SYMMETRY AND INVARIANT BASES IN FINITE ELEMENT EXTERIOR CALCULUS

MARTIN W. LICHT

Abstract. We study the symmetry of bases and spanning sets in finite element exterior calculus using representation theory. The affine symmetries of a simplex form a group that is isomorphic to a permutation group and that is represented on simplicial finite element spaces by the pullback operation. We are interested in whether a vector-valued finite element space has a basis that is invariant under permutation of indices. We determine a natural notion of invariance and determine sufficient conditions on the dimension and polynomial degrees for the existence of invariant bases. It is conjectured that these conditions are necessary too. We utilize Djoković and Malzan's classification of monomial irreducible representations of the symmetric group, and the permutation invariance of the geometric decomposition of finite element spaces. Invariant bases are constructed in dimensions two and three for different spaces of finite element differential forms.

1. Introduction

The Lagrange finite element space over a simplex is a canonical example of a finite element space that can easily be defined for arbitrary polynomial degree. The literature knows several common examples of bases for the higher-order Lagrange space, including the standard nodal basis, the barycentric bases, and the Bernstein bases [2, 1, 29]. A convenient feature of these canonical bases is their invariance under re-numbering of the vertices: the basis does not change if we re-number the vertices of the simplex, or equivalently, if we transport the basis functions along an affine automorphism of the simplex.

While this convenient feature might easily be taken for granted, it fails to hold for vector-valued finite element spaces, such as the Raviart-Thomas spaces, Brezzi-Douglas-Marini spaces, and the Nédeléc spaces of first and second kind [41, 12, 37]. Indeed, even finding explicit bases for these vector-valued finite element spaces is a non-trivial topic which has only been addressed after the turn of the century [6, 7, 24, 20, 9, 30]. Whether an invariant basis exists seems to be an intricate question: while no such basis exists for the space of constant vector fields over a triangle, one easily finds such a basis for the linear vector fields of a triangle.

2000 *Mathematics Subject Classification.* 65N30.

Key words and phrases. barycentric coordinates, finite element exterior calculus, monomial representation, representation theory of the symmetric group.

This research was supported by the European Research Council through the FP7-IDEAS-ERC Starting Grant scheme, project 278011 STUCCOFIELDS. This research was supported in part by NSF DMS/RTG Award 1345013 and DMS/CM Award 1262982. The author would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme "Geometry, compatibility and structure preservation in computational differential equations" where work on this paper was undertaken. This work was supported by EPSRC grant no EP/K032208/1.

Generally speaking, the existence of such a basis for each family of vector-valued finite element spaces seems to depend on the polynomial degree and the dimension.

The purpose of this article is to address this algebraic aspect of finite element differential forms: we present a natural notion of invariance and show the existence of invariant bases for certain finite element spaces. In particular, we give sufficient conditions on the polynomial degree and the dimension for each family of finite element spaces. We conjecture that these conditions are also necessary. This work continues prior study of bases in finite element exterior calculus [30].

In order to achieve the aim of this article, we adopt the framework of finite element exterior calculus (FEEC, [6]), which translates the vector-valued finite element spaces into the calculus of differential forms. A peculiar feature of FEEC is that it generalizes many results in finite element theory previously known only for special cases and puts these into a common framework: this includes convergence results [8, 5, 4, 3], approximation theory [16, 26, 32, 31], and a posteriori error estimation [18].

We use the representation theory of the permutation group in order to address the question of invariant bases in finite element exterior calculus. Affine automorphisms of a simplex correspond to permutations reordering the vertices of that simplex. These automorphisms constitute a finite symmetric group, and they transform finite element differential forms via the pullback operation. Thus representation theory emerges naturally in our study, because these pullbacks are a linear representations over finite element spaces. These automorphisms constitute a finite group isomorphic to the group of permutations of the simplex vertices,

It turns out that we need to study finite element spaces with complex coefficients in order to develop a satisfying theory of invariant bases. Our notion of invariance in this article is *invariant under the action of the symmetric group up to multiplication by complex units*. In the language of representation theory, we are interested under which circumstances the action of the symmetric group over a finite element space can be represented by a *monomial matrix group* with real or complex coefficients [38]. The transition to complex numbers reveals interesting structures: for example, the constant complex vector fields over a triangle have a basis invariant up to multiplication by complex roots of unity. The calculus of differential forms is essential in order to understand the relations within our theoretical framework.

We construct invariant bases for finite element spaces of higher polynomial order by a reduction to the case of lower polynomial degree. Towards that aim, we analyze the interaction of simplicial symmetries with two concepts in the theory of finite element exterior calculus. On the one hand, we recall the geometric decomposition of the finite element spaces [7]

$$\mathcal{P}_r\Lambda^k(T) = \bigoplus_{F\subseteq T} \operatorname{ext}^{r,k}_{F,T}\, \mathring{\mathcal{P}}_r\Lambda^k(F), \quad \mathcal{P}_r^-\Lambda^k(T) = \bigoplus_{F\subseteq T} \operatorname{ext}^{r,k,-}_{F,T}\, \mathring{\mathcal{P}}_r^-\Lambda^k(F).$$

This decomposition involves extension operators that we show to preserve invariant bases, so a geometrically decomposed invariant basis for finite element methods can be constructed from invariant bases for the finite element spaces with boundary conditions. On the other hand, we recall the canonical isomorphisms [6] over an

n-dimensional simplex T

$$\mathring{\mathcal{P}}_r^- \Lambda^k(T) \simeq \mathcal{P}_{r-n+k-1}\Lambda^{n-k}(T), \quad \mathring{\mathcal{P}}_r \Lambda^k(T) \simeq \mathcal{P}^-_{r-n+k}\Lambda^{n-k}(T).$$

These isomorphisms are natural for the algebraic theory of finite element exterior calculus in the sense that they preserve the canonical spanning sets [30]. We show that they commute with the simplicial symmetries and thus preserve invariant bases. Hence, in order to construct invariant bases for the finite element spaces with boundary conditions it suffices to find invariant bases for finite element spaces of lower polynomial degree and over lower-dimensional simplices. The former two observations combined enable a recursive construction of invariant, under the precondition that invariant bases are available in the base cases.

The aforementioned base case refers to the finite element spaces of differential forms of polynomial order zero, that is, constants. The theory of invariant bases for spaces of constant differential forms over a simplex derives from classification of monomial irreducible representations of the symmetric group due to Djoković and Malzan [19]. Specifically, invariant bases for spaces of constant differential forms exist only in the case of scalar and volume forms, the case of differential forms up to dimension 3, and the case of constant 2-forms over 4-simplices. This completely determines under which conditions on the polynomial degree our recursive construction facilitates the construction of invariant bases up from the base cases.

We outline the invariant bases for constant fields in vector calculus notation and using barycentric coordinates of the simplex. Over a tetrahedron, for example, the three vector fields

$$\begin{gathered}\psi_w = \nabla\lambda_0 - \nabla\lambda_1 + \nabla\lambda_2 - \nabla\lambda_3, \quad \psi_p = \nabla\lambda_0 + \nabla\lambda_1 - \nabla\lambda_2 - \nabla\lambda_3, \\ \psi_k = \nabla\lambda_0 - \nabla\lambda_1 - \nabla\lambda_2 + \nabla\lambda_3\end{gathered} \tag{1}$$

are a basis for the constant vector fields, and that basis is invariant under renumbering of vertices up to signs. Similarly, the three constant cross products

$$\psi_w \times \psi_p, \quad \psi_w \times \psi_k, \quad \psi_p \times \psi_k \tag{2}$$

are a basis for the constant pseudovector fields over a tetrahedron and that basis is again invariant under renumbering up to signs. Over a triangle, the transition to complex coefficients reveals the following observation which may come surprising to some readers: the two constant vector fields

$$\begin{gathered}\phi_0 = \nabla\lambda_0 + e^{2i\pi/3}\nabla\lambda_1 + e^{-2i\pi/3}\nabla\lambda_2, \\ \phi_1 = \nabla\lambda_0 + e^{-2i\pi/3}\nabla\lambda_1 + e^{2i\pi/3}\nabla\lambda_2\end{gathered} \tag{3}$$

are a basis for the complex constant vector fields over a triangle that is invariant under renumbering of vertices up to complex units, more specifically, up to cubic roots of unity. Lastly, we mention that quartic roots of unity appear in the construction of an invariant bases for the bivector fields over a four-dimensional hypertetrahedron.

This allows the construction of bases for finite element spaces that are invariant up to complex roots of unity. Whether these complex roots of unity are real, that is, the basis is invariant up to sign changes, depends on the simplex dimension and the polynomial degree. We conjecture that our construction is exhaustive: no finite element spaces in finite element exterior calculus have bases invariant up to real

and complex units except for the one discussed in this article.

As a convenience for the reader, we summarize the application of our theory to common (real-valued) finite element spaces below. We use the language of vector analysis and the notation as in the article. The following finite element spaces have bases that invariant up to sign changes under reordering of the vertices:

- The Brezzi-Douglas-Marini space of degree r over a triangle T,
$$\mathbf{BDM}_r(T) := \operatorname{span}\{\ \lambda^\alpha \nabla\lambda_i \mid \alpha \in A(r, 0:2), 0 \leq i \leq 2\ \}$$
if r is not divisible by 3.
- The Raviart-Thomas space of degree r over a triangle T,
$$\mathbf{RT}_r(T) := \operatorname{span}\{\ \lambda^\alpha \phi_{ij} \mid \alpha \in A(r-1, 0:2), 0 \leq i < j \leq 2\ \}$$
if $r-2$ is not divisible by 3.
- The divergence-conforming Brezzi-Douglas-Marini space of degree r over a tetrahedron T,
$$\mathbf{BDM}_r(T) := \operatorname{span}\{\ \lambda^\alpha \nabla\lambda_i \times \nabla\lambda_j \mid \alpha \in A(r, 0:3), 0 \leq i < j \leq 3\ \}$$
if $r \in \{1, 2, 4, 5, 8\}$.
- The divergence-conforming Raviart-Thomas space of degree r over a tetrahedron T,
$$\mathbf{RT}_r(T) := \operatorname{span}\{\ \lambda^\alpha \phi_{ijk} \mid \alpha \in A(r-1, 0:3), 0 \leq i < j < k \leq 3\ \}$$
if $r \in \{2, 3, 4, 6, 7, 9\}$.
- The curl-conforming Nédélec space of the first kind degree r over a tetrahedron T,
$$\mathbf{Nd}_r^{\mathrm{fst}}(T) := \operatorname{span}\{\ \lambda^\alpha \phi_{ij} \mid \alpha \in A(r-1, 0:3), 0 \leq i < j \leq 3\ \}$$
if $r \in \{0, 1, 3, 4, 7\}$.
- The curl-conforming Nédélec space of the second kind of degree r over a tetrahedron T,
$$\mathbf{Nd}_r^{\mathrm{snd}}(T) := \operatorname{span}\{\ \lambda^\alpha \nabla\lambda_i \mid \alpha \in A(r, 0:3), 0 \leq i \leq 3\ \}$$
if $r \in \{0, 1, 2, 4, 5, 8\}$.

However, the complex-valued versions of these finite element spaces have bases invariant up to multiplication by cubic roots of unity, irrespective of the polynomial degree. We conjecture that the for remaining polynomial degrees not covered above, no basis invariant up to sign changes exists.

This article utilizes representation theory for a theoretical contribution to numerical analysis, and some aspects can be of broader interest in representation theory. Our action of the symmetric group on finite element spaces is fully specified by action of the symmetric group on the barycentric coordinates, which is a rewriting of the *standard representation* of the symmetric group. The author suggests this as another interesting aspect of the representation theory of the symmetric group. The notion of monomial representation is central to our contribution. However, monomial representations do not seem to be a standard topic in introductory textbooks on representation theory, and only a few articles approach constructive aspects of monomial representations (see [39, 40]). We also remark that groups of

monomial matrices over finite fields have found use in cryptography and coding theory [23]. The author suggests a comprehensive study of the category of monomial representations of finite groups. Such study may verify or refute our conjecture that there are no monomial representations on finite element spaces not covered by our recursive construction. Moreover, such research would provide foundations to analyze finite element spaces over polytopes with more general symmetry groups using generalized barycentric coordinates [35, 21, 33, 13].

The representation theory of groups has had various applications throughout numerical and computational mathematics, such as in geometric integration theory [14, 45, 36] and artificial neural networks [11]. Our application in finite element methods adds new application of representation theory to that list.

Bases for finite element spaces have been subject of research for a long time under various perspectives. The choice of bases influences the condition numbers and sparsity properties of the global finite element matrices [2, 42, 10, 28]. Bases for vector-valued finite element spaces, such as Brezzi-Douglas-Marini spaces, Raviart-Thomas spaces, or Nédélec spaces have been stated explicitly relatively recently [6, 7, 24, 20, 9, 30]. The invariance of bases under renumbering of the vertices of a simplex is not an issue for scalar-valued finite element spaces but becomes a highly nontrivial topic for vector-valued finite element spaces. To the author's best knowledge, the questions addressed in this article have been in informal circulation for quite some time but no research results have been published before. We remark that the seminal article of Arnold, Falk, and Winther [6] utilized techniques of representation theory to classify the affinely invariant finite-dimensional vector spaces of polynomial differential forms.

The remainder of this work is structured as follows. Important preliminaries on combinatorics, exterior calculus, and polynomial differential forms are summarized in Section 2. We review elements of representation theory in Section 3. In Section 4 we establish first results on the coordinate transformation of polynomial differential forms. In Section 5 we study invariant bases and spanning sets for lowest-order finite element spaces. We discuss the symmetry properties of the canonical isomorphisms in Section 6. We discuss extension operators, geometric decompositions, and their symmetry properties in Section 7. Putting these results together, the recursive construction of invariant bases and applications are discussed in Section 8.

2. Notation and Definitions

We introduce and review notions from combinatorics, simplicial geometry, and differential forms over simplices. Much of this section, though not everything, is a summary of results in [30]. We refer to Arnold, Falk, and Winther [6, 7] and to Hiptmair [25] for further background on polynomial differential forms.

2.1. **Combinatorics.** We let $\delta_{m,n}$ be the Kronecker delta for any $m, n \in \mathbb{Z}$. For $m, n \in \mathbb{Z}$ we write $[m : n] = \{i \in \mathbb{Z} \mid m \leq i \leq n\}$ and let $\epsilon(m, n) = 1$ if $m < n$ and $\epsilon(m, n) = -1$ if $m > n$. The set of all permutations of $[m : n]$ is written $\mathrm{Perm}(m : n)$ and we abbreviate $\mathrm{Perm}(n) = \mathrm{Perm}(0 : n)$. We let $\mathrm{sgn}(\pi) \in \{-1, 1\}$ be the *sign* of any permutation $\pi \in \mathrm{Perm}(m : n)$.

We write $A(m:n)$ for the set of multiindices over $[m:n]$. For any $\alpha \in A(m:n)$,

$$|\alpha| := \sum_{i=m}^{n} \alpha(i), \quad [\alpha] := \{\, i \in [m:n] \mid \alpha(i) > 0 \,\}, \quad \lfloor \alpha \rfloor := \min[\alpha].$$

We let $A(r, m:n)$ be the set of those $\alpha \in A(m:n)$ for which $|\alpha| = r$, and we abbreviate $A(r,n) := A(r, 0:n)$. The sum $\alpha + \beta$ of $\alpha, \beta \in A(r, m:n)$ is defined in the obvious manner. We let $\delta_p : \mathbb{Z} \to \mathbb{N}$ be the function that equals 1 at p and is zero otherwise. When $\alpha \in A(r, m:n)$ and $p \in [m:n]$, then $\alpha + p \in A(r+1, m:n)$ is notation for $\alpha + \delta_p$. Similarly, when $p \in [\alpha]$, then $\alpha - p$ is notation for $\alpha - \delta_p$.

For $a, b, m, n \in \mathbb{Z}$, we let $F(a:b, m:n)$ be the set of functions from $[a:b]$ to $[m:n]$. For any $\tau \in F(a:b, m:n)$ we let $\epsilon(\tau) \in \{-1, 1\}$ be the sign of the permutation that orders the sequence $\tau(a), \ldots, \tau(b)$ in ascending order.

We let $\Sigma(a:b, m:n)$ be the set of strictly ascending mappings from $[a:b]$ to $[m:n]$. We call those mappings also *alternator indices*. By convention, $\Sigma(a:b, m:n) := \{\emptyset\}$ whenever $a > b$. For any $\sigma \in \Sigma(a:b, m:n)$ we let

$$[\sigma] := \{\, \sigma(i) \mid i \in [a:b] \,\},$$

and we write $\lfloor \sigma \rfloor$ for the minimal element of $[\sigma]$ provided that $[\sigma]$ is not empty, and $\lfloor \sigma \rfloor = \infty$ otherwise. Furthermore, if $q \in [m:n] \setminus [\sigma]$, then we write $\sigma + q$ for the unique element of $\Sigma(a : b+1, m:n)$ with image $[\sigma] \cup \{q\}$. In that case, we also write $\epsilon(q, \sigma)$ for the sign of the permutation that orders the sequence $q, \sigma(a), \ldots, \sigma(b)$ in ascending order, and we write $\epsilon(\sigma, q)$ for the sign of the permutation that orders the sequence $\sigma(a), \ldots, \sigma(b), q$ in ascending order. Note also that $\epsilon(\sigma, q) = (-1)^{b-a}\epsilon(q, \sigma)$. Similarly, if $p \in [\sigma]$, then we write $\sigma - p$ for the unique element of $\Sigma(a : b-1, m:n)$ with image $[\sigma] \setminus \{p\}$.

We abbreviate $\Sigma(k,n) = \Sigma(1:k, 0:n)$ and $\Sigma_0(k,n) = \Sigma(0:k, 0:n)$. If n is understood and $k, l \in [0:n]$, then for any $\sigma \in \Sigma(k,n)$ we define $\sigma^c \in \Sigma_0(n-k, n)$ by the condition $[\sigma] \cup [\sigma^c] = [0:n]$, and for any $\rho \in \Sigma_0(l,n)$ we define $\rho^c \in \Sigma(n-l, n)$ by the condition $[\rho] \cup [\rho^c] = [0:n]$. In particular, $\sigma^{cc} = \sigma$ and $\rho^{cc} = \rho$. We emphasize that σ^c and ρ^c depend on n, which we suppress in the notation.

When $\sigma \in \Sigma(k,n)$ and $\rho \in \Sigma_0(l,n)$ with $[\sigma] \cap [\rho] = \emptyset$, then $\epsilon(\sigma, \rho)$ denotes the sign of the permutation ordering the sequence $\sigma(1), \ldots, \sigma(k), \rho(0), \ldots, \rho(l)$ in ascending order.

2.2. **Simplices.** Let $n \in \mathbb{N}_0$. An n-dimensional simplex T is the convex closure of pairwise distinct affinely independent points $v_0^T, \ldots, v_n^T$ in Euclidean space, called the *vertices* of T. We call $F \subseteq T$ a *subsimplex* of T if the set of vertices of F is a subset of the set of vertices of T. We write $\imath(F,T) \colon F \to T$ for the set inclusion of F into T.

As an additional structure, we assume that the vertices of all simplices are ordered. For simplicity, we assume that all simplices have vertices ordered compatibly to the order of vertices on their subsimplices. Suppose that F is an m-dimensional subsimplex of T with ordered vertices $v_0^F, \ldots, v_m^F$. With a mild abuse of notation, we let $\imath(F,T) \in \Sigma_0(m,n)$ be defined by $v^T_{\imath(F,T)(i)} = v_i^F$.

2.3. **Barycentric Coordinates and Differential Forms.** Let T be a simplex of dimension n. Following the notation of [6], we denote by $\Lambda^k(T)$ the space of *differential k-forms* over T with smooth bounded real coefficients of all orders,

where $k \in \mathbb{Z}$. Recall that these mappings take values in the k-th exterior power of the dual of the tangential space of the simplex T. In the case $k = 0$, the space $\Lambda^0(T) = C^\infty(T)$ is just the space of smooth functions over T with uniformly bounded derivatives. Furthermore, $\Lambda^k(T) = \{0\}$ unless $0 \le k \le n$.

We write $\mathbb{R}\Lambda^k(T) = \Lambda^k(T)$ and let $\mathbb{C}\Lambda^k(T)$ denote the complexification of $\mathbb{R}\Lambda^k(T)$. All the algebraic operations defined in the following apply to $\mathbb{C}\Lambda^k(T)$ completely analogously.

We recall the *exterior product* $\omega \wedge \eta \in \Lambda^{k+l}(T)$ for $\omega \in \Lambda^k(T)$ and $\eta \in \Lambda^l(T)$ and that it satisfies $\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega$. We let $\mathsf{d}\colon \Lambda^k(T) \to \Lambda^{k+1}(T)$ denote the *exterior derivative*. It satisfies $\mathsf{d}\,(\omega \wedge \eta) = \mathsf{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathsf{d}\eta$ for $\omega \in \Lambda^k(T)$ and $\eta \in \Lambda^l(T)$. We also recall that the integral $\int_T \omega$ of a differential n-form over T is well-defined.

Let F be an m-dimensional subsimplex of T. The inclusion $\imath(F,T)\colon F \to T$ naturally induces a mapping $\mathrm{tr}_{T,F}\colon \Lambda^k(T) \to \Lambda^k(F)$ by taking the pullback, which we call the *trace* from T onto F. It is well-known that the exterior derivative commutes with taking traces, that is, $\mathsf{d}\,\mathrm{tr}_{T,F}\,\omega = \mathrm{tr}_{T,F}\,\mathsf{d}\omega$ for all $\omega \in \Lambda^k(T)$.

The *barycentric coordinates* $\lambda_0^T, \dots, \lambda_n^T \in \Lambda^0(T)$ are the unique affine functions over T that satisfy the *Lagrange property*

$$\lambda_i^T(v_j) = \delta_{ij}, \quad i,j \in [0:n]. \tag{4}$$

The barycentric coordinate functions of T are linearly independent and constitute a partition of unity:

$$1 = \lambda_0^T + \cdots + \lambda_n^T. \tag{5}$$

We write $\mathsf{d}\lambda_0^T, \mathsf{d}\lambda_1^T, \dots, \mathsf{d}\lambda_n^T \in \Lambda^1(T)$ for the exterior derivatives of the barycentric coordinates. The exterior derivatives are differential 1-forms and constitute a partition of zero:

$$0 = \mathsf{d}\lambda_0^T + \cdots + \mathsf{d}\lambda_n^T. \tag{6}$$

It can be shown that this is the only linear independence between the exterior derivatives of the barycentric coordinate functions.

Several classes of differential forms over T that are expressed in terms of the barycentric polynomials and their exterior derivatives. When $r \in \mathbb{N}_0$ and $\alpha \in A(r,n)$, then the corresponding *barycentric polynomial* over T is

$$\lambda_\alpha^T := \prod_{i=0}^{n} (\lambda_i^T)^{\alpha(i)}. \tag{7}$$

When $a, b \in \mathbb{N}_0$ and $\sigma \in \Sigma(a:b, 0:n)$, the corresponding *barycentric alternator* is

$$\mathsf{d}\lambda_\sigma^T := \mathsf{d}\lambda_{\sigma(a)}^T \wedge \cdots \wedge \mathsf{d}\lambda_{\sigma(b)}^T. \tag{8}$$

Here, we treat the special case $\sigma = \emptyset$ by defining $\mathsf{d}\lambda_\emptyset^T = 1$.

Whenever $a, b \in \mathbb{N}_0$ and $\rho \in \Sigma(a:b, 0:n)$, then the corresponding *Whitney form* is

$$\phi_\rho^T := \sum_{p \in [\rho]} \epsilon(p, \rho - p) \lambda_p^T \mathsf{d}\lambda_{\rho-p}^T. \tag{9}$$

In the special case that $\rho_T\colon [0:n] \to [0:n]$ is the single member of $\Sigma_0(n,n)$, then we write $\phi_T := \phi_{\rho_T}$ for the associated Whitney form. In the sequel, we call the differential forms (7), (8), (9), and their sums and exterior products, *barycentric differential forms* over T.

For notational convenience in some statements, we will also allow $\sigma \in F(a:b,0:n)$ and $\rho \in F(a:b,0:n)$ to be arbitrary functions and then define

$$\mathsf{d}\lambda_\sigma^T := \mathsf{d}\lambda_{\sigma(a)}^T \wedge \cdots \wedge \mathsf{d}\lambda_{\sigma(b)}^T,$$
$$\phi_\rho^T := \sum_{i\in[a:b]} (-1)^{i-a} \lambda_{\rho(i)}^T \mathsf{d}\lambda_{\rho(a)}^T \wedge \cdots \wedge \mathsf{d}\lambda_{\rho(i-1)}^T \wedge \mathsf{d}\lambda_{\rho(i+1)}^T \wedge \cdots \wedge \mathsf{d}\lambda_{\rho(b)}^T.$$

Lemma 2.1.
Let $\tau \in F(a:b,0:n)$ be injective. Then there exist a unique $\rho \in \Sigma(a:b,0:n)$ and a unique permutation $\pi : [a:b] \to [a:b]$ such that $\rho = \tau\pi$. Moreover

$$\mathsf{d}\lambda_\rho = \epsilon(\tau)\mathsf{d}\lambda_\tau, \quad \phi_\rho = \epsilon(\tau)\phi_\tau.$$

Proof. For every injective $\tau \in F(a:b,0:n)$ there exist a unique $\rho \in \Sigma(a:b,0:n)$ and $\pi \in \mathrm{Perm}(a:b)$ such that $\rho = \tau\pi$. We have $\mathsf{d}\lambda_\rho = \epsilon(\pi)\mathsf{d}\lambda_\tau$ and $\epsilon(\tau) = \epsilon(\pi)$, and π is the permutation that orders the sequence $\tau(a),\ldots,\tau(b)$ into ascending order.

Consider any $p \in [\tau]$ such that $p = \tau(i)$ for some $i \in [a:b]$; we let $t_{p,0}$ be the minimal number of transpositions necessary to bring the sequence $\tau(a),\ldots,\tau(b)$ into order $\tau(i),\tau(a),\ldots,\tau(i-1),\tau(i+1),\ldots,\tau(b)$ and let $t_{p,1}$ be the minimal number of transpositions necessary to bring the sequence $\tau(a),\ldots,\tau(i-1),\tau(i+1),\ldots,\tau(b)$ into ascending order. We now see that $(-1)^{t_{p,0}+t_{p,1}} = \epsilon(\pi)\epsilon(p,\rho-p)$. It follows that

$$\epsilon(\pi)\phi_\rho = \sum_{p\in[\rho]} \epsilon(\pi)\epsilon(p,\rho-p)\lambda_p^T \mathsf{d}\lambda_{\rho-p}^T = \sum_{p\in[\rho]} (-1)^{t_{p,0}+t_{p,1}} \lambda_p^T \mathsf{d}\lambda_{\rho-p}^T = \phi_\tau.$$

This had to be shown. □

2.4. **Finite Element Spaces over Simplices.** Consider an n-dimensional simplex T, a polynomial degree $r \in \mathbb{N}_0$, and a form degree $k \in [0:n]$. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We introduce the sets of polynomial differential forms

$$\mathcal{SP}_r\Lambda^k(T) := \left\{ \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \;\middle|\; \alpha \in A(r,n),\ \sigma \in \Sigma(k,n) \right\}, \tag{10a}$$
$$\mathcal{S}\mathring{\mathcal{P}}_r\Lambda^k(T) := \left\{ \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \;\middle|\; \begin{array}{l} \alpha \in A(r,n),\ \sigma \in \Sigma(k,n), \\ [\alpha]\cup[\sigma] = [0:n] \end{array} \right\}, \tag{10b}$$
$$\mathcal{SP}_r^-\Lambda^k(T) := \left\{ \lambda_T^\alpha \phi_\rho^T \;\middle|\; \alpha \in A(r-1,n),\ \rho \in \Sigma_0(k,n) \right\}, \tag{10c}$$
$$\mathcal{S}\mathring{\mathcal{P}}_r^-\Lambda^k(T) := \left\{ \lambda_T^\alpha \phi_\rho^T \;\middle|\; \begin{array}{l} \alpha \in A(r-1,n),\ \rho \in \Sigma_0(k,n), \\ [\alpha]\cup[\rho] = [0:n] \end{array} \right\}, \tag{10d}$$

and their linear hulls

$$\begin{aligned} \mathbb{K}\mathcal{P}_r\Lambda^k(T) &:= \mathrm{span}_{\mathbb{K}}\, \mathcal{SP}_r\Lambda^k(T), & \mathbb{K}\mathcal{P}_r^-\Lambda^k(T) &:= \mathrm{span}_{\mathbb{K}}\, \mathcal{SP}_r^-\Lambda^k(T), \\ \mathbb{K}\mathring{\mathcal{P}}_r\Lambda^k(T) &:= \mathrm{span}_{\mathbb{K}}\, \mathcal{S}\mathring{\mathcal{P}}_r\Lambda^k(T), & \mathbb{K}\mathring{\mathcal{P}}_r^-\Lambda^k(T) &:= \mathrm{span}_{\mathbb{K}}\, \mathcal{S}\mathring{\mathcal{P}}_r^-\Lambda^k(T). \end{aligned} \tag{11}$$

The sets (10) are called the *canonical spanning sets*. Their linear hulls give rise to the standard finite element spaces (11) of finite element exterior calculus.

These canonical spanning sets are generally not linearly independent and so it remains to state an explicit bases for the spaces (11). We can give some simply examples whenever $r \geq 1$. We define the sets of barycentric differential forms

$$\mathcal{BP}_r\Lambda^k(T) := \left\{ \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \;\middle|\; \alpha \in A(r,n),\ \sigma \in \Sigma(k,n),\ \lfloor \alpha \rfloor \notin [\sigma] \right\}, \tag{12a}$$

$$\mathcal{B}\mathring{\mathcal{P}}_r\Lambda^k(T) := \left\{ \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \;\middle|\; \begin{array}{l} \alpha \in A(r,n),\ \sigma \in \Sigma(k,n), \\ \lfloor \alpha \rfloor \notin [\sigma],\ [\alpha] \cup [\sigma] = [0:n] \end{array} \right\}. \tag{12b}$$

$$\mathcal{BP}_r^-\Lambda^k(T) := \left\{ \lambda_T^\alpha \phi_\rho^T \;\middle|\; \begin{array}{l} \alpha \in A(r-1,n),\ \rho \in \Sigma_0(k,n), \\ \lfloor \alpha \rfloor \geq \lfloor \rho \rfloor \end{array} \right\}, \tag{12c}$$

$$\mathcal{B}\mathring{\mathcal{P}}_r^-\Lambda^k(T) := \left\{ \lambda_T^\alpha \phi_\rho^T \;\middle|\; \begin{array}{l} \alpha \in A(r-1,n),\ \rho \in \Sigma_0(k,n), \\ \lfloor \rho \rfloor = 0,\ [\alpha] \cup [\rho] = [0:n] \end{array} \right\}. \tag{12d}$$

A particular feature of these bases and spanning sets are their inclusion relations. On the one hand, the bases are subsets of the spanning sets,

$$\mathcal{BP}_r\Lambda^k(T) \subseteq \mathcal{SP}_r\Lambda^k(T), \quad \mathcal{BP}_r^-\Lambda^k(T) \subseteq \mathcal{SP}_r^-\Lambda^k(T),$$
$$\mathcal{B}\mathring{\mathcal{P}}_r\Lambda^k(T) \subseteq \mathcal{S}\mathring{\mathcal{P}}_r\Lambda^k(T), \quad \mathcal{B}\mathring{\mathcal{P}}_r^-\Lambda^k(T) \subseteq \mathcal{S}\mathring{\mathcal{P}}_r^-\Lambda^k(T).$$

On the other hand, the generators for the spaces with boundary conditions are contained in the generators for the unconstrained spaces,

$$\mathcal{S}\mathring{\mathcal{P}}_r\Lambda^k(T) \subseteq \mathcal{SP}_r\Lambda^k(T), \quad \mathcal{S}\mathring{\mathcal{P}}_r^-\Lambda^k(T) \subseteq \mathcal{SP}_r^-\Lambda^k(T),$$
$$\mathcal{B}\mathring{\mathcal{P}}_r\Lambda^k(T) \subseteq \mathcal{BP}_r\Lambda^k(T), \quad \mathcal{B}\mathring{\mathcal{P}}_r^-\Lambda^k(T) \subseteq \mathcal{BP}_r^-\Lambda^k(T).$$

We remark that

$$\mathring{\mathcal{P}}_r\Lambda^k(T) = \left\{ \omega \in \mathcal{P}_r\Lambda^k(T) \;\middle|\; \forall F \subsetneq T : \operatorname{tr}_{T,F}\omega = 0 \right\}, \tag{13}$$

$$\mathring{\mathcal{P}}_r^-\Lambda^k(T) = \left\{ \omega \in \mathcal{P}_r^-\Lambda^k(T) \;\middle|\; \forall F \subsetneq T : \operatorname{tr}_{T,F}\omega = 0 \right\}, \tag{14}$$

and we could have defined equivalently

$$\mathcal{S}\mathring{\mathcal{P}}_r\Lambda^k(T) \subseteq \mathcal{SP}_r\Lambda^k(T) \cap \mathring{\mathcal{P}}_r\Lambda^k(T), \quad \mathcal{S}\mathring{\mathcal{P}}_r^-\Lambda^k(T) \subseteq \mathcal{SP}_r^-\Lambda^k(T) \cap \mathring{\mathcal{P}}_r^-\Lambda^k(T),$$
$$\mathcal{B}\mathring{\mathcal{P}}_r\Lambda^k(T) \subseteq \mathcal{BP}_r\Lambda^k(T) \cap \mathring{\mathcal{P}}_r\Lambda^k(T), \quad \mathcal{B}\mathring{\mathcal{P}}_r^-\Lambda^k(T) \subseteq \mathcal{BP}_r^-\Lambda^k(T) \cap \mathring{\mathcal{P}}_r^-\Lambda^k(T).$$

For any $\sigma \in \Sigma(k,n)$ and $\rho \in \Sigma_0(k,n)$ we let

$$\lambda_{\sigma^c}^T := \prod_{i \in [\sigma^c]} \lambda_i^T \in \mathcal{P}_{n-k+1}(T), \quad \lambda_{\rho^c}^T := \prod_{i \in [\rho^c]} \lambda_i^T \in \mathcal{P}_{n-k}(T).$$

We can thus write

$$\mathcal{S}\mathring{\mathcal{P}}_r\Lambda^k(T) = \left\{ \lambda_T^\beta \lambda_{\sigma^c}^T \mathsf{d}\lambda_\sigma^T \;\middle|\; \beta \in A(r-n+k-1,n),\ \sigma \in \Sigma(k,n), \right\}, \tag{15}$$

$$\mathcal{S}\mathring{\mathcal{P}}_r^-\Lambda^k(T) = \left\{ \lambda_T^\beta \lambda_{\rho^c}^T \phi_\rho^T \;\middle|\; \beta \in A(r-n+k-1,n),\ \rho \in \Sigma_0(k,n) \right\}, \tag{16}$$

$$\mathcal{B}\mathring{\mathcal{P}}_r\Lambda^k(T) = \left\{ \lambda_T^\beta \lambda_{\sigma^c}^T \mathsf{d}\lambda_\sigma^T \;\middle|\; \begin{array}{l} \beta \in A(r-n+k-1,n),\ \sigma \in \Sigma(k,n) \\ \lfloor \beta \rfloor = \lfloor \sigma^c \rfloor, \end{array} \right\}, \tag{17}$$

$$\mathcal{B}\mathring{\mathcal{P}}_r^-\Lambda^k(T) = \left\{ \lambda_T^\beta \lambda_{\rho^c}^T \phi_\rho^T \;\middle|\; \beta \in A(r-n+k-1,n),\ \rho \in \Sigma_0(k,n) \right\}. \tag{18}$$

These identities will be used in Section 6 but we note that they also simplify indexing the basis forms, which is an auxiliary result in its own right.

3. Elements of Representation Theory

In this section we gather element of the representation theory of finite groups. We keep this rather concise and refer to the literature [43, 17, 44, 27, 22] for a more thorough introduction. We introduce the relevant definitions and results so that the reader can follow the literature references upon which we build later in this exposition, which include the notions of irreducible representations, induced representations, and monomial representations. While the first two concepts are all but standard material in expositions on representation theory, the notion of monomial representation does not seem to have attracted much attention yet.

Throughout this section we fix a finite group G. The binary operation of the group is written multiplicatively. We let $e \in G$ denote the identity element of G and we let $g^{-1} \in G$ be the inverse of any $g \in G$. Furthermore, we fix $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ in this section to be either the field of real numbers or the field of complex numbers. For any vector space V over $\mathbb{K}$ we write $\mathrm{GL}(V)$ for its general linear group.

A *representation of* G is a group homomorphism $r : G \to \mathrm{GL}(V)$ from G into the general linear group of a vector space V. Definitions imply that $r(e) = \mathrm{Id}_V$ and that for all $g, h \in G$ we have

$$r(gh) = r(g)r(h), \quad r(g)^{-1} = r(g^{-1}).$$

The *dimension* of any vector space V is denoted by $\dim V$. The *dimension* of r is defined as the dimension of V, and the representation r is called *finite-dimensional* if $\dim V < \infty$.

Example 3.1.
The most important example of a group in this article is the group $\mathrm{Perm}(a : b)$ of permutations of the set $[a : b]$ for some $a, b \in \mathbb{Z}$. The composition is the binary operation of that group. We also recall the cycle notation: when $x_1, x_2, \ldots, x_m \in [a : b]$ are pairwise distinct, then $\pi := (x_1 x_2 \ldots x_m) \in \mathrm{Perm}(a : b)$ is the unique permutation that satisfies

$$\pi(x_1) = x_2, \quad \pi(x_2) = x_3, \quad \ldots \quad \pi(x_m) = x_1$$

and leaves all other members of $[a : b]$ invariant.

Example 3.2.
For any group G and any vector space V over some field $\mathbb{K}$ the mapping $r : G \to \mathrm{GL}(V)$ that assumes the constant value Id_V is a representation of G. This simple but important example is the *trivial representation of* G. For another basic example, recall that every group G generates the vector space $V = G^{\mathbb{K}}$ over $\mathbb{K}$. The mapping $r : G \to \mathrm{GL}(V)$ such that $r(g)h = gh$ for all $g, h \in G$ is a representation of G.

The matrices representing a finite group are all unitary. Indeed, since every $g \in G$ satisfies $g^{|G|} = e$, where $|G|$ denotes the cardinality of G, we also have $r(g)^{|G|} = \mathrm{Id}$. Consequently, the determinant of every $r(g)$ is a complex unit.

The representation r is called *faithful* if it is a group monomorphism, that is, only the unit of the group is mapped onto the identity.

We call two representations $r : G \to V$ and $r' : G \to V$ *equivalent* if there exists an isomorphism $T : V \to V$ such that $r'(g) = T^{-1} r(g) T$ for all $g \in G$. In

many circumstances, we are only interested in features of representations up to equivalence.

3.1. **Direct sums, subrepresentations, and irreducible representations.** We are interested in composing new representations from old representations. One way of doing so is the direct sum. Let $r : G \to \mathrm{GL}(V)$ and $s : G \to \mathrm{GL}(W)$ be two representations of G. Their *direct sum*

$$r \oplus s : G \to V \oplus W$$

is another representation of G and is defined by

$$(r \oplus s)(g)(v, w) = (r(g)v, s(g)w), \quad g \in G, \quad (v, w) \in V \oplus W.$$

The definition of the direct sum extends to the case of several summands in the obvious manner.

We are interested in how to conversely decompose a representation into direct summands. To study that question, we introduce further terminology.

Let $r : G \to \mathrm{GL}(V)$ be a representation. A subspace $A \subseteq V$ is called *r-invariant* if $r(g)A = A$ for all $g \in G$. Examples of r-invariant subspaces are V itself and the zero vector space. We call the representation r *irreducible* if the only r-invariant subspaces of V are V itself and the zero vector space, and otherwise we call r *reducible*.

Suppose that $A \subseteq V$ is an r-invariant subspace. Then there exists a representation $r^A : G \to \mathrm{GL}(A)$ in the obvious way. We call r^A a *subrepresentation* of r.

The following result is well-known in the literature of representation theory, and is known as Maschke's theorem [34].

Lemma 3.3.
Let $r : G \to \mathrm{GL}(V)$ be a finite-dimensional representation of G. Then there exist r-invariant subspaces $V_1, \dots, V_m \subseteq V$ such that

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_m, \quad r = r^{V_1} \oplus r^{V_2} \oplus \cdots \oplus r^{V_m},$$

and such that r^{V_i} is irreducible.

Proof. If r is irreducible, then there is nothing to show. Otherwise, there exists an r-invariant subspace $W \subset V$ that is neither V nor trivial. We let $P : V \to V$ be any projection of V onto W. Since G is finite, we can define the linear mapping

$$S : V \to V, \quad v \mapsto |G|^{-1} \sum_{z \in G} r(z)^{-1} P(r(z)v).$$

One verifies that S is again a projection onto W. Furthermore, we see that $S(r(g)v) = r(g)S(v)$ for all $g \in G$ and $v \in V$. So $\ker S$ is r-invariant. Since $V = W \oplus \ker S$ by linear algebra, we have a decomposition of V as the direct sum of two non-trivial r-invariant subspaces. One then sees that r is the direct sum of the representations of G over these spaces. The claim follows by an induction argument over the dimension of V. $\square$

3.2. **Restrictions and Induced Representations.** Let $H \subset G$ be a subgroup of G. We recall that the cardinality of H divides the cardinality of G, and that the quotient $|G|/|H|$ is called the *index* of H in G. Then we have a representation $r_H : H \to V$ that is called the *restriction* of r to the subgroup H. We generally cannot recover the original representation from its restriction to a subgroup, but there exists canonical way of inducing a representation of a group from a representation of a subgroup.

Suppose that we have a representation $s : H \to W$ of the subgroup H over the vector space W. First, we let $\mathcal{G} = (g_1, g_2, \ldots, g_M)$ be the list of representatives of the left cosets of H in G, where necessarily $M = |G|/|H|$ is in the index of H in G. We recall that for every $g \in G$ there exists a unique permutation $\tau_g \in \mathrm{Perm}(1 : M)$ such that $gg_i \in g_{\tau(i)}H$. More specifically, there exists a unique $h_{g,i} \in H$ such that $gg_i = g_{\tau(i)}h_{g,i}$. We now define the vector space

$$V = \bigoplus_{i=1}^{M} W$$

and define a representation $r : G \to \mathrm{GL}(V)$ by setting

$$r(g)(w_1, \ldots, w_M)_{\tau(i)} := s(h_{g,i})w_i, \quad 1 \leq i \leq M, \quad w_1, \ldots, w_M \in W.$$

In other words,

$$r(g)(w_1, \ldots, w_M) = \left(s(h_{g,\tau^{-1}(1)})w_{\tau^{-1}(1)}, \ldots, s(h_{g,\tau^{-1}(M)})w_{\tau^{-1}(M)}\right).$$

We call this the *induced representation*. Conceptually, V consists of M copies of W, each of which is associated to a coset representative g_i, and the induced representation applies the representation of H componentwise and then permutates the components.

We remark that the induced representation as defined above depends on the choice of representatives of the left cosets, which we have encoded in the set $\mathcal{G}$. However, different choices of representatives will yield representations that are equivalent; we refer to [43, Chapter 12.5] for the details.

3.3. **Monomial representations and invariant sets.** An square matrix is called *monomial* or a *generalized permutation matrix* if it is the product of a permutation matrix and an invertible diagonal matrix. A group representation $r : G \to \mathrm{GL}(V)$ is called *monomial* if there exists a basis of V with respect to which $r(g)$ is a monomial matrix for each $g \in G$.

A representation of G is called *induced monomial* if it is induced by a one-dimensional representation of a subgroup H of G. It is easy to see that every induced monomial representation is monomial. We remark that many authors use the term *monomial* for what we call *induced monomial*. For irreducible representations, being monomial and being induced monomial are equivalent [17, Corollary 50.6].

Lemma 3.4.
If the representation ρ is irreducible and induced monomial, then ρ is monomial.

We now introduce the notion of invariance that is the central object to study in this exposition. This notion is not standard in the literature of representation theory to the authors best knowledge.

Let $\mathcal{Q} \subseteq V$ be a finite set of cardinality M,

$$\mathcal{Q} = \{\omega_1, \dots, \omega_M\}$$

We say that $\mathcal{Q}$ is $\mathbb{K}$-*invariant* if for every $g \in G$ there exists a permutation $\tau \in \mathrm{Perm}(1:M)$ and a sequence of non-zero numbers $\chi_1, \dots, \chi_M \in \mathbb{K}$ such that

$$r(g)\omega_i = \chi_i \omega_{\tau(i)}, \quad 1 \le i \le M.$$

Since finite groups have unitary representations, $\chi_1, \dots, \chi_M \in \mathbb{K}$ must be units in $\mathbb{K}$ provided that $\omega_1, \dots, \omega_M$ are unit vectors.

We remark that any $\mathbb{R}$-invariant subset of a real vector space gives rise to an $\mathbb{R}$-invariant subset of the complexification of that vector space.

4. Notions of Invariance

In this section we study the pullback of polynomial differential forms under affine transformations between simplices in greater detail. We then introduce the simplicial symmetry group and its action on finite element spaces.

Suppose that T and T' are n-simplices and write $v_0, \dots, v_n$ and $v'_0, \dots, v'_n$ for the ordered list of vertices of T and T', respectively. For any permutation $\pi \in \mathrm{Perm}(n)$ there exists a unique affine diffeomorphism $S_\pi : T \to T'$ such that

$$S_\pi(v_i) = v'_{\pi^{-1}(i)}.$$

We now study the pullback operation on barycentric differential forms along S_π. We begin with the observation that

$$\left(S_\pi^* \lambda_i^{T'}\right)(v_j) = \lambda_i^{T'}\left(S_\pi(v_j)\right) = \lambda_i^{T'}\left(v'_{\pi^{-1}(j)}\right) = \delta_{i,\pi^{-1}(j)} = \delta_{\pi(i),j}, \quad i,j \in [0:n].$$

Consequently,

$$S_\pi^* \lambda_i^{T'} = \lambda_{\pi(i)}^T, \quad S_\pi^* \mathsf{d}\lambda_i^{T'} = \mathsf{d} S_\pi^* \lambda_i^{T'} = \mathsf{d}\lambda_{\pi(i)}^T. \tag{19}$$

It follows that for any multiindex $\alpha \in A(n)$ we have

$$S_\pi^* \lambda_{T'}^\alpha = \prod_{i=0}^n \left(\lambda_{\pi(i)}^T\right)^{\alpha(i)} = \prod_{i=0}^n \left(\lambda_i^T\right)^{\alpha(\pi^{-1}(i))} = \lambda_T^{\alpha\pi^{-1}}. \tag{20}$$

For any $\sigma, \sigma' \in \Sigma(1:k, 0:n)$ such that $[\sigma'] = [\pi\sigma]$ we observe

$$S_\pi^* \mathsf{d}\lambda_{T'}^\sigma = \epsilon(\pi\sigma)\mathsf{d}\lambda_T^{\sigma'}. \tag{21}$$

Similarly, for any $\rho, \rho' \in \Sigma(0:k, 0:n)$ such that $[\rho'] = [\pi\rho]$ we calculate

$$S_\pi^* \phi_\rho^{T'} = \epsilon(\pi\rho)\phi_{\rho'}^T. \tag{22}$$

Both (21) and (22) are a consequence of Lemma 2.1. These elementary observations suffice to completely describe the transformation of barycentric polynomial differential forms along affine diffeomorphisms.

With these technical preparations complete, we apply the theoretical framework of Section 3 We are particularly interested in the affine automorphisms of a simplex. Let T be an n-dimensional simplex with ordered list of vertices $v_0, \dots, v_n$. We let $\mathrm{Sym}(T)$ denote the *symmetry group* of T, which is the group of all affine

automorphisms of T and whose members we call *simplicial symmetries*. For any permutation $\pi \in \mathrm{Perm}(n)$ there exists a unique $S_\pi \in \mathrm{Sym}(T)$ such that

$$S_\pi(v_i) = v_{\pi^{-1}(i)}.$$

Similar as above, we say that π *induces* the simplicial symmetry S. Note that this defines a group isomorphism $\mathrm{Perm}(n) \simeq \mathrm{Sym}(T)$ by mapping $\pi \in \mathrm{Perm}(n)$ to the induced simplicial symmetry S_π.

Let now $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. In the terminology of representation theory, we have representations

$$r : \mathrm{Sym}(T) \to \mathrm{GL}\left(\mathbb{K}\mathcal{P}_r\Lambda^k(T)\right), \quad S_\pi \mapsto S_\pi^*,$$

We say that a set $\mathcal{Q} \subseteq \mathbb{K}\mathcal{P}_r\Lambda^k(T)$ is $\mathbb{K}$-invariant if it is $\mathbb{K}$-invariant under the representation r. The following observation is a simple but helpful auxiliary result.

Lemma 4.1.
Let $S : T \to T'$ be an affine diffeomorphism between n-simplices T and T'. If $\mathcal{Q} \subseteq \mathbb{K}\mathcal{P}_r\Lambda^k(T')$ is $\mathbb{K}$-invariant, then so is $S^*\mathcal{Q}$.

Proof. We write $\mathcal{Q} := S^*\mathcal{Q}$. For every $S_\pi \in \mathrm{Sym}(T)$ there exists $S'_\pi \in \mathrm{Sym}(T')$ such that $S_\pi = S^{-1}S'_\pi S$. Consequently, $S_\pi^*\mathcal{Q}' = S^*S_\pi^*S^{-*}\mathcal{Q}' = S^*S_\pi^*\mathcal{Q} = S^*\mathcal{Q} = \mathcal{Q}'$, where we have used $\mathbb{K}$-invariance of $\mathcal{Q}$. □

The following observations summarize a few examples.

Lemma 4.2.
The canonical spanning sets

$$\mathcal{SP}_r\Lambda^k(T), \quad \mathcal{SP}_r^-\Lambda^k(T), \quad \mathcal{S}\mathring{\mathcal{P}}_r\Lambda^k(T), \quad \mathcal{S}\mathring{\mathcal{P}}_r^-\Lambda^k(T)$$

are $\mathbb{R}$-invariant.

Proof. This follows from the definitions of these sets together with (20), (21), and (22). □

Lemma 4.3.
The basis $\mathcal{BP}_1^-\Lambda^k(T)$ of the lowest-order Whitney k-form is $\mathbb{R}$-invariant.

Proof. This is a direct consequence of the results in Section 4. □

Lemma 4.4.
The bases

$$\mathcal{BP}_r\Lambda^0(T), \quad \mathcal{B}\mathring{\mathcal{P}}_r\Lambda^0(T), \quad \mathcal{BP}_r\Lambda^n(T),$$
$$\mathcal{BP}_{r+1}^-\Lambda^0(T), \quad \mathcal{B}\mathring{\mathcal{P}}_{r+1}^-\Lambda^0(T), \quad \mathcal{BP}_r^-\Lambda^n(T)$$

are $\mathbb{R}$-invariant.

Proof. For the case of 0-forms, we verify the identities

$$\mathcal{BP}_r\Lambda^0(T) = \mathcal{SP}_r\Lambda^0(T) = \mathcal{SP}_r^-\Lambda^0(T) = \mathcal{BP}_r^-\Lambda^0(T),$$
$$\mathcal{B}\mathring{\mathcal{P}}_r\Lambda^0(T) = \mathcal{S}\mathring{\mathcal{P}}_r\Lambda^0(T) = \mathcal{S}\mathring{\mathcal{P}}_r^-\Lambda^0(T) = \mathcal{B}\mathring{\mathcal{P}}_r^-\Lambda^0(T),$$

and use Lemma 4.2. For the case of n-forms, we observe that

$$\mathcal{BP}_r^-\Lambda^n(T) = \mathcal{B}\mathring{\mathcal{P}}_r^-\Lambda^n(T), \quad \mathcal{BP}_r\Lambda^n(T) = \mathcal{B}\mathring{\mathcal{P}}_r\Lambda^n(T).$$

The $\mathbb{R}$-invariance of those sets follows by elementary calculations. $\square$

Needless to say that this examples are intuitively clear and do not cover the entire range of finite element spaces. The remainder of the exposition will address the question under which circumstances finite element spaces of differential forms allow for a $\mathbb{K}$-invariant basis in the sense of this subsection.

5. Invariant Bases of lowest polynomial Order

We commence our study of invariant bases with an analysis of the case of lowest order, that is, the constant differential forms over a simplex. The lowest-order case already exhibits non-trivial features and will serve as the base case for a recursive construction of invariant bases for spaces of higher-order polynomial differential forms. Moreover, in this section we determine the irreducible group action of the permutation group equivalent to the pullback group action on the finite element space. We take a closer look at the spaces of constant k-forms over n-simplices. We first review the following fact from representation theory.

We let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be arbitrary unless mentioned otherwise. We first recall the following result from the literature.

Lemma 5.1.
Let T be an n-simplex and $k \in \mathbb{N}_0$. The action of $\mathrm{Perm}(n)$ on $\mathbb{K}\mathcal{P}_0\Lambda^k(T)$ is irreducible. It is faithful for $0 < k < n$.

Example 5.2.
If $0 < k < n$, it is easily seen that the group action is faithful since only the identity element of $\mathrm{Perm}(n)$ acts as the identity on $\mathbb{K}\mathcal{P}_0\Lambda^k(T)$. To see that the group action is irreducible we refer to [22, Proposition 3.12].

We first consider the tetrahedron. We build an $\mathbb{R}$-invariant basis of $\mathbb{K}\mathcal{P}_0\Lambda^1(T)$, and then construct an $\mathbb{R}$-invariant basis for $\mathbb{K}\mathcal{P}_0\Lambda^2(T)$ by taking the exterior power.

Lemma 5.3.
Let T be a 3-simplex. An $\mathbb{R}$-invariant basis of $\mathbb{R}\mathcal{P}_0\Lambda^1(T)$ is

$$\begin{aligned}
\psi_w &= \mathsf{d}\lambda_0 - \mathsf{d}\lambda_1 + \mathsf{d}\lambda_2 - \mathsf{d}\lambda_3,\\
\psi_p &= \mathsf{d}\lambda_0 + \mathsf{d}\lambda_1 - \mathsf{d}\lambda_2 - \mathsf{d}\lambda_3,\\
\psi_k &= \mathsf{d}\lambda_0 - \mathsf{d}\lambda_1 - \mathsf{d}\lambda_2 + \mathsf{d}\lambda_3.
\end{aligned}$$

Proof. An elementary calculation verifies that the set is a basis. The permutation group $\mathrm{Perm}(3)$ is generated by the three cycles (01), (02), and (03). Direct computation shows that

$$\begin{aligned}
S^*_{(01)}\psi_w &= -\psi_k, & S^*_{(01)}\psi_p &= +\psi_p, & S^*_{(01)}\psi_k &= -\psi_w,\\
S^*_{(02)}\psi_w &= +\psi_w, & S^*_{(02)}\psi_p &= -\psi_k, & S^*_{(02)}\psi_k &= -\psi_p,\\
S^*_{(03)}\psi_w &= -\psi_p, & S^*_{(03)}\psi_p &= -\psi_w, & S^*_{(03)}\psi_k &= +\psi_k.
\end{aligned}$$

So this set is $\mathbb{R}$-invariant. $\square$

Lemma 5.4.
Let T be a 3-simplex. An $\mathbb{R}$-invariant basis of $\mathbb{R}\mathcal{P}_0\Lambda^2(T)$ is

$$\psi_w \wedge \psi_p, \quad \psi_w \wedge \psi_k, \quad \psi_p \wedge \psi_k.$$

Proof. We immediately see that these three 2-forms are a basis of $\mathbb{R}\mathcal{P}_0\Lambda^2(T)$. Using the cycles (01), (02), and (03) as in the previous proof, direct computation shows

$$\begin{aligned} S^*_{(01)}(\psi_w \wedge \psi_p) &= \psi_p \wedge \psi_k, & S^*_{(01)}(\psi_w \wedge \psi_k) &= -\psi_w \wedge \psi_k, \\ S^*_{(02)}(\psi_w \wedge \psi_p) &= -\psi_w \wedge \psi_k, & S^*_{(02)}(\psi_w \wedge \psi_k) &= -\psi_w \wedge \psi_p, \\ S^*_{(03)}(\psi_w \wedge \psi_p) &= -\psi_w \wedge \psi_p, & S^*_{(03)}(\psi_w \wedge \psi_k) &= -\psi_p \wedge \psi_k, \end{aligned}$$

$$\begin{aligned} S^*_{(01)}(\psi_p \wedge \psi_k) &= \psi_w \wedge \psi_p, \\ S^*_{(02)}(\psi_p \wedge \psi_k) &= -\psi_p \wedge \psi_k, \\ S^*_{(03)}(\psi_p \wedge \psi_k) &= -\psi_w \wedge \psi_k, \end{aligned}$$

The claim follows analogously as in the previous proof. □

Next we inspect the triangle, where the situation is more complicated. Here we need to consider not only real but also complex coefficients.

Lemma 5.5.
Let T be a 2-simplex. An $\mathbb{C}$-invariant basis of $\mathbb{C}\mathcal{P}_0\Lambda^1(T)$ is

$$\phi_0 = \mathsf{d}\lambda_0 + \xi_3 \mathsf{d}\lambda_1 + \xi_3^2 \mathsf{d}\lambda_2, \quad \phi_1 = \mathsf{d}\lambda_0 + \xi_3^2 \mathsf{d}\lambda_1 + \xi_3 \mathsf{d}\lambda_2,$$

where $\xi_3 = \exp(2i\pi/3)$ is the cubic root of unity. $\mathbb{R}\mathcal{P}_0\Lambda^1(T)$ has no $\mathbb{R}$-invariant basis.

Proof. We easily check that the two vectors constitute a basis and that

$$\begin{aligned} S^*_{(01)}\phi_0 &= \xi_3\phi_1, \quad S^*_{(01)}\phi_1 = \xi_3^2\phi_0, \\ S^*_{(02)}\phi_0 &= \xi_3^2\phi_1, \quad S^*_{(02)}\phi_1 = \xi_3\phi_0, \end{aligned}$$

where the cycles $(01), (02) \in \mathrm{Perm}(2)$ are generators of $\mathrm{Perm}(2)$. Hence it follows that $\{\phi_0, \phi_1\}$ is a $\mathbb{C}$-invariant basis of $\mathbb{C}\mathcal{P}_0\Lambda^1(T)$.

Assume that $\mathbb{C}\mathcal{P}_0\Lambda^1(T)$ has an $\mathbb{R}$-invariant basis. Since our representation of $\mathrm{Sym}(T)$ over $\mathbb{C}\mathcal{P}_0\Lambda^1(T)$ is faithful by Lemma 5.1, it then follows that $\mathrm{Sym}(T)$ is isomorphic to a subgroup of the group of 2×2 signed permutation matrices. The latter group has order 8 whereas $\mathrm{Sym}(T)$ has order 6. This contradicts the well-known fact that the order of a subgroup divides the order of any supergroup. So $\mathbb{C}\mathcal{P}_0\Lambda^1(T)$ has no $\mathbb{R}$-invariant basis. □

Lemma 5.6.
Let T be a 4-simplex. Define $\tau, \kappa \in \mathrm{Perm}(4)$ by $\tau := (01)$ and $\kappa := (01234)$. We abbreviate $\mathsf{d}\lambda_{ij} := \mathsf{d}\lambda_i \wedge \mathsf{d}\lambda_j$ for $0 \leq i < j \leq 4$. An $\mathbb{C}$-invariant basis of $\mathbb{C}\mathcal{P}_0\Lambda^2(T)$ is given by

$$\zeta_0 := \mathsf{d}\lambda_{01} + \mathsf{i}\mathsf{d}\lambda_{02} - \mathsf{i}\mathsf{d}\lambda_{03} - \mathsf{d}\lambda_{04} + \mathsf{d}\lambda_{12} + \mathsf{i}\mathsf{d}\lambda_{13} - \mathsf{i}\mathsf{d}\lambda_{14} + \mathsf{d}\lambda_{23} + \mathsf{i}\mathsf{d}\lambda_{24} + \mathsf{d}\lambda_{34},$$
$$\zeta_1 = S^*_\tau\zeta_0, \quad \zeta_2 = S^*_\kappa\zeta_1, \quad \zeta_3 = S^*_\kappa\zeta_2, \quad \zeta_4 = S^*_\kappa\zeta_3, \quad \zeta_5 = S^*_\kappa\zeta_4.$$

Proof. Recall that τ and κ are generators of the group Perm(4). One easily checks that

$$\zeta_0 = S_\kappa^*\zeta_0, \quad \zeta_1 = S_\kappa^*\zeta_5, \quad \mathrm{i}\zeta_3 = S_\tau^*\zeta_2, \quad -\mathrm{i}\zeta_2 = S_\tau^*\zeta_3, \quad -\mathrm{i}\zeta_5 = S_\tau^*\zeta_4, \quad \mathrm{i}\zeta_4 = S_\tau^*\zeta_5.$$

It follows that these vectors are a $\mathbb{C}$-invariant set. That they are a basis is verified by elementary calculations. □

Remark 5.7.
The existence of such a representation is implied by results of Djoković and Malzan [19]. Using the ansatz that the monomial matrices have coefficients in the quartic roots of unity, the basis above can be found as follows. The 5-cycle (01234) is represented by a generalized permutation matrix of size 6×6, and so that matrix has the non-zero structure of a 6×6 permutation matrix of order 5. Consequently, one of the $\mathbb{C}$-invariant basis vectors must be left invariant by the cyclic vertex permutation. Via machine assisted search one finds 4 different such vectors, up to multiplication by complex units. With the ansatz that the 2-cycle (01) maps these invariant forms into the orbit of the aforementioned 5-cycle, we can construct the desired basis.

We have already pointed out the relevance of Djoković and Malzan's contribution [19] on monomial representations of the symmetric group several times: The invariant bases constructed in this section concretize their results. Apart from the constant scalar and volume forms, for which $\mathbb{R}$-invariant bases are obvious, the bases found above are already exhaustive examples: no other spaces of constant differential forms over simplices of any dimension allows for a $\mathbb{C}$-invariant basis.

Lemma 5.8.
There does not exist a $\mathbb{C}$-invariant basis of $\mathcal{P}_0\Lambda^1(T)$ unless $k = 0$ or $k = n$ or $\dim T \leq 3$ or $k = 2$ with $\dim(T) = 4$.

Proof. We recall that the representations of Perm(n) over $\mathcal{P}_0\Lambda^k(T)$ are irreducible. Djoković and Malzan have shown [19, Theorem 1] that the only induced monomial irreducible representation of the group Perm(n) over spaces of constant differential forms are the trivial and the alternating representations, which corresponds to the group action on the space of constant functions and constant volume forms, the irreducible representations of Perm(2) and Perm(3), and an irreducible representation of Perm(4) on the space $\mathcal{P}_0\Lambda^2(T)$ for any 4-dimensional simplex T. These are the cases covered by Lemma 4.4, and by Lemmas 5.3, 5.4, 5.5, and 5.6. All other irreducible representations of Perm(n) are not induced monomial. Since induced monomial irreducible representations are monomial, the desired claim follows. □

6. Canonical Isomorphisms

In this section we review two fundamental classes of isomorphisms in finite element exterior calculus and show that they preserve $\mathbb{K}$-invariance of sets. These isomorphisms were discussed in [6] and also [15]; we follow the discussion in [30], where it is shown that these isomorphisms can be described in terms of the canonical spanning sets. In that sense, they are the natural isomorphisms for the algebraic theory of finite element exterior calculus.

Recall that the canonical isomorphisms

$$\mathcal{I}_{k,r} : \mathbb{K}\mathcal{P}_r\Lambda^k(T) \to \mathbb{K}\mathring{\mathcal{P}}^-_{r+k+1}\Lambda^{n-k}(T), \tag{23}$$

$$\mathcal{J}_{k,r} : \mathbb{K}\mathring{\mathcal{P}}_{r+n-k+1}\Lambda^k(T) \to \mathbb{K}\mathcal{P}^-_{r+1}\Lambda^{n-k}(T) \tag{24}$$

are uniquely defined by the identities

$$\mathcal{I}_{k,r}\left(\lambda^\alpha \mathsf{d}\lambda_\sigma\right) = \epsilon(\sigma,\sigma^c)\lambda^\alpha\lambda_\sigma\phi_{\sigma^c}, \tag{25}$$

$$\mathcal{J}_{k,r}\left(\lambda^\alpha\lambda_{\sigma^c}\mathsf{d}\lambda_\sigma\right) = \epsilon(\sigma,\sigma^c)\lambda^\alpha\phi_{\sigma^c}. \tag{26}$$

Note that these two identities prescribe the values of $\mathcal{I}_{k,r}$ and $\mathcal{J}_{k,r}$ over the canonical spanning sets. One can show that this results in well-defined $\mathbb{K}$-linear mappings [30].

Remark 6.1.
The underlying idea of these isomorphisms is multiplication or division by a monomial "bubble" function to map between finite element spaces with and without boundary conditions. For example, in the case $k = 0$, we have $\mathcal{I}^0_r(f) = \lambda_0\lambda_1\cdots\lambda_n\cdot f$ for all $f \in \mathcal{P}_r(T)$. This basic is modified for general k-forms such that the target space has minimal polynomial degree.

The following lemma shows that the isomorphisms commute with the simplicial symmetries up to sign changes.

Theorem 6.2.
Let $\pi \in S(0:n)$ and $S_\pi \in \mathrm{Sym}(T)$. Then

$$S^*_\pi\mathcal{I}_{k,r} = \epsilon(\pi)\mathcal{I}_{k,r}S^*_\pi, \quad S^*_\pi\mathcal{J}_{k,r} = \epsilon(\pi)\mathcal{J}_{k,r}S^*_\pi,$$
$$S^*_\pi\mathcal{I}^{-1}_{k,r} = \epsilon(\pi)\mathcal{I}^{-1}_{k,r}S^*_\pi, \quad S^*_\pi\mathcal{J}^{-1}_{k,r} = \epsilon(\pi)\mathcal{J}^{-1}_{k,r}S^*_\pi.$$

Proof. Let $\alpha \in A(r,n)$ and $\sigma \in \Sigma(1:k, 0:n)$. We write $\epsilon(\pi\sigma,\pi\sigma^c) \in \{-1,1\}$ for the sign of the permutation that reorders the sequence $\pi(0),\pi(1),\ldots,\pi(n)$ into the sequence

$$\pi\sigma(1),\ldots,\pi\sigma(k),\pi\sigma^c(0),\pi\sigma^c(1),\ldots,\pi\sigma^c(n-k).$$

We then observe the combinatorial identity

$$\epsilon(\pi\sigma,\pi\sigma^c)\epsilon(\pi\sigma)\epsilon(\pi\sigma^c)\epsilon(\sigma,\sigma^c) = \epsilon(\pi).$$

Using Lemma 2.1 and the results of Section 4, direct calculation now shows that

$$\begin{aligned}
S^*_\pi\mathcal{I}_{k,r}\left(\lambda^\alpha\mathsf{d}\lambda_\sigma\right) &= \epsilon(\sigma,\sigma^c)S^*_\pi\left(\lambda^\alpha\lambda_\sigma\phi_{\sigma^c}\right)\\
&= \epsilon(\sigma,\sigma^c)\epsilon(\pi\sigma^c)\lambda^{\alpha\pi^{-1}}\lambda_{\pi\sigma}\phi_{\pi\sigma^c}\\
&= \epsilon(\pi)\epsilon(\pi\sigma,\pi\sigma^c)\epsilon(\pi\sigma)\lambda^{\alpha\pi^{-1}}\lambda_{\pi\sigma}\phi_{\pi\sigma^c}\\
&= \epsilon(\pi)\epsilon(\pi\sigma)\mathcal{I}_{k,r}\left(\lambda^{\alpha\pi^{-1}}\mathsf{d}\lambda_{\pi\sigma}\right)\\
&= \epsilon(\pi)\mathcal{I}_{k,r}S^*_\pi\left(\lambda^\alpha\mathsf{d}\lambda_\sigma\right).
\end{aligned}$$

Analogous derivations work for the other isomorphism:

$$\begin{aligned}
S_\pi^* \mathcal{J}_{k,r}\left(\lambda^\alpha \lambda_{\sigma^c} \mathsf{d}\lambda_\sigma\right) &= \epsilon(\sigma,\sigma^c) S_\pi^*\left(\lambda^\alpha \phi_{\sigma^c}\right) \\
&= \epsilon(\sigma,\sigma^c)\epsilon(\pi\sigma^c)\lambda^{\alpha\pi^{-1}}\phi_{\pi\sigma^c} \\
&= \epsilon(\pi)\epsilon(\pi\sigma,\pi\sigma^c)\epsilon(\pi\sigma)\lambda^{\alpha\pi^{-1}}\phi_{\pi\sigma^c} \\
&= \epsilon(\pi)\epsilon(\pi\sigma)\mathcal{J}_{k,r}\left(\lambda^{\alpha\pi^{-1}}\lambda_{\sigma^c}\mathsf{d}\lambda_{\pi\sigma}\right) \\
&= \epsilon(\pi)\mathcal{J}_{k,r} S_\pi^*\left(\lambda^\alpha \lambda_{\sigma^c}\mathsf{d}\lambda_\sigma\right)
\end{aligned}$$

Finally, we find that

$$\begin{aligned}
\mathcal{I}_{k,r}^{-1} S_\pi^* &= \mathcal{I}_{k,r}^{-1} S_\pi^* \mathcal{I}_{k,r}\mathcal{I}_{k,r}^{-1} = \epsilon(\pi)\mathcal{I}_{k,r}^{-1}\mathcal{I}_{k,r} S_\pi^* \mathcal{I}_{k,r}^{-1} = \epsilon(\pi) S_\pi^* \mathcal{I}_{k,r}^{-1}, \\
\mathcal{J}_{k,r}^{-1} S_\pi^* &= \mathcal{J}_{k,r}^{-1} S_\pi^* \mathcal{J}_{k,r}\mathcal{J}_{k,r}^{-1} = \epsilon(\pi)\mathcal{J}_{k,r}^{-1}\mathcal{J}_{k,r} S_\pi^* \mathcal{J}_{k,r}^{-1} = \epsilon(\pi) S_\pi^* \mathcal{J}_{k,r}^{-1}.
\end{aligned}$$

This completes the proof. □

An important corollary is that the canonical isomorphisms and their inverses map $\mathbb{K}$-invariant sets to $\mathbb{K}$-invariant sets. We will use this fact to construct $\mathbb{K}$-invariant bases.

Corollary 6.3.
Let T be an n-simplex. Then

- $\mathcal{Q} \subseteq \mathbb{K}\mathcal{P}_r\Lambda^k(T)$ is $\mathbb{K}$-invariant if and only if $\mathcal{I}_k^r\mathcal{Q} \subseteq \mathbb{K}\mathring{\mathcal{P}}_{r+k+1}^-\Lambda^{n-k}(T)$ is $\mathbb{K}$-invariant.
- $\mathcal{Q} \subseteq \mathbb{K}\mathring{\mathcal{P}}_{r+n-k+1}\Lambda^k(T)$ is $\mathbb{K}$-invariant if and only if $\mathcal{J}_k^r\mathcal{Q} \subseteq \mathbb{K}\mathcal{P}_{r+1}^-\Lambda^{n-k}(T)$ is $\mathbb{K}$-invariant.

7. Traces and Extension Operators

In this section we study the interaction of simplicial symmetries with trace and extension operators. On the one hand, the trace operation preserves $\mathbb{K}$-invariant sets. On the other hand, there exist extension operators that preserve $\mathbb{K}$-invariant sets. The consequence of the latter is that a $\mathbb{K}$-invariant geometrically decomposed basis can be constructed if such bases are known for each component in the geometric decomposition.

We first introduce some additional notation. Let $F \subseteq T$ be a subsimplex and let $\pi \in \mathrm{Perm}(n)$ be a permutation. Then $S_\pi F$ is another subsimplex of T of the same dimension. With some abuse of notation, we have affine diffeomorphisms

$$S_\pi : F \to S_\pi F, \quad S_\pi^{-1} : S_\pi F \to F. \tag{27}$$

We first prove the result concerning the traces of $\mathbb{K}$-invariant sets.

Lemma 7.1.
Let T be an n-dimensional simplex and let $F \subseteq T$ be a subsimplex. If $\mathcal{Q} \subseteq \mathbb{K}\Lambda^k(T)$ is $\mathbb{K}$-invariant, then $\mathrm{tr}_{T,F}\,\mathcal{Q} \subseteq \mathbb{K}\Lambda^k(F)$ is $\mathbb{K}$-invariant.

Proof. Let $\mathcal{Q} = \{\omega_1, \ldots, \omega_M\}$, where M denotes the size of Q. Let $S \in \mathrm{Sym}(T)$ such that $S(F) = F$. Then there exist a permutation $\tau \in \mathrm{Perm}(1:M)$ and a sequence of numbers $\chi_1, \ldots, \chi_M \in \mathbb{K}$ such that $S^*\omega_i = \chi_i\omega_{\tau(i)}$ for $1 \le i \le M$. We

have

$$S^* \operatorname{tr}_{T,F} \omega_i = \operatorname{tr}_{T,F} S^*\omega_i = \operatorname{tr}_{T,F} \chi_i \omega_{\tau(i)} = \chi_i \operatorname{tr}_{T,F} \omega_{\tau(i)} \in \operatorname{tr}_{T,F} \mathcal{Q}.$$

This shows the desired result. □

Extension operators that facilitate a geometric decomposition have been described for several spaces of polynomial differential forms in finite element exterior calculus [6, 7, 30]. For our purpose, we utilize the extension operators given in [7].

Let T be an n-dimensional simplex and let $F \subseteq T$ be an m-dimensional subsimplex, and let $k \in \mathbb{N}_0$ and $r \in \mathbb{N}$. The extension operator for the $\mathcal{P}_r^- \Lambda^k$-family of spaces

$$\operatorname{ext}_{F,T}^{r,k,-} : \mathbb{K}\mathring{\mathcal{P}}_r^- \Lambda^k(F) \to \mathbb{K}\mathcal{P}_r^- \Lambda^k(T) \tag{28}$$

is uniquely defined by setting

$$\operatorname{ext}_{F,T}^{r,k,-} \lambda_F^\alpha \phi_\rho^F = \lambda_T^{\tilde{\alpha}} \phi_{\tilde{\rho}}^T, \tag{29}$$

for all $\alpha \in A(r-1,m)$, $\rho \in \Sigma_0(k,m)$, where $\tilde{\alpha} \in A(r-1,n)$ satisfies $\tilde{\alpha} = \alpha \circ \imath(F,T)^\dagger$ over $[\imath(F,T)]$ and is zero otherwise, and where $\tilde{\rho} = \imath(F,T) \circ \rho \in \Sigma_0(k,n)$. This prescribes the extension of a spanning set of $\mathbb{K}\mathcal{P}_r\Lambda^k$, and it follows from [7, Section 7] that this defines a linear operator.

The definition of the extension operators in the $\mathcal{P}_r\Lambda^k$-family

$$\operatorname{ext}_{F,T}^{r,k} : \mathbb{K}\mathring{\mathcal{P}}_r \Lambda^k(F) \to \mathbb{K}\mathcal{P}_r\Lambda^k(T) \tag{30}$$

is slightly more intricate. For any $\alpha \in A(r,n)$ and $\sigma \in \Sigma(1:k, 0:n)$ we define

$$\Psi_i^{\alpha,F,T} := \mathsf{d}\lambda_i^T - \frac{\alpha_i}{|\alpha|} \sum_{j \in [\imath(F,T)]} \mathsf{d}\lambda_j^T \tag{31}$$

and

$$\Psi_\sigma^{\alpha,F,T} := \Psi_{\sigma(1)}^{\alpha,F,T} \wedge \cdots \wedge \Psi_{\sigma(k)}^{\alpha,F,T}. \tag{32}$$

As described in [7, Section 8], the extension operator is well-defined by setting

$$\operatorname{ext}_{F,T}^{r,k} \lambda_F^\alpha \mathsf{d}\lambda_\sigma^F = \lambda_T^{\tilde{\alpha}} \Psi_{\tilde{\sigma}}^{\tilde{\alpha},F,T} \tag{33}$$

for all $\alpha \in A(r,m)$, $\sigma \in \Sigma(1:k, 0:m)$, where $\tilde{\alpha} \in A(r,n)$ satisfies $\tilde{\alpha} = \alpha \circ \imath(F,T)^\dagger$ over $[\imath(F,T)]$ and is zero otherwise, and where $\tilde{\sigma} = \imath(F,T) \circ \sigma \in \Sigma(k,n)$.

These operators are called *extension* operators because they are right-inverses of the trace,

$$\begin{aligned} \operatorname{tr}_{T,F} \operatorname{ext}_{F,T}^{r,k,-} \omega &= \omega, \quad \omega \in \mathbb{K}\mathring{\mathcal{P}}_r^- \Lambda^k(F), \\ \operatorname{tr}_{T,F} \operatorname{ext}_{F,T}^{r,k} \omega &= \omega, \quad \omega \in \mathbb{K}\mathring{\mathcal{P}}_r \Lambda^k(F). \end{aligned}$$

Moreover, if $F, G \subseteq T$ are subsimplices such that $F \nsubseteq G$, then

$$\begin{aligned} \operatorname{tr}_{T,G} \operatorname{ext}_{F,T}^{r,k,-} \omega &= 0, \quad \omega \in \mathbb{K}\mathring{\mathcal{P}}_r^- \Lambda^k(F), \\ \operatorname{tr}_{T,G} \operatorname{ext}_{F,T}^{r,k,-} \omega &= 0, \quad \omega \in \mathbb{K}\mathring{\mathcal{P}}_r \Lambda^k(F). \end{aligned}$$

We refer to prior publications [7] for detailed discussion of these extension operators. The central result is the following decomposition.

Theorem 7.2.
Let T be an n-simplex and let $r, k \in \mathbb{N}_0$ with $r > 0$. Then

$$\mathcal{P}_r\Lambda^k(T) = \bigoplus_{F\subseteq T} \operatorname{ext}^{k,r}_{F,T} \mathring{\mathcal{P}}_r\Lambda^k(F), \quad \mathcal{P}_r^-\Lambda^k(T) = \bigoplus_{F\subseteq T} \operatorname{ext}^{k,r,-}_{F,T} \mathring{\mathcal{P}}_r^-\Lambda^k(F).$$

We want to show that $\mathbb{K}$-invariant bases for the components in that geometric decomposition yield $\mathbb{K}$-invariant bases for the entire finite element space over the simplex. We begin with analyzing the how these extension operators interact with simplicial symmetries.

Theorem 7.3.
Let T be an n-dimensional simplex and let $F \subseteq T$ be an m-dimensional subsimplex. Let $r \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $S_\pi \in \operatorname{Sym}(T)$. Then

$$\operatorname{ext}^{r,k}_{F,T} S_\pi^* = S_\pi^* \operatorname{ext}^{r,k}_{S_\pi F,T}, \quad \operatorname{ext}^{r,k,-}_{F,T} S_\pi^* = S_\pi^* \operatorname{ext}^{r,k,-}_{S_\pi F,T}.$$

Proof. We will prove both identities over the canonical spanning sets, which will be sufficient.

Similarly, it is easily verified that for all $\alpha \in A(r,n)$ and $\sigma \in \Sigma(k,n)$ we have

$$S_\pi^* \operatorname{ext}^{r,k}_{S_\pi F,T} \lambda^\alpha_{S_\pi F} \mathsf{d}\lambda^{S_\pi F}_\sigma = S_\pi^* \lambda^{\tilde\alpha}_T \Psi^{\tilde\alpha,F,T}_{\tilde\sigma} = \lambda^{\tilde\alpha\pi}_T \Psi^{\tilde\alpha\pi,F,T}_{\pi\tilde\sigma} = \operatorname{ext}^{r,k}_{F,T} S_\pi^* \lambda^\alpha_{S_\pi F} \mathsf{d}\lambda^{S_\pi F}_\sigma.$$

As above, $\tilde\alpha \in A(r-1,n)$ satisfies $\tilde\alpha = \alpha \circ \imath(F,T)^\dagger$ over $[\imath(F,T)]$ and is zero otherwise, and $\tilde\sigma = \imath(F,T) \circ \sigma \in \Sigma(k,n)$.

Similarly, for all $\alpha \in A(r-1,n)$ and $\rho \in \Sigma_0(k,n)$ we have

$$S_\pi^* \operatorname{ext}^{r,k,-}_{S_\pi F,T} \lambda^\alpha_{S_\pi F} \phi^{S_\pi F}_\rho = S_\pi^* \lambda^{\tilde\alpha}_T \phi^T_{\tilde\rho} = \lambda^{\tilde\alpha\pi}_T \phi^T_{\pi\tilde\rho} = \operatorname{ext}^{r,k,-}_{F,T} S_\pi^* \lambda^\alpha_{S_\pi F} \phi^{S_\pi F}_\rho.$$

As above, $\tilde\alpha \in A(r-1,n)$ satisfies $\tilde\alpha = \alpha \circ \imath(F,T)^\dagger$ over $[\imath(F,T)]$ and is zero otherwise, and $\tilde\rho = \imath(F,T) \circ \rho \in \Sigma_0(k,n)$. □

We now work along the following idea: we obtain a $\mathbb{K}$-invariant bases of some finite element space if can find $\mathbb{K}$-invariant bases for each components in the geometric decomposition of that space. This is formalized in the following theorem and its proof.

Theorem 7.4.
Let $r \in \mathbb{N}_0$ and $r, k \in \mathbb{N}$, and let T be an n-simplex. Let $\mathcal{Q}\mathring{\mathcal{P}}_r\Lambda^k(F)$ be a $\mathbb{K}$-invariant basis of $\mathring{\mathcal{P}}_r\Lambda^k(F)$ for each subsimplex $F \subseteq T$ such that for all $\pi \in \operatorname{Perm}(0:n)$ we have

$$S_\pi^* \mathcal{Q}\mathring{\mathcal{P}}_r\Lambda^k(F) = \mathcal{Q}\mathring{\mathcal{P}}_r\Lambda^k(S_\pi F).$$

Then the union

$$\mathcal{Q}\mathcal{P}_r\Lambda^k(T) := \bigcup_{F\subseteq T} \operatorname{ext}^{r,k}_{F,T} \mathcal{Q}\mathring{\mathcal{P}}_r\Lambda^k(F)$$

is a $\mathbb{K}$-invariant basis of $\mathcal{P}_r\Lambda^k(T)$.

Proof. The union is disjoint. For every $\pi \in \mathrm{Perm}(n)$ we observe

$$\begin{aligned} S_\pi^* \mathcal{QP}_r \Lambda^k(T) &= \bigcup_{F \subseteq T} S_\pi^* \operatorname{ext}_{F,T}^{r,k} \mathcal{Q}\mathring{\mathcal{P}}_r \Lambda^k(F) \\ &= \bigcup_{F \subseteq T} \operatorname{ext}_{F,T}^{r,k} S_\pi^* \mathcal{Q}\mathring{\mathcal{P}}_r \Lambda^k(F) = \bigcup_{F \subseteq T} \operatorname{ext}_{S_\pi F,T}^{r,k} \mathcal{Q}\mathring{\mathcal{P}}_r \Lambda^k(S_\pi F). \end{aligned}$$

The desired claim is now evident. □

Theorem 7.5.
Let $r \in \mathbb{N}_0$ and $r, k \in \mathbb{N}$, and let T be an n-simplex. Let $\mathcal{Q}\mathring{\mathcal{P}}_r \Lambda^k(F)$ be a $\mathbb{K}$-invariant basis of $\mathring{\mathcal{P}}_r \Lambda^k(F)$ for each subsimplex $F \subseteq T$ such that for all $\pi \in \mathrm{Perm}(0:n)$ we have

$$S_\pi^* \mathcal{Q}\mathring{\mathcal{P}}_r^- \Lambda^k(F) = \mathcal{Q}\mathring{\mathcal{P}}_r^- \Lambda^k\left(S_\pi F\right).$$

Then the union

$$\mathcal{QP}_r^- \Lambda^k(T) := \bigcup_{F \subseteq T} \operatorname{ext}_{F,T}^{r,k,-} \mathcal{Q}\mathring{\mathcal{P}}_r^- \Lambda^k(F)$$

is a $\mathbb{K}$-invariant basis of $\mathcal{P}_r \Lambda^k(T)$.

Proof. This is proven completely analogously as the preceding theorem. □

8. Recursive Basis Construction

In this section we describe a general method of recursively constructing geometrically decomposed bases for higher-order finite element spaces over simplices in finite element exterior calculus. The recursion starts with bases for the constant k-forms over simplices. If $\mathbb{K}$-invariant bases are given in the base cases, then the higher-order bases will be $\mathbb{K}$-invariant as well.

We commence the construction as follows. For every simplex T we fix a basis $\mathcal{AP}_0 \Lambda^k(T)$ for the space $\mathcal{AP}_0 \Lambda^k(T)$, arbitrary for the time being. Formally, we also let $\mathcal{AP}_{-r} \Lambda^k(T)$ and $\mathcal{AP}_{-r}^- \Lambda^k(T)$ be the empty sets.

Second, along the canonical isomorphisms

$$\begin{aligned} \mathring{\mathcal{P}}_r^- \Lambda^k(T) &\simeq \mathcal{P}_{r-n+k-1} \Lambda^{n-k}(T), \\ \mathring{\mathcal{P}}_r \Lambda^k(T) &\simeq \mathcal{P}_{r-n+k}^- \Lambda^{n-k}(T) \end{aligned}$$

we define bases for the finite element spaces with homogeneous boundary conditions:

$$\mathcal{A}\mathring{\mathcal{P}}_r^- \Lambda^k(T) := \mathcal{I}_{n-k,r-n+k-1} \mathcal{AP}_{r-n+k-1} \Lambda^{n-k}(T), \tag{34}$$

$$\mathcal{A}\mathring{\mathcal{P}}_r \Lambda^k(T) := \mathcal{J}_{k,r}^{-1} \mathcal{AP}_{r-n+k}^- \Lambda^{n-k}(T). \tag{35}$$

As has been shown in Corollary 6.3. the bases $\mathcal{AP}_r^- \Lambda^k(T)$ and $\mathcal{AP}_r \Lambda^k(T)$ are $\mathbb{K}$-invariant if and only if the bases $\mathcal{AP}_{r-n+k-1} \Lambda^{n-k}(T)$ and $\mathcal{AP}_{r-n+k}^- \Lambda^{n-k}(T)$ are $\mathbb{K}$-invariant, respectively. Putting it the other way around, $\mathbb{K}$-invariant bases for finite element spaces with boundary conditions can be constructed from $\mathbb{K}$-invariant bases of finite element spaces without boundary conditions over lower polynomial degree.

Third, taking into account the geometric decomposition

$$\mathcal{P}_r\Lambda^k(T) = \bigoplus_{F\subseteq T} \operatorname{ext}_{F,T}^{r,k}\, \mathring{\mathcal{P}}_r\Lambda^k(F),$$
$$\mathcal{P}_r^-\Lambda^k(T) = \bigoplus_{F\subseteq T} \operatorname{ext}_{F,T}^{r,k,-}\, \mathring{\mathcal{P}}_r^-\Lambda^k(F),$$

we define bases

$$\mathcal{A}\mathcal{P}_r\Lambda^k(T) = \bigcup_{F\subseteq T} \operatorname{ext}_{F,T}^{r,k}\, \mathcal{A}\mathring{\mathcal{P}}_r\Lambda^k(F), \tag{36}$$

$$\mathcal{A}\mathcal{P}_r^-\Lambda^k(T) = \bigcup_{F\subseteq T} \operatorname{ext}_{F,T}^{r,k,-}\, \mathcal{A}\mathring{\mathcal{P}}_r^-\Lambda^k(F). \tag{37}$$

If the bases $\mathcal{A}\mathring{\mathcal{P}}_r\Lambda^k(F)$ for every F satisfy the conditions of Theorem 7.4, for which it suffices to have a $\mathbb{K}$-invariant basis of $\mathcal{A}\mathring{\mathcal{P}}_r\Lambda^k(F)$ for *any* F, then the basis $\mathcal{A}\mathcal{P}_r\Lambda^k(T)$ is $\mathbb{K}$-invariant too. Analogously, if the bases $\mathcal{A}\mathring{\mathcal{P}}_r^-\Lambda^k(F)$ for every F satisfy the conditions of Theorem 7.5, for which it suffices to have a $\mathbb{K}$-invariant basis of $\mathcal{A}\mathring{\mathcal{P}}_r^-\Lambda^k(F)$ for *any* F, then the basis $\mathcal{A}\mathcal{P}_r^-\Lambda^k(T)$ is $\mathbb{K}$-invariant too. Putting it the other way around, $\mathbb{K}$-invariant bases for finite element spaces can be constructed from $\mathbb{K}$-invariant bases of finite element spaces with boundary conditions over simplices of the same or lower dimension.

A $\mathbb{K}$-invariant basis for some given finite element space can be constructed recursively in the manner outlined above under the provision that $\mathbb{K}$-invariant bases are available for the lowest-order finite element spaces.

We apply this to the cases when the simplex T has low (and practically relevant) dimensions. We know that $\mathbb{R}$-invariant bases exist in the "endpoints" of the de Rham complex, hence we focus on k-forms with $0 < k < \dim(T)$. Note that the construction of a $\mathbb{K}$-invariant basis rests upon the existence of such a basis in the base cases of lowest polynomial order, and in the light of discussion in Section 5 we now understand that the construction will depend on conditions on k, r, and $\dim(T)$.

Here and below, we additionally constraint the choice of bases that lay the foundation for $\mathbb{K}$-invariance in our bases. We assume that $\mathcal{A}\mathcal{P}_0\Lambda^1(T)$ is the $\mathbb{C}$-invariant basis of Lemma 5.5 if T is a triangle, that $\mathcal{A}\mathcal{P}_0\Lambda^1(T)$ and $\mathcal{A}\mathcal{P}_0\Lambda^2(T)$ are the $\mathbb{R}$-invariant bases of Lemmas 5.3 and 5.4 if T is a tetrahedron, and that that $\mathcal{A}\mathcal{P}_0\Lambda^2(T)$ is the $\mathbb{C}$-invariant basis of Lemma 5.6 if T is a 4-simplex.

Theorem 8.1.
Let T be a triangle, and let $r \in \mathbb{N}_0$. Then the following holds:

- The bases $\mathcal{A}\mathcal{P}_r\Lambda^1(T)$ and $\mathcal{A}\mathcal{P}_r^-\Lambda^1(T)$ are $\mathbb{C}$-invariant.
- The basis $\mathcal{A}\mathcal{P}_r\Lambda^1(T)$ is $\mathbb{R}$-invariant if and only if $r \notin 3\mathbb{N}_0$.
- The basis $\mathcal{A}\mathcal{P}_r^-\Lambda^1(T)$ is $\mathbb{R}$-invariant if and only if $r \notin 3\mathbb{N}_0 + 2$.

Proof. Let E_0, E_1, E_2 be an enumeration of the edges of T. Recall that

$$\mathcal{AP}_r\Lambda^1(T) = \mathring{\mathcal{AP}}_r\Lambda^1(T) \cup \bigcup_{i=0}^{2} \operatorname{ext}^{k,r}_{E_i,T} \mathring{\mathcal{AP}}_r\Lambda^1(E_i),$$

$$\mathcal{AP}^-_r\Lambda^1(T) = \mathring{\mathcal{AP}}^-_r\Lambda^1(T) \cup \bigcup_{i=0}^{2} \operatorname{ext}^{k,r,-}_{E_i,T} \mathring{\mathcal{AP}}^-_r\Lambda^1(E_i),$$

and that $\mathring{\mathcal{AP}}_r\Lambda^1(E_i) = \mathring{\mathcal{AP}}^-_{r+1}\Lambda^1(E_i) = \mathcal{AP}_r\Lambda^1(E_i)$ is an $\mathbb{R}$-invariant basis. By an application of the canonical isomorphisms,

$$\mathring{\mathcal{AP}}_r\Lambda^1(T) = \mathcal{I}_{r-1,1}\mathcal{AP}^-_{r-1}\Lambda^1(T), \quad \mathring{\mathcal{AP}}^-_r\Lambda^1(T) = \mathcal{J}^{-1}_{r,1}\mathcal{AP}_{r-2}\Lambda^1(T).$$

It is clear from Lemma 5.5 that the bases $\mathcal{AP}_r\Lambda^1(T)$ and $\mathcal{AP}^-_r\Lambda^1(T)$ are $\mathbb{C}$-invariant. We prove the remaining two claims by induction.

For the base case, we observe that two claims are obvious if $r = 0$. For the induction step, assume that $r \geq 1$ and that the theorem is true for all $s \in \mathbb{N}_0$ with $s < r$.

In the case that $r \notin 3\mathbb{N}_0$ we have $r-1 \notin 3\mathbb{N}_0+2$. Using the induction assumption, $\mathcal{AP}_r\Lambda^1(T)$ is $\mathbb{R}$-invariant if and only if $\mathcal{AP}^-_{r-1}\Lambda^1(T)$ is $\mathbb{R}$-invariant. Similarly, in the case that $r \notin 3\mathbb{N}_0 + 2$ we have $r - 2 \notin 3\mathbb{N}_0$. Using the induction assumption, $\mathcal{AP}^-_r\Lambda^1(T)$ is $\mathbb{R}$-invariant if and only if $\mathcal{AP}_{r-2}\Lambda^1(T)$ is $\mathbb{R}$-invariant. The proof is complete. □

Corollary 8.2.
Let T be a triangle, and let $r \in \mathbb{N}_0$. Then the following holds:

- The bases $\mathring{\mathcal{AP}}_r\Lambda^1(T)$ and $\mathring{\mathcal{AP}}^-_r\Lambda^1(T)$ are $\mathbb{C}$-invariant.
- The basis $\mathring{\mathcal{AP}}_r\Lambda^1(T)$ is $\mathbb{R}$-invariant if and only if $r \notin 3\mathbb{N}$.
- The basis $\mathring{\mathcal{AP}}^-_r\Lambda^1(T)$ is $\mathbb{R}$-invariant if and only if $r \notin 3\mathbb{N} + 2$.

Remark 8.3.
The basic idea of the above proof is that the bases $\mathcal{AP}_r\Lambda^k(T)$ and $\mathcal{AP}^-_r\Lambda^k(T)$ are not only $\mathbb{C}$-invariant but also $\mathbb{R}$-invariant, unless the non-$\mathbb{R}$-invariant component $\mathcal{AP}_0\Lambda^1(T)$ appears in the construction.

Example 8.4.
Let us restate this result in the language of vector analysis: On any triangle T and any polynomial degree, the Raviart-Thomas space $\mathbf{RT}_r(T)$ and the Brezzi-Douglas-Marini space $\mathbf{BDM}_r(T)$ have geometrically decomposed $\mathbb{C}$-invariant bases. In addition to that, that basis of $\mathbf{RT}_r(T)$ is $\mathbb{R}$-invariant if $r \notin 3\mathbb{N}_0$, and that basis of $\mathbf{BDM}_r(T)$ is $\mathbb{R}$-invariant if $r \notin 3\mathbb{N}_0 + 2$.

We apply the same proof technique for the finite element spaces over a tetrahedron. The details, however, are more complicated in this case.

Theorem 8.5.
Let T be a tetrahedron, and let $r \in \mathbb{N}_0$. Then the following holds:

- The following bases are $\mathbb{C}$-invariant:
$$\mathcal{AP}_r\Lambda^1(T), \quad \mathcal{AP}^-_r\Lambda^1(T), \quad \mathcal{AP}_r\Lambda^2(T), \quad \mathcal{AP}^-_r\Lambda^2(T).$$
- The basis $\mathcal{AP}_r\Lambda^1(T)$ is $\mathbb{R}$-invariant if and only if $r \in \{0, 1, 2, 4, 5, 8\}$.

- The basis $\mathcal{AP}_r^-\Lambda^1(T)$ is $\mathbb{R}$-invariant if and only if $r \in \{0,1,3,4,7\}$.
- The basis $\mathcal{AP}_r\Lambda^2(T)$ is $\mathbb{R}$-invariant if if and only if $r \in \{1,2,4,5,8\}$.
- The basis $\mathcal{AP}_r^-\Lambda^2(T)$ is $\mathbb{R}$-invariant if and only if $r \in \{2,3,4,6,7,9\}$.

Proof. We $E_0, E_1, \dots, E_5$ be an enumeration of the edges of T and let F_0, F_1, F_2, F_3 be an enumeration of the faces of T. Recall that

$$\mathcal{AP}_r\Lambda^1(T) = \mathcal{A}\mathring{\mathcal{P}}_r\Lambda^1(T) \cup \bigcup_{i=0}^{3} \operatorname{ext}^{k,r}_{F_i,T}\mathcal{A}\mathring{\mathcal{P}}_r\Lambda^1(F_i) \cup \bigcup_{i=0}^{5} \operatorname{ext}^{k,r}_{E_i,T}\mathcal{A}\mathring{\mathcal{P}}_r\Lambda^1(E_i)$$

$$\mathcal{AP}_r\Lambda^1(T) = \mathcal{A}\mathring{\mathcal{P}}_r^-\Lambda^2(T) \cup \bigcup_{i=0}^{3} \operatorname{ext}^{k,r,-}_{F_i,T}\mathcal{A}\mathring{\mathcal{P}}_r^-\Lambda^2(F_i) \cup \bigcup_{i=0}^{5} \operatorname{ext}^{k,r,-}_{E_i,T}\mathcal{A}\mathring{\mathcal{P}}_r^-\Lambda^1(E_i)$$

$$\mathcal{AP}_r\Lambda^2(T) = \mathcal{A}\mathring{\mathcal{P}}_r\Lambda^2(T) \cup \bigcup_{i=0}^{3} \operatorname{ext}^{k,r}_{F_i,T}\mathcal{A}\mathring{\mathcal{P}}_r\Lambda^2(F_i)$$

$$\mathcal{AP}_r\Lambda^2(T) = \mathcal{A}\mathring{\mathcal{P}}_r^-\Lambda^2(T) \cup \bigcup_{i=0}^{3} \operatorname{ext}^{k,r,-}_{F_i,T}\mathcal{A}\mathring{\mathcal{P}}_r^-\Lambda^2(F_i)$$

As in the previous proof, we see that $\mathbb{R}$-invariant bases exist for the terms associated to edges. By Theorem 8.1 and Corollary 8.2, There exist $\mathbb{C}$-invariant bases for the terms associated to the faces, Recall furthermore that

$$\mathcal{A}\mathring{\mathcal{P}}_r\Lambda^1(T) = \mathcal{J}_{r,2}^{-1}\mathcal{AP}_{r-2}^-\Lambda^2(T), \quad \mathcal{A}\mathring{\mathcal{P}}_r^-\Lambda^1(T) = \mathcal{I}_{r-3,2}\mathcal{AP}_{r-3}\Lambda^2(T),$$
$$\mathcal{A}\mathring{\mathcal{P}}_r\Lambda^2(T) = \mathcal{J}_{r,1}^{-1}\mathcal{AP}_{r-1}^-\Lambda^1(T), \quad \mathcal{A}\mathring{\mathcal{P}}_r^-\Lambda^2(T) = \mathcal{I}_{r-2,1}\mathcal{AP}_{r-2}\Lambda^1(T)$$

by the canonical isomorphisms. The existence of a $\mathbb{C}$-invariant basis is clear from Lemma 5.3 and Lemma 5.4. In the light of Theorem 8.1, it only remains to study the terms associated to the faces in order to determine whether there exists an $\mathbb{R}$-invariant basis. Explicit expansion of the dependencies reveals that $\mathbb{R}$-invariant bases appear for the polynomial degrees mentioned in the theorem.

We only need to consider bases $\mathcal{A}\mathring{\mathcal{P}}_r\Lambda^1(T)$ and $\mathcal{A}\mathring{\mathcal{P}}_r^-\Lambda^1(T)$ for the 1-forms to exclude the other polynomial degrees. These spaces enter the construction of the spaces $\mathcal{A}\mathring{\mathcal{P}}_{r+4}\Lambda^1(T)$ and $\mathcal{A}\mathring{\mathcal{P}}_{r+4}^-\Lambda^1(T)$, respectively. For any $r \in \mathbb{N}_0$ there is $b \in \{0,1,2,3\}$ such that $r \in b + 4\mathbb{N}_0$. We now apply Corollary 8.2: $\mathcal{A}\mathring{\mathcal{P}}_r\Lambda^1(T)$ is $\mathbb{R}$-invariant if $r < \min(b + \mathbb{N}_0) \cap 3\mathbb{N}$, and $\mathcal{A}\mathring{\mathcal{P}}_r^-\Lambda^1(T)$ is $\mathbb{R}$-invariant if $r < \min(b + \mathbb{N}_0) \cap (3\mathbb{N}_0 + 2)$. The eliminates the remaining polynomial degrees. □

Corollary 8.6.
Let T be a tetrahedron, and let $r \in \mathbb{N}_0$. Then the following holds:

- The following bases are $\mathbb{C}$-invariant:
$$\mathcal{A}\mathring{\mathcal{P}}_r\Lambda^1(T), \quad \mathcal{A}\mathring{\mathcal{P}}_r^-\Lambda^1(T), \quad \mathcal{A}\mathring{\mathcal{P}}_r\Lambda^2(T), \quad \mathcal{A}\mathring{\mathcal{P}}_r^-\Lambda^2(T).$$
- The basis $\mathcal{A}\mathring{\mathcal{P}}_r\Lambda^1(T)$ is $\mathbb{R}$-invariant if and only if $r \in \{0,1,2,4,5,8\}$.
- The basis $\mathcal{A}\mathring{\mathcal{P}}_r^-\Lambda^1(T)$ is $\mathbb{R}$-invariant if and only if $r \in \{0,1,3,4,7\}$.
- The basis $\mathcal{A}\mathring{\mathcal{P}}_r\Lambda^2(T)$ is $\mathbb{R}$-invariant if if and only if $r \in \{0,1,2,4,5,8\}$.
- The basis $\mathcal{A}\mathring{\mathcal{P}}_r^-\Lambda^2(T)$ is $\mathbb{R}$-invariant if and only if $r \in \{0,1,2,3,4,6,7,9\}$.

Example 8.7.
Let us restate this result in the language of vector analysis: On any triangle T

and for any polynomial degree r, the Raviart-Thomas space $\mathbf{RT}_r(T)$, the Brezzi-Douglas-Marini space $\mathbf{BDM}_r(T)$ and the Nedélec spaces of the first kind $\mathbf{Nd}_r^{\mathrm{fst}}(T)$ and the second kind $\mathbf{Nd}_r^{\mathrm{snd}}(T)$ have geometrically decomposed $\mathbb{C}$-invariant bases. Moreover,

- that basis of $\mathbf{RT}_r(T)$ is $\mathbb{R}$-invariant if $r \in \{2, 3, 4, 6, 7, 9\}$,
- that basis of $\mathbf{BDM}_r(T)$ is $\mathbb{R}$-invariant if $r \in \{1, 2, 4, 5, 8\}$,
- that basis of $\mathbf{Nd}_r^{\mathrm{fst}}(T)$ is $\mathbb{R}$-invariant if $r \in \{0, 1, 3, 4, 7\}$,
- that basis of $\mathbf{Nd}_r^{\mathrm{snd}}(T)$ is $\mathbb{R}$-invariant if $r \in \{0, 1, 2, 4, 5, 8\}$.

Remark 8.8.
The $\mathbb{C}$-invariant bases $\mathcal{AP}_r\Lambda^k(T)$ and $\mathcal{AP}_r^-\Lambda^k(T)$ are $\mathbb{R}$-invariant unless the non-$\mathbb{R}$-invariant space $\mathcal{AP}_0\Lambda^1(F)$ over some face F enters the recursive construction. This constraint turns out to be quite eliminative: only finitely many polynomial degrees remain for each space.

Theorem 8.9.
Let T be a 4-simplex and let $r \in \mathbb{N}_0$. Then the geometrically decomposed bases of $\mathbb{C}\mathcal{P}_r\Lambda^1(T)$ and $\mathbb{C}\mathcal{P}_r^-\Lambda^1(T)$ are $\mathbb{C}$-invariant.

Proof. This uses the same recursive construction as in the proofs of Theorem 8.1 and Theorem 8.5. The base case is addressed by Lemma 5.6. □

Remark 8.10.
Our search for invariant bases has essentially led to positive results: Theorem 8.1, Theorem 8.5, and Theorem 8.9 explicitly construct a $\mathbb{C}$-invariant geometrically decomposed basis. We have pointed out conditions on the polynomial degree for the existence of bases that are $\mathbb{R}$-invariant, that is, invariant under permutation of indices up to sign change. These conditions on $\mathbb{R}$-invariance are sufficient and we conjecture that they are necessary as well.

Acknowledgement. The author would like to thank Dragomir Djoković for helpful advice relating to Lemma 5.6. Moreover, the author acknowledges the hospitality of the University of Minnesota, where parts of this research was done.

References

[1] M. Ainsworth, G. Andriamaro, and O. Davydov, *Bernstein-Bézier finite elements of arbitrary order and optimal assembly procedures*, SIAM Journal on Scientific Computing, 33 (2011), pp. 3087–3109.

[2] M. Ainsworth and J. Coyle, *Hierarchic finite element bases on unstructured tetrahedral meshes*, International journal for numerical methods in engineering, 58 (2003), pp. 2103–2130.

[3] D. Arnold and G. Awanou, *Finite element differential forms on cubical meshes*, Mathematics of Computation, (2013).

[4] D. N. Arnold, G. Awanou. and W. Qiu, *Mixed finite elements for elasticity on quadrilateral meshes*, arXiv preprint arXiv:1306.6821, (2013).

[5] D. N. Arnold, R. S. Falk, and J. Gopalakrishnan, *Mixed finite element approximation of the vector Laplacian with Dirichlet boundary conditions*, Mathematical Models and Methods in Applied Sciences, 22 (2012), p. 1250024.

[6] D. N. Arnold, R. S. Falk, and R. Winther, *Finite element exterior calculus, homological techniques, and applications*, Acta Numerica, 15 (2006), pp. 1–155.

[7] ———, *Geometric decompositions and local bases for spaces of finite element differential forms*, Computer Methods in Applied Mechanics and Engineering, 198 (2009), pp. 1660–1672.

[8] ———, *Finite element exterior calculus: from Hodge theory to numerical stability*, Bulletin of the American Mathematical Society, 47 (2010), pp. 281–354.

[9] A. Bentley, *Explicit construction of computational bases for RT_k and BDM_k spaces in R^3*, Computers & Mathematics with Applications, 73 (2017), pp. 1421–1432.

[10] S. Beuchler, V. Pillwein, and S. Zaglmayr, *Sparsity optimized high order finite element functions for $H(\mathrm{div})$ on simplices*, Numerische Mathematik, 122 (2012), pp. 197–225.

[11] B. Bloem-Reddy and Y. W. Teh, *Probabilistic symmetry and invariant neural networks*, arXiv preprint arXiv:1901.06082, (2019).

[12] F. Brezzi, J. Douglas, and L. D. Marini, *Two families of mixed finite elements for second order elliptic problems*, Numerische Mathematik, 47 (1985), pp. 217–235.

[13] A. Cangiani, O. J. Sutton, V. Gyrya, and G. Manzini, *Virtual element methods for elliptic problems on polygonal meshes*, in Generalized Barycentric Coordinates in Computer Graphics and Computational Mechanics, CRC Press, 2017, pp. 263–279.

[14] S. H. Christiansen, H. Z. Munthe-Kaas, and B. Owren, *Topics in structure-preserving discretization*, Acta Numerica, 20 (2011), pp. 1–119.

[15] S. H. Christiansen and F. Rapetti, *On high order finite element spaces of differential forms*, Mathematics of Computation, (2015). electronically published on July 10, 2015.

[16] S. H. Christiansen and R. Winther, *Smoothed projections in finite element exterior calculus*, Mathematics of Computation, 77 (2008), pp. 813–829.

[17] C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, vol. 356, American Mathematical Soc., 1966.

[18] A. Demlow and A. Hirani, *A posteriori error estimates for finite element exterior calculus: The de Rham complex*, Foundations of Computational Mathematics, (2014), pp. 1–35.

[19] D. u. Djoković and J. Malzan, *Monomial irreducible characters of the symmetric group*, Journal of Algebra, 35 (1975), pp. 153–158.

[20] V. Ervin, *Computational bases for RT_k and BDM_k on triangles*, Computers & Mathematics with Applications, 64 (2012), pp. 2765–2774.

[21] M. S. Floater, K. Hormann, and G. Kós, *A general construction of barycentric coordinates over convex polygons*, Advances in Computational Mathematics, 24 (2006), pp. 311–331.

[22] W. Fulton and J. Harris, *Representation theory: a first course*, vol. 129, Springer Science & Business Media, 2013.

[23] M. I. García-Planas, M. D. Magret, and L. E. Um, *Monomial codes seen as invariant subspaces*, Open Mathematics, 15, pp. 1099–1107.

[24] J. Gopalakrishnan, L. E. García-Castillo, and L. F. Demkowicz, *Nédélec spaces in affine coordinates*, Computers & Mathematics with Applications, 49 (2005), pp. 1285–1294.

[25] R. Hiptmair, *Finite elements in computational electromagnetism*, Acta Numerica, 11 (2002), pp. 237–339.

[26] M. Holst and A. Stern, *Geometric variational crimes: Hilbert complexes, finite element exterior calculus, and problems on hypersurfaces*, Arxiv preprint arXiv:1005.4455, (2010).

[27] J. C. Jantzen, *Representations of algebraic groups*, no. 107 in Mathematical Surveys and Monographs, American Mathematical Soc., 2007.

[28] R. C. Kirby, *Low-complexity finite element algorithms for the de Rham complex on simplices*, SIAM Journal on Scientific Computing, 36 (2014), pp. A846–A868.

[29] R. C. Kirby and K. T. Thinh, *Fast simplicial quadrature-based finite element operators using Bernstein polynomials*, Numerische Mathematik, 121 (2012), pp. 261–279.

[30] M. Licht, *On basis constructions in finite element exterior calculus*. Submitted.

[31] ———, *Smoothed projections and mixed boundary conditions*, Mathematics of Computation, 88 (2019), pp. 607–635.

[32] ———, *Smoothed projections over weakly Lipschitz domains*, Mathematics of Computation, 88 (2019), pp. 179–210.

[33] G. Manzini, A. Russo, and N. Sukumar, *New perspectives on polygonal and polyhedral finite element methods*, Mathematical Models and Methods in Applied Sciences, 24 (2014), pp. 1665–1699.

[34] H. Maschke, *Beweis des Satzes, dass diejenigen endlichen linearen Substitutionsgruppen, in welchen einige durchgehends verschwindende Coefficienten auftreten, intransitiv sind*, Mathematische Annalen, 52 (1899), pp. 363–368.

[35] M. Meyer, A. Barr, H. Lee, and M. Desbrun, *Generalized barycentric coordinates on irregular polygons*, Journal of Graphics Tools, 7 (2002), pp. 13–22.

[36] H. Z. Munthe-Kaas, *Groups and symmetries in numerical linear algebra*, in Exploiting Hidden Structure in Matrix Computations: Algorithms and Applications, Springer, 2016, pp. 319–406.

[37] J.-C. Nédélec, *A new family of mixed finite elements in R^3*, Numerische Mathematik, 50 (1986), pp. 57–81.

[38] O. Ore, *Theory of monomial groups*, Transactions of the American Mathematical Society, 51 (1942), pp. 15–64.

[39] M. Püschel, *Constructive representation theory and fast discrete signal transforms*, tech. rep., Drexel University, 1999. Technical Report Drexel-MCS-1999-1.

[40] ———, *Decomposing monomial representations of solvable groups*, Journal of Symbolic Computation, 34 (2002), pp. 561–596.

[41] P.-A. Raviart and J.-M. Thomas, *A mixed finite element method for 2-nd order elliptic problems*, in Mathematical aspects of finite element methods, Springer, 1977, pp. 292–315.

[42] J. Schöberl and S. Zaglmayr, *High order Nédélec elements with local complete sequence properties*, COMPEL-The international journal for computation and mathematics in electrical and electronic engineering, 24 (2005), pp. 374–384.

[43] W. R. Scott, *Group theory*, Courier Corporation, 2012.

[44] J.-P. Serre, *Linear representations of finite groups*, vol. 42, Springer, 77.

[45] E. Stiefel and A. Fässler, *Group theoretical methods and their applications*, Springer Science & Business Media, 2012.

UCSD Department of Mathematics, 9500 Gilman Drive MC0112, La Jolla, CA 92093-0112, USA

Email address: mlicht@math.ucsd.edu