

Matricial Potentiation

By

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Abstract

In this short note we introduce the potentiation of matrices of the same size. We study some simple properties and some example.

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Introduction

We introduce the matrix potentiation. The problem was pose byGuametti.

Consider two matrices A , of size $m \times n$ and B , $m \times n$. We wish to define the potentiation

$$C = A^B$$

For this purpose. We take the logarithm

$$\ln C = B \ln A,$$

This is valid if C and A are not singular. From now on we assume that when we take \ln the argument is not singular. The $\ln C$ is well defined for the matrices A and B real or complex. In the second case we have that it is a multivalued funchon. This assuming that he $\ln A$ is a converging sequence. Consider the matrix $D = A - I$ where I stands for the identity matrix. Then

$$\ln A = \ln (D + I) = D - \frac{D^2}{2} + \frac{D^3}{3} - \frac{D^4}{4} + \frac{D^5}{5} + (-)^{i+1} \frac{D^i}{i}$$

From here, it is inmediated that if $m = n$

$$C = e^{B \ln A}$$

can be well defined. The only condition that is necessary the convergence of $\ln A$, or

$$\begin{aligned} \ln C = B \ln A &= B \left[(A - I) - \frac{(A - I)^2}{2} + \frac{(A - I)^3}{3} \dots \dots \dots \right] \\ &= B \left[\sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \left(\frac{A - I}{i} \right)^i \right] = B \left[\sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \sum_{j=0}^i \binom{i}{j} A^i (-I)^{i-j} \right] \end{aligned}$$

If $A - I$ is diagonalizable then

$$A - I = Q \Lambda Q^{-1}$$

where the matrix Λ is diagonal with all the eigenvalues in the main diagonal, and Q is formed by the ligenvectors.

Therefore

$$\ln C = B \ln A = B Q \left[\sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \Lambda^i \right] Q^{-1}$$

or

$$C = e^B \quad \ln A = e^B Q \left[\sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \Lambda^i \right] Q^{-1} = \sum_{t=0}^{\infty} \frac{1}{t!} [B \quad Q \quad K(A) \quad Q^{-1}]^t$$

where

$$K(A) = \sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \Lambda^i$$

We have

$$\ln A = \sum_{o=1}^{\infty} \frac{(-)^{o+1}}{o} \sum_{j=0}^o \binom{o}{j} (-)^{o-j} A^j$$

and it easy to see that A is diagonalizable in the following way

$$A = Q (\Lambda + I) Q^{-1}$$

Therefore

$$\begin{aligned} \ln A &= \sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \sum_{j=0}^i (-)^{i-j} \binom{i}{j} Q (\Lambda + I)^j Q^{-1} \\ &= \sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} Q \left\{ \sum_{j=0}^i (-)^{i-j} \binom{i}{j} (\Lambda + I)^j \right\} Q^{-1} \end{aligned}$$

As an example if A is diagonal: $A = \text{diag}(x_1, x_2, \dots, x_m)$ them $A^j = \text{diag}(x_1^j, x_2^j, \dots, x_m^j)$. replacing we see that $\ln A$ is also diagonal

$$\begin{aligned} \ln A (r,r) &= \sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \sum_{j=0}^i (-)^i \binom{i}{j} (\lambda_r + 1)^j = \sum_{i=1}^{\infty} (\lambda_r)^i \frac{(-)^{i+1}}{i} \\ &= \sum_{i=1}^{\infty} (\lambda_r + 1 - 1)^i \frac{(-)^i}{i} = \ln \lambda_r \end{aligned}$$

$$\ln A (r,s) = 0 \quad r \neq s$$

them $\ln A$ is diagonal and it is converging

$$\ln \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_m \end{pmatrix} = \begin{pmatrix} \ln \lambda_1 & & 0 \\ & \ln \lambda_2 & \\ 0 & & \ln \lambda_m \end{pmatrix}$$

Next case, we have that A is diagonalizable

$$A = P \Lambda P^{-1}$$

Then the

$$\ln A = \sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} p^{+1} \Lambda^i p^{-1} = p^{+1} \sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} (\Lambda)^i p^{-1} = p^{-1} \left[\sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} (\Lambda)^i \right] p^{-1}$$

But $Q = p^{-1}$

$$Q \Lambda^i = \lambda_1^i \begin{bmatrix} q_{11} \\ q_{21} \\ q_{m1} \end{bmatrix} + \lambda_2^i \begin{bmatrix} q_{12} \\ q_{22} \\ q_{m2} \end{bmatrix} + \dots = \sum_{r=1}^n \lambda_r^i \begin{bmatrix} q_{1r} \\ q_{2r} \\ q_{nr} \end{bmatrix}$$

therefore

$$Q \Lambda^i P^{-1}(r, s) = \sum_{k=1}^n \lambda_k^i q_{rk} p_{ks}$$

and replacing into equation (1), it turns out that

$$\begin{aligned} (\ln A)^i(r, s) &= \sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \left(\sum_{k=1}^n (\lambda_k - 1) \right)^i q_{rk} p_{ks} \\ &= \sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \left(\sum_{k=1}^n (\lambda_k - 1)^i \right) q_{rk} p_{ks} \\ &= \sum_{k=1}^n q_{rk} \left[\sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} (\lambda_k - 1)^i \right] p_{ks} = \sum_{k=1}^n q_{rk} (\ln \lambda_k) p_{ks} \\ &= [p^{+1} (\ln \Lambda) p^{-1}]^i(r, s) \end{aligned}$$

As a conclusion we have for $A = p^{+1} \Lambda p^{-1}$ diagonalizable with the eigenvalues for $0 < \lambda_r < 2$ cocient

$$\ln A = p^{+1} (\ln \Lambda) p^{-1}$$

and

$$\ln \Lambda = \begin{pmatrix} \ln \lambda_1 & & c \\ & \ddots & \\ 0 & & \ln \lambda_n \end{pmatrix}$$

Then we have proved the following result.

Theorem: Given two diagonalizable matrices $m \times n$: A and B , where $A - I$ has eigenvalues x_r : $0 < x_r < 2$, then the matrix

$$C = A^B$$

is well defines and it has the form

$$C = \sum_{t=0}^{\infty} \frac{(B\bar{A})^t}{t!}$$

where

$$\bar{A} = \ln A$$

This we have been successful in the definition of matrix potentiation.

Properties

In this section we are going to for diagonalizable matrices some the first one, already proved is

$$A^B \leftrightarrow \exp(B \ln A)$$

Now we study

$$(A^B)^C$$

We have

$$(A^B)^C = D^C = \exp(C \ln D) = \exp(C \ln A^B) = \exp(C B \ln A) = A^{CB}$$

wher $D = A^B$.

On the other hand

$$\begin{aligned} A^B A^C &= \exp(B \ln A) \exp(C \ln A) = \exp(B \ln A + C \ln A) \\ &= \exp((B + C) \ln A) = A^{B+C} \end{aligned}$$

We follow with

$$\exp(\ln A) = A$$

We know

$$e^x = \sum_k \frac{x^k}{k!}$$

an if A_h^k diagonalizable we have

$$\ln A = P \ln \Lambda p^{-1}$$

where the diagonal matrix Λ is formed by the eigenvalues by the eigenvector as columns. Then

$$\begin{aligned}
\ln(\exp A) &= \ln \left(\sum_{k=0}^{\infty} \frac{P \Lambda^k P^{-1}}{k!} \right) = \ln \left(P \sum_{k=0}^{\infty} \frac{\Lambda^k}{k!} P^{-1} \right) \\
&= \sum_{k=1}^{\infty} \frac{(-)^{k-1}}{k} (P^{-1} \exp(\Lambda) P)^k = \sum_{k=1}^{\infty} \frac{(-)^{k-1}}{k} P^{-1} (\exp \Lambda)^k P \\
&= P^{-1} \left(\sum_{k=1}^{\infty} \frac{(-)^{k-1}}{k} (\exp \Lambda)^k \right) P = P^{-1} \Lambda P = A
\end{aligned}$$

and in this way we have proved the property.

On the other hand, we have, another basic property, for diagonalizable matrices namely.

Others properties are

$$\exp(A + B) = \exp A \exp B$$

Consider

$$e^{(A+B)} = \sum_{k=0}^{\infty} \frac{(A+B)^k}{k!} = \sum_{k=0}^{\infty} \sum_{t=0}^k \binom{k}{t} \frac{A^t B^{k-t}}{k!} = \sum_{k=0}^{\infty} \sum_{t=0}^k \frac{A^t B^{k-t}}{t! (k-t)!}$$

And now by Fubini property and the standardamegament

$$= \sum_{i=0}^{\infty} \sum_{k=i}^{\infty} \frac{A^i B^{k-i}}{i! (k-i)!}$$

and by a variable change $k - i = j$ and $i = i$ the

$$= \sum_{i=0}^{\infty} \sum_{k=i}^{\infty} \frac{A^i B^j}{i! j!} = \left(\sum_{i=0}^{\infty} \frac{A^i}{i!} \right) \left(\sum_{j=0}^{\infty} \frac{B^j}{j!} \right) = \frac{\exp A}{\exp B}$$

and is this way the property is proved.

On the other hand, we now prove

$$\ln(AB) = \ln A + \ln B$$

always in the case that $AB = BA$, or they comments let

$$\ln(AB) = C$$

$$\exp C = \exp(\ln AB) = AB$$

On the other hand if we call

$$D = \ln A + \ln B$$

then

$$\exp D = \exp(\ln A + \ln B) = \exp(\ln A) \cdot \exp(\ln B) = AB$$

then $C = D$.

Now we present

$$A^B A^C = A^{B+C}$$

Let

$$\begin{aligned} A^B A^C &= \exp(B \ln A) \cdot \exp(C \ln A) = \exp(B \ln A + C \ln A) \\ &= \exp((B + C) \ln A) = \exp \ln (A)^{B+C} = A^{B+C} \end{aligned}$$

Next

$$(A^B)^C = A^{B+C}$$

consider the first term. Calling $D = A^B$ then

$$(A^B)^C = D^C = \exp(C \ln D) = \exp(C \ln A^B) = \exp(C B \ln A) = A^{CB}$$

Next we consider a property about the determining namely

$$\det(\ln A) = \det(\ln \Lambda)$$

When $A = P^{-1} \Lambda P$ which is immediate by the decomposition.

Consider the transpose of $A : A^t$, then

$$\exp(A^t) = (\exp A)^t$$

Let $A = P^{-1} \Lambda P$ then

$$A^t = (P^{-1} \Lambda P)^t = P^t \Lambda (P^{-1})^t = P^t \Lambda (P^t)^{-1}$$

On the other hand

$$\exp(A^t) = \sum_{i=0}^{\infty} \frac{(A^t)^i}{i!} = P^t \left(\sum_{i=0}^{\infty} \frac{\Lambda^i}{i!} \right) (P^t)^{-1} = \left(P^{-1} \sum_{i=0}^{\infty} \frac{\Lambda^i}{i!} P \right)^t = (e^A)^t$$

Now we consider another property, namely

$$\det \exp A = \exp \operatorname{tra} A$$

Let

$$e^A = P^{-1} \left(\sum_{k=0}^{\infty} \frac{\Lambda^k}{k!} \right) P$$

$$\det(e^A) = \det \left(\sum_{i=0}^{\infty} \frac{\Lambda^i}{i!} \right) = \det(e^A) = \prod_{j=1}^n e^{\lambda_j} = e^{\lambda_1} e^{\lambda_2} \dots e^{\lambda_n} = e^{\lambda_1 + \lambda_2 + \dots + \lambda_n} = e^{\text{tra } A} \\ = e^{\text{tra } A}$$

where tra, indicate the trace of the matrix.

Now we consider an example.

$$\begin{pmatrix} \sum_{i=1}^{\infty} \frac{(-)^{i-1}}{i} & \sum_{i=1}^{\infty} \frac{(-)^{i-1}}{i} \left(\frac{1}{2}\right)^i \\ 2 \sum_{i=1}^{\infty} \frac{(-)^{i-1}}{i} & \sum_{i=1}^{\infty} \frac{(-)^{i-1}}{i} \left(\frac{1}{2}\right)^i \end{pmatrix} = \ln C = \begin{pmatrix} 0 & \ln 1/2 \\ 0 & \ln 1/2 \end{pmatrix}$$

Ej. 2

$$I + \frac{\ln C}{1!} + \frac{(\ln C)^2}{2!} + \frac{(\ln C)^3}{3!}$$

$$\begin{pmatrix} 0 & \ln 1/2 \\ 0 & \ln 1/2 \end{pmatrix} \begin{pmatrix} 0 & \ln 1/2 \\ 0 & \ln 1/2 \end{pmatrix} = \begin{pmatrix} 0 & (\ln 1/2)^2 \\ 0 & (\ln 1/2)^2 \end{pmatrix}$$

$$(\ln C)^n = \begin{pmatrix} 0 & (\ln 1/2)^n \\ 0 & (\ln 1/2)^n \end{pmatrix} \begin{pmatrix} 0 & (\ln 1/2)^n \\ 0 & (\ln 1/2)^n \end{pmatrix} \begin{pmatrix} 0 & \ln 1/2 \\ 0 & \ln 1/2 \end{pmatrix} \\ = \begin{pmatrix} 0 & (\ln 1/2)^{nH} \\ 0 & (\ln 1/2)^{nH} \end{pmatrix}$$

$$C = e^{\ln C} = I + \begin{pmatrix} 0 & \ln 1/2 \\ 0 & \ln 1/2 \end{pmatrix} + \begin{pmatrix} 0 & (\ln 1/2)^2 \\ 0 & (\ln 1/2)^2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \sum_{n=1}^{\infty} \frac{(\ln 1/2)^n}{n^1} \\ 0 & \sum_{n=1}^{\infty} \frac{(\ln 1/2)^n}{n^1} \end{pmatrix} = \begin{pmatrix} 1 & e^{\ln 1/2 - 1} \\ 0 & 1/2 \end{pmatrix}$$

$$1 + \sum = e^{\ln 1/2} = \begin{pmatrix} 1 & -1/2 \\ 0 & 1/2 \end{pmatrix}$$

Example

We wish to compute

$$C = A^B$$

Consider as example

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1/2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$$

knowing

$$\ln C = B \ln A = B \ln P \Lambda P^{-1} = B P \ln \Lambda P^{-1}$$

them

$$\ln \Lambda = \begin{pmatrix} \ln 1 & 0 \\ 0 & \ln 1/2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\ln 2 \end{pmatrix}$$

and where

$$P = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}$$

Therefore

$$\ln A = P \ln \Lambda P^{-1} = \begin{pmatrix} 0 & \ln 16 \\ 0 & -\ln 2 \end{pmatrix}$$

hence

$$\ln C = B \ln A = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 0 & \ln 16 \\ 0 & -\ln 2 \end{pmatrix} = \begin{pmatrix} 0 & \ln 16 \\ 0 & \ln 64 \end{pmatrix}$$

from here

$$\begin{aligned}
 C = e^{B \ln A} &= \sum_{l=0}^{\infty} \frac{(B \ln A)^l}{l!} = \sum_i \frac{1}{i!} \begin{pmatrix} 0 & 2^{i+1} & 3^{i-1} & \ln 2 \\ 0 & & 3^i & \ln 2 \end{pmatrix} \\
 &= \ln 2 \begin{pmatrix} 0 & \sum_i \frac{2 \cdot 2^i \cdot 3^i \cdot 3^{-1}}{i!} \\ 0 & \sum_i \frac{3^i}{i!} \end{pmatrix} \\
 C &= \begin{pmatrix} 0 & 2/3 & e^6 \ln 2 \\ 0 & e^3 & \ln 2 \end{pmatrix}
 \end{aligned}$$

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