tau_n \cos \psi_n - \tilde{\tau}_n \cos \tilde{\psi}_n\}_{n \geq 0} \in l_+^2.$$

Then (6.2) implies  $\{\tau_n - \tilde{\tau}_n\}_{n \geq 0} \in l_+^2$ . Now, we have another two conditions

$$\{\tau_n \sin \psi_n - \sin \tilde{\psi}_n\}_{n \geq 0} \in l_+^2 \quad \text{and} \quad \{\sin \psi_n - \tilde{\tau}_n \sin \tilde{\psi}_n\}_{n \geq 0} \in l_+^2.$$

Therefore,

$$\sin \psi_n - \tau_n \tilde{\tau}_n \sin \psi_n - \tilde{\tau}_n (\sin \tilde{\psi}_n - \tau_n \sin \psi_n)$$

belongs to  $l_+^2$ , that is,  $\{\sin \psi_n (1 - \tau_n \tilde{\tau}_n)\}_{n \geq 0} \in l_+^2$ . Thus,  $(\tau_n^2 - 1) \sin \psi_n$  forms an  $l_+^2$ -sequence, as well as  $(\tau_n - 1) \sin \psi_n$ . Finally, since

$$\sin \psi_n - \sin \tilde{\psi}_n = \tau_n \sin \psi_n - \sin \tilde{\psi}_n - (\tau_n - 1) \sin \psi_n,$$

both  $\{\sin \psi_n - \sin \tilde{\psi}_n\}_{n \geq 0}$  and  $\{\cos \psi_n - \cos \tilde{\psi}_n\}_{n \geq 0}$  are  $l_+^2$ -sequences.  $\square$

*Proof of Lemma 6.1.* The first relation (1.31) follows immediately from Lemma 5.5.

Let  $\tilde{A} = \mathcal{O}(A)$ , see (4.1). We use tilde for all entries related to  $\tilde{A}$  and  $\Delta(\tilde{A})$  (5.8), respectively. The entries of  $A(n)$  we denote by  $\{p_j^{(k)}(n), q_j^{(k)}(n)\}$  and we use a similar notation for the entries of  $\Delta(A(n))$  and  $\Delta(\tilde{A}(n))$ . Due to Definition 4.1,

$$\begin{bmatrix} v_{g-1, g-1}^{(0)}(n) & 0 \\ v_{g, g-1}^{(0)}(n) & \lambda_0 p_g^{(0)}(n) \end{bmatrix} \mathbf{o}(\phi_g^{(0)}(n)) = \mathbf{o}(\phi_g^{(-1)}(n)) \begin{bmatrix} \lambda_0 \tilde{p}_g^{(0)}(n) & 0 \\ \tilde{w}_{0, g}^{(0)}(n) & \tilde{v}_{0, 0}^{(1)}(n) \end{bmatrix}. \quad (6.3)$$

Applying Lemma 5.5 to the matrix  $A$ , we obtain

$$\{\lambda_0 p_g^{(0)}(n) - 1\}_{n \geq 0} \in l_+^2, \quad \{v_{g, g-1}^{(0)}(n)\}_{n \geq 0} \in l_+^2.$$

Similarly for the entries related to  $\tilde{A}$  we have  $\{\lambda_0 \tilde{p}_g^{(0)}(n) - 1\}_{n \geq 0} \in l_+^2$ ,  $\{\tilde{w}_{0,g}^{(0)}(n)\}_{n \geq 0} \in l_+^2$ . Thus, we can apply Lemma 6.2 with respect to (6.3). We get  $\{\sin \phi^{(-1)}(n) - \sin \phi^{(0)}(n)\}$  belongs to  $l_+^2$ . That is,  $\{p_{g-1}^{(-1)}(n) - p_{g-1}^{(0)}(n)\}_{n \geq 0} \in l_+^2$ .

Using (4.2), (4.3), we get similar relations for all others  $j$ 's. Using (4.4), we prove the second part of (1.31).  $\square$

*Proof of Theorem 1.21.* Lemma 6.2 implies that  $(v_{g-1,g-1}^{(-1)}(n) - 1) \sin \phi_g^{(-1)}(n)$  form an  $l_+^2$ -sequence, or, equivalently, see (3.1),

$$\{(\Lambda_{-1,g}^\#(n) - \lambda_g) p_{g-1}^{(-1)}(n)\}_{n \geq 0} \in l_+^2. \quad (6.4)$$

Since  $p_{g-1}^{(-1)}(n)$  may approach to zero, it does not imply yet that  $\{\Lambda_{-1,g}^\#(n) - \lambda_g\}$  belongs to  $l_+^2$ . Let us show that

$$\{(\Lambda_{-1,g}^\#(n) - \lambda_g) q_{g-1}^{(-1)}(n)\}_{n \geq 0} \in l_+^2. \quad (6.5)$$

Since  $\inf_n \left( (q_{g-1}^{(-1)}(n))^2 + (p_{g-1}^{(-1)}(n))^2 \right) > 0$ , both (6.4) and (6.5) give us (1.33) for  $m = g$ .

To this end, we note that

$$\Lambda_{-1,g}^\#(n+1) = \frac{\cos \phi_g^{(-1)}(n)}{\cos \phi_g^{(-2)}(n)} \Lambda_{-1,g}^\#(n). \quad (6.6)$$

Indeed, by definition of the Jacobi flow

$$U(\vec{p}_{-2}(n)) \begin{bmatrix} v_{g,g}^{(-2)} & & & \\ * & v_{0,0}^{(-1)} & & \\ * & * & \ddots & \\ * & * & * & v_{g-1,g-1}^{(-1)} \end{bmatrix} (n+1) = \mathbf{v}_{-1}(n) U(\vec{p}_{-1}(n))$$

the second from below entry in the last column in this matrix identity means exactly (6.6). Therefore, by Lemma 6.2, we get

$$\{\Lambda_{-1,g}^\#(n+1) - \Lambda_{-1,g}^\#(n)\}_{n \geq 0} \in l_+^2. \quad (6.7)$$

Now, by (4.11)

$$(\Lambda_{-1,g}^\#(n) - \lambda_g) p_{g-1}^{(-1)}(n) = -(\Lambda_{-1,g}^\#(n) - \lambda_g) q_{g-1}^{(-1)}(n+1) \frac{p_g^{(-1)}(n) \|d_{g-1} \vec{p}_{-1}(n)\|}{\|\vec{p}_{-1}(n)\|}.$$

In combination with (6.7) we have (6.5), and therefore (1.33) for  $k = g$ .

The same arguments with respect to  $\mathcal{O}^k A$ ,  $k = 1, \dots, g-1$ , in a combination with (4.7), give (1.33) for all other  $k$ .

To show the opposite direction, we evaluate the entries of  $\Delta(A)e_{-1}$ . Let  $h(\mathbf{c}_k) = (\mathbf{c}_k - A)^{-1}e_{-1}$ . We use notations from Lemma 3.2. Due to this lemma, we have

$$(h_{-1}(\mathbf{c}_k))_l = \frac{(f_{-1})_l}{(f_{-1})_{k-1}} (h_{-1})_{k-1} = \frac{(\mathbf{p}_{k-1}^{(-1)})^* \mathbf{j}}{\mathbf{c}_{l+1} - \mathbf{c}_k} \prod_{j=k}^{l-1} \mathbf{a}(\mathbf{c}_k, \mathbf{c}_{j+1}, \mathbf{p}_j^{(-1)}) \mathbf{p}_l^{(-1)} \frac{\tilde{\rho}_{k-1}^{(0)}}{\Lambda_{-1,k}^{\#}},$$

for  $k < l + 1$  and  $(h_{-1}(\mathbf{c}_{l+1}))_l = -\frac{\tilde{\rho}_{k-1}^{(0)}}{\Lambda_{-1,k}^{\#}}$ . By the definition

$$w_{l,g}^{(-1)} = p_g^{(-1)} q_l^{(-1)} \lambda_0 + \sum_{k=1}^{l+1} (h_{-1}(\mathbf{c}_k))_l \lambda_k.$$

We substitute in this expression  $\tilde{\rho}_{k-1}^{(0)}$  from (3.7). We use (1.31), (1.32), (1.33), and the identity

$$\prod_{j=0}^{l-1} \mathbf{a}(z, \mathbf{c}_{j+1}; \mathbf{p}_j) = I + \sum_{k=1}^l \frac{\text{Res}_{\mathbf{c}_k} \prod_{j=0}^{l-1} \mathbf{a}(z, \mathbf{c}_{j+1}; \mathbf{p}_j)}{z - \mathbf{c}_k}$$

evaluated at the point  $\mathbf{c}_{l+1}$ . As the result, we obtain  $w_{l,g}^{(-1)}(n) \in l_+^2$  for all  $0 \leq l < g$ . To prove that  $w_{g,g}^{(-1)}(n) \in l_+^2$  we use (3.6) and a little bit more involved identity (2.26) shown in Lemma 2.18. Similarly, one can prove that  $\{v_{g,g}^{(-1)}(n) - 1\}$  and  $v_{g,l}^{(0)}(n)$ , for  $0 \leq l < g$ , form  $l_+^2$  sequences.  $\square$

## 7 Proof of the main Theorem 1.5

### 7.1 From GSMP to Jacobi

Assume that  $A \in \text{GSMP}(\mathbf{C})$ . Let  $A(n) = \mathcal{J}^{on}A$ . Recall that the coefficients of the Jacobi matrix  $J = \mathcal{F}A$  are given by (1.29) and  $l^2$  properties of the coefficients  $\{\vec{p}_{-1}(n), \vec{q}_{-1}(n), \vec{p}_0(n), \vec{q}_0(n)\}$  are given in Theorem 1.21. We consider the isospectral surface  $\mathcal{IS}_E$  given by (1.30), see (2.25), with the identification  $(p_j, q_j) \equiv (-p_j, -q_j)$ ,  $j = 0, \dots, g - 1$ . Note that this is a  $g$  dimensional torus, which we can parametrize by  $\alpha \in \mathbb{R}^g / \mathbb{Z}^g$  according to Theorem 2.8. Moreover, for the given manifold

$$0 < \inf_{\{\vec{\mathbf{p}}\} \in \mathcal{IS}_E} \|(T(\vec{\mathbf{p}})^* T(\vec{\mathbf{p}}))^{-1}\|^{-1} \leq \sup_{\{\vec{\mathbf{p}}\} \in \mathcal{IS}_E} \|T(\vec{\mathbf{p}})^* T(\vec{\mathbf{p}})\| < \infty, \quad (7.1)$$

where

$$T(\vec{\mathbf{p}}) = \begin{bmatrix} \frac{\partial \Lambda_1}{\partial p_0} & \cdots & \frac{\partial \Lambda_g}{\partial p_0} \\ \frac{\partial \Lambda_1}{\partial q_0} & \cdots & \frac{\partial \Lambda_g}{\partial q_0} \\ \vdots & \cdots & \vdots \\ \frac{\partial \Lambda_1}{\partial p_{g-1}} & \cdots & \frac{\partial \Lambda_g}{\partial p_{g-1}} \\ \frac{\partial \Lambda_1}{\partial q_{g-1}} & \cdots & \frac{\partial \Lambda_g}{\partial q_{g-1}} \end{bmatrix}. \quad (7.2)$$

We define a periodic GSMP matrix  $A(\alpha_n)$  generated by  $\{\overset{\circ}{\mathbf{p}}(\alpha_n)\} \in \mathcal{IS}_E$  such that

$$\text{dist}(\vec{\mathbf{p}}_0(n), \mathcal{IS}_E) = \text{dist}(\vec{\mathbf{p}}_0(n), \overset{\circ}{\mathbf{p}}(\alpha_n)). \quad (7.3)$$

Using the standard Lagrange multipliers method, we can estimate the distance from  $\vec{\mathbf{p}}_0(n)$  to the isospectral set in terms of  $\|(T(\vec{\mathbf{p}})^*T(\vec{\mathbf{p}}))^{-1}\|$ . Then, by (1.32), (1.33) and (7.1), we have

$$\sum_{n=0}^{\infty} \text{dist}^2(\vec{\mathbf{p}}_0(n), \overset{\circ}{\mathbf{p}}(\alpha_n)) < \infty \quad (7.4)$$

and also, see (1.29),

$$a(n)^2 - \mathcal{A}(\alpha_n) \in l^2, \quad b(n) - \mathcal{B}(\alpha_n) \in l^2. \quad (7.5)$$

On the other hand, by (1.31)-(1.33) and the uniform smoothness of the Jacobi flow transform (4.11), (4.12)

$$\text{dist}(\vec{\mathbf{p}}_0(n+1), \overset{\circ}{\mathbf{p}}(\alpha_n - \mu)) \leq C(E, J) \{ \text{dist}(\vec{\mathbf{p}}_0(n), \overset{\circ}{\mathbf{p}}(\alpha_n)) + \text{dist}(\vec{\mathbf{p}}_0(n), \vec{\mathbf{p}}_1(n)) \}.$$

That is,

$$\begin{aligned} \text{dist}(\overset{\circ}{\mathbf{p}}(\alpha_{n+1}), \overset{\circ}{\mathbf{p}}(\alpha_n - \mu)) &\leq C(E, J) \{ \text{dist}(\vec{\mathbf{p}}_0(n), \overset{\circ}{\mathbf{p}}(\alpha_n)) + \text{dist}(\vec{\mathbf{p}}_0(n), \vec{\mathbf{p}}_1(n)) \} \\ &\quad + \text{dist}(\vec{\mathbf{p}}_0(n+1), \overset{\circ}{\mathbf{p}}(\alpha_{n+1})). \end{aligned}$$

Since

$$\|\alpha - \beta\| \leq C_1(E) \text{dist}(\overset{\circ}{\mathbf{p}}(\alpha), \overset{\circ}{\mathbf{p}}(\beta)),$$

(7.4) and (1.31) imply

$$\sum_{n=0}^{\infty} \|\epsilon_\alpha(n)\|^2 < \infty, \quad \text{where } \epsilon_\alpha(n) := \alpha_{n+1} - (\alpha_n - \mu).$$

In combination with (7.5), we obtain (1.8).

**Remark 7.1.** Of course in this proof it is not necessary to choose  $\alpha_n$  as the best approximation to  $\vec{\mathbf{p}}_0(n)$ , see (7.3). It is enough to have this distance under an appropriate control. This explains a certain ambiguity in the representation (1.8).

## 7.2 From Jacobi to GSMP

In this section our goal is to estimate  $p_j(n) - \overset{\circ}{p}_j(\alpha_n)$  and  $q_j(n) - \overset{\circ}{q}_j(\alpha_n)$  by means of the related distances  $\text{dist}((S^{-n}JS^n)_+, J(E)) < \infty$ . In fact, we prove the following lemma.

**Lemma 7.2.** *Let  $J$  be of the form (1.8). Then*

$$\sum_{n=0}^{\infty} \text{dist}_\eta^2((S^{-n}JS^n)_+, J(\alpha_n)_+) < \infty, \quad \alpha_n = \sum_{k=0}^n \epsilon_\alpha(k) - \mu n. \quad (7.6)$$



Note that (7.6) evidently implies a word-by-word counterpart of (1.3) in DKST, see Remark 1.7.

*Proof of Lemma 7.2.* We have

$$|b(k+n) - \mathcal{B}(\alpha_n - \mu k)| \leq |\epsilon_b(k+n)| + C_1(E) \left\| \sum_{j=n+1}^{n+k} \epsilon_\alpha(j) \right\|,$$

where  $C_1(E) = \sup_{\alpha \in \mathbb{R}^g / \mathbb{Z}^g} \|\text{grad } \mathcal{B}(\alpha)\|$ . For  $\eta < 1$ , we have

$$\begin{aligned} \sum_{n \geq 0} \left( \sum_{k \geq 1} \left\| \sum_{j=n+1}^{n+k} \epsilon_\alpha(j) \right\|^2 \eta^{2k} \right) &\leq \sum_{n \geq 0} \sum_{k \geq 1} \sum_{j=n+1}^{n+k} \|\epsilon_\alpha(j)\|^2 k \eta^{2k} = \sum_{k \geq 1} \sum_{n \geq 0} \sum_{j=n+1}^{n+k} \|\epsilon_\alpha(j)\|^2 k \eta^{2k} \\ &\leq \sum_{k \geq 1} k \eta^{2k} \sum_{j \geq 1} k \|\epsilon_\alpha(j)\|^2 \leq \sum_{j \geq 1} \|\epsilon_\alpha(j)\|^2 \sum_{k \geq 1} k^2 \eta^{2k}. \end{aligned}$$

Making a similar estimation for  $|a(k+n)^2 - \mathcal{A}(\alpha_n - \mu k)|$  we obtain (7.6).  $\square$

Now, let  $J = \mathcal{F}A$ , see (1.25)-(1.26), that is,  $J = F^*AF$ , where  $F : l^2 \rightarrow l^2$  is the unitary map such that  $Fe_{-1} = e_{-1}$  and  $FP_+ = P_+F$ , in particular,  $Fe_0 = \tilde{e}_0 = \frac{1}{a(0)}P_+Ae_{-1}$ . We note that

$$\{h = (A - \mathbf{c}_1)f : f \in l^2_+, \langle f, \tilde{e}_0 \rangle = 0\} = \{h \in l^2_+, \langle h, e_0 \rangle = 0\}. \quad (7.7)$$

Thus,  $F^*e_0$  can be described by means of an orthogonal complement in the following construction.

Let  $\mathbf{c} \notin \sigma(J)$ . We assume that  $\mathbf{c}$  is real. We define

$$l^2_{+, \mathbf{c}} := \{h = (J - \mathbf{c})f : f \in l^2_+, \langle f, e_0 \rangle = 0\}. \quad (7.8)$$

Recall that  $r_+(z) = \langle (J_+ - z)^{-1}e_0, e_0 \rangle$ .

**Lemma 7.3.** *Let  $\mathfrak{K}_{\mathbf{c}} = l^2_+ \ominus l^2_{+, \mathbf{c}}$ . This is a one dimensional space, i.e.,  $\mathfrak{K}_{\mathbf{c}} = \{\varkappa_{\mathbf{c}}\}$ . Moreover, we can choose*

$$\varkappa_{\mathbf{c}} = (J - \mathbf{c})^{-1}(e_{-1}a(0) \sin \varphi + e_0 \cos \varphi), \quad (7.9)$$

where

$$\tan \varphi = \tan \varphi(\mathbf{c}) = r_+(\mathbf{c}), \quad -\frac{\pi}{2} < \varphi \leq \frac{\pi}{2}, \quad (7.10)$$

including  $\varphi = \frac{\pi}{2}$  if  $r_+(\mathbf{c}) = \infty$ , that is,  $\mathbf{c}$  is a pole of this function. In this notations

$$\|\varkappa_{\mathbf{c}}\|^2 = \frac{r'_+(\mathbf{c})}{1 + r_+(\mathbf{c})^2} = \varphi'(\mathbf{c}). \quad (7.11)$$

Moreover, the following two-sided estimation holds

$$\frac{\min\{a(0)^2, 1\}}{(|\mathbf{c}| + \|J\|)^2} \leq \varphi'(\mathbf{c}) \leq \frac{\max\{a(0)^2, 1\}}{\text{dist}^2(\mathbf{c}, \sigma(J))}. \quad (7.12)$$

*Proof.* If  $r_+(\mathbf{c}) \neq \infty$ , we have  $\varkappa_{\mathbf{c}} = (J_+ - \mathbf{c})^{-1}e_0 \cos \varphi$ . Otherwise  $\varkappa_{\mathbf{c}}$  is collinear to the corresponding eigenvector of  $J_+$ . These prove (7.9), (7.10).

Further, we have

$$\|\varkappa_{\mathbf{c}}\|^2 = \langle (J_+ - \mathbf{c})^{-2}e_0, e_0 \rangle \cos^2 \varphi = \frac{r'_+(\mathbf{c})}{1 + r_+(\mathbf{c})^2},$$

which proves (7.11). We use (5.1). Since  $\int d\Sigma = I$ , we have

$$\frac{1}{(|\mathbf{c}| + \|J\|)^2} \leq R'(\mathbf{c}) = \int \frac{d\Sigma}{(x - \mathbf{c})^2} \leq \frac{1}{\text{dist}^2(\mathbf{c}, \Sigma(J))}.$$

Using (5.1), we obtain

$$\frac{1}{(|\mathbf{c}| + \|J\|)^2} \leq R(\mathbf{c}) \begin{bmatrix} \frac{r'_-(\mathbf{c})}{r_-(\mathbf{c})^2} & 0 \\ 0 & \frac{r'_+(\mathbf{c})}{r_+(\mathbf{c})^2} \end{bmatrix} R(\mathbf{c}) \leq \frac{1}{\text{dist}^2(\mathbf{c}, \sigma(J))},$$

or

$$\frac{R(\mathbf{c})^{-2}}{(|\mathbf{c}| + \|J\|)^2} \leq \begin{bmatrix} \frac{r'_-(\mathbf{c})}{r_-(\mathbf{c})^2} & 0 \\ 0 & \frac{r'_+(\mathbf{c})}{r_+(\mathbf{c})^2} \end{bmatrix} \leq \frac{R(\mathbf{c})^{-2}}{\text{dist}^2(\mathbf{c}, \sigma(J))}.$$

Comparing the values in the lower corner of these matrices, we get

$$\frac{1 + a_0^2 r_+(\mathbf{c})^2}{(|\mathbf{c}| + \|J\|)^2} \leq r'_+(\mathbf{c}) \leq \frac{1 + a_0^2 r_+(\mathbf{c})^2}{\text{dist}^2(\mathbf{c}, \sigma(J))}.$$

Thus, (7.12) is also proved.  $\square$

Defining  $\varkappa_{\mathbf{c}}$  by (7.7) and (7.8), we obtain  $F^*e_0 = \frac{1}{\|\varkappa_{\mathbf{c}_1}\|} \varkappa_{\mathbf{c}_1}$ . Therefore,

$$p_0^{(0)}(0) = \langle Ae_{-1}, e_0 \rangle = \langle Je_{-1}, \frac{\varkappa_{\mathbf{c}_1}}{\|\varkappa_{\mathbf{c}_1}\|} \rangle = \frac{a(0) \sin \varphi(\mathbf{c}_1)}{\varphi'(\mathbf{c}_1)}.$$

**Remark 7.4.** Note that  $\mathfrak{K}_z$  is well define for all  $z \in \mathbb{C} \setminus \sigma(J)$ , that is, in fact, we have a Hermitian analytic vector bundle in this domain. Its fundamental characteristic, the so-called curvature, is of the form  $\Delta \log \langle \varkappa_z, \varkappa_z \rangle$ , see e.g. [8]. Being restricted on the real axis, it represents the Schwarzian derivative of  $r_+(z)$ . Our further considerations are based on estimations of related expressions and involve derivatives of exactly this level, see Lemma 7.5 below. We can conjecture that a certain Hermitian analytic vector bundle model, which generalized the model described in Section 2, is possible for operators of Killip-Simon class. Under more restrictive assumptions, when the absolutely continuous part of the spectral measure satisfies the Szegő condition and positions of the eigenvalues outside  $E$  obey the Blaschke condition, such model does exist. This is the so-called scattering model for the given operator [28, 41, 27].

**Lemma 7.5.** Let  $\overset{\circ}{J} \in J(E)$ . In the previous notations,

$$\langle (J - \overset{\circ}{J})\varkappa_{\mathbf{c}}, \overset{\circ}{\varkappa}_{\mathbf{c}} \rangle = \sin(\overset{\circ}{\varphi}(\mathbf{c}) - \varphi(\mathbf{c})). \quad (7.13)$$

Consequently, there exists  $C = C(\sigma(J), \mathbf{c}) < \infty$  such that

$$|\sin(\overset{\circ}{\varphi}(\mathbf{c}) - \varphi(\mathbf{c}))| \leq C \text{dist}_{\eta}(J_+, \overset{\circ}{J}_+) \quad (7.14)$$

and simultaneously for the derivatives

$$|(\overset{\circ}{\varphi})^{(m)}(\mathbf{c}) - \varphi^{(m)}(\mathbf{c})| \leq C \text{dist}_{\eta}(J_+, \overset{\circ}{J}_+) \quad (7.15)$$

for  $m = 1, 2, 3$  and  $\eta > |b(\mathbf{c})|$ .

*Proof.* We have

$$\langle (J - \overset{\circ}{J})\varkappa_{\mathbf{c}}, \overset{\circ}{\varkappa}_{\mathbf{c}} \rangle = \langle (J - \mathbf{c})\varkappa_{\mathbf{c}}, \overset{\circ}{\varkappa}_{\mathbf{c}} \rangle - \langle (\overset{\circ}{J} - \mathbf{c})\varkappa_{\mathbf{c}}, \overset{\circ}{\varkappa}_{\mathbf{c}} \rangle.$$

We simplify the first term

$$\langle e_{-1}a(0) \sin \varphi + e_0 \cos \varphi, \overset{\circ}{\varkappa}_{\mathbf{c}} \rangle = \frac{1}{\overset{\circ}{a}(0)} \langle (\overset{\circ}{J} - c)e_{-1}, \overset{\circ}{\varkappa}_{\mathbf{c}} \rangle \cos \varphi = \sin \overset{\circ}{\varphi} \cos \varphi.$$

Thus,  $\langle (J - \overset{\circ}{J})\varkappa_{\mathbf{c}}, \overset{\circ}{\varkappa}_{\mathbf{c}} \rangle = \sin \overset{\circ}{\varphi} \cos \varphi - \sin \varphi \cos \overset{\circ}{\varphi}$  and (7.13) is proved.

The upper estimation in (7.12) in combination with (7.13), (7.11) implies

$$|\sin(\overset{\circ}{\varphi}(\mathbf{c}) - \varphi(\mathbf{c}))| \leq \sqrt{\varphi'(\mathbf{c})(\overset{\circ}{\varphi})'(\mathbf{c})} \frac{\|(J - \overset{\circ}{J})\varkappa_{\mathbf{c}}\|}{\|\varkappa_{\mathbf{c}}\|} \leq C \frac{\|(J - \overset{\circ}{J})\varkappa_{\mathbf{c}}\|}{\|\varkappa_{\mathbf{c}}\|}. \quad (7.16)$$

Now, the vector  $\frac{1}{\|\varkappa_{\mathbf{c}}\|} \overset{\circ}{\varkappa}_{\mathbf{c}}$  in the functional model for  $\overset{\circ}{J} = J(\alpha)$  corresponds to the normalized reproducing kernel  $\frac{1}{\|k_{\zeta_{\mathbf{c}}}^{\alpha}\|} k_{\zeta_{\mathbf{c}}}^{\alpha}$ , where  $\zeta_{\mathbf{c}} \in \mathbb{D}$  is such that  $\mathfrak{z}(\zeta_{\mathbf{c}}) = \mathbf{c}$ . The components of this vector were estimated in (2.13). Thus,

$$\frac{1}{\|\varkappa_{\mathbf{c}}\|} \|(J - \overset{\circ}{J})\varkappa_{\mathbf{c}}\| \leq C(E) \text{dist}_{\eta}(J_+, \overset{\circ}{J}_+), \quad |b(\mathbf{c})| < \eta < 1,$$

and (7.16) implies (7.14).

To get (7.15) we differentiate (7.13) with respect to  $\mathbf{c}$

$$\cos(\overset{\circ}{\varphi}(\mathbf{c}) - \varphi(\mathbf{c}))((\overset{\circ}{\varphi})'(\mathbf{c}) - \varphi'(\mathbf{c})) = \langle (J - \overset{\circ}{J})\varkappa'_{\mathbf{c}}, \overset{\circ}{\varkappa}_{\mathbf{c}} \rangle + \langle (J - \overset{\circ}{J})\varkappa_{\mathbf{c}}, (\overset{\circ}{\varkappa}_{\mathbf{c}})' \rangle. \quad (7.17)$$

Since  $\sin(\overset{\circ}{\varphi}(\mathbf{c}) - \varphi(\mathbf{c}))$  was estimated from above, we have a uniform estimation for  $|\cos(\overset{\circ}{\varphi}(\mathbf{c}) - \varphi(\mathbf{c}))|$  from below. Using (7.9), we evaluate  $\varkappa'_{\mathbf{c}}$ . Based on its explicit form

and the estimation for  $\varphi'_{\mathbf{c}}$ , we obtain that  $\|\mathcal{X}'_{\mathbf{c}}\|$  is also bounded by the distance from  $\mathbf{c}$  to  $\sigma(J)$ . Evidently, the coefficients of  $(\overset{\circ}{\mathcal{X}}_{\mathbf{c}})'$  also satisfies (2.13). Thus,

$$|(\overset{\circ}{\varphi})'(\mathbf{c}) - \varphi'(\mathbf{c})| = \frac{\|(J - \overset{\circ}{J})\overset{\circ}{\mathcal{X}}_{\mathbf{c}}\|\|\mathcal{X}'_{\mathbf{c}}\| + \|(J - \overset{\circ}{J})(\overset{\circ}{\mathcal{X}}_{\mathbf{c}})'\|\|\mathcal{X}_{\mathbf{c}}\|}{|\cos(\overset{\circ}{\varphi}(\mathbf{c}) - \varphi(\mathbf{c}))|}$$

implies (7.15). Taking the second and third derivatives in (7.17), we obtain (7.15) for  $m = 2, 3$ .  $\square$

**Corollary 7.6.** *If  $J$  is of the form (1.8) and  $A(n) = \mathcal{F}^{-1}(S^{-n}JS^n)$ , then*

$$\sum_{n=0}^{\infty} |p_0^{(0)}(n) - p_0(\alpha_n)|^2 < \infty, \quad \alpha_n = \sum_{k=0}^n \epsilon_{\alpha}(k) - \mu n. \quad (7.18)$$

*Proof.* By (7.14), (7.15) we can estimate the difference

$$p_0^{(0)}(n) - p_0(\alpha_n) = \frac{a(n) \sin \varphi(\mathbf{c}_1)}{\varphi'(\mathbf{c}_1)} - \frac{\overset{\circ}{a}(0) \sin \overset{\circ}{\varphi}(\mathbf{c}_1)}{(\overset{\circ}{\varphi})'(\mathbf{c}_1)}, \quad \overset{\circ}{J} = J(\alpha_n),$$

by means of  $\text{dist}((S^{-n}JS^n)_+, J(\alpha_n)_+)$ . Due to (7.6), we have (7.18).  $\square$

*Finishing the proof of Theorem 1.5.* It remains to show that (1.8) imply (1.31)-(1.33).

Consider the ordered system of vectors

$$e_{-1}, \mathcal{X}_{\mathbf{c}_1}, \dots, \mathcal{X}_{\mathbf{c}_g}, e_0, \mathcal{X}'_{\mathbf{c}_1}, \dots, \mathcal{X}'_{\mathbf{c}_g}, e_1. \quad (7.19)$$

Let us point out that the orthogonalization of the system

$$e_{-1}, \overset{\circ}{\mathcal{X}}_{\mathbf{c}_1}, \dots, \overset{\circ}{\mathcal{X}}_{\mathbf{c}_g}, e_0, (\overset{\circ}{\mathcal{X}}_{\mathbf{c}_1})', \dots, (\overset{\circ}{\mathcal{X}}_{\mathbf{c}_g})', e_1 \quad (7.20)$$

leads to the family  $\{f_j^{\alpha}\}_{j=-1}^{2g+2}$ , see (2.6), where  $\overset{\circ}{J} = J(\alpha)$ .

To evaluate the Gram-Schmidt matrix of the system (7.19) we use

$$\langle \mathcal{X}_{\mathbf{c}_j}, \mathcal{X}_{\mathbf{c}_m} \rangle = \frac{r_+(\mathbf{c}_j) - r_+(\mathbf{c}_m)}{\mathbf{c}_j - \mathbf{c}_m} \cos \varphi(\mathbf{c}_j) \cos \varphi(\mathbf{c}_m) = \frac{\sin(\varphi(\mathbf{c}_j) - \varphi(\mathbf{c}_m))}{\mathbf{c}_j - \mathbf{c}_m}.$$

Therefore,

$$\langle \mathcal{X}'_{\mathbf{c}_j}, \mathcal{X}_{\mathbf{c}_m} \rangle = \frac{\cos(\varphi(\mathbf{c}_j) - \varphi(\mathbf{c}_m))}{\mathbf{c}_j - \mathbf{c}_m} \varphi'(\mathbf{c}_j) - \frac{\sin(\varphi(\mathbf{c}_j) - \varphi(\mathbf{c}_m))}{(\mathbf{c}_j - \mathbf{c}_m)^2}$$

and

$$\begin{aligned} \langle \mathcal{X}'_{\mathbf{c}_j}, \mathcal{X}'_{\mathbf{c}_m} \rangle &= \frac{\cos(\varphi(\mathbf{c}_j) - \varphi(\mathbf{c}_m))}{(\mathbf{c}_j - \mathbf{c}_m)^2} (\varphi'(\mathbf{c}_j) + \varphi'(\mathbf{c}_m)) \\ &+ \frac{\sin(\varphi(\mathbf{c}_j) - \varphi(\mathbf{c}_m))}{\mathbf{c}_j - \mathbf{c}_m} \varphi'(\mathbf{c}_j) \varphi'(\mathbf{c}_m) - 2 \frac{\sin(\varphi(\mathbf{c}_j) - \varphi(\mathbf{c}_m))}{(\mathbf{c}_j - \mathbf{c}_m)^3}, \quad j \neq m. \end{aligned}$$

Having uniform estimations from below for all Gram-Schmidt determinants of the system (7.20), from (7.14), (7.15), similarly to (7.18), we obtain

$$\sum_{n=0}^{\infty} |p_j^{(m)}(n) - p_j(\alpha_n)|^2 < \infty, \quad \sum_{n=0}^{\infty} |q_j^{(m)}(n) - q_j(\alpha_n)|^2 < \infty, \quad m = 0, 1, \quad j = 0, \dots, g.$$

This implies (1.31)-(1.33), in particular,  $\sum_{n=0}^{\infty} |p_j^{(1)}(n) - p_j^{(0)}(n)|^2 < \infty$ ,  $j = 0, \dots, g - 1$ .  $\square$

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