KCL-TH-97-29
CTP TAMU-24/97
NI-97027-NQF
hep-th/9705093
June 4, 1997

Aspects of Superembeddings

P.S. Howe [1], E. Sezgin [2†] and P.C. West [3‡]

[1] Department of Mathematics, King's College, London, UK

[2] Center for Theoretical Physics, Texas A&M University, College Station, TX 77843, USA

[3] Isaac Newton Institute for Mathematical Sciences, Cambridge, UK

Abstract

Some aspects of the geometry of superembeddings and its application to supersymmetric extended objects are discussed. In particular, the embeddings of (3|16) and (6|16) dimensional superspaces into (11|32) dimensional superspace, corresponding to supermembranes and superfivebranes in eleven dimensions, are treated in some detail.

A contribution to the D.V. Volkov memorial volume

† Research supported in part by NSF Grant PHY-9411543

‡ Permanent Address: Department of Mathematics, King's College, London, UK

1 Introduction

One of the many contributions that D.V. Volkov made to modern theoretical physics was the realisation that supersymmetric particles, moving in three or four dimensional spacetimes, can be described using a formalism which has both worldline and spacetime supersymmetries built in [1, 2]. Up until then it had been thought that, although the superstring can be described with either worldsheet [3, 4, 5] or spacetime supersymmetry [6], all other extended supersymmetric objects, including the superparticle [7], could only be written with manifest spacetime supersymmetry [8, 9]. The spacetime supersymmetric formalism does involve a local fermionic symmetry, called κ-symmetry [10], but the geometric nature of this symmetry remained obscure until the work of the Kharkov group showed that it can be derived from, and is equivalent to, local worldsurface supersymmetry. Subsequently the formalism has been developed by the Kharkov group and others, and has been applied to various other supersymmetric extended objects (see [11] for a full list of references). It also gradually became clear that the formalism can be understood in terms of the embedding of one superspace, the worldsurface, into another, the target superspace. Although this point of view was implicit in the early papers it was made explicit in a study of the heterotic string [12] and was further developed in [11]. More recently, in [13], it was shown that all supersymmetric extended objects can be understood in this way, including objects such as the Dirichlet branes of string theory which have additional physical worldsurface bosonic fields to the usual transverse coordinate fields. The latter are scalars on the worldsurface whereas the new fields are gauge fields; we shall refer to the two types of object as type I (scalars only) and type II (additional gauge fields). The formalism was applied in particular to construct the full equations of motion of the five-brane in eleven-dimensional superspace [14, 15], an object that plays an important role in M-theory. Moreover, it turns out that all branes are described by the same simple embedding condition which is extremely natural from the point of view of supergeometry [13]. We refer the reader to the literature for discussions of the component (GS) approach to Dirichlet branes [16, 17, 18, 19, 20] and the eleven-dimensional five-brane [21, 22, 23, 24, 25, 26, 27, 28, 29].

Before describing superembeddings in more detail it is perhaps worthwhile recalling the problem that Volkov and his collaborators solved. Consider a superparticle moving on a superworldline parametrised by (even, odd) coordinates (t,τ) in flat $(3|2)$-dimensional superspace coordinatised by (x^a,θ^α). Expanding the supercoordinates describing the particle in τ we have

$$\begin{aligned} x^a(t,\tau) &= x^a(t)+\tau\lambda^a(t) \ , \\ \theta^\alpha(t,\tau) &= \theta^\alpha(t)+\tau u^\alpha(t) \ . \end{aligned} \qquad (1)$$

The problem seems to be that there are two "wrong statistics" fields, λ^a and u^α. The solution to this problem found by the Kharkov group was to identify u as a twistor variable, in fact, as the "square-root" of the lightlike momentum of the particle, and to regard λ as an auxiliary field. This is summarised in the superspace equation

$$Dx^a-\frac{i}{2}D\theta^\alpha(\gamma^a)_{\alpha\beta}\theta^\beta=0 \ , \qquad (2)$$

where $D=\frac{\partial}{\partial\tau}+\frac{i}{2}\tau\frac{\partial}{\partial t}$ is the superworldline covariant derivative. The first component of (2) (in a τ-expansion) allows one to solve for λ while the second component relates $\dot{x}$ to u^2. Volkov and his team were able to find a somewhat unusual Lagrangian which gives rise to these conditions on the fields λ and u, but in what follows we shall not have much to say about actions, rather we focus on the dynamics of the extended objects directly and show how these can be understood from the perspective of superembeddings. It is important to notice that the geometrical interpretation of (2) is that, at any point on the superworldline, the odd tangent space of the superworldline is a subspace of the odd tangent space of the target superspace.

2 Flat Branes

We define a flat brane to be an embedding of a flat superspace, of dimension $(d|\frac{1}{2}D')$ in a flat superspace of dimension $(D|D')$. The existence of such objects determines in which dimensions one can have branes and the structure of the worldsurface multiplets can be obtained by considering small deformations. In fact, the allowed super-dimensions correspond to the points on the modified brane scan [30, 13]. One could also consider branes which preserve fewer than half of the target space supersymmetries, but we shall not do so here. In order for flat branes to exist it is necessary that the Γ-matrices should decompose in the right way. If $(x^{\underline{a}}, \theta^{\underline{\alpha}})$, $\underline{a} = 0, 1, \ldots D-1$; $\underline{\alpha} = 1, \ldots D'$ are coordinates on the target superspace split into (x^a, θ^α), $a = 1, 0, \ldots d-1$; $\alpha = 1, \ldots \frac{1}{2}D'$ and $(x^{a'}, \theta^{\alpha'})$, $a' = d, \ldots D-1$; $\alpha' = \frac{1}{2}D' + 1, \ldots D'$, we require that the Γ-matrices split as follows:

$$(\Gamma^{\underline{a}})_{\underline{\alpha}\underline{\beta}} \to (\Gamma^a)_{\alpha\beta},\ (\Gamma^{a'})_{\alpha\beta'} = (\Gamma^{a'})_{\beta'\alpha},\ (\Gamma^a)_{\alpha'\beta'}\ , \tag{3}$$

with all other components vanishing. If this is the case we have

$$[D_\alpha, D_\beta] = i(\Gamma^a)_{\alpha\beta}\partial_a\ , \tag{4}$$

or, equivalently, there is a subalgebra of the supertranslational algebra of the required dimension. The covariant derivative is defined as usual to be

$$D_{\underline{\alpha}} = \partial_{\underline{\alpha}} + \frac{i}{2}(\Gamma^{\underline{a}})_{\underline{\alpha}\underline{\beta}}\theta^{\underline{\beta}}\partial_{\underline{a}}\ . \tag{5}$$

The brane itself is given by the embedding $(x^a, \theta^\alpha) \mapsto (x^a, \theta^\alpha; 0, 0)$, or, equivalently, as the solution of the equations $x^{a'} = \theta^{\alpha'} = 0$ in the target superspace. The condition that the Γ-matrices split as above tells us when branes preserving half-supersymmetry can exist.

We next consider a small deformation of the flat brane, for which the embedding becomes

$$(x, \theta) \mapsto (x, \theta; x'(x, \theta), \theta'(x, \theta))\ , \tag{6}$$

where x' and θ' are small, so that we only need to work to first order in these variables. The odd basis tangent vectors to the submanifold, collectively denoted E_α, are given as the image of the (worldsurface) D_α under the embedding,

$$E_\alpha = D_\alpha + D_\alpha\theta^{\beta'}D_{\beta'} + (D_\alpha X^{b'} - i(\Gamma^{b'})_{\alpha\beta'}\theta^{\beta'})\partial_{b'}\ . \tag{7}$$

Note that D_α on the target space (which occurs on the right-hand side of the above equation) differs from D_α on the brane; the former, which is the α component of (5), includes a term involving θ' which is absent from the latter. The even basis vectors are

$$E_a = \partial_a + \partial_a X^{b'} + \partial_a\theta^{\beta'}D_{\beta'}\ , \tag{8}$$

where

$$X^{a'} = x^{a'} + \frac{i}{2}\theta^\beta(\Gamma^{a'})_{\beta\gamma'}\theta^{\gamma'}\ . \tag{9}$$

We now impose the requirement that the odd tangent space at any point of the embedded submanifold should be a subspace of the odd tangent space of the target space at that point. This condition is required for all supersymmetric extended objects and implies that we must impose

$$D_\alpha X^{a'} = i(\Gamma^{a'})_{\alpha\beta'}\theta^{\beta'} \ . \tag{10}$$

Computing the commutator of the odd tangent vectors (to first order in the transverse variables) one finds

$$[E_\alpha, E_\beta] = i(\Gamma^a)_{\alpha\beta}E_a + i(2D_{(\alpha}\theta^{\gamma'}(\Gamma^{b'})_{\beta)\gamma'} - (\Gamma^a)_{\alpha\beta}\partial_a X^{b'})\partial_{b'} \ . \tag{11}$$

Since the commutator of two vectors of the submanifold must lead to a third we have

$$2D_{(\alpha}\theta^{\gamma'}(\Gamma^{b'})_{\beta)\gamma'} = (\Gamma^a)_{\alpha\beta}\partial_a X^{b'} \ . \tag{12}$$

In fact, this constraint is not independent; it follows directly from (10) by differentiation, as indeed it must since the algebra of the covariant derivatives on the brane is preserved.

Equation (10) above is the key equation for branes since it determines the structure of the worldsurface supermultiplet. It can be one of three types: on-shell, in which case it leads directly to the dynamics of the physical fields; off-shell Lagrangian, in which case it determines an off-shell multiplet which can be used in a Lagrangian to determine the dynamics; or off-shell non-Lagrangian, in which case the multiplet is off-shell but there is not a Lagrangian, at least of conventional type, which can be constructed which leads to the dynamics. In the third case further conditions are required, but most examples fall into the first two classes.

In eleven dimensions (with 32 odd dimensions), there are two possible branes, the two-brane and the five-brane[1]. In both cases equation (10) defines an on-shell supermultiplet, the $d=3, N=8$ scalar multiplet and the $d=6, N=2$ tensor multiplet, respectively. In both cases the leading components of $X^{a'}$ and $\theta^{\alpha'}$ can be interpreted as Goldstone fields corresponding to the breaking of supertranslational symmetry, but in the five-brane there is an extra component field which appears at leading order in $D_\alpha\theta^{\beta'}$. In general one has

$$D_\alpha\theta^{\beta'} = \frac{1}{2}(\Gamma^{ab'})_\alpha{}^{\beta'}\partial_a X_{b'} + h_\alpha{}^{\beta'} \ , \tag{13}$$

where

$$h_{(\alpha}{}^{\gamma'}(\Gamma^{a'})_{\beta)\gamma'} = 0 \ , \tag{14}$$

but the latter equation only has non-trivial solutions for type II branes. For example, for the eleven-dimensional five-brane one finds

$$h_\alpha{}^{\beta'} = \frac{1}{6}(\Gamma^{abc})_\alpha{}^{\beta'}h_{abc} \ . \tag{15}$$

The field h_{abc} is totally antisymmetric and self-dual, and at the linearised level is closed. Its leading component is therefore the self-dual field strength tensor of a two-form gauge field. The quantity $D_\alpha\theta^{\beta'}$, evaluated at $\theta=0$, is the analogue of u^α in equation (1), at least in the linearised theory. The term involving $\partial_a X^{b'}$ generalises the momentum which arises in the particle case, while the h-term is present only for type II branes.

[1]The possibility that there might be a nine-brane has been raised [13]; such an object, if it exists, would have to have additional worldsurface fermion fields.

3 D = 11 Supergeometry

In the rest of the paper we shall focus on superembeddings in eleven dimensions. We briefly recall the salient features of eleven-dimensional supergeometry. One has a real (11|32)-dimensional supermanifold $\underline{M}$ with a choice of odd tangent bundle $\underline{F} \subset \underline{T}$, the full tangent bundle, such that the associated Frobenius tensor $\underline{\phi}$, defined by

$$\underline{\phi}(X,Y) = [X,Y] \,\mathrm{mod}\, \underline{F} \ , \tag{16}$$

where X and Y are odd vector fields, is invariant under the group $Spin(1,10)\times\mathbb{R}^+$. This implies that there exist local bases $(E_{\underline{\alpha}})$ for $\underline{F}$ and $(E^{\underline{a}})$ for $\underline{B}^*$, where $\underline{B}$ is the quotient of $\underline{T}$ by $\underline{F}$ and the star denotes dual, in which the components of $\underline{\phi}$ are given by

$$\underline{\phi}_{\underline{\alpha}\underline{\beta}}{}^{\underline{c}} = \langle [E_{\underline{\alpha}}, E_{\underline{\beta}}], E^{\underline{c}}\rangle = i(\Gamma^{\underline{c}})_{\underline{\alpha}\underline{\beta}} \ . \tag{17}$$

This set up defines what one might call a special superconformal structure; more generally one can allow for additional terms in $\underline{\phi}$ involving two-index and five-index Γ-matrices and we shall come back to this possibility shortly. If (17) holds it can be shown that it implies the equations of motion of eleven-dimensional supergravity must be satisfied modulo certain topological niceties which we shall ignore here [31]. More precisely, one can show, given (17), that one can find a choice of $\underline{B}$ as a subbundle of $\underline{T}$ and a choice of $Spin(1,10)$ connection such that the torsion and curvature tensors in superspace are related to those of on-shell supergravity by a super-Weyl transformation. One can therefore make such a transformation to eliminate the conformal factor thereby arriving at the standard geometry [32, 33]. This geometry has structure group $Spin(1,10)$ and therefore admits an invariant Lorentzian metric $\underline{g}_{\underline{B}}$ on $\underline{B}$ and also an invariant fermionic metric $\underline{g}_{\underline{F}}$ on $\underline{F}$ whose components in the standard basis are the components of the charge-conjugation matrix.

We note that when a connection is introduced it is natural to equate $\underline{\phi}$ with the dimension zero component of the torsion tensor (with a minus sign), although it is a perfectly well-defined tensor belonging to the space $\wedge^2\underline{F}^*\otimes\underline{B}$ even if a connection is not introduced. Thus we have

$$\underline{\phi}_{\underline{\alpha}\underline{\beta}}{}^{\underline{c}} = -T_{\underline{\alpha}\underline{\beta}}{}^{\underline{c}} \ . \tag{18}$$

4 Embeddings

We consider embeddings $M \xrightarrow{f} \underline{M}$ of the worldsurface M into the target space $\underline{M}$ which we shall take to have dimension (11|32) for definiteness, although the discussion below is applicable more generally with appropriate modifications. It will be assumed that $\underline{M}$ has a superconformal structure but initially at least we shall not suppose that the Frobenius tensor $\underline{\phi}$ is invariant under the structure group $Spin(1,10)\times\mathbb{R}^+$. Without loss of generality we can take it to be of the form

$$\underline{\phi}_{\underline{\alpha}\underline{\beta}}{}^{\underline{c}} = i(\Gamma^{\underline{c}})_{\underline{\alpha}\underline{\beta}} + (\Gamma^{\underline{bc}})_{\underline{\alpha}\underline{\beta}} X_{\underline{bc}}{}^{\underline{a}} + (\Gamma^{\underline{bcdef}})_{\underline{\alpha}\underline{\beta}} Y_{\underline{bcdef}}{}^{\underline{a}} \ , \tag{19}$$

where the antisymmetric components of X and Y vanish, as well as their traces. Both of these restrictions are compatible with the group structure. We shall call such a structure a general superconformal structure.

There is a natural choice of odd tangent bundle F on M given by

$$F = T \cap \underline{F} \; . \tag{20}$$

Dually, one has

$$B^* = T^* \cap \underline{B}^* \; . \tag{21}$$

The only other requirement we need to impose on the embedding is that the metric iduced on B^* (from any of the conformal class of Lorentzian metrics on $\underline{B}^*$) should be Lorentzian with signature $(p-1)$, $p = 2, 5$. Again this condition is automatically conformally invariant.

The Frobenius tensor ϕ of M is defined in the same way as above, namely as the commutator of two odd vector fields modulo the odd tangent bundle. The relation between the worldsurface and target space Frobenius tensors is given by

$$\underline{\phi}(X, Y, \underline{\omega}) = \phi(X, Y, f^*\underline{\omega}) \; , \tag{22}$$

where X and Y are odd vector fields on M (which may be considered as vector fields on $\underline{M}$) and $\underline{\omega}$ is a one-form on $\underline{M}$.

For any embedding one has three natural bundles (on M), the tangent bundle T, the tangent bundle, $\underline{T}$, of $\underline{M}$ restricted to M and the normal bundle T' which fit together in a short exact sequence

$$0 \to T \to \underline{T} \to T' \to 0 \; . \tag{23}$$

However, in the super case we have even and odd tangent bundles which themselves fit into an exact sequence

$$0 \to F \to T \to B \to 0 \; , \tag{24}$$

and similarly for the target space as well as the corresponding normal bundles. In fact there are nine bundles in all and it can be shown that they fit together into the following diagram,

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & F' & \to & T' & \to & B' & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & \underline{F} & \to & \underline{T} & \to & \underline{B} & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & F & \to & T & \to & B & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{25}$$

where each of the rows and columns is exact and where each square is commutative. The proof of these assertions is straightforward. The dual bundles give rise to a similar diagram with the arrows reversed. In practice one wishes to split the sequences, so that the central bundle of each sequence becomes a direct sum of the other two. However, when this is carried out it is important to note that, although B as a quotient bundle is a subbundle of $\underline{B}$ the same is not true when B and $\underline{B}$ are regarded as subbundles of T and $\underline{T}$ respectively.

5 Brane Integrability

The basic embedding condition described above is extremely natural given the geometrical structures that arise in supergeometry. Moreover, it is also extremely restrictive. In fact, in the eleven-dimensional examples we are discussing, one has the following results: if an (11|32)-dimensional supermanifold $\underline{M}$ with a general superconformal structure admits embeddings of the type described in the previous section of either two-branes or five-branes through every point, then:

- any such brane is dynamical, that is the embedding implies that the worldsurface multiplet is on-shell,
- the superconformal structure on the target space must be special, which implies, as we have discussed above, that the equations of motion of eleven-dimensional supergravity must be satisfied,
- the worldsurface supergeometry is completely specified up to gauge freedoms.

In some respects this may not be too surprising as it has been known for some time that the requirement of κ-symmetry in the GS formulation of the membrane forces the target space supergeometry to be equivalent to eleven-dimensional supergravity [9]. However, in the present approach we have achieved this with the bare minimum of assumptions; everything, including κ-symmetry follows from the simple embedding condition (20).

The complete proof of the above assertions is extremely long and rather complicated in terms of details, but it is simple to understand how it comes about in principle. Consider the relation (22) between the Frobenius tensors of the two manifolds. We introduce local bases (E_α), (E^a) for F and B^* respectively and note that (20) implies that

$$E_\alpha = E_\alpha{}^{\underline{\alpha}} E_{\underline{\alpha}} \ , \tag{26}$$

for some 16×32 matrix $E_\alpha{}^{\underline{\alpha}}$, while the dual condition (21) implies

$$f^* E^{\underline{a}} = E^a E_a{}^{\underline{a}} \ . \tag{27}$$

Equation (22) then reads in components with respect to these bases,

$$E_\alpha{}^{\underline{\alpha}} E_\beta{}^{\underline{\beta}} \phi_{\underline{\alpha}\underline{\beta}}{}^{\underline{c}} = \phi_{\alpha\beta}{}^c E_c{}^{\underline{c}} \ . \tag{28}$$

If $(E_{\underline{\alpha}})$ is a spin basis for $\underline{F}$ any other such basis will be related to it by an element u of $Spin(1,10)$ up to a conformal factor which we shall ignore for the moment. We write $u = (u_\alpha{}^{\underline{\alpha}}, u_{\alpha'}{}^{\underline{\alpha}})$, with $\alpha' = 1, \ldots 16$. Since $E_\alpha{}^{\underline{\alpha}}$ has maximal rank there will be a choice of u such that E_α is related to $u_\alpha{}^{\underline{\alpha}} E_{\underline{\alpha}}$ by a non-singular matrix. Hence, without loss of generality, we can write

$$E_\alpha{}^{\underline{\alpha}} = A_\alpha{}^\beta u_\beta{}^{\underline{\alpha}} + B_\alpha{}^{\beta'} u_{\beta'}{}^{\underline{\alpha}} \ , \tag{29}$$

where $\det A \neq 0$. Making a change of basis for F we arrive at

$$E_\alpha{}^{\underline{\alpha}} = u_\alpha{}^{\underline{\alpha}} + h_\alpha{}^{\beta'} u_{\beta'}{}^{\underline{\alpha}} \ . \tag{30}$$

On the bosonic space B^* the situation resembles more closely the case of a Lorentzian embedding and we may choose, again up to a conformal factor

$$E_a{}^{\underline{a}} = u_a{}^{\underline{a}} \ , \tag{31}$$

where $(u_a{}^{\underline{a}}, u_{a'}{}^{\underline{a}})$ is the element of $SO(1,10)$ corresponding to $u = (u_\alpha{}^{\underline{\alpha}}, u_{\alpha'}{}^{\underline{\alpha}}) \in Spin(1,10)$ Thus, at any point $p \in M$, the embedding is specified by $u_a{}^{\underline{a}}$, $u_\alpha{}^{\underline{\alpha}}$ and $h_\alpha{}^{\beta'}$.

We can now decompose equation (28) into components tangent and normal to M. We then find

$$\underline{\phi}_{\alpha\beta}{}^{c'} + 2h_{(\alpha}{}^{\gamma'}\underline{\phi}_{\beta)\gamma'}{}^{c'} + h_\alpha{}^{\gamma'}h_\beta{}^{\delta'}\underline{\phi}_{\gamma'\delta'}{}^{c'} = 0\ , \tag{32}$$

and

$$\underline{\phi}_{\alpha\beta}{}^{c} + 2h_{(\alpha}{}^{\gamma'}\underline{\phi}_{\beta)\gamma'}{}^{c} + h_\alpha{}^{\gamma'}h_\beta{}^{\delta'}\underline{\phi}_{\gamma'\delta'}{}^{c} = \phi_{\alpha\beta}{}^c\ , \tag{33}$$

where

$$\underline{\phi}_{\alpha\beta}{}^{c'} = u_\alpha{}^{\underline{\alpha}}u_\beta{}^{\underline{\beta}}\underline{\phi}_{\underline{\alpha}\underline{\beta}}{}^{\underline{c}}u_{\underline{c}}{}^{c'}\ , \tag{34}$$

and similarly for the other projections of $\underline{\phi}_{\underline{\alpha}\underline{\beta}}{}^{\underline{c}}$. Now in order for there to be embeddings of branes in general we require that these equations be satisfied for arbitrary embeddings passing through a given point $p \in \underline{M}$ and furthermore that this should be true for all points of $\underline{M}$. Since we may vary u and h independently this requires

$$\underline{\phi}_{\alpha\beta}{}^{c'} = 0\ . \tag{35}$$

This can only be satisfied for arbitrary embeddings if the superconformal structure on $\underline{M}$ is special, i.e. if

$$\underline{\phi}_{\underline{\alpha}\underline{\beta}}{}^{\underline{c}} = i(\Gamma^{\underline{c}})_{\underline{\alpha}\underline{\beta}}\ . \tag{36}$$

Given this one finds that equations (32) and (33) are solved by

$$h_\alpha{}^{\beta'} = \begin{cases} 0 & \text{two-brane} \\ \frac{1}{6}(\Gamma^{abc})_\alpha{}^{\beta'}h_{abc} & \text{five-brane} \end{cases} \tag{37}$$

where h_{abc} is self-dual, and

$$\phi_{\alpha\beta}{}^c = \begin{cases} i(\Gamma^c)_{\alpha\beta} & \text{two-brane} \\ i(\Gamma^b)_{\alpha\beta}m_b{}^c & \text{five-brane} \end{cases} \tag{38}$$

where

$$m_a{}^b = \delta_a{}^b - 2h_{acd}h^{bcd}\ . \tag{39}$$

The above argument establishes brane-integrablity; to see that the embedding implies the dynamics it is sufficient to consider the linearised case, i.e. take the target space to be flat and assume that the embedded submanifold is also nearly flat. It is not too difficult to see that in this limit one recovers the equations describing the deformations of flat branes, and hence the worldsurface fields are indeed on-shell. By working to second order in the transverse fields one can quickly see that the worldsurface supergravity fields are also determined.

6 Some Geometrical Aspects of Superembeddings

We briefly recall some aspects of Riemannian embeddings (see, for example, [34]). Let M be a manifold embedded in a Riemannian manifold $(\underline{M}, \underline{g})$. The metric on the target space induces natural metrics on the embedded space and on the normal bundle T' as well as determining a natural orthogonal decomposition of $\underline{T}$ into tangential and normal components. Explicitly

$$\begin{aligned} \underline{g}(X,Y) &= g(X,Y)\ , \\ \underline{g}(X,Y') &= 0\ , \\ \underline{g}(X',Y') &= g'(X',Y')\ , \end{aligned} \tag{40}$$

where X, Y are tangential vector fields, X', Y' normal vector fields, g is the induced metric on M and g' is the metric induced on the normal bundle. Metric connections ∇ and ∇' are determined in T and T' respectively from the metric connection $\underline{\nabla}$ on $\underline{M}$ by the Gauss-Weingarten equations

$$\begin{aligned} \underline{\nabla}_X Y &= \nabla_X Y + K'(X,Y) \ , \\ \underline{\nabla}_X Y' &= \nabla'_X Y' + K(X,Y') \ , \end{aligned} \tag{41}$$

where $K'(X,Y)$ is normal and $K(X,Y')$ tangential. K' is the second fundamental form, and K is related to K' by

$$\underline{g}(K'(X,Y), Z') + \underline{g}(X, K(Y,Z')) = 0 \ . \tag{42}$$

From (41) one can derive the torsion equations

$$\begin{aligned} [\underline{T}(X,Y)]^t &= T(X,Y) \ , \\ [\underline{T}(X,Y)]^n &= K'(X,Y) - K'(Y,X) \ , \end{aligned} \tag{43}$$

where the superscripts t and n denote tangential and normal respectively. Finally, we have the equations of Gauss and Codazzi relating the curvature tensors of T and T' to the Riemann curvature tensor of $\underline{M}$:

$$\begin{aligned} \underline{R}(X,Y,Z,\omega) &= R(X,Y,Z,\omega) + (K(X, K'(Y,Z),\omega) - X \leftrightarrow Y) \ , \\ \underline{R}(X,Y,Z',\omega') &= R'(X,Y,Z',\omega') + (K'(X, K(Y,Z'),\omega') - X \leftrightarrow Y) \ , \end{aligned} \tag{44}$$

where ω and ω' are respectively tangential and normal one-forms.

The above equations can be generalised to the supersymmetric case although the situation is more complicated due to the even-odd split. In view of the discussion of the preceding section we can assume that the target space supergeometry is the standard geometry describing on-shell supergravity. We begin with the membrane. The tensor $\underline{\phi}$ on $\underline{M}$ gives rise to the following tensors via embedding:

$$\begin{aligned} \underline{\phi}(X,Y,\omega) &= \phi(X,Y,\omega) \ , \\ \underline{\phi}(X,Y',\omega') &= \tilde{\phi}(X,Y',\omega') \ , \\ \underline{\phi}(X',Y',\omega) &= \phi'(X',Y',\omega) \ , \end{aligned} \tag{45}$$

while

$$\begin{aligned} \underline{\phi}(X,Y,\omega') &= 0 \ , \\ \underline{\phi}(X,Y',\omega) &= 0 \ , \\ \underline{\phi}(X',Y',\omega') &= 0 \ , \end{aligned} \tag{46}$$

where, in both equations, the vectors are all odd, the forms are even and normal vectors or forms are distinguished by a prime. From the bosonic metric we derive

$$\begin{aligned} \underline{g}_B(X,Y) &= g_B(X,Y) \ , \\ \underline{g}_B(X,Y') &= 0 \ , \\ \underline{g}_B(X',Y') &= g'_B(X,Y) \ , \end{aligned} \tag{47}$$

for even tangential and normal vectors X, Y and X', Y', thus defining induced metrics for B and B'. Starting from the fermionic metric we get

$$\begin{aligned} \underline{g}_F(X,Y) &= g_F(X,Y) \ , \\ \underline{g}_F(X,Y') &= 0 \ , \\ \underline{g}_F(X',Y') &= g'_F(X,Y) \ , \end{aligned} \tag{48}$$

for odd tangential and normal vectors X, Y and X', Y', thus defining induced fermionic metrics for F and F'. We also have

$$
\begin{aligned}
\underline{g}_F(X,Y) &= 0 \ , \\
\underline{g}_F(X,Y') &= 0 \ , \\
\underline{g}_F(X',Y) &= \Lambda(X',Y) \ , \\
\underline{g}_F(X',Y) &= 0 \ ,
\end{aligned} \tag{49}
$$

for odd vectors X, X' and even vectors Y, Y', with the primes denoting normal vectors as usual. The above equations determine a decomposition of $\underline{T}$ with respect to the tangential and normal bundles. Explicitly, we have

$$\underline{F} \cong F \oplus F' \ , \tag{50}$$

while

$$\underline{B} \subset B \oplus B' \oplus F' \ . \tag{51}$$

The field Λ can be thought of as providing a gauge-invariant representation of the worldsurface multiplet. Indeed, in the linearised case it reduces to the even derivative of the transverse odd coordinate functions.

The generalisations of the Gauss-Weingarten equations are

$$
\begin{aligned}
\underline{\nabla}_X Y &= \nabla_X Y + K'(X,Y) + L(X,Y) \ , \\
\underline{\nabla}_X Y' &= \nabla'_X Y' + K(X,Y') + L'(X,Y') \ ,
\end{aligned} \tag{52}
$$

where $K'(X,Y)$ and $L'(X,Y)$ are normal while $K(X,Y')$ and $L(X,Y)$ are tangential. The additional tangential terms are required because even vectors on M have non-vanishing projections on $\underline{F}$. For Y Y' odd $L(X,Y)$ and $L'(X,Y')$ both vanish while $K(X,Y)$ and $K'(X,Y')$ are odd. The torsion equations are

$$
\begin{aligned}
[\underline{T}(X,Y)]^t &= T(X,Y) + L(X,Y) - L(Y,X) \ , \\
[\underline{T}(X,Y)]^n &= K(X,Y) - K(Y,X) \ ,
\end{aligned} \tag{53}
$$

while the Gauss-Codazzi equations have the same form as in the bosonic case despite the presence of the additional terms in the Gauss-Weingarten equations,

$$\underline{R}(X,Y,Z,\omega) = R(X,Y,Z,\omega) + (K'(X,K(Y,Z),\omega) - X \leftrightarrow Y) \ , \tag{54}$$

$$\underline{R}(X,Y,Z',\omega') = R'(X,Y,Z',\omega') + (K(X,K'(Y,Z'),\omega') - X \leftrightarrow Y) \ , \tag{55}$$

where the last two arguments of the curvatures are either both even or both odd.

One can obtain many relations for the tensors defined above by differentiating the invariant tensors. It is straightforward to check that the connections defined on T and T' preserve the induced bosonic and fermionic metrics, and that the tensors constructed from $\underline{\phi}$ are also invariant. The structure groups for F and F' are both $Spin(1,2) \cdot Spin(8)$, although different representations are involved, while the structure groups for B and B' are $SO_o(1,2)$ and $SO(8)$, where the superscript "o" denotes the component connected to the identity.

Many of the above equations can be taken over in the case of the five-brane, but there are some differences. The equations for the bosonic metric remain the same but the fermionic ones change. One finds, instead of (48), the equations

$$\begin{aligned} \underline{g}_F(X,Y) &= h(X,Y) \ , && (56)\\ \underline{g}_F(X,Y') &= g_F(X,Y') \ , && (57)\\ \underline{g}_F(X',Y') &= 0 \ , && (58) \end{aligned}$$

for odd arguments. Note that in decomposing the eleven-dimensional charge conjugation matrix ($\underline{g}_F$) into $6+5$ one does not arrive at fermionic metrics on the tangential and normal subspaces but rather at an off-diagonal tensor which we have called g_F above although it is not a metric but rather determines an isomorphism between F^* and F'. The tensor h departs from this expected behaviour and is a signal of a type II brane. In index notation,

$$h_{\alpha\beta} = \frac{1}{3}(\Gamma^{abc})_{\alpha\beta} h_{abc} \ . \tag{59}$$

For mixed arguments ($X's$ odd Y's even) one has

$$\begin{aligned} \underline{g}_F(X,Y) &= \Lambda(X,Y) \ , \\ \underline{g}_F(X,Y') &= 0 \ , \\ \underline{g}_F(X',Y) &= 0 \ , \\ \underline{g}_F(X',Y) &= 0 \ . \end{aligned} \tag{60}$$

Again the field Λ is related to the worldsurface multiplet. For the Frobenius tensor one finds, as before,

$$\begin{aligned} \underline{\phi}(X,Y,\omega) &= \phi(X,Y,\omega) \ , \\ \underline{\phi}(X,Y',\omega') &= \tilde{\phi}(X,Y',\omega') \ , \\ \underline{\phi}(X',Y',\omega) &= \phi'(X',Y',\omega) \ , \end{aligned} \tag{61}$$

and

$$\begin{aligned} \underline{\phi}(X,Y,\omega') &= 0 \ , \\ \underline{\phi}(X',Y',\omega') &= 0 \ , \end{aligned} \tag{62}$$

where the vectors are odd and the forms even. However, one now has

$$\underline{\phi}(X,Y',\omega) \neq 0 \ . \tag{63}$$

In fact, this tensor is also linearly proportional to h.

One can take over the Gauss-Weingarten, torsion and curvature equations formally without change, but there are differences between the two and five-brane cases. For the five-brane the induced connections for the even tangent and normal bundles correspond to the groups $SO_o(1,5)$ and $SO(5)$ respectively, but the connections for the odd tangent bundles do not give $Spin(1,5)\cdot Spin(5)$ connections. This is again due to the type II embedding structure. One would have had this result if h had been zero, but the intervention of this term complicates matters somewhat. An alternative procedure is to define connections which do preserve the natural groups in both the even and odd sectors, and this is the route that has been taken in the literature [14, 15].

We conclude with a few remarks on Wess-Zumino forms. So far we have made no mention of these, even though they play such a crucial rôle in the GS formalism. The reason for this is that supersymmetry implies that they are present, so that one does not have to introduce them separately by hand in the superspace formalism. In eleven dimensional superspace with the standard constraints it is easy to show that there exists a closed four-form $\underline{H}_4$ which has non-trivial components only at dimension zero and one, the dimension one component reflecting the presence of a non-trivial spacetime three-form potential. The pull-back of this form defines a four-form on M, obviously closed, and which is flat in the case of the membrane, i.e. its only non-vanishing component in a standard basis is a Γ-matrix contribution at dimension zero, and which obeys the equation

$$dH_3 = -\frac{1}{4} f^* \underline{H}_4 \ , \tag{64}$$

in the case of the five-brane. The only non-vanishing component of H_3 is the purely vectorial component which is given by

$$H_{abc} = (m^{-1})_a{}^d h_{bcd} \ . \tag{65}$$

We emphasize that (64) is not a new equation; it is identically true provided that one defines H_3 as above and uses the results which follow from the torsion equations of the embedding. We refer the reader to the literature for more details on how one deduces the full equations of motion describing the brane dynamics from the basic embedding condition (20) using the superspace formalism [14, 15, 35, 36].

References

[1] D. Sorokin, V. Tkach and D.V. Volkov, *Superparticles, twistors and Siegel symmetry*, Mod. Phys. Lett. **A4** (1989) 901.

[2] D. Sorokin, V. Tkach, D.V. Volkov and A. Zheltukhin, *From superparticle Siegel supersymmetry to the spinning particle proper-time supersymmetry*, Phys. Lett. **B259** (1989) 302.

[3] P. Ramond, *Dual theory for free fermions*, Phys. Rev. **D3** (1971) 2415.

[4] A. Neveu and J.H. Schwarz, *Factorizable dual model of pions*, Nucl. Phys. **B31** (1971) 86.

[5] F. Gliozzi, J. Scherk and D. Olive, *Supersymmetry, supergravity theories and the dual spinor model*, Nucl. Phys. **B122** (1977) 253.

[6] M.B. Green and J.H. Schwarz, *Covariant description of superstrings*, Phys. Lett. **136B** (1984) 367.

[7] L. Brink and J.H. Schwarz, *Quantum superspace*, Phys. Lett. **100B** (1981) 310.

[8] J. Hughes, J. Liu and J. Polchinski, *Supermembranes*, Phys. Lett. **B180** (1986) 370.

[9] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Supermembranes and eleven-dimensional supergravity*, Phys. Lett. **189B** (1987) 75.

[10] W. Siegel, *Hidden local supersymmetry in the supersymmetric particle action*, Phys. Lett. **128B** (1983) 397.

[11] I.A. Bandos, D. Sorokin, M. Tonin, P. Pasti and D. Volkov, *Superstrings and supermembranes in the doubly supersymmetric geometrical approach*, Nucl. Phys. **B446** (1995) 79, hep-th/9501113.

[12] F. Delduc, A. Galperin, P.S. Howe and E. Sokatchev, *A twistor formulation of the heterotic D=10 superstring with manifest (8,0) worldsheet supersymmetry*, Phys. Rev. **D47** (1993) 578, hep-th/9207050.

[13] P.S. Howe and E. Sezgin, *Superbranes*, Phys. Lett. **B390** (1997) 133, hep-th/9607227.

[14] P.S. Howe and E. Sezgin, *D=11,p=5*, Phys. Lett. **B394** (1997) 62, hep-th/9611008.

[15] P.S Howe, E. Sezgin and P.C. West, *Covariant field equations of the M-theory five-brane*, hep-th/9702008.

[16] M. Cederwall, A. von Gussich, B.E.W. Nilsson and A. Westerberg, *The Dirichlet super-three-brane in ten-dimensional Type IIB supergravity*, Nucl. Phys. **B490** (1997) 163, hep-th/9610148.

[17] M. Aganagic, C. Popescu and J.H. Schwarz, *D-brane actions with local kappa symmetry*, Phys. Lett. **B393** (1997) 311, hep-th/9610249.

[18] M. Aganagic, C. Popescu and J.H. Schwarz, *Gauge-invariant and gauge-fixed D-brane actions*, hep-th/9612080.

[19] M. Cederwall, A. von Gussich, B.E.W. Nilsson, P. Sundell and A. Westerberg, *The Dirichlet super p-branes in ten dimensional Type IIA and IIB supergravity*, Nucl. Phys. **B490** (1997) 179, hep-th/9611159.

[20] E. Bergshoeff and P.K. Townsend, *Super D-branes*, Nucl. Phys. **B490** (1997) 145, hep-th/9611173.

[21] P.K. Townsend, *D-branes from M-branes*, Phys. Lett. **B373** (1996) 68, hep-th/9512062.

[22] O. Aharony, *String theory dualities from M theory*, Nucl. Phys. **B476** (1996) 470, hep-th/9604103.

[23] E. Bergshoeff, M. de Roo and T. Ortin, *The eleven dimensional fivebrane*, Phys. Lett. **B386** (1996) 85, hep-th/9606118.

[24] E. Witten, *Five-brane effective action in M-theory*, hep-th/9610234.

[25] M. Perry and J.H. Schwarz, *Interacting chiral gauge fields in six dimensions and Born-Infeld theory*, Nucl. Phys. **B489** (1997) 47, hep-th/9611065.

[26] J.H. Schwarz, *Coupling of self-dual tensor to gravity in six dimensions*, Phys. Lett. **B395** (1997) 191, hep-th/9701008.

[27] P. Pasti, D. Sorokin and M. Tonin, *Covariant action for D=11 five-brane with the chiral field*, hep-th/9701037.

[28] I. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D. Sorokin and M. Tonin, *Covariant action for the superfivebrane of M-theory*, hep-th/9701149.

[29] I. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D. Sorokin M. Tonin, *On the equivalence of different formulations of the M theory five–brane*, hep-th/9703127.

[30] M.J. Duff and J.X. Lu, *Type II p-branes: the brane scan revisited*, Nucl. Phys. **B390** (1993) 276, hep-th/9207060.

[31] P.S. Howe, *Weyl superspace*, in preparation.

[32] E. Cremmer and S. Ferrara, *Formulation of 11-dimensional supergravity in superspace*, Phys. Lett. **91B** (1980) 61.

[33] L. Brink and P.S. Howe, *Eleven dimensional supergravity on the mass shell in superspace*, Phys. Lett. **91B** (1980) 384.

[34] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, vol. II* (Wiley Interscience, 1963).

[35] P.S. Howe, E. Sezgin and P.C. West, *The six-dimensional self-dual tensor*, hep-th/9702111.

[36] P.S. Howe, E. Sezgin and P.C. West, *The geometry of super M-branes*, in preparation.