Supersymmetry and the cohomology of (hyper)Kähler manifolds

JM Figueroa-O'Farrill [a,b,1,4], C Köhl [a,2,5] and B Spence [a,3,6]

[a] *Department of Physics, Queen Mary and Westfield College, Mile End Road, London E1 4NS, UK*

[b] *Isaac Newton Institute for Mathematical Sciences, 20 Clarkson Road, Cambridge, CB3 0EH, UK*

Abstract

The cohomology of a compact Kähler (resp. hyperKähler) manifold admits the action of the Lie algebra so(2,1) (resp. so(4,1)). In this paper we show, following an idea of Witten, how this action follows from supersymmetry; concretely from the symmetries of certain supersymmetric sigma models. In addition, many of the fundamental identities in Hodge–Lefschetz theory are also naturally derived from supersymmetry.

1 Introduction

It is a classical result going back to Hodge and Lefschetz [2] that the de Rham cohomology of a compact Kähler manifold admits an action of the Lie algebra $s\ell(2) \cong \mathrm{so}(2,1)$. This action is generated by the operations of exterior product with the Kähler form and its adjoint operation with respect to the Hodge inner product. A more recent result of Verbitsky [5,6] states that if the manifold is hyperKähler, the $\mathrm{so}(2,1)$ action is part of a larger $\mathrm{so}(4,1)$ action, which is now generated by exterior products with each of the three Kähler forms and their adjoints. Recently Witten [9] has pointed out that this can be understood from supersymmetry.

1 mailto:j.m.figueroa@qmw.ac.uk
2 mailto:c.koehl@qmw.ac.uk
3 mailto:b.spence@qmw.ac.uk
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5 Supported by an EPSRC postgraduate studentship.
6 EPSRC Advanced Fellow.

It is well-known that requiring supersymmetry in a sigma model imposes strong restrictions on the geometry of the target manifold X. The earliest result in this direction was the observation [10] that the four-dimensional sigma model can be made supersymmetric if and only if X admits a Kähler metric. Similarly [1] supersymmetry of a six-dimensional sigma model with target space X demands that X be hyperKähler. Upon dimensional reduction to one dimension these supersymmetric sigma models become quantum mechanical systems, introduced to great effect in [8] and much studied since. These quantum mechanical systems have as Hilbert space the square-integrable differential forms on X, and as hamiltonian the Hodge laplacian. By reason of supersymmetry, the energy is non-negative. Therefore the ground states are in one-to-one correspondence with the square-integrable harmonic forms and thus, when X is compact, with the cohomology. Therefore any symmetry of the hamiltonian will preserve the ground states and hence act on the cohomology. On the other hand, a $(d+1)$-dimensional supersymmetric sigma model with a Lorentzian metric has an action which is invariant under the Lorentz group $\mathrm{SO}(d,1)$. Upon dimensional reduction to one (spatial) dimension, this reduces to an $\mathrm{SO}(d-1,1)$ 'internal' symmetry. Applying this to supersymmetric sigma models in 3+1 (resp. 5+1) dimensions with compact target space X yields an $\mathrm{SO}(2,1)$ (resp. $\mathrm{SO}(4,1)$) action on the cohomology of X. This is Witten's argument.

Whereas this beautiful argument explains the existence of these group actions, it does not tell us that they agree with the ones known from geometry. It is the purpose of this paper to prove that they do. In fact, supersymmetry lies at the heart of much of the Hodge–Lefschetz theory and in this paper we will also briefly mention these results. For example, the Hodge identities are what remains after dimensional reduction of the fact that the supersymmetry generators are spinors of the relevant (i.e., four- or six-dimensional) spacetime.

This paper is organised as follows. In the next section we will consider the Kähler case, reviewing the Hodge–Lefschetz construction and then checking explicitly that it agrees with the one coming from supersymmetry. Then in section 3 we will tackle the hyperKähler case, reviewing Verbitsky's construction and checking that it too agrees with that coming from supersymmetry. Finally in Section 4 we offer some closing comments.

2 The Kähler case

In this section we discuss the Kähler case. We first briefly review some facts from the Hodge–Lefschetz theory of a Kähler manifold which we will need. We will then show how these may be understood by considering the dimensional reduction of the four-dimensional supersymmetric sigma model. We follow the notational conventions of [7].

2.1 *Some harmonic theory on Kähler manifolds*

Let X be a compact Kähler manifold of complex dimension n. The complex differential forms on X decompose as

$$A_c(X) = \bigoplus_{p,q=1,\ldots,n} A_c^{p,q} \ ,$$

where $A_c^{p,q}$ is the subspace of (p,q)-forms. The Hodge $\star$-operator maps

$$\star : A_c^{p,q} \to A_c^{n-q,n-p} \ ,$$

and obeys $\star^2 = (-)^{p+q}$ acting on $A_c^{p,q}$. Using this operator we may define a hermitian inner product on the space of forms, called the Hodge inner product:

$$(\alpha, \beta) = \int_X \alpha \wedge \star\bar{\beta} \ . \qquad (1)$$

Notice that because complex conjugation maps $A_c^{p,q} \to A_c^{q,p}$, the Hodge inner product pairs forms of the same bidegree.

The de Rham operator on X decomposes into two operators of different bidegree: $d = \partial + \bar{\partial}$, where $\partial : A_c^{p,q} \to A_c^{p+1,q}$ and $\bar{\partial} : A_c^{p,q} \to A_c^{p,q+1}$. $\bar{\partial}$ is called the Dolbeault operator. From $d^2 = 0$, one reads that $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$. Let ∂^* and $\bar{\partial}^*$ denote their adjoint operators relative to the Hodge inner product. They also satisfy similar identities. Using these operators we can define two laplacians $\Box$ and $\bar{\Box}$ as follows:

$$\Box = \partial\partial^* + \partial^*\partial \qquad \text{and} \qquad \bar{\Box} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} \ .$$

Part of the magic of Kähler manifolds is that these two laplacians agree. In fact, we have

$$\Box = \bar{\Box} = \tfrac{1}{2}\Delta \ , \qquad (2)$$

where Δ is the Hodge laplacian. Other identities obeyed by the differential operators ∂, $\bar{\partial}$, ∂^*, and $\bar{\partial}^*$ are:

$$\partial\bar{\partial}^* = -\bar{\partial}^*\partial \qquad \text{and} \qquad \partial^*\bar{\partial} = -\bar{\partial}\partial^* \ . \qquad (3)$$

Now let $\omega \in A_c^{1,1}$ denote the Kähler form. It is real: $\bar{\omega} = \omega$. Let $L : A_c^{p,q} \to A_c^{p+1,q+1}$ be defined by $L(\alpha) = \alpha \wedge \omega$. Its adjoint relative to the Hodge inner product is denoted $\Lambda : A_c^{p,q} \to A_c^{p-1,q-1}$, and obeys the relations

$$(\Lambda(\alpha), \beta) = (\alpha, L(\beta)) = (\alpha, \beta \wedge \omega) \ .$$

It can be understood as contraction with the Kähler form. Let us now define a third operator $H \equiv [L, \Lambda] : A_c^{p,q} \to A_c^{p,q}$. Actually H acts diagonally with real eigenvalue $p + q - n$ on $A_c^{p,q}$. Furthermore, the following relations hold:

$$[H, L] = 2L \qquad \text{and} \qquad [H, \Lambda] = -2\Lambda \ .$$

In other words, L, Λ and H define a representation of the Lie algebra $sl(2)$.

These operators obey a series of identities, known as the Hodge identities, which relate the Dolbeault operator and its cousins. These identities read:

$$\begin{array}{llll} [L, \partial] = 0 & [L, \bar{\partial}] = 0 & [\Lambda, \partial^*] = 0 & [\Lambda, \bar{\partial}^*] = 0 \\ [L, \partial^*] = i\bar{\partial} & [L, \bar{\partial}^*] = -i\partial & [\Lambda, \partial] = i\bar{\partial}^* & [\Lambda, \bar{\partial}] = -i\partial^* \ . \end{array} \qquad (4)$$

It follows from these identities, that L, Λ and hence H commute with the Hodge laplacian. This means that they act on the space of harmonic forms and, by the Hodge decomposition theorem, on the cohomology. As we will see below, all the identities in this section will follow naturally from supersymmetry.

2.2 *The supersymmetric sigma model in four dimensions*

We will now recover these results using supersymmetry. Let X be a Kähler manifold of complex dimension n. Let z^a and $\bar{z}^{\bar{a}}$ denote local complex coordinates, relative to which the metric has nonzero components $g_{a\bar{b}}$. The nonzero components of the Christoffel symbols are denoted Γ^a_{bc} and $\Gamma^{\bar{a}}_{\bar{b}\bar{c}}$, and those of the fully covariant Riemann curvature tensor by $R_{a\bar{b}c\bar{d}}$.

The N=1 supersymmetric sigma model in four dimensions with target space X is described by the following lagrangian density:

$$\mathcal{L} = -g_{a\bar{b}}\partial_\mu\phi^a\partial^\mu\bar{\phi}^{\bar{b}} - \tfrac{i}{2}g_{a\bar{b}}\bar{\chi}^{\bar{b}}\bar{\sigma}^\mu\overleftrightarrow{D}_\mu\chi^a + \tfrac{1}{4}R_{a\bar{b}c\bar{d}}(\chi^a\chi^c)(\bar{\chi}^{\bar{b}}\bar{\chi}^{\bar{d}}) \ , \qquad (5)$$

where

- ϕ^a is a complex scalar field with $\bar{\phi}^{\bar{a}} = (\phi^a)^*$;
- χ^a_α is a complex Weyl spinor and $\bar{\chi}^{\bar{a}}_{\dot{\alpha}} = (\chi^a_\alpha)^*$;
- $\bar{\sigma}^\mu = (-\mathbb{1}, -\sigma^i)$ and $\sigma^\mu = (-\mathbb{1}, \sigma^i)$, where σ^i are the Pauli matrices; and
- the covariant derivative is defined by

$$D_\mu\chi^a = \partial_\mu\chi^a + \Gamma^a_{bc}\partial_\mu\phi^b\chi^c \ .$$

The lagrangian (5) is invariant under the superpoincaré group, which contains an $\mathrm{SO}(3,1)$ subgroup generated infinitesimally by:

$$\begin{aligned}
\delta_\Lambda \phi^a &= -\Lambda_{\mu\nu} x^\mu \partial^\nu \phi^a \\
\delta_\Lambda \chi^a &= -\Lambda_{\mu\nu} x^\mu \partial^\nu \chi^a + \tfrac{1}{2}\Lambda_{\mu\nu}\sigma^{\mu\nu}\chi^a \\
\delta_\Lambda \bar{\chi}^{\bar{a}} &= -\Lambda_{\mu\nu} x^\mu \partial^\nu \bar{\chi}^{\bar{a}} + \tfrac{1}{2}\Lambda_{\mu\nu}\bar{\chi}^{\bar{a}}\bar{\sigma}^{\mu\nu} ,
\end{aligned}$$

where $\Lambda_{\mu\nu}$ is constant and antisymmetric. The Noether current associated with $\mathrm{so}(3,1)$ transformations can be worked out by letting the parameter $\Lambda_{\mu\nu}$ depend on position and varying the lagrangian. Then up to a total derivative, we find after some σ-algebra:

$$\delta_\Lambda \mathcal{L} = \tfrac{i}{2} g_{a\bar{b}} \partial^\mu \Lambda_{\mu\nu} \bar{\chi}^{\bar{b}} \bar{\sigma}^\nu \chi^a , \tag{6}$$

where we have omitted terms which will not survive the dimensional reduction.

The lagrangian (5) is also invariant under the supersymmetry transformations:

$$\begin{aligned}
\delta_\varepsilon \phi^a &= \varepsilon \chi^a \\
\delta_\varepsilon \chi^a &= i\sigma^\mu \bar{\varepsilon} \partial_\mu \phi^a - \Gamma^a_{bc} \delta_\varepsilon \phi^b \chi^c ,
\end{aligned}$$

where ε is a complex two-component spinor. The supersymmetry currents S^μ_α and $\bar{S}^{\mu\dot{\alpha}}$ can be obtained just as was done for the Lorentz currents, and one finds:

$$S^\mu = -g_{a\bar{b}}\chi^a \partial^\mu \bar{\phi}^{\bar{b}} + \tfrac{i}{2} g_{a\bar{b}} \Gamma^a_{bc} \left(\bar{\chi}^{\bar{b}} \bar{\sigma}^\mu \chi^b\right) \chi^c + \tfrac{1}{2} g_{a\bar{b}} \partial_\nu \bar{\phi}^{\bar{b}} \sigma^\nu \bar{\sigma}^\mu \chi^a , \tag{7}$$

with $\bar{S}^{\mu\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \left(S^\mu_\beta\right)^*$.

We now perform a trivial dimensional reduction to one dimension by simply dropping the dependence on all coordinates but x^0. The dimensionally-reduced lagrangian then becomes:

$$\mathcal{L} = g_{a\bar{b}} \dot{\phi}^a \dot{\bar{\phi}}^{\bar{b}} - \tfrac{i}{2} g_{a\bar{b}} \bar{\chi}^{\bar{b}} \bar{\sigma}^0 \frac{\overleftrightarrow{D}}{dt} \chi^a + \tfrac{1}{4} R_{a\bar{b}c\bar{d}} (\bar{\chi}^{\bar{b}}\chi^a)(\bar{\chi}^{\bar{d}}\chi^c) ,$$

with $\frac{D}{dt} = D_0$. The conserved charges of the reduced lagrangian are now

$$J^{ij} = -g_{a\bar{b}} \bar{\chi}^{\bar{b}} \bar{\sigma}^0 \sigma^{ij} \chi^a, \qquad \text{with } i,j = 1,2,3,$$

for the 'internal' Lorentz generators, and

$$S = -\tfrac{1}{2} g_{a\bar{b}} \chi^a \dot{\bar{\phi}}^{\bar{b}} + \tfrac{i}{2} g_{a\bar{b}} \Gamma^a_{bc} \left(\bar{\chi}^{\bar{b}} \chi^b\right) \chi^c ,$$

for the supercharges.

Defining $J_i = \frac{1}{2}\epsilon_{ijk} J^{jk}$ and after some algebra, we find that

$$J^i = \tfrac{i}{2} g_{a\bar{b}} \bar{\chi}^{\bar{b}} \bar{\sigma}^i \chi^a . \tag{8}$$

The quantisation of this system is well-known. The quantisation of the bosons ϕ^a and $\bar{\phi}^{\bar{a}}$ is straightforward: the bosonic Hilbert space $\mathcal{B}$ is the space of square integrable complex-valued functions $f(\phi,\bar{\phi})$. The canonical anticommutation relations of the fermions are given by the following Clifford algebra:

$$\{\chi^a_\alpha, \bar{\chi}^{\bar{b}}_{\dot{\alpha}}\} = -g^{a\bar{b}}(\phi,\bar{\phi})\sigma^0_{\alpha\dot{\alpha}} .$$

(Note that in our conventions $-\sigma^0_{\alpha\dot{\alpha}}$ coincides with the 2×2 identity matrix.) We can choose a Clifford vacuum $|0\rangle$ by the condition that $\bar{\chi}^{\bar{b}}_{\dot{2}}|0\rangle = \chi^a_1|0\rangle = 0$. Then a typical state in the fermionic Hilbert space $\mathcal{F}$ is a linear combination of monomials of the form:

$$|\psi\rangle = \chi^{a_1}_2 \chi^{a_2}_2 \cdots \chi^{a_p}_2 \bar{\chi}^{\bar{b}_1}_{\dot{1}} \bar{\chi}^{\bar{b}_2}_{\dot{1}} \cdots \bar{\chi}^{\bar{b}_q}_{\dot{1}} |0\rangle .$$

More precisely, states like these for fixed p and q generate a subspace $\mathcal{F}^{p,q}$ of the total fermionic Hilbert space. The inner product in $\mathcal{F}$ is defined as follows: if $|\psi\rangle$ is as above, its norm is given by $\langle\psi|\psi\rangle$ where

$$\langle\psi| = \langle 0| \chi^{b_q}_1 \chi^{b_{q-1}}_1 \cdots \chi^{b_1}_1 \bar{\chi}^{\bar{a}_p}_{\dot{2}} \bar{\chi}^{\bar{a}_{p-1}}_{\dot{2}} \cdots \bar{\chi}^{\bar{a}_1}_{\dot{2}} .$$

Tensoring bosons and fermions together we see that the total Hilbert space $\mathcal{H} = \mathcal{B} \otimes \mathcal{F}$ is isomorphic to the space of square-integrable complex differential forms on X relative to the Hodge metric. Since X is compact, $\mathcal{H}$ has a dense subspace isomorphic to the smooth complex forms $A_c(X)$. Under the isomorphism, $\mathcal{H}$ inherits a bigrading $\mathcal{H} = \bigoplus \mathcal{H}^{p,q}$ which agrees with the one coming from the fermionic Hilbert space.

Under this isomorphism, operators in the quantum theory can be interpreted geometrically as operators acting on $A_c(X)$. It is not hard to show that the following dictionary holds for the supersymmetry generators:

$$S_1 \mapsto \bar{\partial}^* \qquad S_2 \mapsto \partial \qquad \bar{S}_{\dot{1}} \mapsto \bar{\partial} \qquad \bar{S}_{\dot{2}} \mapsto \partial^* . \tag{9}$$

The supersymmetry algebra obeyed by the supercharges, which can easily be worked out by iterating the supersymmetry transformations, then implies the identities (2) and (3). In particular, this shows that up to an inconsequential factor, the hamiltonian can be interpreted as the Hodge laplacian.

Now consider the following linear combination of the Noether charges (8):

$$\hat{L} = J^1 + iJ^2 = -ig_{a\bar{b}} \bar{\chi}^{\bar{b}}_{\dot{1}} \chi^a_2 = \omega_{a\bar{b}} \bar{\chi}^{\bar{b}}_{\dot{1}} \chi^a_2 .$$

Under the isomorphism $\mathcal{H} \cong A_c(X)$, we see that $\hat{L}$ corresponds to the operator L defined in the previous subsection. Its hermitian adjoint is clearly given by

$$\hat{L}^\dagger = -\omega_{b\bar{a}} \bar{\chi}^{\bar{a}}_{\dot{2}} \chi^b_1 = -(J^1 - iJ^2) .$$

Their commutator is given by

$$\hat{H} = [\hat{L}, \hat{L}^\dagger] = 2iJ^3 = g_{a\bar{b}}(\bar{\chi}_1^{\bar{b}}\chi_1^a - \bar{\chi}_2^{\bar{b}}\chi_2^a) ,$$

which when written in terms of normal-ordered quantities becomes:

$$\hat{H} = g_{a\bar{b}}(\bar{\chi}_1^{\bar{b}}\chi_1^a + \chi_2^a\bar{\chi}_2^{\bar{b}}) - n\mathbb{1} ,$$

which acts as $(p+q-n)\mathbb{1}$ on $\mathcal{H}^{p,q}$. Thus the isomorphism $\mathcal{H} \cong A_c(X)$ sends $(\hat{L}, \hat{L}^\dagger, \hat{H})$ to (L, Λ, H). Notice that although the J^i satisfy the Lie algebra of so(3), the expressions for $(\hat{L}, \hat{L}^\dagger, \hat{H})$ involve complex linear combinations of the J^i. The hermiticity conditions are such that we are in effect choosing a different real section of so$(3,\mathbb{C})$, one isomorphic to so$(2,1) \cong \mathfrak{sl}(2)$. Had we kept x^1, say, in the dimensional reduction, we would have obtained so$(2,1)$ on the nose without having to take complex linear combinations; but such a choice would have made the notation a lot more complicated.

Sigma model	Kähler geometry
Hilbert space $\mathcal{H} = \bigoplus_{p,q} \mathcal{H}^{p,q}$	$A_c(X) = \bigoplus_{p,q} A_c^{p,q}$
inner product	Hodge inner product (1)
supercharge $\bar{S}_{\dot\alpha}$	$(\bar\partial, \partial^*)$
supercharge S_α	$(\bar\partial^*, \partial)$
hamiltonian	Δ
ground states	$H^*(X)$
'internal' symmetry J^i	L, Λ, H
supersymmetry algebra	$\begin{cases} \Box = \bar\Box = \frac{1}{2}\Delta \\ \partial\text{-identities (3)} \end{cases}$
$\mathbb{S}$ is a spinor in 3+1: (10)	Hodge identities (4)

Table 1
Sigma models and Kähler geometry.

Finally we remark that the Hodge identities (4) are a consequence of the spinorial nature of the four-dimensional supercharge. If we let $\mathbb{S} = (S_\alpha\ \bar{S}^{\dot\alpha})^t$ denote the four-dimensional supercharge, then $[J^{\mu\nu}, \mathbb{S}] = \Sigma^{\mu\nu}\mathbb{S}$. Upon dimensional reduction, we find that $[J^{ij}, S] = \sigma^{ij}\, S$, or simply that

$$[J^i, S] = \tfrac{i}{2}\sigma^i\bar\sigma^0 S \ . \tag{10}$$

Using the expressions of L and Λ in terms of the J^i and the dictionary (9) it is easy to show that (10) goes over to the Hodge identities (4). More invariantly, notice that $\mathbb{S}$ is a Majorana spinor in 3+1 dimensions, hence under $\mathrm{Spin}^0(3,1) \cong \mathrm{SL}(2,\mathbb{C})$, it transforms according to the real representation $(0,\frac{1}{2}) \oplus (\frac{1}{2},0)$. Under the 'internal' $\mathrm{Spin}^0(2,1) \cong \mathrm{SL}(2,\mathbb{R})$ subgroup of

$\mathrm{Spin}^0(3,1)$ this representation breaks up into two doublets. The Hodge identities simply reiterate this fact. Table 1 summarises the correspondence described in this section.

3 The hyperKähler case

Now comes the turn of the hyperKähler case. As before we first briefly review Verbitsky's construction of the action of so$(4,1)$ on the cohomology of a compact hyperKähler manifold and then show how this can be reproduced via dimensional reduction of the six-dimensional supersymmetric sigma model.

3.1 Verbitsky's construction

Let X be a compact hyperKähler manifold, and let I, J and K be the three complex structures and ω_i, $i = 1,2,3$ be the corresponding Kähler forms. We choose once and for all one of the complex structures, I, say. Ignoring for the moment the other two complex structures, X is a compact Kähler manifold and all of the Hodge–Lefschetz theory goes through. In particular we have operators L_1, Λ_1 and H defined as before but with ω_1 playing the role of ω.

We now bring to play the other two complex structures. We define operators L_2 and L_3 in the obvious way. Their adjoints relative to the Hodge inner product are Λ_2 and Λ_3. Introducing operators $K_i = \frac{1}{2}\epsilon_{ijk}[L_j, \Lambda_k]$, Verbitsky [6] showed that the following algebra is satisfied:

$$\begin{aligned} [L_i, \Lambda_j] &= \epsilon_{ijk}K_k + \delta_{ij}H \qquad & [K_i, K_j] &= \epsilon_{ijk}K_k \\ [H, L_i] &= 2L_i & [H, \Lambda_i] &= -2\Lambda_i \\ [K_i, L_j] &= \epsilon_{ijk}L_k & [K_i, \Lambda_j] &= \epsilon_{ijk}\Lambda_k \end{aligned} \tag{11}$$

with all other brackets zero. Furthermore, Verbitsky also showed that as in the Kähler case, these operators commute with the Hodge laplacian, thus inducing an action on the cohomology.

The Lie algebra above is that of so$(4,1)$. For the purposes of the calculations in the next section, we will utilise the following description of these generators. Let J_{mn}, for $m,n = 1,\ldots,5$, denote the generators of so(5), satisfying the algebra

$$[J_{mn}, J_{pq}] = \delta_{mq}J_{np} - \delta_{mp}J_{nq} + \delta_{np}J_{mq} - \delta_{nq}J_{mp} \ , \tag{12}$$

Now, defining (for $i, j = 1, 2, 3$)

$$
\begin{aligned}
L_i &= -(J_{i3} + iJ_{i4}) \qquad & \Lambda_i &= J_{i5} - iJ_{i4}, \\
H &= 2iJ_{45} & K_{ij} &= 2J_{ij},
\end{aligned}
$$

one finds that the algebra of these generators reproduces (11). Again, taking complex linear combinations we have moved to a different real section of so(5, C), this time so(4, 1) as evinced by the presence of factors of i in connection with the fourth coordinate.

3.2 *The six-dimensional supersymmetric sigma model*

We will now recover the results above using the supersymmetric sigma model in 5+1 dimensions. The target space of this sigma model is a hyperKähler manifold X of (real) dimension $4n$. Normally the fermions in the sigma model would take values in the positive spinor bundle S_+ over spacetime twisted by the pull-back of the tangent bundle T of the target manifold. However in this case, this prescription does not give rise to a match between the bosonic and fermionic degrees of freedom: we must impose a restriction on the fermions which we now detail. The complexified tangent bundle $T_{\mathbb{C}}$ of a hyperKähler manifold decomposes under the maximal subgroup $\mathrm{Sp}(1) \cdot \mathrm{Sp}(n) \subset \mathrm{SO}(4n)$ as $T_{\mathbb{C}} \cong \Sigma \otimes V$, where Σ is a complex two-dimensional Sp(1) bundle and V is a complex $2n$-dimensional Sp(n) bundle. The holonomy being Sp(n) means that the above decomposition is preserved under parallel transport and that in addition Σ is a trivial bundle. The canonical real structure of $T_{\mathbb{C}}$ is the product of the natural quaternionic structures in Σ and V. Because S_+ also possesses a quaternionic structure, the tensor products $S_+ \otimes V$ and $S_+ \otimes \Sigma$ possess real structures. Therefore we will be able to impose reality conditions on the fermions and on the supersymmetry parameters respectively.[7] The bundle $S_+ \otimes V$ is complex $8n$-dimensional. The reality condition leaves $8n$ real components which gives $4n$ physical degrees of freedom, matching the number of bosonic physical degrees of freedom. Similarly $S_+ \otimes \Sigma$ is complex 8-dimensional and the reality condition leaves the expected 8 real components of the supercharge.

We now introduce some notation to describe the fields in the sigma model. First we have $4n$ bosons ϕ^i which are coordinates of the target manifold. The isomorphism $T_{\mathbb{C}} \cong \Sigma \otimes V$ is given explicitly by objects γ^i_{Aa}. Here $A, B, \ldots$ are Sp(1) indices associated with Σ and running from 1 to 2, and $a, b, \ldots$ are Sp(n) indices associated with V and running from 1 to $2n$. The bundle Σ being trivial allows a constant Sp(1)-invariant symplectic form ϵ_{AB}; whereas

[7] Such spinors are known as symplectic Majorana–Weyl spinors, and they exist in spacetimes of signature (s, t) with $s - t = 4 \mod 8$.

V admits an Sp(n)-invariant symplectic form ω_{ab}. In terms of these symplectic forms, the metric g_{ij} on T can be written as:

$$g_{ij} \gamma^i_{Aa} \gamma^j_{Bb} = \epsilon_{AB} \omega_{ab} \,.$$

Because the holonomy lies in Sp(n), not just the metric g, but also the symplectic forms ϵ and ω are parallel; whence so are the maps γ^i_{Aa}.

A final piece of notation is to choose an explicit realisation for the Clifford algebra in 5+1 dimensions (this will determine the explicit form of the reality condition satisfied by the fermions). The metric is $\eta_{mn} = \mathrm{diag}(-1, 1, 1, 1, 1, 1)$. We choose the $(5+1)$–dimensional Gamma matrices to be

$$\Gamma_\mu = \begin{pmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix} \quad , \quad \Gamma_4 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad \text{and} \quad \Gamma_5 = \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix} ,$$

where γ_μ, for $\mu = 0, 1, 2, 3$, are the (3+1)-dimensional gamma matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \text{and} \qquad \gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \,.$$

Finally, we choose $\Gamma_7 = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5$.

We can now write down the following lagrangian (see, e.g., [4]):

$$\mathcal{L} = \tfrac{1}{2} g_{ij} \partial_m \phi^i \partial^m \phi^j + \tfrac{1}{2} \omega_{ab} \bar{\Psi}^a \Gamma^m D_m \Psi^b + \tfrac{1}{40} \Omega_{abcd} (\bar{\Psi}^a \Gamma_m \Psi^b)(\bar{\Psi}^c \Gamma^m \Psi^d) \,.$$

In this expression,

- the Ψ^a are eight-component positive-chirality Weyl spinors: $\Gamma_7 \Psi^a = +\Psi^a$. In the above basis for the Γ-matrices, it means that $\Psi^a = \left(\begin{smallmatrix} \psi^a \\ 0 \end{smallmatrix} \right)$, with ψ^a a four-component complex Dirac spinor, $\psi^a = \left(\begin{smallmatrix} \chi^a_\alpha \\ \bar{\phi}^{a\dot{\alpha}} \end{smallmatrix} \right)$.
- the Ψ^a *also* satisfy the following symplectic Majorana condition:

$$\Psi^*_a \equiv (\Psi^a)^* = \omega_{ab} \mathsf{B} \Psi^b \,,$$

  where the matrix B must satisfy [3] $\Gamma^*_m = \mathsf{B} \Gamma_m \mathsf{B}^{-1}$, $\mathsf{B}^t = -\mathsf{B}$ and $\mathsf{B}^\dagger \mathsf{B} = \mathbb{1}$. In our choice of basis we can take $\mathsf{B} = \Gamma_2 \Gamma_5$. This condition relates further the two-component spinors comprising ψ^a in such a way that $\psi^a = \left(\begin{smallmatrix} \chi^a_\alpha \\ -\omega^{ab} \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\chi}_{b\dot{\beta}} \end{smallmatrix} \right)$, where $\bar{\chi}_{b\dot{\beta}} \equiv \left(\chi^b_\beta \right)^*$.
- the conjugate $\bar{\Psi}^a$ is given by $\bar{\Psi}^a = (\Psi^a)^t \mathsf{C}$, where C is the charge conjugation matrix, which in our basis can (and will) be chosen to be $\mathsf{C} = -\Gamma_2 \Gamma_5 \Gamma_0$;

- the covariant derivative is given by

$$D_m\Psi^a = \partial_m\Psi^a + \hat{\Gamma}^a_{i\ b}\partial_m\phi^i\Psi^b ,$$

  with $\hat{\Gamma}$ the reduction to $\mathrm{Sp}(n)$ of the riemannian connection; and
- Ω_{abcd} is the hyperKähler curvature, a totally symmetric tensor defined by

$$\gamma^i_{Aa}\gamma^j_{Bb}\gamma^k_{Cc}\gamma^\ell_{Dd}R_{ijk\ell} = \epsilon_{AB}\epsilon_{CD}\Omega_{abcd} .$$

The above lagrangian is invariant under supersymmetry transformations:

$$\begin{aligned}\delta_\varepsilon\phi^i &= \gamma^i_{Aa}\bar{\varepsilon}^A\Psi^a\\ \delta_\varepsilon\Psi^a &= \gamma_i^{Aa}\Gamma^m\partial_m\phi^i\varepsilon_A - \hat{\Gamma}^a_{i\ b}\delta_\varepsilon\phi^i\Psi^b\end{aligned}$$

where ε^A is a constant *negative*-chirality Weyl spinor with values in Σ subject to the symplectic Majorana condition:

$$\varepsilon^*_A \equiv \left(\varepsilon^A\right)^* = \epsilon_{AB}\mathrm{B}\varepsilon^B ,$$

and $\bar{\varepsilon}^A = \left(\varepsilon^A\right)^t\mathrm{C}$. The Noether current generating the supersymmetry is given by

$$S^{Am} = \omega_{ab}\gamma_i^{Ab}\partial_n\phi^i\Gamma^n\Gamma^m\Psi^a .$$

The above lagrangian is also invariant under the following infinitesimal Lorentz transformations satisfying the Lie algebra of $\mathrm{so}(5,1)$:

$$\begin{aligned}\delta_\Lambda\phi^i &= -\Lambda_{mn}x^m\partial^n\phi^i\\ \delta_\Lambda\Psi^a &= -\Lambda_{mn}x^m\partial^n\Psi^a + \tfrac{1}{2}\Lambda_{mn}\Sigma^{mn}\Psi^a .\end{aligned}$$

As in the four-dimensional sigma model, we compute the Noether current by letting Λ_{mn} depend on the position and varying the lagrangian. The time (zeroth) component of this current, whose integral over space yields the conserved quantities, is found to be

$$J_{mn} = \tfrac{1}{4}\omega_{ab}\bar{\Psi}^a\Gamma^0\Gamma_{mn}\Psi^b,$$

where $\Gamma_{mn} = \frac{1}{2}(\Gamma_m\Gamma_n - \Gamma_n\Gamma_m)$, and where, in anticipation, we have omitted terms which will not survive the dimensional reduction.

We now retain x^0 as the time and drop all dependence on the other coordinates. It is now a simple matter to impose the symplectic Majorana Weyl condition upon the fermions and use the explicit realisation for the Γ-matrices, to derive dimensionally reduced expressions for the quantities of interest. The 'internal'

symmetry generators are given by the following expressions, where i and j now go from 1 to 3:

$$\begin{aligned}J_{i4} &= \tfrac{i}{4}\omega_{ab}\chi^a\sigma^2\bar{\sigma}^i\chi^b + \tfrac{i}{4}\omega^{ab}\bar{\chi}_a\bar{\sigma}^2\sigma^i\bar{\chi}_b\\ J_{i5} &= -\tfrac{1}{4}\omega_{ab}\chi^a\sigma^2\bar{\sigma}^i\chi^b + \tfrac{1}{4}\omega^{ab}\bar{\chi}_a\bar{\sigma}^2\sigma^i\bar{\chi}_b\\ J_{45} &= -\tfrac{1}{2}\bar{\chi}_a\bar{\sigma}^2\chi^a\\ J_{ij} &= -\tfrac{i}{2}\bar{\chi}_a\bar{\sigma}^2\sigma^{ij}\chi^a - \tfrac{i}{2}\chi^a\sigma^2\bar{\sigma}^{ij}\bar{\chi}_a .\end{aligned} \tag{13}$$

Similarly, the supercharges become:

$$\begin{aligned}S_{A\alpha} &= g_{ij}\gamma^i_{Aa}\dot{\phi}^j\chi^a_\alpha \qquad\qquad \bar{S}^{\dot{\alpha}}_A = \gamma_i^{Ba}\epsilon_{BA}\epsilon^{\dot{\alpha}\dot{\beta}}\dot{\phi}^i\bar{\chi}_{a\dot{\beta}}\\ &= \epsilon_{AB}\gamma_i^{Bb}\omega_{ab}\dot{\phi}^i\chi^a_\alpha\end{aligned}$$

Finally, one finds after a little calculation that the term in the Lagrangian density which is quadratic in the fermions may be written $i\bar{\chi}_a\bar{\sigma}^0\dot{\chi}^a$. After quantisation, one thus finds the anticommutation relations amongst the fermions

$$\{\chi^a_\alpha, \bar{\chi}_{b\dot{\alpha}}\} = -\delta^a_b\sigma^0_{\alpha\dot{\alpha}} ,$$

where again we remind the reader that in our conventions, $\sigma^0 = -\mathbb{1}$. We will choose the χ^a_α as creation operators and the $\bar{\chi}_{a\dot{\alpha}}$ as annihilation operators, acting on the appropriate Clifford vacuum. The bosons are quantised in the usual way, their Hilbert space being the space of square-integrable functions $f(\phi)$. Just as in the Kähler case, the total Hilbert space is (the completion of) the space of smooth complex-valued differential forms $A_c(X)$ on the hyperKähler manifold X. The explicit map is the following:

$$f_{i_1i_2\cdots i_k}(x)dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_k} \leftrightarrow f_{i_1i_2\cdots i_k}(\phi)\chi^{i_1}\chi^{i_2}\cdots\chi^{i_k}|0\rangle , \tag{14}$$

where

$$\chi^i \equiv \gamma^i_{a1}\chi^{a1} + \gamma^i_{a2}\bar{\chi}^{a2} .$$

Notice that in this equation we have identified the index A pertaining to Σ with the index α – both are $\mathrm{Sp}(1) = \mathrm{SU}(2)$ doublets, but the isomorphism is not natural. The inner product is defined as follows: if

$$|\psi\rangle = \chi^{a_1}_{\alpha_1}\chi^{a_2}_{\alpha_2}\cdots\chi^{a_k}_{\alpha_k}|0\rangle$$

its norm is given by $\langle\psi|\psi\rangle$ where

$$\langle\psi| = \langle 0|\bar{\chi}_{a_k\dot{\alpha}_k}\bar{\chi}_{a_{k-1}\dot{\alpha}_{k-1}}\cdots\bar{\chi}_{a_1\dot{\alpha}_1} . \tag{15}$$

In other words, just as in the Kähler case the adjoint is complex conjugation: $(\chi^a_\alpha)^\dagger = \bar{\chi}_{a\dot{\alpha}} = (\chi^a_\alpha)^*$. We see that again the Hilbert space of the quantum mechanical sigma model is (the completion of) the space of complex-valued smooth differential forms relative to the Hodge inner product. It is now possible to quantise the supercharges and compute their algebra. As is well-known, one finds that the hamiltonian agrees (up to a factor) with the Hodge laplacian. Therefore the ground states are once again in one-to-one correspondence with the cohomology.

As in the Kähler case, it is possible to make a dictionary relating the quantities appearing in the sigma model and the geometry of X. However we will simply mention that again the supercharges go to differential operators. Making a choice of a complex structure on X, it is possible to single out a particular linear combination of the supercharges which can be identified with the Dolbeault operator. In fact, on X there is a 2-sphere worth of complex structures, on which the Lie group SO(3) acts transitively. On the sigma model side, this is nothing but the R-symmetry of the six-dimensional supersymmetry algebra. There are also analogues of the Hodge identities. Just as in the Kähler case, these identities reflect the transformation properties of the supercharges under the 'internal' symmetry subgroup $\mathrm{Spin}^0(4,1) \cong \mathrm{Sp}(1,1)$ of $\mathrm{Spin}^0(5,1) \cong \mathrm{SL}(2,\mathbb{H})$. The supercharges, being Weyl spinors, transform as the canonical irreducible representation of $\mathrm{SL}(2,\mathbb{H})$ of quaternionic dimension 2, which under the 'internal' $\mathrm{Sp}(1,1)$ subgroup remains irreducible. We now come to the main point of this section: the supersymmetric origins of the action of $\mathrm{so}(4,1)$ on $H^*(X)$.

As before, this action will come induced from the action of the Lorentz generators under the reduction to one dimension. The generators J_{mn}, for $m, n = 1, ..., 5$, now generate an 'internal' so(5) symmetry of the one-dimensional supersymmetric quantum model obtained by this reduction. By virtue of its six-dimensional origin, this symmetry will furthermore commute with the action of the Hamiltonian, hence will induce an action on the ground states, i.e., on the cohomology of X. It remains to show that out of this so(5) symmetry follows the $\mathrm{so}(4,1)$ symmetry discussed by Verbitsky and summarised in the previous section. We do so now.

It is enough to consider the linear combinations $L_i = J_{i5} + iJ_{i4}$. Using the expressions (13), one finds that

$$L_i = \tfrac{1}{2}\omega_{ab}(\chi^a)^t \sigma^2 \sigma^i \chi^b$$

or more explicitly

$$\begin{aligned} L_1 &= -\tfrac{i}{2}\omega_{ab}\left(\chi_1^a \chi_1^b - \chi_2^a \chi_2^b\right) \\ L_2 &= \tfrac{1}{2}\omega_{ab}\left(\chi_1^a \chi_1^b + \chi_2^a \chi_2^b\right) \\ L_3 &= i\omega_{ab}\chi_1^a \chi_2^b \ . \end{aligned} \tag{16}$$

We define Λ_i as the adjoints of L_i with respect to the inner product defined by the norm (15), and then the other operators are defined using the Lie algebra of $\mathrm{so}(4,1)$. In other words, K_i and H are defined by $K_i = \frac{1}{2}\epsilon_{ijk}[L_j, \Lambda_k]$, and $H = [L_1, \Lambda_1]$, say. One can then check that these operators satisfy the Lie algebra $\mathrm{so}(4,1)$ as written in (11).

To conclude the proof we must show that this $\mathrm{so}(4,1)$ agrees with the one introduced by Verbitsky. Clearly it is sufficient to check that the L_i can be interpreted as exterior product with the three Kähler forms. Under the isomorphism (14), L_i indeed corresponds to exterior product with a 2-form $\omega_i \in \bigwedge^2 T^* \cong \bigwedge^2(\Sigma \otimes V)$ with components $\omega^i_{aAbB} = \omega_{ab} M^i_{AB}$, where M^i are matrices which can be read off from (16). It is clear that the forms ω^i are parallel, since so are ω and any constant section of any power of Σ (recall that Σ is a trivial bundle with the zero connection). It is moreover clear that these forms are linearly independent. Since in an otherwise arbitrary compact hyperkähler manifold X the space of parallel 2-forms is three-dimensional and spanned by the Kähler forms, we conclude that these are them.

4 Conclusions

In this paper we have shown, following an idea of Witten, how the Hodge–Lefschetz theory for (hyper)Kähler manifolds is a natural consequence of supersymmetry. That such a statement can be made should not come as a surprise because Kähler and hyperKähler manifolds can be *defined* by the fact that on them the (3+1)- and (5+1)-dimensional sigma models admit supersymmetry. That this is the case is also linked in a special way to the properties of Clifford algebras in those dimensions: Weyl spinors in (3+1) dimensions are complex, whereas in (5+1) dimensions they are quaternionic. With hindsight, that the dimensionally reduced sigma model should have anything to do with the Hodge–Lefschetz theory (or Verbitsky's extension) is again not surprising, since this theory deals with the cohomology of the manifold, which are the quantum-mechanical ground states of the one-dimensional sigma model.

It is tempting to speculate that other aspects of the geometry of these manifolds also have supersymmetric origins. For example, if X is compact hyperKähler, Verbitsky [6] proved that in general, there is an action of $\mathrm{so}(4, b_2-2)$ on $H^*(X)$, where $b_2 = \dim H^2(X)$. It remains an open problem to find a supersymmetric origin to this symmetry.

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