EXACT COMPLETION OF PATH CATEGORIES AND ALGEBRAIC SET THEORY

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Abstract. In this paper, we introduce the notion of a “category with path objects”, as a slight strengthening of Ken Brown’s classical notion of a “category of fibrant objects”. We develop the basic properties of such a category and its associated homotopy category. Subsequently, we show how the exact completion of this homotopy category can be obtained as the homotopy category associated to a larger category with path objects, obtained by freely adjoining certain homotopy quotients. Next, we prove that if the original category with path objects is equipped with a representable class of fibrations having some basic closure properties, the methods of algebraic set theory imply that the latter homotopy category contains a model of a constructive version of Zermelo-Fraenkel set theory. Although our work is partly motivated by recent developments in homotopy type theory, this paper is written purely in the language of homotopy theory and category theory, and we do not presuppose any familiarity with type theory on the side of the reader.

1. Introduction

1.1. **Summary.** The phrase “path category” in the title is short for “category with path objects” and refers to a modification of Kenneth Brown’s notion of a category of fibrant objects [13], originally meant to axiomatise the homotopical properties of the category of simplicial sheaves on a topological space. Like categories of fibrant objects, path categories are categories equipped with classes of fibrations and weak equivalences, and as such they are closely related to Quillen’s model categories which have an additional class of cofibrations [39, 40, 26]. In particular, the fibrant objects in a Quillen model category form a category of fibrant objects in Brown’s sense. Much of this paper is concerned with a homotopically meaningful way of freely (up to homotopy) adjoining quotients of equivalence relations to a path category, resulting in a larger path category whose homotopy category has nice exactness properties.

We believe this construction is of interest from the point of view of homotopy theory. But we were mainly motivated by the recent discovery of the relation between homotopy theory and type theory. Indeed, the interpretation of the identity types

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from Martin-Löf type theory in Quillen model structures as in [4] has been extended by Voevodsky to an interpretation of the Calculus of Constructions in the category of simplicial sets [33] (see also [12, 18, 23, 43, 44]), while on the other hand, any type theory defines a path category in which the identity types define the homotopy relation and the dependent types play the rôle of the fibrations [3, 22].

It is possible to interpret the more familiar mathematical language of Zermelo-Fraenkel set theory in type theory with a suitable universe [1], and thus the question arises whether it is possible to construct models of set theory out of path categories. A main result of this paper is that this is indeed possible, and the method of adjoining quotients to a path category is an important intermediate step in this construction of such models of set theory. After having adjoined such quotients, the associated homotopy category turns out to be a pretopos, and it becomes possible to apply the methods of algebraic set theory [31].

In this way, we believe that this paper contributes to the further understanding of the many relations between homotopy theory, category theory, type theory and set theory. We should describe some of these relations in more detail, and explain how our current work exactly fits in.

1.2. **Background.** First of all, the relation between Martin-Löf-style dependent type theory and category theory goes back to the work of Robert Seely [42], who pointed out the close relation between locally cartesian closed categories and dependent type theories with constructors for Π- and Σ-types. Indeed, one can obtain such a category from the syntax of type theory. The relation in the other direction is more subtle than originally envisaged, mainly because of a mismatch between the up-to-isomorphism nature of universal constructions in category theory and the strictly equational nature of the corresponding type-theoretic constructions. This mismatch, known as the coherence problem, can be addressed in various ways, for example, by means of Grothendieck's theory of fibred categories [27] (see also [24, 35]).

The relation between Zermelo-Fraenkel-style set theory and category theory goes back to Lawvere and Tierney and the theory of elementary toposes [46]. Elementary toposes per se only model a very restricted form of "bounded" Zermelo-Fraenkel set theory [28], but work of Freyd [21] and Fourman [20] showed that one can construct models of full Zermelo-Fraenkel set theory in any Grothendieck topos. These models of course support a constructive logic and do not validate the Law of the Excluded Middle, but a simple localisation by a finer Grothendieck topology corresponding to Gödel's double negation translation [36] transforms these into models which do satisfy the Law of Excluded Middle, and are in fact very closely related to the original forcing models of Cohen [19]. The construction by Freyd and Fourman of these models was axiomatised and refined in [31], where the authors introduced a categorical notion of a pretopos equipped with a class of "small" maps, which satisfied axioms that permit the construction of an object mimicking the cumulative hierarchy of sets, and satisfying the axioms of **ZF**.

The relation between type theory and set theory described by Aczel in [1] has already been mentioned above. In fact, Aczel's construction for type theory and the algebraic set theory construction for pretoposes are closely related and can be unified in the following way. On the one hand, it is possible to weaken the original axioms of [31] and develop an analogous theory in a more "predicative" setting, which avoids the use of objects representing the power set construction that play such a central rôle in the Lawvere-Tierney theory. On the other hand, it is possible to adjoin quotients of equivalence relations at the level of type theory (known as setoids in the type theory literature) which together with the definable morphisms between them form a pretopos satisfying the axioms of this predicative weakening of algebraic set theory. This theory provides a clear explanation of the relation between algebraic set theory, type theory and category theory, and has shown to have various proof-theoretic applications concerning consistency and the existence of derived rules for the logical theories involved [38, 10].

The construction of adding quotients is well known in category theory, and goes under the name of exact completion. It associates to a given category $\mathcal{C}$ a larger category $\mathcal{C}'$ which is universal with respect to the existence of coequalisers of equivalence relations. If $\mathcal{C}$ is sufficiently nice, for example if $\mathcal{C}$ has finite limits and stable sums, then the same construction results in a category $\mathcal{C}'$ satisfying the axioms for a pretopos, see [14, 15, 17]. Taking the exact completion often improves the properties of a category: for example, $\mathcal{C}'$ will be locally cartesian closed (that is, have internal homs in every slice) whenever $\mathcal{C}$ has this property in a weak form, where weak is meant to indicate that one weakens the usual universal property of the internal hom by dropping the uniqueness requirement, only keeping existence (see [16]). We will see similar phenomena in the present paper.

1.3. **Contents.** Against this background, let us describe the contents of our paper in more detail. In the first section, we review the axioms of Brown, and modify his formulation by adding an axiom which in the Quillen set-up would correspond to the requirement that all objects are cofibrant. (This cannot be said in such a direct way in the Brown formalism, in which there are no cofibrations.) One justification for this modification is that there still are plenty of examples. One source of examples is provided by taking the fibrant objects in a model category in which all objects are cofibrant, such as the category of simplicial sets, or the categories of simplicial sheaves equipped with the injective model structure. More generally, many model categories have the property that objects over a cofibrant object are automatically cofibrant. For example, this holds for familiar model category structures for simplicial sets with the action of a fixed group, for dendroidal sets, and for many more. In such a model category, the fibrations and weak equivalences between objects which are both fibrant and cofibrant satisfy our modification of Brown's axioms. Another justification for this modification of Brown's axioms is that they are satisfied by the syntactic category constructed out of a type theory [3]. In fact, our modified categories of fibrant objects seem to correspond very closely to dependent type theories in which the usual rules for the identity types are weakened and the computation rule is only asked to hold in a propositional form [8].

We will verify in Section 2 that many familiar constructions from homotopy theory can be performed in such modified categories of fibrant objects, or path categories as we will call them, and retain their expected properties. It is necessary for what follows to perform this verification, but there is very little originality in it. An exception is perhaps formed by our construction of suitable path objects carrying a connection structure as in Theorem 2.28 and our statement concerning the existence of diagonal fillers which are half strict, half up-to-homotopy, as in Theorem 2.38 below. We single out these two properties here also because they play an important rôle in later parts of the paper.

In Section 3 we will introduce a notion of exact completion for path categories and prove the following result:

Theorem 1.1. (= Proposition 3.17 and Theorem 3.14 below) *Any path category $\mathcal{C}$ can be completed into a path category* $\mathrm{Ex}(\mathcal{C})$ *with the property that its homotopy category is the exact completion (in the sense of* [17]*) of the homotopy category of* $\mathcal{C}$.

This paper will mainly be concerned with the homotopy category of $\mathrm{Ex}(\mathcal{C})$, which we will denote by $\mathrm{Hex}(\mathcal{C})$. In fact, we will initially construct this category $\mathrm{Hex}(\mathcal{C})$ directly, while we only show later that it is the homotopy category of some other path category (as in Theorem 3.14). We also study the behaviour of these constructions under change of base, that is, the passage from $\mathcal{C}$ itself to the category of fibrations over a base object X.

In Section 4, we show that if $\mathcal{C}$ has homotopy sums which are, in a suitable sense, stable and disjoint, then the exact completion $\mathrm{Hex}(\mathcal{C})$ is a pretopos (see Theorem 4.10).

Once it has been explained how to obtain a pretopos, we can establish a connection with the framework of algebraic set theory. Thus, in Section 5, we introduce the notion of a path category equipped with a special class of "small" fibrations, and show how in its homotopy exact completion this gives rise to a class of small maps. We also give various explicit characterisations of these small maps in terms of their representations in the original path category. These characterisations are then used in the next section, Section 6, where we study fibrewise internal homs, or Π-types. We show that if the path category has weak Π-types then its exact completion has exponentials in every slice, which behave well with respect to the small maps if the original weak Π-types behave well with respect to the small fibrations. In Section 7 we prove similar results for W-types. These W-types are universal inductive types, which intuitively represent objects of well-founded trees with a prescribed branching.

In a final section, we will consider the situation where the class of small fibrations is representable by a "universal fibration" in the familiar sense of homotopy theory, meaning that every small fibration is homotopy equivalent to a pullback of the universal fibration. We will show that from this universal fibration, when moved into the homotopy exact completion, one can construct a W-type a quotient of which is a model of set theory. As a result we establish the following theorem, which is the main result of this paper.

Theorem 1.2. (= Theorem 8.2 below) *Let $\mathcal{C}$ be a homotopy extensive path category with weak homotopy* Π*-types, a homotopy natural numbers object and homotopy W-types. Assume that $\mathcal{F}$ is a class of fibrations in $\mathcal{C}$ satisfying axioms (F1-8) and which contains a small fibration $\pi\colon E \to U$ such that any other element of $\mathcal{F}$ can be obtained as a homotopy pullback of π. Then* $\mathrm{Hex}(\mathcal{C})$ *contains a model of Aczel's constructive set theory* **CZF**.

These last two sections are relatively short, because they rely heavily on earlier work, on the construction of W-types in exact completions in [7], and the construction of models of set theory from universal W-types in [38]. Indeed, using these earlier results, the theory developed in this paper will enable us to quickly draw the conclusion that any path category equipped with a suitable class of small fibrations contains a model of Aczel's constructive version of Zermelo-Fraenkel set theory.

For the convenience of the reader we have also included an appendix where we list in one place: the axioms for a path category, the relevant axioms for a class of small fibrations and a class of small maps, as well as a recap of the notion of an exact category and a pretopos.

1.4. **Our approach and related work.** At this point it is probably good to add a few words about our approach and how it relates to some of the work that is currently being done at the intersection between type theory and homotopy theory.

First of all, we take a resolutely categorical approach; in particular, no knowledge of the syntax of type theory is required to understand this paper. As a result, we expect our paper to be readable by homotopy theorists.

Despite being inspired by homotopy type theory, the additions to Martin-Löf type theory suggested by its homotopy-theoretic interpretation play no rôle in this paper. In particular, we will not use univalence, higher-inductive types or even functional extensionality. Indeed, all the definitions and theorems have been formulated in such a way that they will apply to the syntactic category of (pure, intensional) Martin-Löf type theory.

In fact, we expect that our definitions remain applicable to the syntactic category of type theory even when all its computation rules are formulated as propositional equalities. This was already mentioned in connection with the identity types above, but we firmly believe that it applies to all type constructors. This idea has guided us in setting up many of the definitions of this paper. This includes, for example, the definition of a (weak) homotopy Π-type as in Definition 6.3 below, or the definition of a universal small fibration using homotopy pullbacks (as in Proposition 8.8 below; see also [12]).

In addition to the reasons already mentioned in Section 1.3, these considerations have determined our choice to work in the setting of path categories. As said, our path categories are related to categories of fibrant objects *à la* Brown, or fibration categories as they have been called by other authors. Structures similiar to fibration categories or their duals have been studied by Baues [6] and Waldhausen [47], for homotopy-theoretic purposes. For a survey and many basic properties, we refer to [41].

More recently, several authors have also considered such axiomatisations in order to investigate the relation between homotopy theory and type theory. For instance, Joyal (unpublished) and Shulman [44] have considered axiomatisations in terms of a weak factorisation system for fibrations and acyclic cofibrations, a set-up which is somewhat stronger than ours. In our setting we do not have such a weak factorisation system, and the lifting properties that we derive in our path categories yield diagonals that make lower triangles strictly commutative, while upper triangles need only commute up to (fibrewise) homotopy. Our reasons for deviating from Joyal and Shulman are that in the setting of the weak rules for the identity types such weak liftings seem to be the best possible; in addition, the category $\mathrm{Ex}(\mathcal{C})$ only seems to be a path object category in our sense, even when $\mathcal{C}$ is a type-theoretic fibration category in the sense of Shulman.

In this paper we have not entered into any ∞-categorical aspects. For readers interested in the use of fibration categories in ∞-category theory and its relation to type theory, we refer to the work of Kapulkin and Szumiło [32, 34, 45].

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2. Path categories

2.1. **Axioms.** Throughout this paper we work with path categories, a modification of Brown's notion of a category of fibrant objects [13]. We will start by recalling Brown's definition.

The basic structure is that of a category $\mathcal{C}$ together with two classes of maps in $\mathcal{C}$ called the *weak equivalences* and the *fibrations*, respectively. Morphisms which belong to both classes of maps will be called *acylic fibrations*. A *path object* on an object B is a factorisation of the diagonal $\Delta\colon B \to B \times B$ as a weak equivalence $r\colon B \to PB$ followed by a fibration $(s,t)\colon PB \to B \times B$.

Definition 2.1. [13] The category $\mathcal{C}$ is called a *category of fibrant objects* if the following axioms are satisfied:

(1) Fibrations are closed under composition.
(2) The pullback of a fibration along any other map exists and is again a fibration.
(3) The pullback of an acylic fibration along any other map is again an acyclic fibration.
(4′) Weak equivalence satisfy 2-out-of-3: if $gf = h$ and two of f, g, h are weak equivalences then so is the third.
(5′) Isomorphisms are acyclic fibrations.
(6) For any object B there is a path object PB (not necessarily functorial in B).
(7) $\mathcal{C}$ has a terminal object 1 and every map $X \to 1$ to the terminal object is a fibration.

We make two modifications to Brown's definition, the first of which is relatively minor. Instead of the more familiar 2-out-of-3 property we demand that the weak equivalences satisfy 2-out-of-6:

(4) Weak equivalence satisfy 2-out-of-6: if $f\colon A \to B$, $g\colon B \to C$, $h\colon C \to D$ are three composable maps and both gf and hg are weak equivalences, then so are f, g, h and hgf.

It is not hard to see that this implies 2-out-of-3. We have decided to stick with the stronger property, as it is something which is both useful and true in all the examples we are interested in. (See also Remark 2.17 below.)

A more substantial change is that we will add an axiom saying that every acyclic fibration has a section (this is sometimes expressed by saying that "every object is cofibrant"). To be precise, we will modify (5′) to:

(5) Isomorphisms are acyclic fibrations and every acyclic fibration has a section.

As discussed in the introduction, one reason we have made this change is that it is satisfied in the syntactic category associated to type theory [3] and in many situations occurring in homotopy theory. In fact, axiom (5) will be used throughout this paper and in this section we will investigate, somewhat systematically, the consequences of this axiom.

To summarise:

Definition 2.2. The category $\mathcal{C}$ will be called a *category with path objects*, or a *path category* for short, if the following axioms are satisfied:

(1) Fibrations are closed under composition.
(2) The pullback of a fibration along any other map exists and is again a fibration.
(3) The pullback of an acylic fibration along any other map is again an acyclic fibration.

(4) Weak equivalence satisfy 2-out-of-6: if $f: A \to B$, $g: B \to C$, $h: C \to D$ are three composable maps and both gf and hg are weak equivalences, then so are f, g, h and hgf.
(5) Isomorphisms are acyclic fibrations and every acyclic fibration has a section.
(6) For any object B there is a path object PB (not necessarily functorial in B).
(7) $\mathcal{C}$ has a terminal object 1 and every map $X \to 1$ to the terminal object is a fibration.

We have chosen the name path category because its homotopy category is completely determined by the path objects (as every object is cofibrant).

Examples are:

(1) The syntactic category associated to type theory [3]. In fact, to prove that the syntactic category is an example, it suffices to assume that the computation rule for the identity type holds only in a propositional form (see [8]).
(2) Let $\mathcal{M}$ be a Quillen model category. If every object is cofibrant in $\mathcal{M}$, then the full subcategory of fibrant objects in $\mathcal{M}$ is a path category in our sense. More generally, if any object over a cofibrant object is also cofibrant, then the full subcategory of fibrant-cofibrant objects in $\mathcal{M}$ is a path category.
(3) In addition, there is the following trivial example: if $\mathcal{C}$ is a category with finite limits, it can be considered as a path category in which every morphism is a fibration and only the isomorphisms are weak equivalences. By considering this trivial situation, it can be seen that our theory of the homotopy exact completion in the next section generalises the classical theory of exact completions of categories with finite limits.

2.2. **Basic properties.** We start off by making some basic observations about path categories, all of which are due to Brown in the context of categories of fibrant objects ([13]; see also [41]). First of all, note that the underlying category $\mathcal{C}$ has finite products and all projection maps are fibrations. From this it follows that if $(f, g): P \to X \times X$ is a fibration, then so are f and g.

Proposition 2.3. *In a path category any map $f: Y \to X$ factors as $f = p_f w_f$ where p_f is a fibration and w_f is a section of an acylic fibration (and hence a weak equivalence).*

Proof. This is proved on page 421 of [13]. Since the factorisation will be important in what follows, we include the details here. First observe that if PX is a path object for X with weak equivalence $r: X \to PX$ and fibration $(s, t): PX \to X \times X$, then it follows from 2-out-of-3 for weak equivalences and $sr = tr = 1$ that both $s, t: PX \to X$ are acyclic fibrations. So for any map $f: Y \to X$ the following pullback

$$\begin{tikzcd} P_f \arrow[r, "p_2"] \arrow[d, "p_1"'] & PX \arrow[d, "s"] \\ Y \arrow[r, "f"'] & X, \end{tikzcd}$$

exists with p_1 being an acyclic fibration. We set $w_f := (1, rf): Y \to P_f$ and $p_f := tp_2: P_f \to X$. Then $p_f w_f = f$ and w_f is a section of p_1. Moreover, the following square

$$\begin{tikzcd} P_f \arrow[r, "p_2"] \arrow[d, "(p_1, p_f)"'] & PX \arrow[d, "(s,t)"] \\ Y \times X \arrow[r, "f \times 1"'] & X \times X. \end{tikzcd}$$

is a pullback, so (p_1, p_f) is fibration, which implies that p_f is a fibration as well. □

Corollary 2.4. *Any weak equivalence $f: Y \to X$ factors as $f = p_f w_f$ where p_f is an acylic fibration and w_f is a section of an acyclic fibration.*

Definition 2.5. If $\mathcal{C}$ is a path category and A is any object in $\mathcal{C}$ we can define a new path category $\mathcal{C}(A)$, as follows: its underlying category is the full subcategory of $\mathcal{C}/A$ whose objects are the fibrations with codomain A. This means that its objects are fibrations $X \to A$, while a morphism from $q: Y \to A$ to $p: X \to A$ is a map $f: Y \to X$ in $\mathcal{C}$ such that $pf = q$; such a map f is a fibration or a weak equivalence in $\mathcal{C}(A)$ precisely when it is a fibration or a weak equivalence in $\mathcal{C}$.

Clearly, $\mathcal{C}(1) \cong \mathcal{C}$. Observe that for any $f: B \to A$ there is a pullback functor $f^*: \mathcal{C}(A) \to \mathcal{C}(B)$, since pullbacks of fibrations always exist and are again fibrations.

Proposition 2.6. *For any morphism $f: B \to A$ the functor $f^*: \mathcal{C}(A) \to \mathcal{C}(B)$ preserves both fibrations and weak equivalences.*

Proof. This is proved on page 428 of [13] and the proof method is often called Brown's Lemma. The idea is that Axiom 3 for path categories tells us that f^* preserves acyclic fibrations. But then it follows from the previous corollary and 2-out-of-3 for weak equivalences that f^* preserves weak equivalences as well. □

This proposition can be used to derive:

Proposition 2.7. *The pullback of a weak equivalence $w: A' \to A$ along a fibration $p: B \to A$ is again a weak equivalence.*

Proof. See pages 428 and 429 of [13]. □

2.3. **Homotopy.** In any path category we can define an equivalence relation on the hom-sets: the homotopy relation.

Definition 2.8. Two parallel arrows $f, g: Y \to X$ are *homotopic*, if there is a path object PX for X with fibration $(s, t): PX \to X \times X$ and a map $h: Y \to PX$ (the *homotopy*) such that $f = sh$ and $g = th$. In this case, we write $f \simeq g$, or $h: f \simeq g$ if we wish to stress the homotopy h.

At present it is not clear that this definition is independent of the choice of path object PX, or that it defines an equivalence relation. In order to prove this, we use the following lemma, which is a consequence of (and indeed equivalent to) the axiom that every acyclic fibration has a section.

Lemma 2.9. *Suppose we are given a commutative square*

$$\begin{tikzcd} D \arrow[r, "g"] \arrow[d, "w"'] & C \arrow[d, "p"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

in which w is a weak equivalence and p is a fibration. Then there is a map $l\colon B \to C$ such that $pl = f$ (for convenience, we will call such a map a lower filler*).*

Proof. Let $k\colon D \to B \times_A C$ be the map to the pullback with $p_1 k = w$ and $p_2 k = g$, and factor k as $k = qi$ where i is a weak equivalence and q is a fibration. Then $p_1 q$ is an acyclic fibration and hence has a section a. So if we put $l := p_2 qa$, then $pl = pp_2 qa = fp_1 qa = f$, as desired. □

Just in passing we should note that a statement much stronger than Lemma 2.9 is true, but that in order to state and prove it we need to develop a bit more theory (see Theorem 2.38 below).

Corollary 2.10. *If PX is a path object for X and PY is a path object for Y and $f\colon X \to Y$ is any morphism, then there is a map $Pf\colon PX \to PY$ such that*

$$\begin{tikzcd} PX \arrow[r, "Pf"] \arrow[d, "{(s,t)}"'] & PY \arrow[d, "{(s,t)}"] \\ X \times X \arrow[r, "f \times f"'] & Y \times Y \end{tikzcd}$$

commutes. In particular, if PX and $P'X$ are two path objects for X then there is a map $f\colon PX \to P'X$ which commutes with the source and target maps of PX and $P'X$.

Proof. We obtained the desired maps as a lower filler in:

$$\begin{tikzcd} X \arrow[r, "rf"] \arrow[d, "r"'] & PY \arrow[d, "{(s,t)}"] \\ PX \arrow[r, "{(fs,ft)}"'] & Y \times Y. \end{tikzcd}$$

□

The second statement in the previous corollary implies that if two parallel maps $f, g\colon X \to Y$ are homotopic relative to one path object PY on Y, then they are homotopic with respect to any path object on Y; so in the definition of the homotopy relation nothing depends on the choice of the path object.

In order to show that the homotopy relation is an equivalence relation, and indeed a congruence, we introduce the following definition, which will also prove useful later.

Definition 2.11. A fibration $p = (p_1, p_2): R \to X \times X$ is a *homotopy equivalence relation*, if the following three conditions are satisfied:

(1) There is a map $\rho: X \to R$ such that $p\rho = \Delta_X$.
(2) There is a map $\sigma: R \to R$ such that $p_1\sigma = p_2$ and $p_2\sigma = p_1$.
(3) For the pullback

$$\begin{tikzcd} R \times_X R \arrow[r, "q_2"] \arrow[d, "q_1"'] & R \arrow[d, "p_1"] \\ R \arrow[r, "p_2"'] & X \end{tikzcd}$$

there is a map $\tau: R \times_X R \to R$ such that $p_1 q_1 = p_1\tau$ and $p_2 q_2 = p_2\tau$.

Proposition 2.12. *If PX is a path object with fibration $p = (s,t): PX \to X \times X$ and weak equivalence $r: X \to PX$, then p is a homotopy equivalence relation.*

Proof. (1) We put $\rho = r$.
(2) The map σ is obtained as a lower filler in:

$$\begin{tikzcd} X \arrow[r, "r"] \arrow[d, "r"'] & PX \arrow[d, "{(s,t)}"] \\ PX \arrow[r, "{(t,s)}"'] & X \times X. \end{tikzcd}$$

(3) Let α be the unique map filling

$$\begin{tikzcd} X \arrow[dr, dotted, "\alpha"] \arrow[drr, bend left, "r"] \arrow[ddr, bend right, "r"'] & & \\ & PX \times_X PX \arrow[r, "q_2"'] \arrow[d, "q_1"] & PX \arrow[d, "s"] \\ & PX \arrow[r, "t"'] & X. \end{tikzcd}$$

The maps s and t are acyclic fibrations, and therefore their pullbacks q_1 and q_2 are acyclic fibrations as well; in particular, they are weak equivalences. Since r is also a weak equivalence, the map α is a weak equivalence by 2-out-of-3. Therefore a suitable τ can be obtained as the lower filler of

$$\begin{tikzcd} X \arrow[r, "r"] \arrow[d, "\alpha"'] & PX \arrow[d, "{(s,t)}"] \\ PX \times_X PX \arrow[r, "{(sq_1, tq_2)}"'] & X \times X. \end{tikzcd}$$

□

In a way PX is the least homotopy equivalence relation on X.

Lemma 2.13. *If $p = (p_1, p_2): R \to X \times X$ is a homotopy equivalence relation on X, then there is a map $h: PX \to R$ such that $p_1 h = s$ and $p_2 h = t$. More generally, any map $f: Y \to X$ gives rise to a morphism $h: PY \to R$ such that $p_1 h = fs$ and $p_2 h = ft$.*

Proof. We obtain h as a lower filler of

$$\begin{tikzcd} Y \arrow[r, "\rho f"] \arrow[d, "r"'] & R \arrow[d, "p"] \\ PY \arrow[r, "{(fs, ft)}"'] & X \times X. \end{tikzcd}$$

□

Theorem 2.14. *The homotopy relation $\simeq$ defines an congruence relation on $\mathcal{C}$.*

Proof. We have already seen that if P is a path object on X and there is a suitable homotopy connecting f and g relative to P, then there is such a homotopy relative to any path object Q for X. Therefore the statement that $\simeq$ defines an equivalence relation on each hom-set follows from Proposition 2.12.

For showing that $\simeq$ is a congruence (i.e., that $f \simeq g$ and $k \simeq l$ imply $kf \simeq gl$), it suffices to prove that $f \simeq g$ implies $fk \simeq gk$ and $lf \simeq lg$; the former, however, is immediate, while the latter follows from Corollary 2.10. □

The previous theorem means that we can quotient $\mathcal{C}$ by identifying homotopic maps and obtain a new category. The result is the *homotopy category* of $\mathcal{C}$ and will be denoted by $\mathrm{Ho}(\mathcal{C})$.

Definition 2.15. A map $f: X \to Y$ is a *homotopy equivalence* if it becomes an isomorphism in $\mathrm{Ho}(\mathcal{C})$ or, in other words, if there is a map $g: Y \to X$ (a *homotopy inverse*) such that the composites fg and gf are homotopic to the identities on Y and X, respectively. If such a homotopy equivalence $f: X \to Y$ exists, we say that X and Y are *homotopy equivalent*.

Theorem 2.16. *Weak equivalences and homotopy equivalences coincide.*

Proof. First note that any section of a weak equivalence $f: Y \to X$ is a homotopy inverse. The reason is that if $g: X \to Y$ is a section with $fg = 1$, then g is a weak equivalence as well. Therefore we can find a homotopy $h\colon gf \simeq 1$ as a lower filler of

$$\begin{tikzcd} X \arrow[r, "rg"] \arrow[d, "g"'] & PY \arrow[d, "{(s,t)}"] \\ Y \arrow[r, "{(gf,1)}"'] & Y \times Y. \end{tikzcd}$$

Since every acyclic fibration has a section, it now follows that acyclic fibrations are homotopy equivalences. But then Corollary 2.4 implies that every weak equivalence is a homotopy equivalence.

For the converse direction we also need to make a preliminary observation: if $f, g\colon A \to B$ are homotopic and f is a weak equivalence, then so is g. To see this suppose that f is a weak equivalence and there is a map $h\colon A \to PB$ such that $sh = f$ and $th = g$. Since s and t are weak equivalences, it follows from the first equality that h is a weak equivalence and hence from the second equality that g is a weak equivalence.

Now suppose $f\colon A \to B$ is a homotopy equivalence with homotopy inverse $g\colon B \to A$. Then in

$$A \xrightarrow{f} B \xrightarrow{g} A \xrightarrow{f} B$$

both gf and fg are homotopic to the identity. Therefore both gf and fg are weak equivalences by the previous observation; but then we can use 2-out-of-6 to deduce that f is a weak equivalence. □

Remark 2.17. Other authors who work in categorical frameworks similar to ours often call categories of fibrant objects "saturated" if they have the property that every homotopy equivalence is a weak equivalence. In our set-up this is derivable, so our path categories are always saturated in their sense. Note that in order to prove this we have made our first genuine use of the 2-out-of-6 axiom as opposed to the weaker 2-out-of-3 axiom: this is no coincidence, as relative to the 2-out-of-3 axiom the statement that every homotopy equivalence is a weak equivalence is equivalent to the 2-out-of-6 property (this observation is due to Cisinski; see [41, p. 82–4]). Using the theory that we will develop in the next subsection it will also not be hard to show that if a path category only satisfies 2-out-of-3 one can obtain a "saturated" path category from it by taking the same underlying category and the same fibrations, while enlarging the class of weak equivalences to include all homotopy equivalences. This means that restricting to saturated path categories is no real loss of generality; moreover, all the examples we are interested in are saturated, including the syntactic category associated to type theory. For these reasons we have decided to restrict our attention to path categories that are saturated.

Corollary 2.18. *Weak equivalences are closed under retracts.*

Corollary 2.19. *The quotient functor $\gamma\colon \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ is the universal solution to inverting the weak equivalences.*

Proof. We have just seen that this functor inverts the weak equivalences; conversely, any functor $\delta: \mathcal{C} \to \mathcal{D}$ which sends weak equivalences to isomorphisms must identify homotopic maps, for if PX is a path object with $r: X \to PX$ and $(s,t): PX \to X \times X$, then $\delta(s) = \delta(r)^{-1} = \delta(t)$. □

2.4. **Homotopy pullbacks.** Path categories need not have pullbacks; what they do have are homotopy pullbacks, a notion that we will now recall (see, for example, [41]).

Given two arrows $f: A \to I$ and $g: B \to I$ one can take the pullback

$$\begin{tikzcd} A \times_I^h B \arrow[r] \arrow[d, "(p_1,p_2)"'] & PI \arrow[d, "(s,t)"] \\ A \times B \arrow[r, "f \times g"'] & I \times I. \end{tikzcd}$$

This object $A \times_I^h B$ fits in a square

$$\begin{tikzcd} A \times_I^h B \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & I, \end{tikzcd}$$

which commutes up to homotopy.

Definition 2.20. Suppose

$$\begin{tikzcd} C \arrow[r, "q_2"] \arrow[d, "q_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & I \end{tikzcd}$$

is a square which commutes up to homotopy. If there is a homotopy equivalence $h: C \to A \times_I^h B$ such that $q_i = p_i h$ for $i \in \{1,2\}$, then the square above is called a *homotopy pullback square* and C is a *homotopy pullback* of f and g.

Remark 2.21. Clearly, a homotopy pullback is unique up to homotopy, but there are different ways of constructing it. For example, the homotopy pullback can be obtained by taking the fibrant replacement of either f or g (or both) and then taking the actual pullback. Using this one easily checks that the following hold:

Lemma 2.22. (i) *If*

$$\begin{tikzcd} D \arrow[r] \arrow[d, "g"'] & B \arrow[d, "f"] \\ C \arrow[r] & A \end{tikzcd}$$

is a homotopy pullback and f is a homotopy equivalence, then so is g.

(ii) *If*

$$\begin{tikzcd} F \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & B \arrow[d] \\ E \arrow[r] & C \arrow[r] & A \end{tikzcd}$$

commutes and the square on the right is a homotopy pullback, then the square on the left is a homotopy pullback if and only if the outer rectangle is a homotopy pullback.

2.5. **Connections and transport.** One key fact about fibrations in path categories is that they have a path lifting property and allow for what the type-theorists call transport. The aim of this subsection is to show these facts, starting with the latter.

To formulate the notion of transport we need some additional terminology.

Definition 2.23. Suppose $f, g: Y \to X$ are parallel arrows and X comes with a fibration $p: X \to I$. If $pf = pg$, then we can compare $f: Y \to X$ and $g: Y \to X$ with respect to the path object $(s,t): P_I(X) \to X \times_I X$ of X in $\mathcal{C}(I)$: one calls f and g *fibrewise homotopic* if there is a map $h: Y \to P_I(X)$ such that $sh = f$ and $th = g$, and write $f \simeq_I g$, or $h: f \simeq_I g$, if we wish to stress the homotopy. (If $pf = pg$ is a fibration, then this is just the homotopy relation in $\mathcal{C}(I)$; but, and this will be important later, this definition makes sense even when $pf = pg$ is not a fibration.)

Recall from Proposition 2.3 that any map $f: Y \to X$ can be factored as a weak equivalence $w_f: Y \to P_f$ followed by a fibration $p_f: P_f \to X$, where $P_f = Y \times_X PX$ is the pullback

$$\begin{tikzcd} P_f \arrow[r, "p_2"] \arrow[d, "p_1"'] & PX \arrow[d, "s"] \\ Y \arrow[r, "f"'] & X, \end{tikzcd}$$

while $w_f = (1_Y, rf)$ and $p_f = tp_2$. If f is a fibration, then we can regard both Y and P_f as objects in $\mathcal{C}(X)$ via f and p_f, respectively, and w_f as a morphism between them in $\mathcal{C}(X)$.

Definition 2.24. Let $f: Y \to X$ be a fibration. A *transport structure* on f is a morphism $\Gamma: P_f \to Y$ such that $f\Gamma = p_f$ and $\Gamma w_f \simeq_X 1_Y$.

The idea behind transport is this: given an element $y \in Y$ and a path $\alpha: x \to x'$ in X with $f(y) = x$, one can transport y along α to obtain an element y' with $f(y') = x'$; in addition, we demand that in case α is the identity path on x, then the element y' should be connected to y by a path which lies entirely in the fibre over x. In order to show that every fibration carries a transport structure, we need to strengthen Lemma 2.9 to:

Lemma 2.25. *Suppose we are given a commutative square*

$$\begin{tikzcd} D \arrow[r, "g"] \arrow[d, "w"'] & C \arrow[d, "p"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

in which w is a weak equivalence and p is a fibration. Then there is a map $l\colon B \to C$, unique up to homotopy, such that $pl = f$ and $lw \simeq g$.

Proof. We repeat the earlier proof: let $k\colon D \to B \times_A C$ be the map to the pullback with $p_1 k = w$ and $p_2 k = g$, and factor k as $k = qi$ where i is a weak equivalence and q is a fibration. Then $p_1 q$ is an acyclic fibration, so has a section a. So if we put $l := p_2 qa$, then $pl = pp_2 qa = fp_1 qa = f$. But (the proof of) Theorem 2.16 implies that $ap_1 q \simeq 1$, so that

$$lw = p_2 qaw = p_2 qap_1 k = p_2 qap_1 qi \simeq p_2 qi = p_2 k = g.$$

To see that l is unique up to homotopy, note that, more generally, the fact that weak equivalences are homotopy equivalences implies that if $lw \simeq l'w$ and w is a weak equivalence, then $l \simeq l'$. □

Theorem 2.26. *Every fibration $f\colon Y \to X$ carries a transport structure. Moreover, transport structures are unique up to fibrewise homotopy over X.*

Proof. If $f\colon Y \to X$ is a fibration then the commuting square

$$\begin{tikzcd} Y \arrow[r, "1"] \arrow[d, "w_f"'] & Y \arrow[d, "f"] \\ P_f \arrow[r, "p_f"'] & X \end{tikzcd}$$

does not only live in $\mathcal{C}$, but also in $\mathcal{C}(X)$. Applying the previous lemma to this square in $\mathcal{C}(X)$ gives one the desired transport structure. □

Definition 2.27. Let $f\colon Y \to X$ be a fibration. A *connection* on f consists of a path object PY for Y, a fibration $Pf\colon PY \to PX$ commuting with the r, s and t-maps on PX and PY, together with a morphism $\nabla\colon P_f \to PY$ such that $Pf \circ \nabla = p_2$ and $s\nabla = p_1$.

The idea behind a connection is this: given an element $y \in Y$ and a path $\alpha\colon x \to x'$ in X with $f(y) = x$, the connection finds a path $\beta\colon y \to y'$ with $f(\beta) = \alpha$.

Theorem 2.28. *Let $f\colon Y \to X$ be a fibration in a path category $\mathcal{C}$ and assume that PX is a path object on X and $\Gamma\colon Y \times_X PX \to Y$ is a transport structure on f. Then we can construct a path object PY on Y and a fibration $Pf\colon PY \to PX$ with the following properties:*

(i) *Pf commutes with the r, s and t-maps on PX and PY.*

(ii) *There exists a connection structure* $\nabla: P_f \to PY$ *with* $t\nabla = \Gamma$.

In particular, every fibration $f: Y \to X$ *carries a connection structure.*

Proof. The proof will make essential use of the path object $P_X(Y)$ of Y in $\mathcal{C}(X)$. We will write $\rho: X \to P_X(Y)$ and $(\sigma, \tau): P_X(Y) \to Y \times_X Y$ for the factorisation of $Y \to Y \times_X Y$ as a weak equivalence followed by a fibration.

The idea is to construct PY as P_Γ in $\mathcal{C}(X)$, that is, as the following pullback:

$$\begin{tikzcd} PY \arrow[r, "q_2"] \arrow[d, "q_1"'] & P_X(Y) \arrow[d, "\sigma"] \\ P_f \arrow[r, "\Gamma"'] & Y. \end{tikzcd}$$

Since $\Gamma: P_f \to Y$ is a transport structure, there is a homotopy $h: \Gamma w_f \simeq 1$ in $\mathcal{C}(X)$. This allows us to factor the diagonal $Y \to Y \times Y$ as $(w_f, h): Y \to PY$ followed by $(p_1 q_1, \tau q_2): PY \to Y \times Y$, so to prove that this defines a path object on Y we need to show that the first map is a weak equivalence and the second a fibration. For the former, note that $q_1(w_f, h) = w_f$, where w_f is a weak equivalence and q_1 is an acyclic fibration, as it is the pullback of σ. For the latter, note that $PY = P_\Gamma$ in $\mathcal{C}(X)$ can also be constructed as the pullback

$$\begin{tikzcd} PY \arrow[r, "q_2"] \arrow[d, "(q_1, \tau q_2)"'] & P_X(Y) \arrow[d, "(\sigma, \tau)"] \\ P_f \times_X Y \arrow[r, "\Gamma \times_X Y"'] & Y \times_X Y, \end{tikzcd}$$

as in the proof of Proposition 2.3, so that $(q_1, \tau q_2)$ is a fibration. Moreover,

$$p_1 \times 1: P_f \times_X Y \to Y \times Y$$

is a fibration as well, as it arises in the following pullback:

$$\begin{tikzcd} P_f \times_X Y \arrow[r, "p_2 \pi_1"] \arrow[d, "p_1 \times 1"'] & PX \arrow[d, "(s,t)"] \\ Y \times Y \arrow[r, "f \times f"'] & X \times X. \end{tikzcd}$$

So $(p_1 \times 1)(q_1, \tau q_2) = (p_1 q_1, \tau q_2)$ is a fibration, as desired. In addition, we have a map $Pf := p_2 q_1: PY \to PX$, which is also fibration. We now check points (i) and (ii).

(i) We have to show that Pf commutes with the maps r, s, t on PY and PX.

(1) The r-map on PY is (w_f, h), and we have

$$Pf \circ r_Y = p_2 q_1 (w_f, h) = p_2 w_f = r_X \circ f.$$

(2) The s-map on PY is p_1q_1, and we have
$$s_X \circ Pf = sp_2q_1 = fp_1q_1 = f \circ s_Y.$$
(3) The t-map on PY is τq_2, and we have
$$t_X \circ Pf = tp_2q_1 = p_fq_1 = f\Gamma q_1 = f\sigma q_2 = f\tau q_2 = f \circ t_Y,$$
where we have used that $f\sigma = f\tau$ is the map exhibiting $P_X(Y)$ as an object of $\mathcal{C}(X)$.

(ii): To construct the connection, we simply put $\nabla := (1, \rho\Gamma)$. Then
$$s_Y \nabla = p_1q_1(1, \rho\Gamma) = p_1$$
and
$$Pf \circ \nabla = p_2q_1(1, \rho\Gamma) = p_2,$$
showing that ∇ is indeed a connection. In addition, one has
$$t_Y \nabla = \tau q_2(1, \rho\Gamma) = \tau\rho\Gamma = \Gamma,$$
as desired. □

Remark 2.29. We have just shown that if PX is any path object for X and $f: Y \to X$ is any fibration, one can find a suitable path object PY for Y and a connection map $\nabla: P_f \to PY$ for that particular path object. From this it does not follow that if $P'Y$ is another path object for Y then one can find a connection $\nabla': P_f \to P'Y$ as well: in that sense the notion of connection is not invariant. In view of Corollary 2.10 we will have a map $\nabla': P_f \to P'Y$ with $s\nabla' = p_1$ and $ft\nabla' = tp_2$. We will occassionally meet such *weak connections* as well, where the main point about such weak connections is:

Corollary 2.30. *Let $f: Y \to X$ be a fibration and PY be an arbitrary path object for Y. Then there is a map $\nabla: P_f \to PY$ such that $s\nabla = p_1$ and $tf\nabla = tp_2$.*

We conclude this subsection by noting the following consequence of Theorem 2.28, which we will repeatedly use in what follows.

Proposition 2.31. *If a triangle*

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Z \arrow[ur, "f"] \arrow[r, "g"'] & X \end{tikzcd}$$

with a fibration p on the right commutes up to a homotopy $h: pf \simeq g$, then we can also find a map $f': Z \to Y$, homotopic to f, such that for f' the triangle commutes strictly, that is, $pf' = g$.

Proof. Let $h: pf \simeq g$ be a homotopy and choose a path object PY for Y, a fibration $Pp: PY \to PX$ and a connection structure $\nabla: P_p \to PY$ as in Theorem 2.28. Put $h' = \nabla(f, h)$ and $f' := th'$. One may now calculate that
$$pf' = pt\nabla(f, h) = t \circ Pp \circ \nabla \circ (f, h) = tp_2(f, h) = th = g,$$

so the triangle commutes strictly for f'. Moreover,

$$sh' = s\nabla(f,h) = p_1(f,h) = f,$$

so h' is a homotopy between f and f'. $\square$

2.6. **Lifting properties.** At various points (Lemma 2.9 and Lemma 2.25) we have seen statements to the effect that weak equivalences have a weak lifting property with respect to the fibrations. Lemma 2.9 said that if

$$\begin{tikzcd} A \arrow[r, "m"] \arrow[d, "w"'] & C \arrow[d, "p"] \\ B \arrow[r, "n"'] & D \end{tikzcd}$$

is a commutative square with a weak equivalence w on the left and a fibration p on the right, there is a map $l\colon B \to C$ such that $pl = n$. In Lemma 2.25 we saw that l could be chosen such that $lw \simeq m$. The aim of this subsection is to show that this can be strengthened even further: we can find a map l such that $pl = n$ and $lw \simeq_D m$, where $\simeq_D$ is meant to indicate that lw and m are fibrewise homotopic over D via $p\colon C \to D$. This seems to be the strongest lifting property which could reasonably be expected in our setting.

The proof that this stronger lifting property holds proceeds in several steps. It will be convenient to temporarily call the weak equivalences w with the desired property *good*. So a weak equivalence w will be called good if in any square as the one above with a fibration p on the right, there is a map $l\colon B \to C$ such that $pl = n$ and $lw \simeq_D m$.

Lemma 2.32. (i) *Good weak equivalences are closed under composition.*
(ii) *In order to show that a weak equivalence is good we only need to consider the case where the map n along the bottom of the square is the identity. In other words, a weak equivalence $w\colon A \to B$ is good whenever for any commuting triangle*

$$\begin{tikzcd} & C \arrow[d, "p"] \\ A \arrow[ur, "k"] \arrow[r, "w"'] & B \end{tikzcd}$$

in which p is a fibration, the map p has a section j such that $jw \simeq_B k$.

Proof. (i): Suppose w_1 and w_2 are good weak equivalences and there is a commuting square

$$\begin{tikzcd}
Z \arrow[r, "m"] \arrow[d, "w_1"'] & C \arrow[dd, "p"] \\
Y \arrow[d, "w_2"'] & \\
X \arrow[r, "n"'] & D
\end{tikzcd}$$

with a fibration p on the right. From the fact that w_1 is good we get a map $t_1 \colon Y \to C$ such that $pt_1 = nw_2$ and $tw_1 \simeq_D m$; then, from the fact that w_2 is good we get a map $t \colon X \to C$ such that $pt = n$ and $tw_2 \simeq_D t_1$. Then $tw_2w_1 \simeq_D t_1w_1 \simeq_D m$, so t is as desired.

(ii): Suppose that

$$\begin{tikzcd}
A \arrow[r, "m"] \arrow[d, "w"'] & C \arrow[d, "p"] \\
B \arrow[r, "n"'] & D
\end{tikzcd}$$

is a commuting square with a fibration p on the right. Pulling back p along n we obtain a diagram of the form

$$\begin{tikzcd}
 & E \arrow[r, "n'"] \arrow[d, "p'"] & C \arrow[d, "p"] \\
A \arrow[ur, "{(w,m)}"] \arrow[r, "w"'] & B \arrow[r, "n"'] & D
\end{tikzcd}$$

in which $E = B \times_D C$ and p' is a fibration. Note that Proposition 2.6 implies that we can obtain a path object for E in $\mathcal{C}(B)$ by pulling back the path object for C in $\mathcal{C}(D)$ along $n \colon B \to D$, as in

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & D \arrow[d] \\
P_B(E) \arrow[r] \arrow[d] & P_D(C) \arrow[d] \\
E \times_B E \arrow[r] \arrow[d] & C \times_D C \arrow[d] \\
B \arrow[r] & D,
\end{tikzcd}$$

with all squares being pullbacks.

Now suppose that w is a weak equivalence with property formulated in the lemma. This means that p' has a section j such that $jw \simeq_B (w, m)$, as witnessed by some homotopy $A \to P_B(E)$. Composing this homotopy with the map $P_B(E) \to P_D(C)$ above we obtain a homotopy witnessing that $n'jw \simeq_D n'(w,m) = m$. So putting $l := n'j$, we have $pl = pn'j = np'j = n$ and $lw = n'jw \simeq_D m$, as desired. □

Our strategy for showing that any weak equivalence is good is to use the factorisation of any weak equivalence as a weak equivalence of the form $w_f: Y \to P_f$ followed by an acyclic fibration $p_f: P_f \to Y$. So once we show that any weak equivalence of the form $w_f: Y \to P_f$ is good and any acyclic fibration is good, we are done in view of part (i) of the previous lemma. We do the latter thing first.

Proposition 2.33. *A fibration $f: B \to A$ is acyclic precisely when it has a section $g: A \to B$ with $gf \simeq_A 1_B$.*

Proof. If a fibration $f: B \to A$ has a section $g: A \to B$ with $gf \simeq_A 1_B$, then g is a homotopy inverse. So f is a weak equivalence by Theorem 2.16.

Conversely, if $f: B \to A$ is an acyclic fibration, then it has a section $g: A \to B$. From 2-out-of-3 for weak equivalences and $fg = 1_A$ it follows that g is a weak equivalence. Therefore

$$\begin{tikzcd} A \arrow[r, "rg"] \arrow[d, "g"'] & P_A(B) \arrow[d, "{(s,t)}"] \\ B \arrow[r, "{(gf,1)}"'] & B \times_A B \end{tikzcd}$$

is a commuting square with a weak equivalence on the left and a fibration on the right. A lower filler for this diagram is a fibrewise homotopy showing that $gf \simeq_A 1_B$. □

Corollary 2.34. *Acyclic fibrations are good.*

Proof. We will use part (ii) of Lemma 2.32. So suppose we are given a commuting triangle of the form

$$\begin{tikzcd} & C \arrow[d, "p"] \\ A \arrow[ur, "k"] \arrow[r, "w"'] & B \end{tikzcd}$$

in which p is a fibration and w is an acyclic fibration. The previous proposition tells us that there is a map $a: B \to A$ such that $wa = 1_B$ and $aw \simeq_B 1_A$. But then $j := ka$ is a section of p with $jw = kaw \simeq_B k$. □

To show that weak equivalences of the form $w_f: Y \to P_f$ are good, it will be useful to introduce a bit of terminology.

Definition 2.35. A morphism $f: A \to B$ is a *strong deformation retract* if there are a map $g: B \to A$, a path object PB for B and a homotopy $h: B \to PB$ such that

$$gf = 1_A, \quad sh = fg, \quad th = 1, \quad \text{and} \quad hf = rf.$$

The reason is the following:

Lemma 2.36. *Strong deformation retracts are good weak equivalences.*

Proof. Let $f: A \to B$ be a strong deformation retract and g and h as in the definition. Strong deformation retracts are clearly homotopy equivalences, so they are weak equivalences as well. To show that they are also good, suppose that

$$\begin{tikzcd} & C \arrow[d, "q"] \\ A \arrow[ur, "k"] \arrow[r, "f"'] & B, \end{tikzcd}$$

is a commutative triangle in which q is a fibration. Then consider:

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "f"'] & C \arrow[r, "1"] \arrow[d, "w_q"'] & C \arrow[d, "q"] \\ B \arrow[r, "{(kg,h)}"'] & P_q \arrow[r, "p_q"'] \arrow[ur, dotted, "\Gamma_q"'] & B. \end{tikzcd}$$

The lefthand square commutes, as

$$(kg, h)f = (kgf, hf) = (k, rf) = (k, rqk) = (1, rq)k = w_q k.$$

Moreover, the arrows along the bottom compose to the identity on B, because $p_q(kg, h) = tp_2(kg, h) = th = 1$. This means that we can use the transport structure on q with $q\Gamma_q = p_q$ and $\Gamma_q w_q \simeq_B 1$ to define j as $\Gamma_q(kg, h)$. $\square$

So it remains to show:

Proposition 2.37. *For any morphism $f: Y \to X$ the weak equivalence $w_f: Y \to P_f$ is a strong deformation retract.*

Proof. The main difficulty is to find a suitable path object for P_f. What we will do is take the following pullback:

$$\begin{tikzcd} PX \times_X PY \times_X PX \arrow[r] \arrow[d, "{(\sigma,\tau)}"'] & PY \arrow[d, "{(s,t)}"] \\ P_f \times P_f \arrow[r, "p_1 \times p_1"'] & Y \times Y, \end{tikzcd}$$

where σ and τ intuitively take a triple $(\alpha: f(y) \to x, \gamma: y \to y', \alpha': f(y') \to x')$ and produce (y, α) and (y', α'), respectively. By construction (σ, τ) is a fibration. The reflexivity term $\rho: P_f \to PX \times_X PY \times_X PX$ is given by (p_2, rp_1, p_2); in other words, by sending $(y, \alpha: f(y) \to x)$ to $(\alpha, r(y), \alpha)$. We have $\sigma\rho = \tau\rho = 1$, so to show that ρ is a weak equivalence, it suffices to show this for σ; this map, however, is the pullback of the map on the left in

$$\begin{tikzcd}
PY \times_X PX \arrow[rr] \arrow[d] & & PX \arrow[d, "s"] \\
PY \arrow[r, "t"'] \arrow[d, "s"'] & Y \arrow[r, "f"'] & X \\
Y & &
\end{tikzcd}$$

along $p_1 : P_f \to Y$ and hence an acyclic fibration. So we have described a suitable candidate for PP_f.

We know that $p_1 w_f = 1$, so to prove that w_f is a strong deformation retract we need to find a homotopy $h : P_f \to PP_f$ such that $\sigma h = w_f p_1$, $\tau h = 1$ and $h w_f = \rho w_f$. We set

$$h = (rfp_1, rp_1, p_2).$$

Then we can compute:

$$\sigma h = \sigma(rfp_1, rp_1, p_2) = (srp_1, rfp_1) = (1, rf)p_1 = w_f p_1$$

and

$$\tau h = \tau(rfp_1, rp_1, p_2) = (trp_1, p_2) = (p_1, p_2) = 1.$$

In addition, the equations

$$hw_f = (rfp_1, rp_1, p_2)(1, rf) = (rf, r, rf)$$

and

$$\rho w_f = (p_2, rp_1, p_2)(1, rf) = (rf, r, rf)$$

hold, showing that $hw_f = \rho w_f$. □

We conclude that every weak equivalence is good, which we formulate more explicitly as follows.

Theorem 2.38. *If*

$$\begin{tikzcd}
A \arrow[r, "m"] \arrow[d, "f"'] & C \arrow[d, "p"] \\
B \arrow[r, "n"'] & D
\end{tikzcd}$$

is a commutative square with a weak equivalence f on the left and a fibration p on the right, then there is a filler $l : B \to C$ such that $n = pl$ and $lf \simeq_D m$. Moreover, such a filler is unique up to fibrewise homotopy over D.

Proof. Any weak equivalence $f : A \to B$ can be factored as $w_f : A \to P_f$ followed by an acyclic fibration $p_f : P_f \to B$. The former is good by Lemma 2.36 and Proposition 2.37, while the latter is good by Corollary 2.34; so f is good by part (i) of Lemma 2.32.

It remains to show uniqueness of l: but if both l and l' are as desired, then $lf \simeq_D l'f$, so there is a fibrewise homotopy $h: A \to P_D(C)$ such that

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & P_D(C) \arrow[d, "{(s,t)}"] \\ B \arrow[r, "{(l,l')}"'] & D \times_C D \end{tikzcd}$$

commutes. A lower filler for this square is a fibrewise homotopy showing that $l \simeq_D l'$. □

We are now able to prove that the factorisations of maps as weak equivalences followed by fibrations are unique up to homotopy equivalence.

Corollary 2.39. *If a map $k: Y \to X$ can be written as $k = pa = qb$ where $a: Y \to A$ and $b: Y \to B$ are weak equivalences and $p: A \to X$ and $q: B \to X$ are fibrations, then A and B are homotopy equivalent; moreover, the homotopy equivalence $f: A \to B$ and homotopy inverse $g: B \to A$ can be chosen such that $qf = p$, $pg = q$, $fa \simeq_X b$, $gb \simeq_X a$, $gf \simeq_X 1$ and $fg \simeq_X 1$.*

This means in particular that any two path objects on an object X are homotopy equivalent, where the homotopy equivalence and inverse can be chosen to behave nicely with respect to the r, s, t-maps, as in the statement of Corollary 2.39.

3. Homotopy exact completion

3.1. **Exactness.** This section will be devoted to developing a notion of exact completion for path categories, generalising the exact completion of a category with finite limits, as in [14, 15]. In fact, this homotopy exact completion, as we will call it, will coincide with the ordinary exact completion if we regard a category with finite limits as a path category in which every morphism is a fibration and only the isomorphisms are weak equivalences. Another feature of our account is that the category of setoids, studied in the type-theoretic literature (see, for example, [25, 5]), is the homotopy exact completion of the syntactic category of type theory.

Initially, we will study this homotopy exact completion directly; in later stages we will use that it can also be obtained as the homotopy category of an intermediate path category (see Theorem 3.14 below).

Definition 3.1. Given a path category $\mathcal{C}$ one may construct a new category as follows: its objects are the homotopy equivalence relations, as defined in Definition 2.11, while a morphism from $(X, \rho: R \to X \times X)$ to $(Y, \sigma: S \to Y \times Y)$ is an equivalence class of morphisms $f\colon X \to Y$ for which there is a map $\varphi\colon R \to S$ making

$$\begin{tikzcd} R \arrow[r, dotted, "\varphi"] \arrow[d, "\rho"'] & S \arrow[d, "\sigma"] \\ X \times X \arrow[r, "f \times f"'] & Y \times Y \end{tikzcd}$$

commute, where $f\colon X \to Y$ and $g\colon X \to Y$ are identified in case there is a map $H\colon X \to S$ such that

$$\begin{tikzcd} & S \arrow[d, "\sigma"] \\ X \arrow[ur, dotted, "H"] \arrow[r, "(f,g)"'] & Y \times Y \end{tikzcd}$$

commutes. This new category will be called the *homotopy exact completion* of $\mathcal{C}$ and will be denoted by $\mathrm{Hex}(\mathcal{C})$.

Remark 3.2. In what follows we will often denote objects of $\mathrm{Hex}(\mathcal{C})$ as pairs (X, R), leaving the fibration $\rho\colon R \to X \times X$ implicit. If it is made explicit, then ρ_1 and ρ_2 denote the first and second projection $R \to X$, respectively. Also, we will not distinguish notationally between a morphism $f\colon X \to Y$ in $\mathcal{C}$ which represents a morphism $(X, R) \to (Y, S)$ in $\mathrm{Hex}(\mathcal{C})$ and the morphism thus represented; we do not expect that these conventions will lead to confusion.

Remark 3.3. In this definition we have asked for the existence of fillers making the diagrams commute strictly; however, in view of Proposition 2.31, it suffices if there are dotted arrows making the diagrams commute up to homotopy.

Our first task is to outline a proof that $\mathrm{Hex}(\mathcal{C})$ is exact (the definition of an exact category can be found in section A.3 of the appendix). We do this through a sequence of lemmas.

It will be convenient to first introduce some notation. If $f\colon X \to Y$ is any map and $\sigma\colon S \to Y \times Y$ is a homotopy equivalence relation, then the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & S \arrow[d, "\sigma"] \\ X \times X \arrow[r, "f \times f"'] & Y \times Y \end{tikzcd}$$

is a homotopy equivalence relation on X, which will be denoted by $f^*\sigma\colon f^*S \to X \times X$. Moreover, if $R \to X \times X$ and $S \to X \times X$ are two homotopy equivalence relations, then $R \cap S$ is the homotopy equivalence relation obtained by taking the following pullback:

$$\begin{tikzcd} R \cap S \arrow[r] \arrow[d] & S \arrow[d] \\ R \arrow[r] & X \times X. \end{tikzcd}$$

Lemma 3.4. *The category* $\mathrm{Hex}(\mathcal{C})$ *has finite limits.*

Proof. Since $(1, P1)$ is the terminal object, it suffices to construct pullbacks. If $f\colon (Y, S) \to (X, R)$ and $g\colon (Z, T) \to (X, R)$ are two maps in $\mathrm{Hex}(\mathcal{C})$ its pullback (W, Q) can be constructed by letting W be the pullback

$$\begin{tikzcd} W \arrow[r] \arrow[d] & R \arrow[d] \\ Y \times Z \arrow[r, "f \times g"'] & X \times X \end{tikzcd}$$

and by letting Q be the homotopy equivalence relation on W obtained by pulling back the homotopy equivalence relation $\pi_1^* S \cap \pi_2^* T$ on $Y \times Z$ along the map $W \to Y \times Z$. $\square$

In the previous lemma we have used that if $f\colon X \to Y$ is any map and $\sigma\colon S \to Y \times Y$ is a homotopy equivalence relation, then the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & S \arrow[d, "\sigma"] \\ X \times X \arrow[r, "f \times f"'] & Y \times Y \end{tikzcd}$$

is a homotopy equivalence relation on X, which will be denoted by $f^*\sigma\colon f^*S \to X \times X$.

Lemma 3.5. *A morphism* $f\colon (X, \rho\colon R \to X \times X) \to (Y, \sigma\colon S \to Y \times Y)$ *is monic if and only if there is a morphism* $h\colon f^*S \to R$ *such that* $\rho h = f^*\sigma$*. Therefore every mono is isomorphic to one of the form* $f\colon (X, f^*S) \to (Y, S)$*.*

Proof. Use the description of pullbacks from the previous lemma and the fact that $m\colon A \to B$ is monic if and only if in the pullback

$$\begin{tikzcd} A \times_B A \arrow[r, "p_2"] \arrow[d, "p_1"'] & A \arrow[d, "m"] \\ A \arrow[r, "m"'] & B \end{tikzcd}$$

we have $p_1 = p_2$. $\square$

Lemma 3.6. *The category* $\mathrm{Hex}(\mathcal{C})$ *is regular and the covers are those maps*

$$f\colon (X, \rho\colon R \to X \times X) \to (Y, \sigma\colon S \to Y \times Y)$$

for which there are maps $g\colon Y \to X$ *and* $h\colon Y \to S$ *in* $\mathcal{C}$ *such that* $\sigma h = (1, fg)$*.*

Proof. Let us call maps f as in the statement of the proposition *nice epis*. Then the proposition follows as soon as we show:

(1) Every map factors as a nice epi followed by a mono.
(2) Nice epis are stable under isomorphism.
(3) Nice epis are covers.

(4) Nice epis are stable under pullback.

This is all fairly easy: for example, if $f\colon (X,R) \to (Y,S)$ is any map, then it can be factored as

$$(X,R) \xrightarrow{1} (X, f^*S) \xrightarrow{f} (Y,S),$$

where the first map is a nice epi and the second a mono. We leave it to the reader to check the other properties. □

We record the following corollary for future reference:

Lemma 3.7. *If $f\colon (X,R) \to (Y,S)$ is a cover in* $\mathrm{Hex}(\mathcal{C})$*, then $(X, f^*S) \cong (Y,S)$.*

Proof. If $f\colon (X,R) \to (Y,S)$ is a cover, then (Y,S) is isomorphic to the image of f, which, according to Lemma 3.6, is precisely (X, f^*S). □

Lemma 3.8. *For every map $f\colon (X,R) \to (Y,S)$ there exists a factorisation $(X,R) \to (X',R') \to (Y,S)$ where the first is an iso in* $\mathrm{Hex}(\mathcal{C})$ *and the second is represented by a fibration $X' \to Y$ in $\mathcal{C}$. In particular, every subobject of (Y,S) has a representative via a map $f\colon (X, f^*S) \to (Y,S)$ where $f\colon X \to Y$ is a fibration.*

Proof. The map f can be factored as a homotopy equivalence $w_f\colon X \to X'$ followed by a fibration $p_f\colon X' \to Y$; this means that there is a map $i\colon X' \to X$ such that $w_f i \simeq 1$ and $i w_f \simeq 1$. We obtain R' by pulling back R along i. We leave verification of the details to the reader. □

Theorem 3.9. *The category* $\mathrm{Hex}(\mathcal{C})$ *is exact.*

Proof. In view of the previous lemma it suffices to construct quotients of equivalence relations $f\colon (Y, f^*(R \times R)) \to (X \times X, R \times R)$ where $f\colon Y \to X \times X$ is a fibration and $(X, \rho\colon R \to X \times X)$ is an object in $\mathrm{Hex}(\mathcal{C})$. But in this case one can take $(X, \tau\colon R \times_X Y \times_X R \to X \times X)$, where if we consider $R \times_X Y \times_X R$ heuristically as the set of triples (r, y, r') with $\rho_2(r) = f_1(y)$ and $f_2(y) = \rho_1(r')$, then τ sends such a triple to $(\rho_1(r), \rho_2(r'))$. □

Remark 3.10. The set-theoretic notation that we have used in the description of τ in the previous theorem can be justified in various ways, for example, by using generalised elements. In that case the description can be understood to say that $R \times_X Y \times_X R$ is an object such that maps into it from an object I correspond bijectively to triples of maps $r\colon I \to P, y\colon I \to Y, r'\colon I \to R$ with $\rho_2 r = f_1 y$ and $f_2 y = \rho_1 r'$. In addition, the existence of τ derives from the fact that the operation taking such triples (r, y, r') to $(\rho_1 r, \rho_2 r')$ is a natural operation of the form

$$\mathrm{Hom}(I, R \times_X Y \times_X R) \to \mathrm{Hom}(I, X \times X).$$

But then it follows from the Yoneda Lemma that this operation must be given by postcomposition by some unique map $R \times_X Y \times_X R \to X \times X$. From now on we will increasingly rely on such heuristic set-theoretic descriptions; we trust that the reader can replace these descriptions by diagrammatic ones, if desired.

3.2. **Alternative constructions of the homotopy exact completion.** It turns out that the way we have just described the homotopy exact completion is not always the most convenient when we try to prove things about it. For this reason we will now describe two alternative ways of constructing it, where the first is just a minor variation, while the second is more substantial.

Our first alternative description will make use of the notion of a pseudo-equivalence relation.

Definition 3.11. Let $f = (f_1, f_2): R \to X \times X$ be an arbitrary map (not necessarily a fibration) in a path category $\mathcal{C}$. Then f will be called a *pseudo-equivalence relation*, if there are maps $\rho: X \to R, \sigma: R \to R$ and $\tau: P \to R$ witnessing reflexivity, symmetry and transitivity of this relation, where P is the homotopy pullback of f_1 and f_2.

The alternative definition could now be given as follows: take as objects pairs (X, R), where R is a pseudo-equivalence relation on X, and a morphism

$$(X, \rho: R \to X \times X) \to (Y, \sigma: S \to Y \times Y)$$

is an equivalence class of morphisms $f: X \to Y$ for which there is an arrow $\varphi: R \to S$ making

$$\begin{tikzcd} R \arrow[r, dotted, "\varphi"] \arrow[d, "\rho"'] & S \arrow[d, "\sigma"] \\ X \times X \arrow[r, "f \times f"'] & Y \times Y \end{tikzcd}$$

commute up to homotopy, while two such arrows $f, g: X \to Y$ are equivalent if there is a map $h: X \to S$ such that $(f, g) \simeq \sigma h$.

Proposition 3.12. *The category just described is equivalent to* $\mathrm{Hex}(\mathcal{C})$.

Proof. Any homotopy equivalence relation is also a pseudo-equivalence relation, so $\mathrm{Hex}(\mathcal{C})$ embeds into the category just described. Therefore it remains to check that any pseudo-equivalence relation $\rho: R \to X \times X$ is isomorphic to a homotopy equivalence relation in this category. But it can be shown quite easily using the lifting properties that if ρ is factored as a homotopy equivalence $R \to \hat{R}$ followed by a fibration $\hat{\rho}: \hat{R} \to X \times X$, then $\hat{\rho}: \hat{R} \to X \times X$ is a homotopy equivalence relation. $\square$

More significantly, one may also view the homotopy exact completion $\mathrm{Hex}(\mathcal{C})$ as a homotopy category. Indeed, our second way of looking at the homotopy exact completion is to regard $\mathrm{Hex}(\mathcal{C})$ as the result of a two step procedure, where one first constructs out of any path category $\mathcal{C}$ a new path category $\mathrm{Ex}(\mathcal{C})$, from which $\mathrm{Hex}(\mathcal{C})$ can then be obtained by taking the homotopy category.

The objects of $\mathrm{Ex}(\mathcal{C})$ are again homotopy equivalence relations, as defined in Definition 2.11, while a morphism from $(X, \rho: R \to X \times X)$ to $(Y, \sigma: S \to Y \times Y)$ is a morphisms $f: X \to Y$ for which there exists a map $\varphi: R \to S$ making

$$\begin{tikzcd} R \arrow[r, dotted, "\varphi"] \arrow[d, "\rho"'] & S \arrow[d, "\sigma"] \\ X \times X \arrow[r, "f \times f"'] & Y \times Y \end{tikzcd}$$

commute (we call such a map φ a *tracking*). For any two such arrows $f, g: X \to Y$ we will write $f \sim g$ if there is a map $H: X \to S$ such that $(f, g) = \sigma H: X \to Y \times Y$. This relation defines a congruence on $\mathrm{Ex}(\mathcal{C})$ and we will choose our fibrations and weak equivalences in such a way that this will become the homotopy relation on this path category.

A morphism f as above is said to be a fibration in $\mathrm{Ex}(\mathcal{C})$ if:

(1) f is a fibration in $\mathcal{C}$, and
(2) if $X \times_Y S$ is the pullback

$$\begin{tikzcd} X \times_Y S \arrow[r, "p_2"] \arrow[d, "p_1"'] & S \arrow[d, "\sigma_1"] \\ X \arrow[r, "f"'] & Y, \end{tikzcd}$$

there is a map $\nabla: X \times_Y S \to R$ in $\mathcal{C}$ ("a weak connection structure") such that $\rho_1 \nabla = p_1$ and $f \rho_2 \nabla = \sigma_2 p_2$.

And f will be weak equivalence in $\mathrm{Ex}(\mathcal{C})$ if there is a map $g: (Y, S) \to (X, R)$ such that $fg \sim 1_Y$ and $gf \sim 1_X$.

Lemma 3.13. *A fibration $f: (X, \rho: R \to X \times X) \to (Y, \sigma: S \to Y \times Y)$ in $\mathrm{Ex}(\mathcal{C})$ is acyclic if and only there is a map $a: Y \to X$ in $\mathcal{C}$ such that $fa = 1_Y$ and $af \sim 1_X$. Indeed, such a map $a: Y \to X$ in $\mathcal{C}$ will automatically be a map in $\mathrm{Ex}(\mathcal{C})$.*

Proof. If f is acyclic, there is a map $g: (Y, S) \to (X, R)$ such that $fg \sim 1_Y$ and $gf \sim 1_X$. The former gives one a map $H: Y \to S$ such that $\sigma H = (fg, 1)$. We put $a := \rho_2 \nabla(g, H)$. Then

$$fa = f\rho_2 \nabla(g, H) = \sigma_2 p_2(g, H) = \sigma_2 H = 1_Y$$

and $\nabla(g, H)$ witnesses that $g \sim a$, so $af \sim gf \sim 1_X$.

Conversely, if $a: Y \to X$ is such that $fa = 1_Y$ and $af \sim 1_X$, then a can be regarded as a map $(Y, S) \to (X, R)$. To show this, we use set-theoretic notation: for if $s \in S$ connects y_0 and y_1, that is, if $\sigma_1(s) = y_0$ and $\sigma_2(s) = y_1$, then $t_0 := \nabla(a(y_0), s)$ connects $a(y_0)$ with some point x such that $f(x) = y_1$. But then from the witness of $1_X \sim af$ we find a t_1 connecting x and $a(f(x)) = a(y_1)$. So in order to obtain a tracking for a we should send s to the composition of t_0 and t_1, using the transitivity of R. $\square$

Theorem 3.14. *The category* $\mathrm{Ex}(\mathcal{C})$ *is a path category whose homotopy category is equivalent to* $\mathrm{Hex}(\mathcal{C})$.

Proof. We check the axioms.

(1) *Fibrations are closed under composition.* If $f\colon (X, \rho\colon R \to X \times X) \to (Y, \sigma\colon S \to Y \times Y)$ and $g\colon (Y, \sigma\colon S \to Y \times Y) \to (Z, \tau\colon T \to Z \times Z)$ are fibrations with weak connections $\nabla^f\colon X \times_Y S \to R$ and $\nabla^g\colon Y \times_Z T \to S$, respectively, then gf is a fibration with weak connection $\nabla^{gf}\colon X \times_Z T \to R$ defined by $\nabla^{gf}(x,t) := \nabla^f(x, \nabla^g(f(x), t))$.

(2) *The pullback of a fibration along any map exists and is again a fibration.* If $f\colon (X, \rho\colon R \to X \times X) \to (Y, \sigma\colon S \to Y \times Y)$ is a fibration with weak connection ∇^f and $g\colon (Z, \tau\colon T \to Z \times Z) \to (Y, \sigma\colon S \to Y \times Y)$ is tracked by φ, then we can construct its pullback by taking $X \times_Y Z$ together with the homotopy equivalence relation $\pi_1^* R \cap \pi_2^* S$. The projection $X \times_Y Z \to Z$ has a weak connection structure: given a pair (x_0, z_0) with $f(x_0) = g(z_0)$ and an element $t \in T$ from z_0 to z_1, the element $s := \varphi(t) \in S$ connects $f(x_0) = g(z_0)$ and $g(z_1)$. So by the weak connection on f one obtains an element $r \in R$ connecting x_0 to some x_1 with $f(x_1) = g(z_1)$. So (r, t) connects (x_0, z_0) to some point (x_1, z_1) in $X \times_Y Z$ above z_1.

(3) *The pullback of an acyclic fibrations along any map is again an acyclic fibration.* If in the situation as in (2) the map f has a section a with $af \sim 1$, then the projection $\pi_2\colon X \times_Y Z \to Z$ has a section b defined by $b(z) = (a(g(z)), z)$. It is clear that $b\pi_2 \sim 1$.

(4) *Weak equivalence satisfy 2-out-of-6.* This follows from the fact that $\sim$ is a congruence.

(5) *Isomorphisms are acyclic fibrations and every acyclic fibration has a section.* Immediate from the previous lemma.

(6) *The existence of path objects.* If $(X, \rho\colon R \to X \times X)$ is a homotopy equivalence relation with $e\colon X \to R$ witnessing reflexivity (so $\rho e = \Delta_X$), then we can factor the diagonal on X as:

$$(X, R) \xrightarrow{e} (R, \rho_1^* R) \xrightarrow{\rho} (X \times X, R \times R),$$

where $\rho_1\colon R \to X$ is an acyclic fibration left inverse to the first map. We leave the verifications to the reader.

(7) *The category has a terminal object and every map to the terminal object is a fibration.* The terminal object is $(1, P1)$. The verification that the unique map $X \to 1$ is always a fibration $(X, R) \to (1, P1)$ in $\mathrm{Ex}(\mathcal{C})$ is trivial.

Note that it follows from the description of the path objects in (6) that the relation $\sim$ is precisely the homotopy relation in $\mathrm{Ex}(\mathcal{C})$: for if $f, g\colon (Y, \sigma\colon S \to Y \times Y) \to (X, \rho\colon R \to X \times X)$ are two parallel maps and $H\colon Y \to R$ witnesses that $f \sim g$, then H is also map $(Y, S) \to (R, \rho_1^* R)$ which is tracked by any tracking of f. Therefore $\mathrm{Ex}(\mathcal{C})$ is a path category whose homotopy category is equivalent to $\mathrm{Hex}(\mathcal{C})$. $\square$

3.3. **The embedding.** In the theory of exact completions of categories with finite limits the embedding of the original category into the exact completion plays an important rôle. For homotopy exact completions there is a similar functor

$$i: \mathcal{C} \to \mathrm{Hex}(\mathcal{C})$$

obtained by sending X to (X, PX). In this subsection we will try to determine which properties from the theory of ordinary exact completions continue to hold and which ones seem to break down.

First of all, we should note that the functor i is full, but not faithful: indeed, its image is equivalent to the homotopy category $\mathrm{Ho}(\mathcal{C})$.

In the ordinary theory of exact completion the functor i preserves finite limits. Here one has:

Proposition 3.15. *The functor $i: \mathcal{C} \to \mathrm{Hex}(\mathcal{C})$ sends homotopy pullback squares to pullback squares. In particular, it sends pullbacks of fibrations to pullbacks.*

Proof. Since i sends homotopy equivalences to isomorphisms, it suffices to check that i sends any pullback square in $\mathcal{C}$

$$\begin{tikzcd} C \times_A B \arrow[r, "g"] \arrow[d, "q"'] & B \arrow[d, "p"] \\ C \arrow[r, "f"'] & A \end{tikzcd}$$

in which all maps are fibrations to a pullback square in $\mathrm{Hex}(\mathcal{C})$.

Using the description of pullbacks in $\mathrm{Ex}(\mathcal{C})$ in the proof of Theorem 3.14 the pullback of the images of f and g there is $C \times_A B$ together with the homotopy equivalence relation $g^*PB \cap q^*PC$. Since weak equivalences are stable under pullback along fibrations and g and q are fibrations, both g^*PB and q^*PB are homotopic to $P(C \times_A B)$. Therefore the image in $\mathrm{Ex}(\mathcal{C})$ of a pullback involving only fibrations remains a pullback involving only fibrations in $\mathrm{Ex}(\mathcal{C})$. Since Proposition 2.31 implies that the pullback of a square involving only fibrations remains a pullback square in the homotopy category, the proposition follows. □

In the ordinary theory of exact completions the objects in the image of i are, up to isomorphism, the projectives (an object P in an exact category is projective if any cover $e: X \to P$ has a section). That does not seem to be the case here, but we do have the following useful result:

Proposition 3.16. *The objects in the image of the functor $i: \mathcal{C} \to \mathrm{Hex}(\mathcal{C})$ are projective and each object in $\mathrm{Hex}(\mathcal{C})$ is covered by some object in the image of this functor.*

Proof. It follows immediately from Lemma 2.13 and the characterisation of covers in Lemma 3.6 that objects of the form iX are projective, while maps of the form $1: (X, PX) \to (X, R)$ are covers. □

Proposition 3.17. *The category* $\mathrm{Hex}(\mathcal{C})$ *is the exact completion of* $\mathrm{Ho}(\mathcal{C})$ *as a weakly lex category, as defined in* [17].

Proof. This follows from Theorem 3.9 and Proposition 3.16 above and Theorem 16 in [17]. □

We will also be concerned with objects that are internally projective.

Definition 3.18. Let $\mathcal{E}$ be an exact category. An object P in $\mathcal{E}$ is *internally projective* if for any T in $\mathcal{E}$, any cover $Y \to X$ and any map $T \times P \to X$, there are a cover $e: T' \to T$ and a map $T' \times P \to Y$ making the square

$$\begin{tikzcd} T' \times P \arrow[r] \arrow[d, "e \times 1"'] & Y \arrow[d] \\ T \times P \arrow[r] & X \end{tikzcd}$$

commute. A morphism $f: P \to A$ in $\mathcal{E}$ is a *choice map* if it is internally projective in the slice category $\mathcal{E}/A$.

Lemma 3.19. *Let* $\mathcal{E}$ *be an exact category and* $\mathcal{P}$ *be a collection of objects in* $\mathcal{E}$ *with the following three properties:*

(1) *The objects in* $\mathcal{P}$ *are closed under finite products.*
(2) *The objects in* $\mathcal{P}$ *are projective.*
(3) *For every object* E *in* $\mathcal{E}$ *there is a cover* $P \to E$ *with* $P \in \mathcal{P}$.

Then each object in $\mathcal{P}$ *is internally projective.*

Proof. Suppose we are given an element P in $\mathcal{P}$, a cover $Y \to X$ and a map $T \times P \to X$. Then the third property allows us to choose $T' \in \mathcal{P}$ together with a cover $e: T' \to T$; the first property tells us that $T' \times P$ belongs to $\mathcal{P}$, while the second property tells us that it is also projective. So if we pull back the cover $Y \to X$ along $T' \times P \to T \times P \to X$, the resulting cover will have a section. Therefore there is a map $T' \times P \to Y$ making the resulting square commutative. □

Corollary 3.20. *Objects of the form* iX *are internally projective in* $\mathrm{Hex}(\mathcal{C})$ *and morphisms of the form* $i(f): i(Y) \to i(X)$ *are choice maps in* $\mathrm{Hex}(\mathcal{C})$.

Proof. The first statement follows from the fact that objects of the form iX satisfy all the properties in the previous lemma. In the same fashion one shows that objects of the form $i(g): i(Y) \to i(X)$ where g is a fibration satisfy the properties of the previous lemma in $\mathrm{Hex}(\mathcal{C})/i(X)$. Therefore maps of the form $i(f)$ are choice maps, because the factorisation of maps as homotopy equivalences following by fibrations shows that any such is isomorphic to one of the form $i(g)$ with g being a fibration. □

Another aspect of the theory of exact completions is that the subobject lattices of objects of the form iX can be described concretely as a poset reflection.

Proposition 3.21. *Let X be an object in a path category $\mathcal{C}$. The subobject lattice of iX in* $\mathrm{Hex}(\mathcal{C})$ *is order isomorphic to the poset reflection of $\mathcal{C}(X)$.*

Proof. Lemma 3.8 tells us that every subobject of iX in $\mathrm{Hex}(\mathcal{C})$ has a representative given by a map $f\colon (Y,R) \to (X,PX)$ where f is a fibration and $R = f^*PX$. If $h\colon (Z,S) \to (Y,R)$ is a map over iX between two such representatives $g\colon (Z,S) \to (X,PX)$ and $f\colon (Y,R) \to (X,PX)$, then $fh \simeq g$. But then there is also a map $h'\colon Z \to Y$ homotopic to h such that $fh' = g$. Since h and h' are homotopic, h' also has a tracking as a map $(Z,S) \to (Y,R)$ in $\mathrm{Hex}(\mathcal{C})$ and as such h and h' represent the same map. In fact, any map h' such that $fh' = g$ will have tracking as a map $(Z,S) \to (Y,R)$ because we are assuming that $S = g^*PX$ and $R = f^*PX$. It follows that the subobject lattice of iX in $\mathrm{Hex}(\mathcal{C})$ is the poset reflection of $\mathcal{C}(X)$, as claimed. □

Another aspect of the classical theory of exact completions is that exact completion and slicing commute. That fails for path categories; in fact, we only have the following.

Proposition 3.22. *Let $\mathcal{C}$ be a path category and X be an object in $\mathcal{C}$. Then* $\mathrm{Hex}(\mathcal{C})/i(X)$ *is a reflective subcategory of* $\mathrm{Hex}(\mathcal{C}(X))$.

Proof. Let us first take a closer look at $\mathrm{Hex}(\mathcal{C})/i(X)$. Objects in this category are morphisms $f\colon (Y,S) \to (X,PX)$ in $\mathrm{Hex}(\mathcal{C})$, that is, homotopy classes of arrows $f\colon Y \to X$ with a tracking $S \to PX$. Using the factorisation of arrows as homotopy equivalences followed by fibrations in $\mathrm{Ex}(\mathcal{C})$, we may assume that f is an $\mathrm{Ex}(\mathcal{C})$-fibration. This means that we may assume that the objects in this category are pairs consisting of a fibration $f\colon Y \to X$ and a homotopy equivalence relation $\sigma\colon S \to Y \times Y$ for which there is a weak connection structure $\nabla\colon Y \times_X PX \to S$ as well as a map $S \to PX$ making

$$\begin{tikzcd} S \arrow[r, dotted] \arrow[d, "\sigma"'] & PX \arrow[d, "{(s,t)}"] \\ Y \times Y \arrow[r, "f \times f"'] & X \times X \end{tikzcd}$$

commute.

Furthermore, the morphisms in $\mathrm{Hex}(\mathcal{C})/i(X)$ are equivalence classes of arrows

$$\varphi\colon (g\colon Z \to X, R) \to (f\colon Y \to X, S)$$

such that $f \circ \varphi \simeq g$ and for which a tracking $R \to S$ exists, while φ and φ' are equivalent in case there is map H making

$$\begin{tikzcd} & S \arrow[d] \\ Z \arrow[ur, dotted, "H"] \arrow[r, "{(\varphi,\varphi')}"'] & Y \times Y \end{tikzcd}$$

commute. Since we are assuming that f is a fibration, it follows from Proposition 2.31 that we may just as well assume that φ satisfies $f\varphi = g$. If both φ and φ' are such representations, then they represent the same arrow in $\mathrm{Hex}(\mathcal{C})/i(X)$ if there is a dotted filler as in

$$\begin{tikzcd}
 & T \arrow[r] \arrow[d] & S \arrow[d] \\
Z \arrow[ur, dotted] \arrow[r, "(\varphi,\varphi')"'] & Y \times_X Y \arrow[r] & Y \times Y,
\end{tikzcd}$$

where the square is a pullback.

This suggests the correct definition of the embedding $\rho\colon \mathrm{Hex}(\mathcal{C})/i(X) \to \mathrm{Hex}(\mathcal{C}(X))$. Note that objects in $\mathrm{Hex}(\mathcal{C}(X))$ consist of pairs $(f, Y \to X, T \to Y \times_X Y)$, where f is a fibration and $T \to Y \times_X Y$ is a homotopy equivalence relation in $\mathcal{C}(X)$. So we can define a functor $\rho\colon \mathrm{Hex}(\mathcal{C})/i(X) \to \mathrm{Hex}(\mathcal{C}(X))$ by sending $(f\colon Y \to X, S)$ to f together with the homotopy equivalence relation in $\mathcal{C}(X)$ obtained as the pullback

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & S \arrow[d] \\
Y \times_X Y \arrow[r] & Y \times Y.
\end{tikzcd}$$

This functor ρ has a left adjoint $\lambda\colon \mathrm{Hex}(\mathcal{C}(X)) \to \mathrm{Hex}(\mathcal{C})/i(X)$. The quickest way to define it is to use the factorisation in $\mathcal{C}$: starting from a pair $(f\colon Y \to X, T \to Y \times_X Y)$ we can factor the composition of $T \to Y \times_X Y$ with the inclusion $Y \times_X Y \to Y \times Y$ as a homotopy equivalence followed by a fibration:

$$\begin{tikzcd}
T \arrow[r, "\sim"] \arrow[d] & S \arrow[d] \\
Y \times_X Y \arrow[r] & Y \times Y.
\end{tikzcd}$$

Using the lifting properties one can now show that $S \to Y \times Y$ is a homotopy equivalence relation and that λ defines a left adjoint to ρ.

To complete the proof we have to show that $\lambda\rho \cong 1$. So suppose we are given a fibration $f\colon Y \to X$ and a homotopy equivalence relation $\sigma\colon S \to Y \times Y$ for which there are a weak connection $\nabla\colon Y \times_X PX \to S$ as well as a tracking $S \to PX$. Construct the following four pullbacks:

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & S^* \arrow[r] \arrow[d] & S \arrow[d] \\
Y \times_X Y \arrow[r] \arrow[d] & Y \times_X PX \times_X Y \arrow[r] \arrow[d] & Y \times Y \arrow[d] \\
X \arrow[r, "r"'] & PX \arrow[r, "(s,t)"'] & X \times X.
\end{tikzcd}$$

Note that all four arrows in the lower righthand square are fibrations; since $S \to Y \times Y$ is a fibration, the maps $S^* \to Y \times Y$ and $S^* \to PX$ are fibrations as well. From the latter it follows that $T \to S^*$ is a weak equivalence since $r: X \to PX$ is. Therefore applying ρ to (f, S) yields $T \to Y \times_X Y$ and the result of applying λ to that is $S^* \to Y \times Y$. Therefore it remains to construct a suitable map $S \to S^*$ over $Y \times Y$: but the existence of such a map follows from the universal property of S^* and the existence of a tracking $S \to PX$. □

Remark 3.23. The adjunction $\lambda \dashv \rho$ in the proof above is not an equivalence: indeed, consider the category of topological spaces, and take for $f: Y \to X$ the universal cover $\mathbb{R} \to S^1$ of the circle and let $\Delta: Y \to Y \times_X Y$ be the diagonal. Then $\lambda(f, \Delta) \cong (\mathbb{R}, P\mathbb{R}) \cong 1$ and $\rho\lambda(f, \Delta) \cong 1$, but $(f, \Delta) \not\cong 1$. In the same way one can show that λ does not preserve finite products (if it would our treatment of Π-types below could have been simplified considerably). For $\lambda(f, \Delta) \times \lambda(f, \Delta) \cong 1$, while $(f, \Delta) \times (f, \Delta)$ is $Y \times_X Y$ with the diagonal, so

$$\lambda((f, \Delta) \times (f, \Delta)) = (Y \times_X Y, P(Y \times_X Y)),$$

which is isomorphic to $\mathbb{Z}$ with the discrete topology.

In the remainder of this section we will try to characterise the image of ρ in $\mathrm{Hex}(\mathcal{C}(X))$. In order to do this, we introduce the following notion.

For the moment, fix a fibration $f: Y \to X$ and a homotopy equivalence relation $\tau: T \to Y \times_X Y$ in $\mathcal{C}(X)$; so, in effect, we are fixing an object in $\mathrm{Hex}(\mathcal{C}(X))$.

Definition 3.24. A *transport structure relative to* T, or a T*-transport*, is a map $\Gamma: Y \times_X PX \to Y$ such that:

(1) $f\Gamma = tp_2$, and
(2) there is a map $L: Y \to T$ such that $\tau L = (1, \Gamma(1, rf))$.

Proposition 3.25. *T-transports exist and are unique up to T-equivalence; more precisely, if Γ and Γ' are two T-transports, there will be a map $H: Y \times_X PX \to T$ such that $\tau H = (\Gamma, \Gamma')$.*

Proof. For $T = P_X(Y)$ a T-transport structure is the same thing as an ordinary transport structure. Because there will always be a map $P_X(Y) \to T$ over $Y \times_X Y$, every ordinary transport structure is also a transport structure relative to T. In particular, transport structure relative to T exist since ordinary ones do.

To show essential uniqueness, let Γ and Γ' be two T-transports. Then $\Gamma(1, rf)$ and $\Gamma'(1, rf)$ will be T-equivalent, as they are both T-equivalent to the identity on Y. This means that there is a map K making

$$\begin{tikzcd} Y \arrow[r, "K"] \arrow[d, "{(1,rf)}"'] & T \arrow[d, "\tau"] \\ Y \times_X PX \arrow[r, "{(\Gamma,\Gamma')}"'] & Y \times_X Y \end{tikzcd}$$

commute. But since τ is a fibration and $(1, rf)$ is a weak equivalence, we get the desired map H from the usual lifting properties. $\square$

Proposition 3.26. *T-transports preserve T-equivalence. More precisely, if Γ is a T-transport, there will be a map $H: T \times_X PX \to T$ such that*

$$\tau_1 H = \Gamma(\tau_1 p_1, p_2) \quad \textit{and} \quad \tau_2 H = \Gamma(\tau_2 p_1, p_2).$$

Proof. If Γ is a T-transport, then

$$\Gamma(1, rf)\tau_1 \simeq_T \tau_1 \simeq_T \tau_2 \simeq_T \Gamma(1, rf)\tau_2 : T \to Y.$$

Therefore there is a map K making

$$\begin{tikzcd}
T \arrow[rrr, "K"] \arrow[d, "{(1, rf\tau_1)}"'] & & & T \arrow[d, "\tau"] \\
T \times_X PX \arrow[r, "\tau \times_X 1"'] & Y \times_X Y \times_X PX \arrow[rr, "{(\Gamma(p_1, p_3), \Gamma(p_2, p_3))}"'] & & Y \times_X Y
\end{tikzcd}$$

commute, and H is obtained as a lower filler of this diagram. $\square$

Definition 3.27. Let (f, T) be an element of $\mathrm{Hex}(\mathcal{C}(X))$, so $f: Y \to X$ is a fibration and $T \to Y \times_X Y$ is a homotopy equivalence relation in $\mathcal{C}(X)$. We call such an object *stable* if the action of loops in X on the fibres of f by the (essentially unique) T-transport $\Gamma: Y \times_X PX \to Y$ is T-trivial: so if $f(y) = x$ and α is a loop at x, then $\Gamma_\alpha(y) \simeq_T y$.

Theorem 3.28. *Let $\mathcal{C}$ be a path category and X be an object in $\mathcal{C}$. Then $\mathrm{Hex}(\mathcal{C})/i(X)$ is equivalent to the full subcategory of $\mathrm{Hex}(\mathcal{C}(X))$ consisting of the stable objects.*

Proof. Note that objects in the image of ρ are always stable: for suppose T is the restriction to $Y \times_X Y$ of some homotopy equivalence relation $\sigma: S \to Y \times Y$ over (X, PX). We may assume that $f: (Y, S) \to (X, PX)$ is an $\mathrm{Ex}(\mathcal{C})$-fibration, so that there is a weak connection structure $\nabla: Y \times_X PX \to S$. From this we obtain a T-transport Γ given by $\Gamma = \sigma_2 \nabla$. If $f(y) = x$ and α is a loop at x, the weak connection ∇ tells us that $\Gamma_\alpha(y) \simeq_S y$; but then also $\Gamma_\alpha(y) \simeq_T y$, by definition of T.

Conversely, let $(f: Y \to X, \tau: T \to Y \times_X Y)$ be an element of $\mathrm{Hex}(\mathcal{C}(X))$, and let $\Gamma: Y \times_X PX \to Y$ be the essentially unique T-transport. Compute the following pullbacks:

$$\begin{tikzcd}
T \arrow[d, "\tau"'] & S \arrow[l] \arrow[d] & T^* \arrow[l] \arrow[d, "\tau^*"] \\
Y \times_X Y & Y \times_X PX \times_X Y \arrow[l, "\Gamma \times_X 1"'] \arrow[dl] \arrow[d] & Y \times_X Y \arrow[l] \arrow[d] \\
Y \times Y \arrow[d, "f \times f"'] & PX \arrow[dl, "(s,t)"] & X \arrow[l, "r"] \\
X \times X & &
\end{tikzcd}$$

From the fact that Γ is a T-transport it follows that that there are maps between T and T^* which commute over $Y \times_X Y$. In other words, (Y, T) and (Y, T^*) are isomorphic in $\mathrm{Hex}(\mathcal{C})$. But since $T^* \to S$, as a pullback of $r: X \to PX$ along a fibration, is a weak equivalence, we see that $\lambda(Y, T)$ is $(Y, S \to Y \times Y)$. Therefore $\rho\lambda(Y, T)$ is the element in $\mathrm{Hex}(\mathcal{C}(X))$ consisting of $f: Y \to X$ together with the following homotopy equivalence relation in $\mathcal{C}(X)$: y_1 and y_2 over the same x are related if there is a loop α on x such that $\Gamma_\alpha(y_1) \simeq_T y_2$. But if (Y, T) is stable, this is equivalent to $y_1 \simeq_T y_2$; so in this case $\rho\lambda(Y, T) \cong (Y, T)$. □

This theorem gives us a useful way of thinking about the slice category $\mathrm{Hex}(\mathcal{C})/i(X)$: especially when we have to deal with small maps, it is more convenient to think about the stable elements in $\mathrm{Hex}(\mathcal{C}(X))$.

4. Sums

4.1. **Definition.** This section will be devoted to a study of homotopy initial objects and homotopy sums in a path category. These can be defined quite simply as objects that become initial objects and sums in the homotopy category. That is:

Definition 4.1. An object 0 is *homotopy initial* if for any object A there is a map $f: 0 \to A$ and any two such maps are homotopic. A *homotopy sum* or *homotopy coproduct* of two objects A and B is an object $A+B$ together with two maps $i_A: A \to A + B$ and $i_B: B \to A + B$ such that for any pair of maps $f: A \to X$ and $g: B \to X$ there is a map $h: A+B \to X$, unique up to homotopy, such that $hi_A \simeq f$ and $hi_B \simeq g$.

This is not quite what the type theorist would expect: the type-theoretic axiom for the initial object, for example, says that any fibration $A \to 0$ has a section. However, this condition turns out to be equivalent.

Proposition 4.2. *In a path category an object* 0 *is homotopy initial if and only if any fibration* $f: A \to 0$ *has a section.*

Proof. Suppose we are given a fibration $f: A \to 0$. If 0 is homotopy initial, then there is a map $g: 0 \to A$ with $fg \simeq 1$. So by Proposition 2.31 there is a map $g': 0 \to A$ such that $fg' = 1$.

Conversely, suppose 0 is such that any fibration $A \to 0$ has a section. For any object B the second projection $\pi_2 \colon B \times 0 \to 0$ is a fibration, so there is a map $a \colon 0 \to B \times 0$ such that $\pi_2 a = 1$; but then $f = \pi_1 a$ is a map $0 \to B$. In addition, if $g \colon 0 \to B$ is another map, then we can take the pullback

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & PB \arrow[d, "{(s,t)}"] \\ 0 \arrow[r, "{(f,g)}"'] & B \times B \end{tikzcd}$$

giving rise to a fibration $Q \to 0$. This map has a section, and composing this section with the map $Q \to PB$ gives rise to a homotopy between f and g. □

In the same way one has:

Proposition 4.3. *An object $A + B$ together with maps $i_A \colon A \to A + B$ and $i_B \colon B \to A + B$ is the homotopy sum of A and B if and only if for any fibration $p \colon C \to A + B$ and any pair of maps $a \colon A \to C$ and $b \colon B \to C$ such that $pa = i_A$ and $pb = i_B$, there is a map $\sigma \colon A + B \to C$ such that $p\sigma = 1, \sigma i_A \simeq a$ and $\sigma i_B \simeq b$.*

Proof. $\Rightarrow$: Suppose we are given a fibration $p \colon C \to A + B$ together with maps $a \colon A \to C$ and $b \colon B \to C$ such that $pa = i_A$ and $pb = i_B$. We know that there is a map $h \colon A + B \to C$ such that $hi_A \simeq a$ and $hi_B \simeq b$. In addition, we must have $ph \simeq 1$, so by Proposition 2.31 there is a map $\sigma \colon A + B \to C$ such that $p\sigma = 1$ and $\sigma \simeq h$; hence $\sigma i_A \simeq hi_A \simeq a$ and $\sigma i_B \simeq hi_B \simeq b$.

$\Leftarrow$: Let $f \colon A \to X$ and $g \colon B \to X$ be two maps. We want to show that there is a map $h \colon A + B \to X$, unique up to homotopy, such that $hi_A \simeq f$ and $hi_B \simeq g$. Put $C = X \times (A + B)$ and consider the projection $\pi_2 \colon C \to A + B$ together with the maps $(f, i_A) \colon A \to C$ and $(g, i_B) \colon B \to C$. By assumption, there is a map $\sigma \colon A + B \to C$ such that $\pi_2 \sigma = 1$, $\sigma i_A \simeq (f, i_A)$, $\sigma i_B \simeq (g, i_B)$. So if we put $h = \pi_1 \sigma \colon A + B \to X$, then $hi_A \simeq f$ and $hi_B \simeq g$, as desired. If $h' \colon A + B \to X$ satisfies the same equations, then we can take the following pullback:

$$\begin{tikzcd} Q \arrow[r, "p_2"] \arrow[d, "p_1"'] & PX \arrow[d, "{(s,t)}"] \\ A + B \arrow[r, "{(h,h')}"'] & X \times X \end{tikzcd}$$

giving rise to a fibration $Q \to A + B$. In addition, since $(h, h')i_A \simeq (f, f) = (s, t)rf$, there is a map $u \colon A \to PX$ such that $(s, t)u = (h, h')i_A$ and a map $k \colon A \to Q$ such that $p_1 k = i_A$; similarly, there is a map $l \colon B \to Q$ such that $p_1 l = i_B$. So p_1 has a section and composing this section with p_2 yields the desired homotopy between h and h'. □

Proposition 4.4. *Suppose $\mathcal{C}$ is a path category with homotopy sums.*

(i) *If 0 is homotopy initial, then $0 + X \simeq X$ for any object X.*

(ii) *$f + g\colon X + Y \to A + B$ will be a homotopy equivalence if both $f\colon X \to A$ and $g\colon Y \to B$ are.*
(iii) *$P(A + B) \simeq PA + PB$.*

Proof. Parts (i) and (ii) are immediate consequences of the fact homotopy equivalences are precisely those maps which become isomorphisms in the homotopy category, while homotopy initial objects become initial objects and homotopy sums become ordinary sums in the homotopy category.

(iii): It follows from (ii) that the canonical map $A + B \to PA + PB$ is a weak equivalence. So the lifting properties give us a map $PA + PB \to P(A + B)$ making the top triangle in

$$\begin{tikzcd} A + B \arrow[rr] \arrow[d] & & P(A+B) \arrow[d] \\ PA + PB \arrow[r] \arrow[urr, dotted, squiggly] & A \times A + B \times B \arrow[r] & (A+B) \times (A+B) \end{tikzcd}$$

commute up to homotopy. Since the map along the top is a homotopy equivalence, so is $PA + PB \to P(A + B)$. □

4.2. **Homotopy extensive path categories.** For later purposes we do not only need homotopy sums to exist, but they should also have properties like disjointness and stability, as ordinary categorical sums have in an extensive category. So we need a suitable notion of extensivity for path categories.

Definition 4.5. Suppose $\mathcal{C}$ is a path category.

(1) A homotopy sum $A + B$ in $\mathcal{C}$ is *stable*, if for any diagram of the form

$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \arrow[d] & D \arrow[l] \arrow[d] \\ A \arrow[r] & A + B & B, \arrow[l] \end{tikzcd}$$

the top row is a homotopy coproduct whenever both squares are homotopy pullbacks.
(2) A homotopy sum $A + B$ is *disjoint* if the square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & A + B \end{tikzcd}$$

is a homotopy pullback.
(3) If $\mathcal{C}$ has a homotopy inital object and homotopy sums which are both stable and disjoint, then $\mathcal{C}$ will be called *homotopy extensive.*

Proposition 4.6. *Let $\mathcal{C}$ be a path category with stable homotopy sums and a homotopy initial object.*

(i) *The distributive law* $X \times (A + B) \simeq X \times A + X \times B$ *holds.*
(ii) *The homotopy initial object* 0 *is strict: any map* $X \to 0$ *is a homotopy equivalence.*
(iii) *The functor* $\mathcal{C}(A + B) \to \mathcal{C}(A) \times \mathcal{C}(B)$ *is homotopy conservative (i.e., detects homotopy equivalences).*

Proof. Property (i) is a special case of stability, as applied to the following diagram:

$$\begin{tikzcd} X \times A \arrow[r] \arrow[d] & X \times (A + B) \arrow[d] & X \times B \arrow[l] \arrow[d] \\ A \arrow[r] & A + B & B. \arrow[l] \end{tikzcd}$$

To prove (ii), note that given any arrow $f\colon X \to 0$ the diagram

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "f"'] & X \arrow[d, "f"] & X \arrow[l, "1"'] \arrow[d, "f"] \\ 0 \arrow[r, "1"'] & 0 & 0 \arrow[l, "1"] \end{tikzcd}$$

consists of two (homotopy) pullbacks. So the top row is homotopy coproduct diagram by stability and therefore any two parallel arrows with domain X are homotopic. This, in combination with the existence of a map $f\colon X \to 0$, implies that X is a homotopy initial object and f is a homotopy equivalence.

To prove (iii), suppose $f\colon Y \to X$ is a map in $\mathcal{C}(A + B)$ and let $f_A\colon Y_A \to X_A$ and $f_B\colon Y_B \to X_B$ be the pullbacks of f along $A \to A+B$ and $B \to A+B$, respectively. If both f_A and f_B are homotopy equivalences, then so is $f_A + f_B\colon Y_A + Y_B \to X_A + X_B$ by Proposition 4.4.(ii). But if the sums in $\mathcal{C}$ are stable, then $Y_A + Y_B \simeq Y$ and $X_A + X_B \simeq X$, so f is a homotopy equivalence, as desired. □

Proposition 4.7. *Suppose* $\mathcal{C}$ *is a path category which has a homotopy initial object and homotopy sums. Then* $\mathcal{C}$ *is homotopy extensive if and only if the following two conditions are satisfied:*

(i) *If* $C \to A$ *and* $D \to B$ *are two maps, then*

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C + D \arrow[d] & D \arrow[l] \arrow[d] \\ A \arrow[r] & A + B & B, \arrow[l] \end{tikzcd}$$

consists of two homotopy pullbacks.
(ii) *The functor* $\mathcal{C}(A + B) \to \mathcal{C}(A) \times \mathcal{C}(B)$ *is homotopy conservative.*

Proof. $\Rightarrow$: In view of Proposition 4.6.(iii) it remains to show that (i) holds in all homotopy extensive path categories. To this purpose consider a homotopy pullback of the form

$$\begin{tikzcd} C' \arrow[r] \arrow[d] & C + D \arrow[d] \\ A \arrow[r] & A + B. \end{tikzcd}$$

We would like to show that $C' \simeq C$, and since we have already shown that the functor $\mathcal{C}(C + D) \to \mathcal{C}(C) \times \mathcal{C}(D)$ is homotopy conservative, it suffices to prove that the following two squares are homotopy pullbacks:

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C \arrow[d] & 0 \arrow[r] \arrow[d] & D \arrow[d] \\ C' \arrow[r] & C + D & C' \arrow[r] & C + D \end{tikzcd} \tag{1}$$

By pasting of homotopy pullbacks, the second square is a homotopy pullback if and only if

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & D \arrow[d] \\ A \arrow[r] & A + B \end{tikzcd}$$

is. But the latter square can be decomposed as

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & D \arrow[d] \\ 0 \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & A + B. \end{tikzcd}$$

Here the bottom square is a homotopy pullback by the disjointness of the homotopy sums and the top square is a homotopy pullback by Proposition 4.6.(ii). We conclude that the second square in (1) is a homotopy pullback.

In the same way one can show that

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & C \arrow[d] \\ D' \arrow[r] & C + D \end{tikzcd}$$

is a homotopy pullback, where D' is the homotopy pullback in

$$\begin{tikzcd} D' \arrow[r] \arrow[d] & C + D \arrow[d] \\ B \arrow[r] & A + B. \end{tikzcd}$$

Now consider

$$\begin{tikzcd}
C'' \arrow[r] \arrow[d] & C \arrow[d] & 0 \arrow[l] \arrow[d] \\
C' \arrow[r] \arrow[d] & C + D \arrow[d] & D' \arrow[l] \arrow[d] \\
A \arrow[r] & A + B & B \arrow[l]
\end{tikzcd}$$

in which all squares are homotopy pullbacks. By stability of sums we have that $C \simeq C'' + 0 \simeq C''$. This shows that also the first square in (1) is a homotopy pullback.

$\Leftarrow$: Suppose (i) and (ii) are satisfied. To show that the homotopy sums are stable, suppose that

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & X \arrow[d] & D \arrow[l] \arrow[d] \\
A \arrow[r] & A + B & B, \arrow[l]
\end{tikzcd}$$

consists of two homotopy pullbacks. We have to show $C + D \simeq X$. Without loss of generality we may assume that both $X \to A + B$ and $C + D \to A + B$ are fibrations. Therefore it suffices to prove that $C + D$ and X are homotopy equivalent after pulling back along $A \to A + B$ and $C \to C + D$. But for both $C + D$ and X the results are homotopy equivalent to C and D, respectively, so $C + D \simeq X$.

To see that homotopy sums are disjoint, note that (i) implies that

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 + B \arrow[d] \\
A \arrow[r] & A + B
\end{tikzcd}$$

is a homotopy pullback. □

Proposition 4.8. *Let $\mathcal{C}$ be a homotopy extensive path category. If the following squares*

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & X \arrow[d] & Y' \arrow[r] \arrow[d] & Y \arrow[d] \\
A' \arrow[r] & A & B' \arrow[r] & B
\end{tikzcd}$$

are homotopy pullbacks in $\mathcal{C}$, then so is

$$\begin{tikzcd}
X' + Y' \arrow[r] \arrow[d] & X + Y \arrow[d] \\
A' + B' \arrow[r] & A + B.
\end{tikzcd}$$

Proof. Let P be such that

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X+Y \arrow[d] \\ A'+B' \arrow[r] & A+B \end{tikzcd}$$

is a homotopy pullback. To show that P is a homotopy sum of X' and Y' it suffices, by stability, to show that both

(2)
$$\begin{tikzcd} X' \arrow[r] \arrow[d] & P \arrow[d] & Y' \arrow[r] \arrow[d] & P \arrow[d] \\ A' \arrow[r] & A'+B' & B' \arrow[r] & A'+B' \end{tikzcd}$$

are homotopy pullbacks. To see this for the first square, note that we have a commuting cube

$$\begin{tikzcd} & P \arrow[rr] \arrow[dd] & & X+Y \arrow[dd] \\ X' \arrow[ur] \arrow[rr] \arrow[dd] & & X \arrow[ur] \arrow[dd] & \\ & A'+B' \arrow[rr] & & A+B. \\ A' \arrow[ur] \arrow[rr] & & A \arrow[ur] & \end{tikzcd}$$

Since the front, the back and the right face are homotopy pullbacks, the same holds for the left face. A similar cube shows that the second square in (2) is a homotopy pullback as well. □

4.3. **Homotopy exact completion.** If $\mathcal{C}$ is a homotopy extensive, then $\mathrm{Hex}(\mathcal{C})$ will not only be exact: it will be a pretopos. This subsection will be devoted to a direct proof of this fact. (Alternatively, we could have appealed to Proposition 3.17 above and Section 3.4 in [17]. For the definition of a pretopos, see Section A.3 of the appendix.)

Proposition 4.9. *Homotopy initial objects become initial objects in the homotopy exact completion.*

Proof. Let (X, R) be an arbitrary object in the homotopy exact completion with a fibration $\rho\colon R \to X \times X$. If 0 is homotopy initial, there will be maps $f\colon 0 \to X$ and $g\colon 0 \to R$. Now $(f, f) \simeq \rho g$, so by Proposition 2.31 there is also a map $g'\colon 0 \to R$ such that $(f, f) = \rho g'$. Hence the square

$$\begin{tikzcd} 0 \arrow[r, "g'"] \arrow[d, "r"'] & R \arrow[d, "\rho"] \\ P0 \arrow[r, "{(fs,ft)}"'] & X \times X \end{tikzcd}$$

commutes. Since it has a weak equivalence on the left and a fibration on the right, there is a map $P0 \to R$ to track f, showing the existence of a morphism $(0, P0) \to (X, R)$ in the homotopy exact completion. To prove uniqueness, note that if there are two maps $f, f' \colon 0 \to X$ then $(f, f') \simeq \rho g$, which shows that f and f' are identical as maps in the homotopy exact completion (see Remark 3.3). □

Theorem 4.10. *If $\mathcal{C}$ is a homotopy extensive path category, then its homotopy exact completion* $\mathrm{Hex}(\mathcal{C})$ *is a pretopos.*

Proof. It will be convenient to use the first alternative description of $\mathrm{Hex}(\mathcal{C})$ in terms of pseudo-equivalence relations. So let $R \to X \times X$ and $S \to Y \times Y$ be two pseudo-equivalence relations.

If $R + S$ and $X + Y$ are the homotopy sums, then from the maps $X \times X \to (X + Y) \times (X + Y)$ and $Y \times Y \to (X + Y) \times (X + Y)$ and the universal property of $R + S$ we obtain a map

$$R + S \to (X + Y) \times (X + Y).$$

Using the properties of homotopy extensive categories that we have established, one can show that this map is a pseudo-equivalence relation and indeed the sum of $R \to X \times X$ and $S \to Y \times Y$ in the homotopy exact completion. The (easy) verification that these sums are stable and disjoint is left to the reader. □

5. Small maps

5.1. **Small fibrations and small maps.** The main purpose of this section is to show that a class of small fibrations in a path category $\mathcal{C}$ gives rise to a class of small maps in the pretopos $\mathrm{Hex}(\mathcal{C})$ having properties similar to those in [31]. In order to describe how we do this, we will need the following definition taken from [9]:

Definition 5.1. A commuting square in an exact category of the form

$$\begin{tikzcd} D \arrow[r, "q"] \arrow[d, "g"'] & C \arrow[d, "f"] \\ B \arrow[r, "p"'] & A \end{tikzcd}$$

will be called a *covering square* if both p and the inscribed map $D \to B \times_A C$ to the pullback are covers. If g and f fit into a covering square as shown, then we will say that *g covers f* or that *f is covered by g*.

So let $\mathcal{C}$ be a path category and $\mathcal{F}$ be a class of fibrations in $\mathcal{C}$. We will refer to the elements of $\mathcal{F}$ as the *small fibrations* and assume that $\mathcal{F}$ satisfies the following axioms:

(F1) If in a homotopy pullback square

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "p"'] & X \end{tikzcd}$$

both f and f' are fibrations, then f' belongs to $\mathcal{F}$ whenever f does.
(F2) $\mathcal{F}$ contains all isomorphisms.
(F3) $\mathcal{F}$ is closed under composition.
(F4) If $Y \to X$ belongs to $\mathcal{F}$, then so does $P_X(Y) \to Y \times_X Y$.

Note that it follows from (F1) that the property of being a small fibration is homotopy invariant, meaning that if $h = g'fg$ and h and f are fibrations and g and g' are homotopy equivalences, then f will belong to $\mathcal{F}$ if and only if h does. Therefore the validity of (F4) is independent of the particular choice of path object.

Consider the homotopy exact completion $\mathrm{Hex}(\mathcal{C})$ of $\mathcal{C}$ and the functor

$$i: \mathcal{C} \to \mathrm{Hex}(\mathcal{C}).$$

A map $f: Y \to X$ in $\mathrm{Hex}(\mathcal{C})$ will be called *quasi-small* if it is covered by a map of the form $i(g)$ where g belongs to $\mathcal{F}$; and f will be called *small* if both f itself and $Y \to Y \times_X Y$ are quasi-small.

Proposition 5.2. *If f is small fibration, then $i(f)$ is small.*

Proof. If $f: Y \to X$ is a small fibration, then $i(f)$ is clearly quasi-small; so it remains to show that $i(Y) \to i(Y) \times_{i(X)} i(Y)$ is quasi-small. First of all, i preserves pullbacks along fibrations, so $i(Y \times_X Y) \cong i(Y) \times_{i(X)} i(Y)$. In addition, i turns homotopy equivalences into isomorphisms, so $i(Y) \cong i(P_X(Y))$. Therefore $i(Y) \to i(Y) \times_{i(X)} i(Y)$ is isomorphic to $i(P_X(Y)) \to i(Y \times_X Y)$, which is quasi-small by axiom (F4). □

In order to say something more about the small maps, we first need to obtain some results about the class of quasi-small maps.

Lemma 5.3. *The following are equivalent for a map $f: (Y, S) \to (X, R)$ in* $\mathrm{Hex}(\mathcal{C})$*:*

(i) *f is quasi-small.*
(ii) *For any epi $i(A) \to (X, R)$ there is a covering square of the form*

$$\begin{tikzcd} iB \arrow[r] \arrow[d, "i(g)"'] & (Y, S) \arrow[d, "f"] \\ iA \arrow[r] & (X, R) \end{tikzcd}$$

with $g: B \to A$ a small fibration in $\mathcal{C}$.

(iii) *There is a covering square of the form*

$$\begin{tikzcd} iZ \arrow[r] \arrow[d, "i(g)"'] & (Y,S) \arrow[d, "f"] \\ iX \arrow[r, "1"'] & (X,R) \end{tikzcd}$$

with $g\colon Z \to X$ a small fibration in $\mathcal{C}$.

Proof. Since (ii) ⇒ (iii) and (iii) ⇒ (i) are obvious, we only have to prove (i) ⇒ (ii).

So suppose

$$\begin{tikzcd} iD \arrow[r, "q"] \arrow[d, "i(h)"'] & (Y,S) \arrow[d, "f"] \\ iC \arrow[r, "p"'] & (X,R) \end{tikzcd}$$

is a covering square with $h\colon D \to C$ a small fibration in $\mathcal{C}$ and assume in addition that $e\colon i(A) \to (X,R)$ is epi. Since $i(A)$ is projective and the functor i is full, there is a map $v\colon A \to C$ such that $e = pi(v)$. If g is the result of pulling back h along v in $\mathcal{C}$, then g is a small fibration by (F1); as i preserves pullbacks of fibrations, $i(g)$ will cover f, as desired. □

Proposition 5.4. *Let $\mathcal{C}$ be a path category and $\mathcal{F}$ be a class of fibrations in $\mathcal{C}$ satisfying (F1-4). If $\mathcal{S}$ is the class of quasi-small maps in* Hex$(\mathcal{C})$ *determined by $\mathcal{F}$, then $\mathcal{S}$ satisfies the following axioms:*

(S1) *In a pullback square*

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "p"'] & X \end{tikzcd}$$

f' belongs to $\mathcal{S}$ whenever f does.

(S2) *If in a pullback square as the one above, the map p is a cover, then f belongs to $\mathcal{S}$ whenever f' does.*

(S3) *$\mathcal{S}$ is closed under composition.*

(S4) *$\mathcal{S}$ contains all isomorphisms.*

(Q) *In a commutative triangle*

$$\begin{tikzcd} Z \arrow[rr, "p"] \arrow[dr, "g"'] & & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

in which g belongs to $\mathcal{S}$ and p is a cover, f also belongs to $\mathcal{S}$.

Proof. It is immediate from the definition that the quasi-small maps satisfy both (S2) and (Q). So we check the other axioms.

(S1): Suppose

$$\begin{tikzcd}
Y' \arrow[r] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\
X' \arrow[r, "p"'] & X
\end{tikzcd}$$

is a pullback in which f is covered by a map of the form $i(g)$, where $g\colon B \to A$ is a small fibration in $\mathcal{C}$. We can construct a cube

$$\begin{tikzcd}
W \arrow[rr] \arrow[dr] \arrow[dd] & & iB \arrow[dr] \arrow[dd, "i(g)" near end] & \\
& Y' \arrow[rr] \arrow[dd, "f'" near end] & & Y \arrow[dd, "f"] \\
V \arrow[rr] \arrow[dr] & & iA \arrow[dr] & \\
& X' \arrow[rr] & & X
\end{tikzcd}$$

in which the bottom and top face are pullbacks, so that the face at the back becomes a pullback as well and the face on the left becomes a covering square. We can cover V by an object in the image of i, via $i(A') \to V$ say, and, since i is full, the composed map $i(A') \to i(A)$ is of the form $i(h)$ for some $h\colon A' \to A$ in $\mathcal{C}$. Since i preserves pullbacks of fibrations, we may compute the pullback of $i(g)$ along $i(h)$ by first taking the pullback of g along h in $\mathcal{C}$ and then applying i. The result will be some $i(g')$ where $g\colon B' \to A'$ is a small fibration in $\mathcal{C}$ by axiom (F1): since f' will be covered by this map, f' will be quasi-small, as desired.

(S3): Suppose $f\colon Y \to X$ is covered by $i(f')\colon i(B') \to i(A')$ and $g\colon Z \to Y$ is covered by $i(g')\colon i(C') \to i(B'')$, as in:

$$\begin{tikzcd}
i(B') \arrow[r, "p"] \arrow[d, "i(f')"'] & Y \arrow[d, "f"] & i(C'') \arrow[r] \arrow[d, "i(g'')"'] & Z \arrow[d, "g"] \\
i(A') \arrow[r] & X & i(B'') \arrow[r, "q"'] & Y
\end{tikzcd}$$

In fact, Lemma 5.3 tells us that we may assume that $B' = B''$, so that f' and g' can be composed. But then $f'g'$ is a small fibration, by axiom (F3), and $i(f'g')$ covers fg, as desired.

(S4): The functor i preserves the terminal object, so (F2) and Proposition 5.2 imply that $1 \to 1$ is quasi-small in $\mathrm{Hex}(\mathcal{C})$. But if $f\colon Y \to X$ is an isomorphism, then

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] & 1 \arrow[d] \\ X \arrow[r] & 1 \end{tikzcd}$$

is a pullback. So (S1) implies that f is quasi-small. □

From now on we will assume that $\mathcal{C}$ is a homotopy extensive path category, so that $\mathcal{C}$ has well-behaved homotopy sums and $\mathrm{Hex}(\mathcal{C})$ is a pretopos (see Theorem 4.10). In that case it makes sense to require the following additional properties for our class of small fibrations $\mathcal{F}$:

- (F5) If two maps $Y \to X$ and $Y' \to X'$ both belong to $\mathcal{F}$ then so does a fibrant replacement of $Y + Y' \to X + X'$.
- (F6) The maps $0 \to 1$ and $1 + 1 \to 1$ belong to $\mathcal{F}$.

Note that since smallness is a homotopy invariant property of fibrations, axiom (F5) is unambiguous and does not depend on the particular choice of a fibrant replacement.

Proposition 5.5. *Let $\mathcal{C}$ be a homotopy extensive path category and $\mathcal{F}$ be a class of fibrations in $\mathcal{C}$ satisfying axioms (F1-6). If $\mathcal{S}$ is the class of quasi-small maps determined by $\mathcal{F}$ in* $\mathrm{Hex}(\mathcal{C})$*, then $\mathcal{S}$ satisfies axioms (S1-4) and (Q), as well as:*

- (S5) *If two maps $Y \to X$ and $Y' \to X'$ both belong to $\mathcal{S}$ then so does their sum $Y + Y' \to X + X'$.*
- (S6) *For any object X the map $0 \to X$ belongs to $\mathcal{S}$.*
- (S7) *If $Y \to X$ and $Z \to X$ belong to $\mathcal{S}$, then so does $Y + Z \to X$.*

Proof. It remains to check axioms (S5-7).

(S5): This is a consequence of the combination of the following: axiom (F5), the fact that under the functor i homotopy sums become sums, and the fact that in a pretopos the sum of two covering squares is again a covering square.

(S6): The functor i maps homotopy initial objects to initial objects, so $0 \to 1$ is quasi-small in $\mathrm{Hex}(\mathcal{C})$ by Proposition 5.2 and (F6). For any object X the square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ X \arrow[r] & 1 \end{tikzcd}$$

is a pullback, so $0 \to X$ is quasi-small by (S1).

(S7): For any two quasi-small maps $Y \to X$ and $Z \to X$ the map $Y + Z \to X$ is the composition of $Y + Z \to X + X$ and $X + X \to X$, so to prove that $Y + Z \to X$

is quasi-small it suffices by (S3) and (S5) to prove that $X + X \to X$ is quasi-small. For this it suffices to prove that $1 + 1 \to 1$ is quasi-small, because

$$\begin{tikzcd} X + X \arrow[r] \arrow[d] & 1 + 1 \arrow[d] \\ X \arrow[r] & 1 \end{tikzcd}$$

is a pullback. But from the fact that the functor i maps homotopy sums to sums and the fact that (F6) holds for $\mathcal{F}$, one obtains that $1 + 1 \to 1$ is quasi-small in $\mathrm{Hex}(\mathcal{C})$. □

Theorem 5.6. *Suppose that $\mathcal{C}$ is a homotopy extensive path category and $\mathcal{F}$ is a class of small fibrations in $\mathcal{C}$ satisfying axioms (F1-6). If $\mathcal{S}$ is the class of small maps in* $\mathrm{Hex}(\mathcal{C})$ *determined by $\mathcal{F}$, then $\mathcal{S}$ satisfies axioms (S1-7) from the previous proposition, as well as:*

(S8) *Suppose $R \to Y \times_X Y$ is an equivalence relation in $\mathcal{E}/X$ with quotient $Y \to Q$ as in*

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & Y \arrow[r] \arrow[d] & Q \arrow[dl] \\ & X. & \end{tikzcd}$$

If $R \to X$ and $Y \to X$ belong to $\mathcal{S}$, then so does $Q \to X$.

(S9) *Suppose*

$$\begin{tikzcd} Z \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

is a commutative triangle. If both g and h belong to $\mathcal{S}$, then so does f.

Proof. It is not hard to verify that any class of "open maps" satisfying (S1-7) and (Q) in a pretopos determines a class of "étale maps" satisfying (S1-9) by declaring a map $f: Y \to X$ to be étale if both f and $Y \to Y \times_X Y$ are open (compare [30, Proposition 1.6]). □

Definition 5.7. In general, any class of maps $\mathcal{S}$ in a pretopos $\mathcal{E}$ satisfying axioms (S1-9) will be called a *class of small maps*; if $\mathcal{S}$ is such a class, its elements will be called *small*.

Remark 5.8. If $\mathcal{S}$ is a class of small maps in a pretopos $\mathcal{E}$, it also has the following property:

> If a map $h: Z \to X \in \mathcal{S}$ is written as $h = gf$ with $g: Y \to X$ a mono and $f: Z \to Y$ an epi, then g belongs to $\mathcal{S}$.

To see this, assume that $h: Z \to X$ is small; then $Z \times_X Z \to X$ is small by (S1) and (S3). Since g is monic, the kernel pairs of h and f coincide and $Z \times_Y Z \to X$ is small as well. In a pretopos every epi is the coequalizer of its kernel pair, so the fact that g is small now follows from (S8).

5.2. **Small maps characterised in terms of Ex.** We continue to work in the setting of Theorem 5.6; that is, $\mathcal{C}$ is a homotopy extensive path category equipped with a class of small fibrations $\mathcal{F}$ satisfying axioms (F1-6).

In our discussion of Π-types below it will be convenient to have a more explicit description of the small maps in $\mathrm{Hex}(\mathcal{C})$. As we will show in this subsection, this can be done using the fibrations in $\mathrm{Ex}(\mathcal{C})$.

Lemma 5.9. *A map $\varphi\colon (Y,S) \to (X,R)$ in $\mathrm{Hex}(\mathcal{C})$ is quasi-small if and only if there are*

(i) *a fibration $f\colon (Z,T) \to (X,R)$ in $\mathrm{Ex}(\mathcal{C})$ whose underlying map $f\colon Z \to X$ is a small fibration, and*
(ii) *an isomorphism $(Y,S) \cong (Z,T)$ in $\mathrm{Hex}(\mathcal{C})$ making*

$$\begin{tikzcd} (Y,S) \arrow[rr, "\cong"] \arrow[dr, "\varphi"'] & & (Z,T) \arrow[dl, "f"] \\ & (X,R) & \end{tikzcd}$$

commute.

Proof. Recall from Theorem 3.14 that the pullback of a fibration $f\colon (Z,T) \to (X,R)$ along the projective cover $iX \to (X,R)$ is (Z,T') with $T' = T \cap f^*PX$. This means that if (i) and (ii) hold, then

$$\begin{tikzcd} iZ \arrow[r] \arrow[d, "i(f)"'] & (Z,T) \arrow[d, "f"] \\ iX \arrow[r] & (X,R) \end{tikzcd}$$

is a covering square, showing that both f and φ are quasi-small.

To show the converse, assume without loss of generality that φ is a fibration $(Y,S) \to (X,R)$ in $\mathrm{Ex}(\mathcal{C})$. It follows from Lemma 5.3 that φ is quasi-small if and only if there exists a covering square of the form

$$\begin{tikzcd} iZ \arrow[r] \arrow[d, "if"'] & (Y,S) \arrow[d] \\ iX \arrow[r] & (X,R) \end{tikzcd}$$

with f a small fibration in $\mathcal{C}$. Such a covering square can be decomposed as a pullback and an epi, as in

$$\begin{tikzcd} iZ \arrow[r, two heads] \arrow[dr, "i(f)"'] & (Y,S') \arrow[r] \arrow[d] & (Y,S) \arrow[d, "g"] \\ & iX \arrow[r] & (X,R) \end{tikzcd}$$

where $S' = S \cap g^*PX$. Let $k\colon Z \to Y$ be a morphism representing the epimorphism $i(Z) \to (Y,S)$ in $\mathrm{Hex}(\mathcal{C})$. Since the diagram commutes, the maps kg and f are R-equivalent; so, using the weak connection structure on g, we can replace k by an S-equivalent map k' such that $gk' = f$. In other words, we may assume that $gk = f$ and from now on we will.

Put $T = k^*S$. Then $(Z,T) \cong (Y,S)$ (see Lemma 3.7), so the proof will be finished once we show that $f\colon (Z,T) \to (X,R)$ is a fibration in $\mathrm{Ex}(\mathcal{C})$. Therefore it remains to construct a weak connection structure $Z \times_X R \to T$.

To avoid having to draw cumbersome diagrams, we will describe this weak connection using the language of elements. So suppose $f(z_0) = x_0$ and $r \in R$ connects x_0 and x_1. Using the weak connection structure on g we find $y_1 \in Y$ with $g(y_1) = x_1$ and an element $s_1 \in S$ connecting $k(z_0)$ and y_1. From the fact that $i(Z) \to (Y,S')$ is an epimorphism, one obtains an element $z_2 \in Z$, an element $s_2 \in S$ and a path α in PX with s_2 connecting y_1 and $k(z_2)$ and α connecting x_1 and $f(z_2)$. Using α and the fact that f is a fibration in $\mathcal{C}$ we find an element $z_1 \in Z$ with $f(z_1) = x_1$ and a path β in Z connecting z_2 and z_1. It remains to check that z_0 and z_1 are T-equivalent, in other words, that $k(z_0)$ and $k(z_1)$ are S-equivalent: note that there is a path $k(\beta)$ connecting $k(z_2)$ and $k(z_1)$ and therefore there is also an element $s_3 \in S$ connecting them. So the following chain

$$k(z_0) \xrightarrow{s_1} y_1 \xrightarrow{s_2} k(z_2) \xrightarrow{s_3} k(z_1)$$

gives us what we want. □

Lemma 5.10. (i) *Monomorphisms are small if and only if they are quasi-small.*
(ii) *A monomorphism $A \to i(X)$ in* $\mathrm{Hex}(\mathcal{C})$ *is small if and only if under the correspondence with the poset reflection of $\mathcal{C}(X)$ (see Proposition 3.21) the corresponding equivalence class contains a small fibration $Y \to X$.*

Proof. (i): If $m\colon A \to X$ is a monomorphism, then the diagonal $\Delta\colon A \to A \times_X A$ is an isomorphism and any isomorphism is quasi-small.

(ii): In view of Lemma 3.5 we may assume that $A \to iX$ is of the form $(Z, m^*PX) \to (X, PX)$ for a fibration $m\colon Z \to X$ in $\mathcal{C}$. Lemma 5.3 tells us that this map is (quasi-)small if and only if there is small fibration $f\colon Y \to X$ in $\mathcal{C}$ and an epi $e\colon (Y, PY) \to (Z, m^*PX)$ such that $me \simeq f$. Since m is a fibration, we may assume that $me = f$. Then from the fact that e is an epi we have $(Y, f^*PX) \cong (Z, m^*PX)$ (see Lemma 3.7). In particular, there is a map $d\colon Z \to Y$ with $fd \simeq m$ and hence also a map $d'\colon Z \to Y$ with $fd' = m$ (because f is a fibration too). In other words, m and f are identified in the poset reflection of $\mathcal{C}(X)$. □

Theorem 5.11. *A map $\varphi\colon (Y,S) \to (X,R)$ in $\mathrm{Hex}(\mathcal{C})$ is small if and only if there are*

(i) *a fibration $f\colon (Z,T) \to (X,R)$ in $\mathrm{Ex}(\mathcal{C})$ whose underlying map $f\colon Z \to X$ is a small fibration, and*
(ii) *a small fibration $U \to Z \times_X Z$ which becomes identified with the pullback*

$$\begin{tikzcd} V \arrow[r] \arrow[d] & T \arrow[d] \\ Z \times_X Z \arrow[r] & Z \times Z \end{tikzcd}$$

in the poset reflection of $\mathcal{C}(Z \times_X Z)$, and
(iii) *an isomorphism $(Y,S) \cong (Z,T)$ in $\mathrm{Hex}(\mathcal{C})$ making*

$$\begin{tikzcd} (Y,S) \arrow[rr, "\cong"] \arrow[dr, "\varphi"'] & & (Z,T) \arrow[dl, "f"] \\ & (X,R) & \end{tikzcd}$$

commute.

Proof. By definition, a map $\varphi\colon (Y,S) \to (X,R)$ is small if and only if both φ itself and its diagonal $(Y,S) \to (Y,S) \times_{(X,R)} (Y,S)$ are quasi-small. If we first replace φ by a fibration $(Z,T) \to (X,R)$ in $\mathrm{Ex}(\mathcal{C})$ and observe that $(Z,T) \times_{(X,R)} (Z,T) \cong (Z \times_X Z, T \times T)$, the latter is equivalent to saying that the map m in the pullback

$$\begin{tikzcd} (P,E) \arrow[r, "n"] \arrow[d, "m"'] & (Z,T) \arrow[d] \\ i(Z \times_X Z) \arrow[r] & (Z \times_X Z, T \times T) \end{tikzcd}$$

is quasi-small. If we construct the pullback (P,E) in the canonical manner, P will consist of 5-tuples (z_0, z_1, z, t_0, t_1) where z_0 and z_1 are elements of Z living over the same element in X, z is another element of Z and $t_0 \in T$ connects z_0 and z, while $t_1 \in T$ connects z_1 and z; moreover, m projects onto the first two coordinates, n projects onto the third and $E = m^*P(Z \times_X Z) \cap n^*T$. It is not hard to see that (P,E) is isomorphic to (V,Q), where V is the pullback

$$\begin{tikzcd} V \arrow[r] \arrow[d, "p"'] & T \arrow[d] \\ Z \times_X Z \arrow[r] & Z \times Z \end{tikzcd}$$

and $Q = p^*P(Z \times_X Z)$. So the previous lemma implies that m is quasi-small if and only if in the poset reflection $p\colon V \to Z \times_X Z$ becomes identified with some small map $U \to Z \times_X Z$. The theorem now follows. $\square$

This theorem has the following consequence which will be especially useful in our subsequent discussion of Π-types.

Corollary 5.12. *A morphism $\varphi\colon A \to i(X)$ in $\mathrm{Hex}(\mathcal{C})$ is small if and only if under the equivalence with the stable objects in $\mathrm{Hex}(\mathcal{C}(X))$ it is isomorphic to an object of the form $(f\colon Z \to X, \tau\colon U \to Y \times_X Y)$ where both f and τ are small fibrations.*

6. Π-types

In this section, we study a form of function space in path categories and the structure these function spaces induce on the homotopy exact completion. We are guided by the relevant properties of the classical exact completion of categories with finite limits, where the existence of a weak kind of internal hom-object in every slice of the original category implies that every slice of the exact completion has actual internal homs, i.e., is a locally cartesian closed category [16]. These weak internal hom-objects enjoy the existence condition for the internal hom in the sense that any map $A \times B \to C$ gives a map $A \to \mathrm{Hom}(B, C)$, but the latter is not required to be unique. A similar situation arises in type theory, and the path categories constructed as syntactic categories of dependent type theories only possess such weak internal homs. It is important to realise that for these type-theoretic categories there is a priori no uniqueness condition involved at all, not even in a up-to-homotopy sense. (Uniqueness up to homotopy is related to an additional property of type theory called function extensionality, see Remark 6.2 below.)

More generally, dependent type theories usually include a type constructor for Π-types. For a path category $\mathcal{C}$ arising as the syntactic category of such a type theory, the pullback functors $\mathcal{C}(B) \to \mathcal{C}(A)$ along fibrations $B \to A$ have a weak kind of right adjoint (weakness here is meant in the same sense as for internal homs above).

In this section, we will define notions of weak homotopy exponential and weak homotopy Π-type in the context of an arbitrary path category $\mathcal{C}$. The notion of weak homotopy Π-type is sufficiently strong to ensure that the homotopy exact completion $\mathrm{Hex}(\mathcal{C})$ is locally cartesian closed if $\mathcal{C}$ has weak homotopy Π-types. In particular, for the special case where $\mathcal{C}$ is a category with finite limits and every map in $\mathcal{C}$ is a fibration and every weak equivalence in $\mathcal{C}$ is an isomorphism, our notion of having weak homotopy Π-types corresponds to the notion of weak local cartesian closure from [16]. In addition, these notions are sufficiently weak to ensure that these structures exist in the syntactic path category obtained from a type theory possessing the corresponding type constructions, even if in the type theory the computation rules would hold only in a propositional form.

6.1. Definition and properties.

Throughout this section $\mathcal{C}$ will be a path category.

Definition 6.1. For objects X and Y in $\mathcal{C}$ a *weak homotopy exponential* is an object X^Y together with a map $\mathrm{ev}\colon X^Y \times Y \to X$ such that for any map $h\colon A \times Y \to X$

there is a map $H\colon A \to X^Y$ such that

$$\begin{tikzcd} X^Y \times Y \arrow[r, "\mathrm{ev}"] & X \\ A \times Y \arrow[u, "H\times 1"] \arrow[ur, "h"'] & \end{tikzcd}$$

commutes up to homotopy. If such a map H is unique up to homotopy, then X^Y is a *homotopy exponential*.

Remark 6.2. Suppose $\mathcal{C}$ has weak homotopy exponentials. One can prove that a weak homotopy exponential X^Y in $\mathcal{C}$ is an ordinary homotopy exponential precisely when there is a morphism $e\colon (PX)^Y \to P(X^Y)$ making both these squares commute up to homotopy:

$$\begin{tikzcd} (PX)^Y \times Y \arrow[r, "se\times 1"] \arrow[d, "\mathrm{ev}"'] & X^Y \times Y \arrow[d, "\mathrm{ev}"] \\ PX \arrow[r, "s"'] & X \end{tikzcd} \qquad \begin{tikzcd} (PX)^Y \times Y \arrow[r, "te\times 1"] \arrow[d, "\mathrm{ev}"'] & X^Y \times Y \arrow[d, "\mathrm{ev}"] \\ PX \arrow[r, "t"'] & X \end{tikzcd}$$

In other words, ordinary homotopy exponentials are those weak homotopy exponentials that satisfy what type-theorists call function extensionality (indeed, in the syntactic category the morphism e would be a proof term for the type-theoretic translation of the statement that two functions $f, g\colon Y \to X$ are equal if $f(y)$ and $g(y)$ are equal for every $y \in Y$). This principle is not valid in the syntactic category associated to type theory, and for this reason the homotopy exponentials in the syntactic category are only weak. The same applies to the homotopy Π-types that we will define below: the syntactic category only has these in the weak form.

Definition 6.3. The category $\mathcal{C}$ has *weak homotopy* Π*-types* if for any two fibrations $f\colon X \to J$ and $\alpha\colon J \to I$ there is a an object $\Pi_\alpha X = \Pi_\alpha f$ in $\mathcal{C}(I)$, that is, a fibration $\Pi_\alpha X \to I$, together with an evaluation map $\mathrm{ev}\colon \alpha^*\Pi_\alpha X \to X$ over I, with the following weak universal property: for any map $g\colon Y \to I$ and $m\colon \alpha^* Y \to X$ over J there is a map $n\colon Y \to \Pi_\alpha X$ over I such that $m\colon \alpha^* Y \to X$ and $\mathrm{ev} \circ \alpha^* n\colon \alpha^* Y \to X$ are fibrewise homotopic over J. If up to fibrewise homotopy over I the map n is unique with this property, we call $\Pi_\alpha f$ and $\mathrm{ev}\colon \alpha^*\Pi_\alpha X \to X$ a *homotopy* Π*-type*.

Remark 6.4. We will not need this observation, but we would like to point out that in the definition above it is sufficient to consider only fibrations $g\colon Y \to I$.

In the proofs of the following two propositions we only give the constructions: verifications are left to the reader.

Proposition 6.5. *If $\mathcal{C}$ has (weak) homotopy Π-types then each $\mathcal{C}(I)$ has (weak) homotopy exponentials.*

Proof. Given $Y, Z \in \mathcal{C}(I)$ one defines Z^Y in $\mathcal{C}(I)$ as $\Pi_\alpha(\pi_2)$, where $\alpha\colon Y \to I$ and $\pi_2\colon Z \times_I Y \to Y$. $\square$

Proposition 6.6. *Let $\mathcal{C}$ be a path category with (weak) homotopy Π-types. Given a fibration $p: Z \to Y$ and a (weak) homotopy exponential (Y^X, ev), there is a (weak) homotopy exponential (Z^X, ev) and a fibration $p^X: Z^X \to Y^X$ such that*

(i) *The diagram*

$$\begin{tikzcd} Z^X \times X \arrow[r, "\mathrm{ev}"] \arrow[d, "p^X \times X"'] & Z \arrow[d, "p"] \\ Y^X \times X \arrow[r, "\mathrm{ev}"'] & Y \end{tikzcd}$$

commutes.

(ii) *For each T the diagram*

$$\begin{tikzcd} \mathrm{Ho}(\mathcal{C})(T, Z^X) \arrow[r] \arrow[d] & \mathrm{Ho}(\mathcal{C})(T \times X, Z) \arrow[d] \\ \mathrm{Ho}(\mathcal{C})(T, Y^X) \arrow[r] & \mathrm{Ho}(\mathcal{C})(T \times X, Y) \end{tikzcd}$$

in $\mathcal{S}ets$ has the property that the map from $\mathrm{Ho}(\mathcal{C})(T, Z^X)$ of the inscribed pullback is an isomorphism in case Z^X is a homotopy exponential, and an epimorphism in case Z^X is a weak homotopy exponential.

Proof. Given Y^X with its evaluation $\mathrm{ev}: Y^X \times X \to Y$ let q be the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d, "q"'] & Z \arrow[d, "p"] \\ Y^X \times X \arrow[r, "\mathrm{ev}"'] & Y, \end{tikzcd}$$

and let Z^X be $\Pi_{\pi_1}(q)$, where $\pi_1: Y^X \times X \to Y^X$. □

Corollary 6.7. *Suppose $p: Z \to Y$ is a fibration and $p^X: Z^X \to Y^X$ is the fibration obtained from it as in the previous proposition. Then any section $s: Y \to Z$ induces a section s^X of p^X such that*

$$\begin{tikzcd} Y^X \times X \arrow[r, "\mathrm{ev}"] \arrow[d, "s^X \times 1_X"'] & Y \arrow[d, "s"] \\ Z^X \times X \arrow[r, "\mathrm{ev}"'] & Z \end{tikzcd}$$

commutes up to homotopy.

Proof. Consider the diagram in (ii) in the previous proposition with $T = Y^X$. Using that the map to the inscribed pullback is an epimorphism, one finds a map $\sigma: Y^X \to Z^X$ that upon postcomposition with p^X is homotopic to the identity and such that

$$\mathrm{ev}(\sigma \times 1_X) \simeq s\,\mathrm{ev}.$$

Using that p^X is a fibration, one may replace σ by a homotopic map s^X such that $p^X s^X = 1$ and $\mathrm{ev}(s^X \times 1_X) \simeq s\,\mathrm{ev}$. $\square$

6.2. **Homotopy exact completion.** The main goal of this section is to show that $\mathrm{Hex}(\mathcal{C})$ is locally cartesian closed, whenever $\mathcal{C}$ has weak homotopy Π-types. We will only outline the constructions here, as a detailed verification that they indeed have the required properties is both straightforward and cumbersome.

Proposition 6.8. *If $\mathcal{C}$ has weak homotopy Π-types, then $\mathrm{Hex}(\mathcal{C})$ has exponentials.*

Proof. Assume $\mathcal{C}$ has weak homotopy Π-types, and let (X, R) and (Y, S) be two objects in $\mathrm{Hex}(\mathcal{C})$; our goal is to construct the exponential $(X, R)^{(Y,S)}$.

The idea is to take (W, Q) where W is the pullback:

$$\begin{tikzcd} W \arrow[rr] \arrow[d, "p"'] & & R^S \arrow[d] \\ X^Y \arrow[r, "\delta"'] & (X \times X)^{Y \times Y} \arrow[r] & (X \times X)^S. \end{tikzcd}$$

Here δ is a map making

$$\begin{tikzcd} X^Y \times Y \times Y \arrow[r, "\delta \times 1"] \arrow[dr, "\epsilon"'] & (X \times X)^{Y \times Y} \times (Y \times Y) \arrow[d, "\mathrm{ev}"] \\ & X \times X \end{tikzcd}$$

commute up to homotopy with $\epsilon = (\mathrm{ev}(p_1, p_2), \mathrm{ev}(p_1, p_3))$, while the map $R^S \to (X \times X)^S$ has the properties from Proposition 6.6; in particular it is a fibration and the pullback W does indeed exist. The object Q is obtained as the pullback

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & R^Y \arrow[d] \\ W \times W \arrow[r, "p \times p"'] & X^Y \times X^Y \cong (X \times X)^Y, \end{tikzcd}$$

where we have used that $X^Y \times X^Y$ acts a suitable weak homotopy exponential $(X \times X)^Y$. In addition, the map on the right is built in accordance with Proposition 6.6; this means in particular that it is a homotopy equivalence relation and therefore the same is true for $Q \to W \times W$. We leave it to the reader to verify that (W, Q) is indeed an exponential. $\square$

Theorem 6.9. *Let $\mathcal{C}$ be a path category. If $\mathcal{C}$ has weak homotopy Π-types, then $\mathrm{Hex}(\mathcal{C})$ is locally cartesian closed.*

Proof. We need to prove that each slice category $\mathrm{Hex}(\mathcal{C})/I$ has exponentials. For this it suffices to consider the case where $I = iZ$: any object in $\mathrm{Hex}(\mathcal{C})$ is covered by such an object (see Proposition 3.16), so the general case follows by descent. Indeed, if $p: P \to I$ and $q: Q \to P \times_I P$ are covers in an exact category $\mathcal{E}$ and both $\mathcal{E}/P$ and $\mathcal{E}/Q$ are cartesian closed, then the exponential of $(X \to I)^{(Y \to I)}$ in $\mathcal{E}/I$ may be computed from two exponentials in $\mathcal{E}/Q$ and $\mathcal{E}/P$ by the taking the coequalizer of the two parallel arrows along the top in the diagram below:

$$\begin{tikzcd} \big((X \times_I Q)^{(Y \times_I Q)}\big)_Q \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \big((X \times_I P)^{(Y \times_I P)}\big)_P \arrow[d] \\ Q \arrow[r, shift left] \arrow[r, shift right] & P. \end{tikzcd}$$

We have proved in Theorem 3.28 that $\mathrm{Hex}(\mathcal{C})/iZ$ is equivalent to the full subcategory of $\mathrm{Hex}(\mathcal{C}(Z))$ on the stable objects. It follows from Proposition 6.5 and Proposition 6.8 that $\mathrm{Hex}(\mathcal{C}(Z))$ has exponentials, so it suffices to prove that if we take an exponential of two stable objects in this category, then the result is again stable.

So let $(f: X \to Z, \rho: R \to X \times_Z X)$ and $(g: Y \to Z, \sigma: S \to Y \times_Z Y)$ be two stable objects in $\mathrm{Hex}(\mathcal{C}(Z))$. These will have two (essentially unique) transport structures $\Gamma_X: X \times_Z PZ \to X$ and $\Gamma_Y: Y \times_Z PZ \to Y$; recall that stability means that $\Gamma(x, \alpha) \simeq_R x$ and $\Gamma(y, \alpha) \simeq_S y$ whenever α is a loop in Z.

So let $(h: W \to Z, q: Q \to W \times_Z W)$ be the result of computing the exponential $(X, R)^{(Y,S)}$ over (Z, PZ) as in the previous proposition. This object has a transport structure as well, which is probably best described in words. What this action should do is to associate to every $w \in W$ living over $z \in Z$ and path α from z to z' a new element $w' \in W$ over z'. Such a w' is intuitively a function, so let $y' \in Y$ be an element over z'. We can transport y' back along the inverse of α to an element y over z; to this y we can apply w and obtain an element x over z. Using transport again, but now on $x \in X$ and α we find an element $x' \in X$ over z'. The idea is to set w' to be the function sending y' to x'. Proposition 3.26 implies that w' will be tracked whenever w is.

If α is a loop, then y' would be S-equivalent to y and x would be T-equivalent to x'. This means that w and w' would be Q-equivalent, showing that (W, Q) is stable, as desired. $\square$

Remark 6.10. One could also have derived Theorem 6.9 from Proposition 3.17 above and the results in [16]. We have included a direct proof of Theorem 6.9 here, for several reasons. The main one is that we need the description of the exponentials in $\mathrm{Hex}(\mathcal{C})$ that this proof provides for our proof of Theorem 6.11 below; moreover, this description only makes sense in the specific context of exact completions of path categories and would not work in the more general context of exact completions of categories with weak finite limits.

In addition, these constructions can also be used to show that $\mathrm{Ex}(\mathcal{C})$ has homotopy Π-types whenever $\mathcal{C}$ has weak homotopy Π-types. In view of Remark 6.2 this means that $\mathrm{Ex}(\mathcal{C})$ satisfies a form of function extensionality even when $\mathcal{C}$ does not. We plan to take this up in future work.

Theorem 6.11. *Let $\mathcal{C}$ be a path category with weak homotopy Π-types, and let $\mathcal{F}$ be a class of fibrations in $\mathcal{C}$ satisfying axioms (F1-4) as well as:*

(F7) *If $f: Y \to X$ and $g: Z \to Y$ belong to $\mathcal{F}$, then so does $\Pi_f(g) \to X$.*

If $\mathcal{S}$ is the class of small maps determined by $\mathcal{F}$ in $\mathrm{Hex}(\mathcal{C})$*, then the class $\mathcal{S}$ satisfies:*

(S10) *For any $Y \to X$ and $Z \to X$ in $\mathcal{S}$, their exponent $(Z^Y)_X \to X$ in $\mathcal{E}/X$ belongs to $\mathcal{S}$.*

Proof. Let $\mathcal{C}$ and $\mathcal{F}$ be as in the theorem. To prove (S10) it again suffices, by descent, to consider the case where $X = iA$. But in that case we use again Theorem 3.28 to identify $\mathrm{Hex}(\mathcal{C})/iA$ with the full subcategory of $\mathrm{Hex}(\mathcal{C}(A))$ on the stable objects.

So let (X, R) and (Y, S) be two stable objects in $\mathrm{Hex}(\mathcal{C}(A))$ and assume that they correspond to small maps $(X, R) \to iA$ and $(Y, S) \to iA$. In view of Corollary 5.12 this means that we may assume that $X \to A$, $R \to X \times_A X$, $Y \to A$ and $Y \to S \times_A S$ are all small fibrations. But if we then follow the construction of the exponential $(X, R)^{(Y,S)}$ over iA as in Proposition 6.8 the result is an object (W, Q) where both $W \to A$ and $Q \to W \times_A W$ are small fibrations by (F6). This proves the theorem. □

7. W-types

What the previous section did for Π-types, this section will do for inductive types; in particular, we will look at the natural numbers and a certain type of well-founded trees called W-types. More concretely, we will formulate a notion of a homotopy natural numbers object and a homotopy W-type and prove that if a path category $\mathcal{C}$ has such structure its homotopy exact completion $\mathrm{Hex}(\mathcal{C})$ has a genuine natural numbers object or genuine W-types; in addition, the definitions are chosen in such a way that they will apply to the syntactic category associated to Martin-Löf type theory with W-types.

For the definition of genuine W-types in a locally cartesian closed category we refer to [37]; we will also make heavy use of the results from [7].

7.1. Homotopy natural numbers object.

A homotopy natural numbers object we define, like a homotopy sum, as a natural numbers object in the homotopy category.

Definition 7.1. An object $\mathbb{N}$ together with maps $0: 1 \to \mathbb{N}$ and $s: \mathbb{N} \to \mathbb{N}$ is a *homotopy natural numbers object* (hnno) if for any pair of maps $y_0: 1 \to Y$ and $g: Y \to Y$ there is a map $h: \mathbb{N} \to Y$, unique up to homotopy, such that $h0 \simeq y_0$ and $hs \simeq gh$.

Proposition 7.2. *An object $\mathbb{N}$ together with maps $0: 1 \to \mathbb{N}$ and $s: \mathbb{N} \to \mathbb{N}$ is a homotopy natural numbers object if and only if for any commuting diagram of the* *form*

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[d, "p"] & X \arrow[d, "p"] \\ 1 \arrow[ur, "x_0"] \arrow[r, "0"'] & \mathbb{N} \arrow[r, "s"'] & \mathbb{N} \end{tikzcd}$$

where p is a fibration, there is a section $a\colon \mathbb{N} \to X$ of p such that $a0 \simeq x_0$ and $as \simeq fa$.

Proof. The argument is very similar to proofs of both Proposition 4.2 and Proposition 4.3, so we will not give many details here. Let us just point out how one proves that if $0\colon 1 \to \mathbb{N}$ and $s\colon \mathbb{N} \to \mathbb{N}$ are as in the statement of the proposition, then for any pair of maps $y_0\colon 1 \to Y$ and $g\colon Y \to Y$ and for any pair of maps $h, h'\colon \mathbb{N} \to Y$ such that $h0 \simeq y_0$ and $h\sigma \simeq gh$ and $h'0 \simeq y_0$ and $h'\sigma \simeq gh'$, one must have $h \simeq h'$. For this one constructs the pullback

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"'] & PY \arrow[d, "(s,t)"] \\ \mathbb{N} \arrow[r, "(h,h')"'] & Y \times Y. \end{tikzcd}$$

Since $h0 \simeq y_0 \simeq h'0$, there is a map $x_0\colon 1 \to X$ such that $px_0 = 0$; in addition there is a map $Pg\colon PY \to PY$ such that $(s,t)Pg = (g \times g)$, which implies that there is also a map $f\colon X \to X$ such that $pf = sp$. It follows that p has a section and hence that h and h' are homotopic. $\square$

Proposition 7.3. *If $\mathcal{C}$ is a path category and $\mathbb{N}$ is a homotopy natural numbers object in $\mathcal{C}$, then it becomes a natural numbers objects in the homotopy exact completion* $\mathrm{Hex}(\mathcal{C})$*. If, in addition, $\mathcal{C}$ comes equipped with a class of small fibrations satisfying axioms (F1-4), as well as*

(F8) *The map $\mathbb{N} \to 1$ belongs to $\mathcal{F}$,*

then the class of small maps in $\mathrm{Hex}(\mathcal{C})$ *determined from $\mathcal{F}$ satisfies:*

(S11) *The map $\mathbb{N} \to 1$ belongs to $\mathcal{S}$.*

Proof. This is a straightforward verification which we leave to the reader. $\square$

7.2. **Homotopy W-types.** In this subsection we will assume that $\mathcal{C}$ comes with a particular choice of (weak) homotopy Π-types. In that case there is for any fibration $f\colon B \to A$ an assigment $X \mapsto P_f(X)$ given by:

$$P_f(X) = \Sigma_A \Pi_f(\pi_1\colon B \times X \to B).$$

Proposition 7.4. *If $P_f(X)$ is a particular choice for X and $p\colon Y \to X$ is a fibration, we can choose $P_f(Y)$ in such a way that $P_f(p)\colon P_f(Y) \to P_f(X)$ is again a fibration and any section s of p induces a section $P_f(s)$ of $P_f(p)$.*

Proof. The proof is analogous to that of Proposition 6.6 and of Corollary 6.7. Any particular choice of $P_f(X)$ comes equipped with an evaluation map $\mathrm{ev}: P_f(X) \times_A B \to X$. If $p: Y \to X$ is a fibration, then one can obtain a choice of $P_f(Y)$ with a fibration $P_f(p): P_f(Y) \to P_f(X)$, as follows. One starts by pulling back $p: Y \to X$ along $\mathrm{ev}: P_f(X) \times_A B \to X$, resulting in some fibration $g: Z \to P_f(X) \times_A B$. Writing $\pi_1: P_f(X) \times_A B \to P_f(X)$ for the first projection, one obtains a suitable choice of $P_f(p)$ as $\Pi_{\pi_1}(g)$. □

Definition 7.5. Let $f: B \to A$ be a fibration. A *homotopy W-type* for f is an object W together with a map $\sup: P_f(W) \to W$ such that for any commuting square of the form

$$\begin{tikzcd} P_f(X) \arrow[r, "a"] \arrow[d, "P_f(p)"'] & X \arrow[d, "p"] \\ P_f(W) \arrow[r, "\sup"'] & W \end{tikzcd}$$

in which $p: X \to W$ is a fibration and $P_f(X)$ and $P_f(p)$ are constructed as in Proposition 7.4, there is a section $s: W \to X$ of p such that $s \circ \sup \simeq a \circ P_f(s)$, where $P_f(s)$ is the section of $P_f(p)$ provided by Proposition 7.4.

Theorem 7.6. *If* $\sup: P_f W \to W$ *is a homotopy W-type associated to a fibration* $f: B \to A$*, then* $\sup$ *is a homotopy equivalence.*

Proof. Our first idea is to regard $W \times P_f(W)$ is a P_f-algebra and $\pi_1: W \times P_f(W) \to W$ as a P_f-algebra morphism. Construct $P_f(\pi_1): P_f(W \times P_f(W)) \to P_f(W)$ in the usual way and let $\mu = (\sup, 1)P_f(\pi_1)$. Then

$$\begin{tikzcd} P_f(W \times P_f(W)) \arrow[r, "\mu"] \arrow[d, "P_f(\pi_1)"'] & W \times P_f(W) \arrow[d, "\pi_1"] \\ P_f W \arrow[r, "\sup"'] & W \end{tikzcd}$$

commutes, hence π_1 has a section $s: W \to W \times P_f(W)$ such that $s \circ \sup \simeq \mu \circ P_f(s)$. Let $\rho := \pi_2 s$. Then:

$$\rho \sup = \pi_2\, s \sup \simeq \pi_2\, \mu\, P_f(s) = P_f(\pi_1)\, P_f(s) = 1.$$

It remains to show that $\sup \rho \simeq 1$. To that purpose consider the pullback

$$\begin{tikzcd} Q \arrow[r, "\gamma"] \arrow[d, "\delta"'] & PW \arrow[d, "(s,t)"] \\ W \arrow[r, "(1, \sup \rho)"'] & W \times W. \end{tikzcd}$$

Our next task is to construct a P_f-algebra structure on Q in such a way that δ becomes a morphism of P_f-algebras: as soon as we do this δ will have a section and we obtain a homotopy between 1 and $\sup \rho$, as desired. But note $\sup \rho \sup \simeq \sup$, so there is a map $H\colon P_f(W) \to PW$ such that

$$\begin{tikzcd} P_f(W) \arrow[r, "H"] \arrow[d, "\sup"'] & PW \arrow[d, "(s,t)"] \\ W \arrow[r, "(1,\sup \rho)"'] & W \times W \end{tikzcd}$$

commutes, and hence there is a map $\beta\colon P_f(W) \to Q$ such that $\delta\beta = \sup$. But this means that

$$\begin{tikzcd} P_f(Q) \arrow[r, "\beta P_f(\delta)"] \arrow[d, "P_f(\delta)"'] & Q \arrow[d, "\delta"] \\ P_f(W) \arrow[r, "\sup"'] & W \end{tikzcd}$$

exhibits Q as an object with a P_f-algebra structure making δ into a morphism of P_f-algebras, as desired. □

7.3. **W-types.** The aim of this subsection is to show that $\mathrm{Hex}(\mathcal{C})$ will inherit W-types if $\mathcal{C}$ has homotopy W-types. We make use of the results from [7], in particular Theorems 29 and 30 therein, which we recall here for the convenience of the reader.

Theorem 7.7. [7, Theorem 29] *Suppose $\mathcal{E}$ is a locally cartesian closed pretopos with a natural numbers object, and suppose*

$$\begin{tikzcd} B \arrow[r, two heads] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ A \arrow[r, two heads] & X \end{tikzcd}$$

is a commutative square in $\mathcal{E}$ such that both the arrow $A \to X$ along the bottom and the inscribed arrow $B \to A \times_X Y$ from B to the pullback are epis. If f is a choice map for which a W-type exists, then there also exists a W-type for g.

Theorem 7.8. [7, Theorem 30] *Let $\mathcal{E}$ be a locally cartesian closed pretopos with a natural numbers object, and $f\colon B \to A$ be a choice map in $\mathcal{E}$. Suppose, moreover, that there is an object V together with an epimorphism $s\colon V \to P_f(V)$ such that V is the only subobject R of V for which the following statement holds in the internal logic of $\mathcal{E}$:*

$$(\forall v \in V, a \in A, t\colon B_a \to V) \text{ if } s(v) = (a,t) \text{ and } tb \in R \text{ for all } b \in B_a, \text{ then } v \in R.$$

Then a W-type for f exists.

But we will also need the following lemma.

Lemma 7.9. *Assume $\mathcal{C}$ is a path category with a particular choice of weak homotopy Π-types. Let $(X, \rho\colon R \to X \times X)$ be an arbitrary object in $\mathrm{Hex}(\mathcal{C})$ and Y be an arbitrary object in $\mathcal{C}$.*

(i) *The object* $X^Y \times X^Y$ *in* $\mathcal{C}$ *can act as a suitable weak homotopy exponential* $(X \times X)^Y$.
(ii) *The fibration*

$$\rho^Y : R^Y \to (X \times X)^Y \cong X^Y \times X^Y$$

computed as in Proposition 6.6 is a homotopy equivalence relation.
(iii) *The object* (X^Y, ρ^Y) *has the universal property of the exponential* $(X, R)^{iY}$ *in* $\mathrm{Hex}(\mathcal{C})$. *In particular, the exponential* $(X, R)^{iY}$ *in* $\mathrm{Hex}(\mathcal{C})$ *is covered by* $i(X^Y)$.
(iv) *If* $\mathcal{C} = \mathcal{D}(I)$ *for some path category* $\mathcal{D}$ *with weak homotopy* Π*-types and both* (X, R) *and* iY *are stable in* $\mathcal{C}$, *then so is* $(X, R)^{iY}$.

Proof. This is shown by a direct verification, which we leave to the reader. Item (iv) follows directly from the proof of Theorem 6.9. □

Lemma 7.10. *Assume* $\mathcal{C}$ *has all the structure discussed so far, so that* $\mathrm{Hex}(\mathcal{C})$ *is a locally cartesian closed pretopos with natural numbers object. Assume* $f : B \to A$ *is a fibration in* $\mathcal{C}$ *whose homotopy W-type exists in* $\mathcal{C}$. *Then the W-type associated to* if *exists in* $\mathrm{Hex}(\mathcal{C})$.

Proof. We work towards applying Theorem 7.8. First of all, maps of the form if where $f : B \to A$ is a fibration in $\mathcal{C}$ are choice maps in $\mathrm{Hex}(\mathcal{C})$ by Corollary 3.20. In addition, item (iii) of the previous lemma implies that $P_f(iX)$ is covered by $iP_f(X)$. So if $V = iW$, where W is the homotopy W-type associated to f in $\mathcal{C}$, then one obtains a cover

$$s : V = iW \cong \longrightarrow iP_f(W) \twoheadrightarrow P_f(iW) = P_f(V)$$

in $\mathrm{Hex}(\mathcal{C})$. Therefore it remains to show that the only subobject R of V for which

$$v \in V, s(v) = (a, t) \text{ and } tb \in R \text{ for all } b \in f^{-1}(a) \text{ imply that } v \in R$$

holds, is the subobject V itself. But any subobject R of V can be covered by an element iX; in addition, we may assume that $p : X \to W$ is a fibration (if not, factor this map as a homotopy equivalence $X \simeq X'$ followed by a fibration $X' \to W$ and replace X by X'). The condition we just stated implies that the image of the map $i(P_f X \to P_f W \to W)$ lands in R, which, by projectivity of $iP_f(X)$, fullness of i and Proposition 2.31, means that there is a map $t : P_f X \to X$ making

$$\begin{tikzcd} P_f X \arrow[r, dotted, "t"] \arrow[d] & X \arrow[d, "p"] \\ P_f W \arrow[r] & W \end{tikzcd}$$

commute in $\mathcal{C}$. So p has a section, which implies $R \cong V$, as desired. □

Theorem 7.11. *Assume* $\mathcal{C}$ *has all the structure discussed so far, so that* $\mathrm{Hex}(\mathcal{C})$ *is a locally cartesian closed pretopos with natural numbers object. If* $\mathcal{C}$ *has homotopy W-types, then* $\mathrm{Hex}(\mathcal{C})$ *has all W-types.*

Proof. The idea is to show that any map $g: Y \to X$ in $\mathrm{Hex}(\mathcal{C})$ fits into a diagram of the form

$$\begin{tikzcd} iB \arrow[r, two heads] \arrow[d, "if"'] & Y \arrow[d, "g"] \\ iA \arrow[r, two heads] & X \end{tikzcd}$$

in which both $iA \to X$ and $iB \to iA \times_X Y$ are epis, and $f: B \to A$ is a fibration in $\mathcal{C}$. Because in that case the map if on the left is a choice map by Corollary 3.20, so a W-type for g will exist by the previous lemma and Theorem 7.7.

But any object X can be covered by an object of the form iA. If pull back $g: Y \to X$ along this cover, then one can cover the resulting pullback by some object iB. Since the functor i is full, the resulting map $iB \to iA$ is of the form if for some map $f: B \to A$. Without loss of generality, we may assume that f is a fibration (if not, factor this map as a homotopy equivalence $B \simeq B'$ followed by a fibration $B' \to A$ and replace B by B'). This proves the theorem. $\square$

Theorem 7.12. *Assume $\mathcal{C}$ has all the structure discussed so far, so that* $\mathrm{Hex}(\mathcal{C})$ *is a locally cartesian closed pretopos with natural numbers object and W-types. Assume also that $\mathcal{C}$ comes equipped with a class of small fibrations $\mathcal{F}$ satisfying axioms (F1-8), as well as:*

(F9) *For a commutative diagram*

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr] & & A \arrow[dl] \\ & X & \end{tikzcd}$$

with all maps in $\mathcal{F}$, the W-type $W_X(f)$ taken in $\mathcal{C}(X)$ (which is a fibration in $\mathcal{C}$ with codomain X) belongs to $\mathcal{F}$.

Then the class of small maps $\mathcal{S}$ in $\mathrm{Hex}(\mathcal{C})$ *determined from $\mathcal{F}$ satisfies axioms (S1-11) as well as*

(S12) *For a commutative diagram*

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr] & & A \arrow[dl] \\ & X & \end{tikzcd}$$

with all maps in $\mathcal{S}$, the W-type $W_X(f)$ taken in $\mathcal{E}/X$ (which is a map in $\mathcal{E}$ with codomain X) belongs to $\mathcal{S}$.

Proof. Let $f: B \to A$ and $A \to X$ be maps in $\mathrm{Hex}(\mathcal{C})$ belonging to $\mathcal{S}$. It follows from the proof of Proposition 5.4 that these maps fit into a double covering square of the form

$$\begin{tikzcd} iB' \arrow[r] \arrow[d, "if'"'] & B \arrow[d, "f"] \\ iA' \arrow[r] \arrow[d, "i\alpha'"'] & A \arrow[d, "\alpha"] \\ iX' \arrow[r, "e"'] & X \end{tikzcd}$$

in which f' and α' are small fibrations in $\mathcal{C}$. Hence e^*f is covered by if'. We will argue that $W_{iX'}(e^*f)$ is small in $\mathrm{Hex}(\mathcal{C})/iX'$, from which it follows by descent that $W_X(f)$ is small in $\mathrm{Hex}(\mathcal{C})/X$.

At this point we need to take a closer look at the proof of Theorem 29 in [7]. This proof constructs the W-type for e^*f as a subquotient of the W-type for if' in $\mathrm{Hex}(\mathcal{C})/iX'$. In fact, it defines, using the internal logic, a symmetric and transitive relation R on $W(if')$ and then constructs $W(e^*f)$ as the R-reflexive elements quotiented by R. By inspection of the formula defining R, and using the closure properties of $\mathcal{S}$, one sees that the map $R \to W(if') \times W(if')$ is small. Because $W(if')$ is small in $\mathrm{Hex}(\mathcal{C})/iX'$ by (F7), the same is true for $W(e^*f)$ by (S8). This finishes the proof. □

8. Models of constructive set theory

As discussed in the introduction, in [1] Aczel gave an interpretation of constructive set theory in Martin-Löf type theory with inductive types and one universe. In a subsequent paper [2] Aczel considered extensions of **CZF** with axioms guaranteeing the existence of inductively defined sets; in particular, he introduced the Regular Extension Axiom (REA) and proved that if the universe in type theory is closed under W-types, his interpretation also validates (REA). An alternative based on W-types and a weak choice principle called the Axiom of Multiple Choice, which is closer to the categorical semantics that we develop here, can be found in [11].

Aczel's interpretation of **CZF** in type theory works more generally, as the authors of [38] show. Using the language of algebraic set theory introduced in [31], they show that if $\mathcal{E}$ is a locally cartesian closed pretopos with W-types and $\mathcal{S}$ is a suitable class of small maps, then one can imitate Aczel's construction to obtain a model of **CZF** inside $\mathcal{E}$. Indeed, the following is essentially Theorem 7.1 from [38]:

Theorem 8.1. *Let $\mathcal{E}$ be a locally cartesian closed pretopos with W-types, and let $\mathcal{S}$ be a representable class of small maps in $\mathcal{E}$ satisfying the Axiom of Multiple Choice, as well as axioms (S10) and (S11). Then $\mathcal{E}$ contains a model of Aczel's constructive set theory* **CZF**. *If $\mathcal{S}$ also satisfies (S12), then in this model the Regular Extension Axiom is valid as well, as is the alternative formulated in* [11].

What we will do in this section is to use this theorem to show that one can also obtain models of **CZF** from path categories $\mathcal{C}$ equipped with a class of small fibrations

$\mathcal{F}$. Of course, we need some conditions on both $\mathcal{C}$ and $\mathcal{F}$ for this to work. In fact, $\mathcal{C}$ should be a homotopy extensive path category with weak homotopy Π-types, a homotopy natural numbers object and homotopy W-types: this will guarantee that $\mathrm{Hex}(\mathcal{C})$ is a locally cartesian closed pretopos with W-types. Also, $\mathcal{F}$ should satisfy the axioms (F1-8), ensuring that the class of small maps derived from $\mathcal{F}$ satisfies axioms (S1-11).

In fact, just two pieces of the puzzle are missing that prevent us from applying Theorem 8.1 directly. First we need to investigate when the Axiom of Multiple Choice holds for a class of small maps in $\mathrm{Hex}(\mathcal{C})$ determined by a class of small fibrations in $\mathcal{C}$. It turns out that this is always the case: we will show this in the first subsection. In the second subsection we will formulate an appropriate notion of universe for $\mathcal{F}$ and show that this leads to a representable class of small maps in $\mathrm{Hex}(\mathcal{C})$. This puts the last piece in place, so that we can derive the main result of this paper.

Theorem 8.2. *Let $\mathcal{C}$ be a homotopy extensive path category with weak homotopy Π-types, a homotopy natural numbers object and homotopy W-types. Assume that $\mathcal{F}$ is a class of fibrations in $\mathcal{C}$ satisfying axioms (F1-8) and which contains a small fibration $\pi \colon E \to U$ such that any other element of $\mathcal{F}$ can be obtained as a homotopy pullback of π. Then* $\mathrm{Hex}(\mathcal{C})$ *contains a model of Aczel's constructive set theory* **CZF**. *If $\mathcal{F}$ satisfies (F9) as well, then in this model the Regular Extension Axiom holds, as does the alternative formulated in* [11].

Throughout this section $\mathcal{E}$ will be a locally cartesian closed pretopos with W-types.

8.1. **Axiom of Multiple Choice.**

In order to formulate the Axiom of Multiple Choice, we first need the notion of a collection square from [9].

Definition 8.3. Recall from Definition 5.1 that a commuting square of the form

$$\begin{tikzcd} D \arrow[r, "q"] \arrow[d, "g"'] & C \arrow[d, "f"] \\ B \arrow[r, "p"'] & A \end{tikzcd}$$

will be called a *covering square* if both p and the inscribed map $D \to B \times_A C$ to the pullback are covers. Such a covering square will be called a *collection square* if the following statement holds in the internal logic of $\mathcal{E}$:

> For any $a \in A$ and epi $e \colon E \to B_a$ there are $c \in p^{-1}(a)$ and $h \colon D_c \to E$ such that $eh(d) = q(d)$ for any $d \in D_c$.

The following is a corrected formulation of the Axiom of Multiple Choice from [38] (see [11]):

Definition 8.4. Let $\mathcal{S}$ be a class of small maps in $\mathcal{E}$. The class $\mathcal{S}$ satisfies the Axiom of Multiple Choice (AMC) if any small map $f\colon Y \to X$ fits into a diagram of the form

$$\begin{tikzcd}
D \arrow[r] \arrow[d] & A \times_X Y \arrow[r] \arrow[d, "q^* f"] & Y \arrow[d, "f"] \\
C \arrow[r] & A \arrow[r, "q"'] & X
\end{tikzcd}$$

in which the square on the right is pullback with a cover q at the bottom, while the left hand square is a collection square in which all maps are small.

Proposition 8.5. *Suppose $\mathcal{C}$ is a homotopy extensive path category with weak homotopy Π-types, a homotopy natural numbers object and homotopy W-types. In addition, let $\mathcal{F}$ be a class of maps satisfying axioms (F1-6). Then the class of small maps $\mathcal{S}$ in* $\mathrm{Hex}(\mathcal{C})$ *determined by $\mathcal{F}$ satisfies the Axiom of Multiple Choice.*

Proof. From the way $\mathcal{S}$ is determined by $\mathcal{F}$ it follows that if $f\colon Y \to X$ is small, then it fits into a diagram

$$\begin{tikzcd}
i(B) \arrow[r, two heads] \arrow[d, "i(g)"'] & i(A) \times_X Y \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
i(A) \arrow[r, "1"'] & i(A) \arrow[r, two heads] & X
\end{tikzcd}$$

in which the left square is covering, the right one is a pullback and g belongs to $\mathcal{F}$. Because $i(g)$ is a choice map by Corollary 3.20, the square on the left is a collection square; and because $i(g)$ is small by Proposition 5.2 and $\mathcal{S}$ satisfies axioms (S1) and (S9), every arrow in the left square is small. □

8.2. **Universes.** The only ingredient for building a model of **CZF** which is missing at this point is a universe. We first formulate an appropriate notion of universe for a class of small maps.

Definition 8.6. A class of small maps $\mathcal{S}$ is *representable* if it contains a map $\pi\colon E \to U$ ("a representation") such that for any $f\colon B \to A \in \mathcal{S}$ there is a diagram of the form

$$\begin{tikzcd}
B \arrow[d, "f"'] & D \arrow[l] \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\
A & C \arrow[l, two heads, "e"] \arrow[r] & U
\end{tikzcd}$$

in which both squares are pullbacks and e is a cover.

Lemma 8.7. *Let $\mathcal{A}$ be a class of "open maps" in $\mathcal{E}$ satisfying axioms (S1-7) and (Q) from Proposition 5.5, and let $\mathcal{S}$ be the class of small maps obtained from $\mathcal{A}$ by declaring a map $f\colon B \to A$ to be small if both f itself and $B \to B \times_A B$ belong to $\mathcal{A}$,* *as in Theorem 5.6. Suppose there is a map $\pi\colon E \to U$ belonging to $\mathcal{A}$ such that for any $f\colon B \to A \in \mathcal{A}$ there is a diagram of the form*

$$\begin{tikzcd} B \arrow[d, "f"'] & D \arrow[l] \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ A & C \arrow[l, two heads, "q"] \arrow[r] & U \end{tikzcd}$$

in which the square on the right is a pullback and the square on the left is covering. Then $\mathcal{S}$ is representable.

Proof. As this is a variation on argument that has appeared in the literature on algebraic set theory before (see, for example, [9, Proposition 4.4]), we only give the construction of the representation $\pi'\colon E' \to U'$ for $\mathcal{S}$. Using the interal logic of $\mathcal{E}$, we define U' to be the collection of triples $(u \in U, v \in U, p\colon E_v \to E_u \times E_u)$ with $\mathrm{Im}(p)$ an equivalence relation on E_u, while the fibre $E'_{u'}$ over such a triple $u' = (u, v, p)$ is defined to be $E_u/\mathrm{Im}(p)$. $\square$

Proposition 8.8. *Suppose $\mathcal{C}$ is a homotopy extensive path category with weak homotopy Π-types, a homotopy natural numbers object and homotopy W-types. In addition, let $\mathcal{F}$ be a class of maps satisfying axioms (F1-6), and $\mathcal{S}$ be the class of small maps determined by $\mathcal{F}$ in $\mathrm{Hex}(\mathcal{C})$. If there is a small fibration $\pi\colon E \to U$ in $\mathcal{C}$ such that any $f\colon B \to A \in \mathcal{F}$ fits into a homotopy pullback of the form*

$$\begin{tikzcd} B \arrow[r] \arrow[d, "f"'] & E \arrow[d, "\pi"] \\ A \arrow[r] & U, \end{tikzcd}$$

then $\mathcal{S}$ is representable.

Proof. Suppose $\mathcal{C}$ and $\mathcal{F}$ have the properties in the statement of the proposition. From the way $\mathcal{F}$ determines the quasi-small maps in $\mathrm{Hex}(\mathcal{C})$ it follows immediately that these quasi-small maps satisfy the hypotheses of the previous lemma. Therefore it follows from that lemma that $\mathcal{S}$ is representable. $\square$

This completes the proof of Theorem 8.2.

Appendix A. Axioms

In this appendix we have collected some definitions and axioms that are used throughout the paper.

A.1. **Path categories.** Usually, the setting of this paper is that we are given a category $\mathcal{C}$ together with two classes of maps called the *weak equivalences* and the *fibrations*, respectively. Morphisms which belong to both classes of maps will be called *acylic fibrations.* A *path object* on an object B is a factorisation of the diagonal $\Delta\colon B \to B \times B$ as a weak equivalence $r\colon B \to PB$ followed by a fibration $(s,t)\colon PB \to B \times B$.

Definition A.1. The category $\mathcal{C}$ will be called a *category with path objects*, briefly a *path category*, if the following axioms are satisfied:

(1) Fibrations are closed under composition.
(2) The pullback of a fibration along any other map exists and is again a fibration.
(3) The pullback of an acylic fibration along any other map is again an acyclic fibration.
(4) Weak equivalence satisfy 2-out-of-6: if $f\colon A \to B$, $g\colon B \to C$, $h\colon C \to D$ are three composable maps and both gf and hg are weak equivalences, then so are f, g, h and hgf.
(5) Isomorphisms are acyclic fibrations and every acyclic fibration has a section.
(6) For any object B there is a path object PB (not necessarily functorial in B).
(7) $\mathcal{C}$ has a terminal object 1 and every map $X \to 1$ to the terminal object is a fibration.

A.2. **Small fibrations.** Let $\mathcal{C}$ be a fibration and $\mathcal{F}$ be a class of fibrations in $\mathcal{C}$. For $\mathcal{F}$ we will consider the following axioms:

(F1) In a homotopy pullback square

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r] & X \end{tikzcd}$$

in which both f' and f are fibrations, f' belongs to $\mathcal{F}$ whenever f does.
(F2) $\mathcal{F}$ contains all isomorphisms.
(F3) $\mathcal{F}$ is closed under composition.
(F4) If $Y \to X$ belongs to $\mathcal{F}$, then so does $P_X(Y) \to Y \times_X Y$.
(F5) If two maps $Y \to X$ and $Y' \to X'$ both belong to $\mathcal{F}$ then so does a fibrant replacement of $Y + Y' \to X + X'$.
(F6) The maps $0 \to 1$ and $1 + 1 \to 1$ belong to $\mathcal{F}$.
(F7) If $f\colon Y \to X$ and $g\colon Z \to Y$ belong to $\mathcal{F}$, then so does $\Pi_f(g) \to X$.
(F8) The map $\mathbb{N} \to 1$ belongs to $\mathcal{F}$.
(F9) For a commutative diagram

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr] & & A \arrow[dl] \\ & X & \end{tikzcd}$$

with all maps in $\mathcal{F}$, the W-type $W_X(f)$ taken in $\mathcal{C}(X)$ (which is a fibration in $\mathcal{C}$ with codomain X) belongs to $\mathcal{F}$.

The axioms (F1-4) make sense in any path category. The natural setting for axioms (F5-6) is that of a homotopy extensive path category, while for (F7), (F8) and (F9) one assumes the existence of (weak) homotopy Π-types, a homotopy natural numbers object and homotopy W-types, respectively.

A.3. **Exact categories and pretoposes.** For more information on the following notions and results, we refer to part A of [29].

Definition A.2. Let $\mathcal{E}$ be a category. A map $f\colon B \to A$ in $\mathcal{E}$ is a *cover* if the only subobject of A through which it factors is the maximal one given by the identity on A. A category $\mathcal{C}$ is *regular* if it has all finite limits, every morphism in $\mathcal{C}$ factors as a cover followed by a mono and covers are stable under pullback.

In a regular category a map is a cover iff it is a regular epi (meaning that it arises as a coequalizer) iff it is the coequalizer of its kernel pair.

Definition A.3. A subobject $R \subseteq X \times X$ is an *equivalence relation* if for any object P in $\mathcal{E}$ the image of the induced map

$$\mathrm{Hom}(P, R) \to \mathrm{Hom}(P, X) \times \mathrm{Hom}(P, X)$$

is an equivalence relation on $\mathrm{Hom}(P, X)$. In a regular category a *quotient* of an equivalence relation is a cover $X \to Q$ such that

$$\begin{tikzcd} R \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & Q \end{tikzcd}$$

is a pullback. (Hence $R \rightrightarrows X$ is the kernel pair of $X \to Q$ and the latter is the coequalizer of the former.)

Definition A.4. A regular category $\mathcal{E}$ is called *exact* (or *effective regular*) if every equivalence relation has a quotient. If $\mathcal{E}$ also has disjoint and stable sums, $\mathcal{E}$ is called a *pretopos*.

In a pretopos covers and epis coincide.

A.4. **Small maps.** Let $\mathcal{E}$ be an exact category and $\mathcal{S}$ be a class of maps in $\mathcal{E}$. In this setting we consider the following axioms:

(S1) In a pullback square

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "p"'] & X \end{tikzcd}$$

f' belongs to $\mathcal{S}$ whenever f does.

(S2) If in a pullback square as the one above, the map p is a cover, then f belongs to $\mathcal{S}$ whenever f' does.
(S3) $\mathcal{S}$ is closed under composition.
(S4) $\mathcal{S}$ contains all isomorphisms.
(S5) If two maps $Y \to X$ and $Y' \to X'$ both belong to $\mathcal{S}$ then so does their sum $Y + Y' \to X + X'$.
(S6) For any object X the map $0 \to X$ belongs to $\mathcal{S}$.
(S7) If $Y \to X$ and $Z \to X$ belong to $\mathcal{S}$, then so does $Y + Z \to X$.
(S8) Suppose $R \to Y \times_X Y$ is an equivalence relation in $\mathcal{E}/X$ with quotient $Y \to Y/R$ as in

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & Y \arrow[r] \arrow[d] & Y/R \arrow[dl] \\ & X. & \end{tikzcd}$$

If $R \to X$ and $Y \to X$ belong to $\mathcal{S}$, then so does $Y/R \to X$.
(S9) Suppose

$$\begin{tikzcd} Z \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

is a commutative triangle. If both g and h belong to $\mathcal{S}$, then so does f.
(S10) For any $Y \to X$ and $Z \to X$ in $\mathcal{S}$, their exponent $(Z^Y)_X \to X$ in $\mathcal{E}/X$ belongs to $\mathcal{S}$.
(S11) The map $\mathbb{N} \to 1$ belongs to $\mathcal{S}$.
(S12) For a commutative diagram

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr] & & A \arrow[dl] \\ & X & \end{tikzcd}$$

with all maps in $\mathcal{S}$, the W-type $W_X(f)$ taken in $\mathcal{E}/X$ (which is a map in $\mathcal{E}$ with codomain X) belongs to $\mathcal{S}$.

Axioms (S1-4) and (S8-9) make sense in any exact category. In a pretopos one may also consider the axioms (S5-7); indeed, a class of maps satisfying axioms (S1-9) will be called a class of *small maps.*

The natural environment to consider the other axioms (S10), (S11) and (S12) is a category $\mathcal{E}$ which is locally cartesian closed, has a natural numbers object or has W-types, respectively.

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