

Discrete series characters for $GL(n, q)$

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Introduction

An ordinary, irreducible character χ of the finite general linear group $GL(n, q)$ is said to belong to the “discrete series” if it is not a constituent of the permutation character induced from the radical U of any proper parabolic subgroup $P = L.U$ of $GL(n, q)$. Such a character χ cannot be obtained by “Harish-Chandra induction” from characters of $GL(n', q)$ for $n' < n$, in fact χ cannot be expressed as a linear combination of induced characters from proper parabolic subgroups of $GL(n, q)$.

Three different methods have been used to calculate the discrete series characters for $GL(n, q)$.

- (1) In [GL], they are constructed using the “Brauer lifts” of natural modular characters of $GL(n, q)$.
- (2) In [L], G. Lusztig constructs a module $D(V)$ which affords a discrete series character for $GL(V) = GL(n, q)$ (V is an n -dimensional vector space over a field of q elements), as an eigenspace of a homology module for a certain simplicial complex made out of affine flags on V .
- (3) In their fundamental work [DL], Deligne and Lusztig use the étale cohomology of certain varieties related to a reductive group \mathbf{G} to construct (generalized) characters of finite subgroups $G = \mathbf{G}^F$ of \mathbf{G} . Taking $\mathbf{G} = \mathbf{GL}_n$, the discrete series characters of $GL(n, q)$ are (up to a sign) Deligne–Lusztig’s $R_{\mathbf{T}}^{\psi}$, where \mathbf{T} is a maximal torus of \mathbf{G} such that $T = \mathbf{T}^F$ is of order $q^n - 1$, and ψ is a character of T in general position.

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It is the purpose of the present work to present the discrete series characters of $GL(n, q)$ in a rather simpler way, namely as \mathbf{Z} -linear combinations of characters induced from linear characters on certain subgroups of $GL(n, q)$. Of course, R. Brauer's theorem ([B], theorem A) shows that it is possible to express any character of any finite group G as \mathbf{Z} -linear combination of characters induced from linear characters on "elementary" subgroups of G . But we are able, in our special situation, to achieve our goal much more economically than would be possible by invoking Brauer's general theorem.

Let k be a field of q elements. The discrete series characters χ are determined by certain class-functions $J_n(\psi)$ on $G = GL(n, q)$ described in section 1 (the parameter ψ is equivalent to a character of Deligne–Lusztig's maximal torus T). $J_n(\psi)$ has "degree" $(1 - q)(1 - q^2) \dots (1 - q^{n-1})$, and has primary support, i.e. $J_n(\psi)(g) \neq 0$ only if the characteristic polynomial $\det(tI_n - g)$ is a power of an irreducible polynomial in $k[t]$. We describe in section 2 a family $F(n)$ of subgroups $H_{d,n}(k)$ of G , one for each divisor d of n . For example $H_{1,n}(k)$ is the product of the centre Z of G with the group P of all upper unitriangular matrices in G , while $H_{n,n}(k)$ is a maximal torus T of order $q^n - 1$. Each element g of each $H_{d,n}(k)$ is primary, and between them, the $H_{d,n}(k)$ meet all the primary conjugacy classes of G . In section 3 we define, for each d and for each partition λ of n/d , a character $X_{d,n}(\psi, \lambda)$ of G , which is induced from a linear character of $H_{d,n}(k)$. Our main theorem (theorem 3.2) states that there exists a family of polynomials $r_\lambda(T) \in \mathbf{Z}[T]$, indexed by the set of all partitions λ (of all positive integers), such that for all n , all ψ and all fields k of order q ,

$$(3.3) \quad J_n(\psi) = \sum_{d|n} \sum_{\lambda \vdash n/d} r_\lambda(q^d) X_{d,n}(\psi, \lambda).$$

Section 4 states without proof some rather technical propositions on the $X_{d,n}(\psi, \lambda)$ and in section 5, the theorem 3.2 is proved on the assumption that these propositions are true. The proofs of the propositions in section 4 require some formulae on the Gelfand–Graev character for $G = GL(n, q)$; these are given in sections 6 and 8 (section 6 is essentially due to Deligne–Lusztig) and may have some interest in their own right. The proofs which were deferred from

section 4 are given in sections 7 and 9. An appendix at the end of the paper gives the polynomials $r_\lambda(T)$ for all partitions $\lambda \vdash n \leq 5$.

1 Notation. The class function $J_n(\psi)$

n is a positive integer, q is a power of a prime p , and \bar{k} is an algebraically closed field of characteristic p . For each positive integer d , k_d is the unique subfield of \bar{k} of order q^d . Write $k = k_1$.

$M_d = k_d^\times$ and $\hat{M}_d = \text{Hom}(M_d, \mathbf{C}^\times)$ are the multiplicative group of k_d , and the character group of M_d , respectively.

From now on, we denote the group $GL(n, q)$ as $G_n(k)$; similarly $GL(n, q^d) = G_n(k_d)$, etc. For any group G , the set of all conjugacy classes of G is denoted $\text{ccl } G$.

t, T are indeterminates over k, \mathbf{Z} , respectively.

A sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_b)$ is called a *composition* of n if $\lambda_1 + \dots + \lambda_b = n$ (notations $\lambda \models n$ and $|\lambda| = n$). If $\lambda_1 \geq \dots \geq \lambda_b$, then λ is a *partition* of n (notation $\lambda \vdash n$). We sometimes use the other standard notation $\lambda = 1^{l_1} 2^{l_2} \dots$ for a partition λ , to indicate that λ has l_1 parts equal to 1, l_2 parts equal to 2, etc. Finally if s is a positive integer, $\lambda \cdot s$ will denote the partition $1^{l_1 s} 2^{l_2 s} \dots$ of $n s$.²

If d and r are positive integers and $X \in GL_d(k)$, then X_r denotes the matrix

$$1.1 \quad X_r = \begin{bmatrix} X & X & 0 & \cdots & 0 & 0 \\ 0 & X & X & \cdots & 0 & 0 \\ 0 & 0 & X & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & X & X \\ 0 & 0 & 0 & \cdots & 0 & X \end{bmatrix} \quad (r \text{ diagonal blocks } X),$$

which is an element of $G_{dr}(k)$. If $\sigma = (\sigma_1, \dots, \sigma_b)$ is a partition of a positive integer e , then $X_{(\sigma)}$ denotes the matrix

² This should not be confused with the partition $s \cdot \pi = s^{p_1} (2s)^{p_2} \dots$ defined in [GL], p.435.

$$1.2 \quad X_{(\sigma)} = X_{\sigma_1} \oplus \dots \oplus X_{\sigma_b}$$

(\oplus means “diagonal sum” of matrices); $X_{(\sigma)}$ is an element of $G_{de}(k)$.

Let $\Phi(k)$ be the set of all monic irreducible polynomials $f(t)$ over k , excepting $f(t) = t$. The degree of $f = f(t)$ is denoted $d(f)$.

For any $f \in \Phi(k)$, $C(f)$ denotes a matrix in $GL_{d(f)}(k)$ having characteristic polynomial f . This determines $C(f)$ only up to conjugacy in $G_{d(f)}(k)$, but this will be sufficient for our purposes.

Definition If $d(f) = d$ divides n , and if $\sigma = (\sigma_1, \dots, \sigma_b)$ is a partition of $e = n/d$, let f^σ be the conjugacy class of $G_n(k)$ which contains the matrix $C(f)_{(\sigma)}$. Every element g of f^σ is primary, and conjugacy classes of form f^σ are called *primary classes*. Every conjugacy class of $G_n(k)$ can be written uniquely as $\bigoplus_{f \in \Phi(k)} f^{\sigma(f)}$, where the partitions $\sigma(f)$ satisfy $\sum_{f \in \Phi(k)} d(f)|\sigma(f)| = n$. In this work we deal only with primary classes.

Jordan factorization An element $g \in G_n(k)$ is *unipotent* of type $\sigma \vdash n$ if it is conjugate in $G_n(k)$ to the matrix $(1)_{(\sigma)}$. Notice that g is unipotent if and only if it has p -power order, i.e. is a p -element. An element g is *semisimple* if it has order prime to p , i.e. is a p' -element. A primary element $g \in G_n(k)$ is semisimple if and only if it is conjugate in $G_n(k)$ to an element of the form $C(f)_{(1^{n/d})}$ for some $f \in \Phi(k)$ of degree d dividing n . Each element $g \in G_n(k)$ has a unique factorization $g = g_p g_{p'} = g_{p'} g_p$ as commuting product of a semisimple element $g_{p'}$ and a unipotent element g_p (see [St], p.25). We call $g = g_{p'} g_p$ the *Jordan factorization* of g , and call $g_{p'}$ and g_p the semisimple and unipotent parts, respectively, of g . The semisimple and unipotent parts of g are both powers of g . The Jordan factorization of the element $C(f)_{(\sigma)}$ in the definition above is $C(f)_{(\sigma)} = C(f)_{(1^{n/d})} (I_d)_{(\sigma)}$, notice that the matrix $(I_d)_{(\sigma)}$ is unipotent, because it is conjugate to $(1)_{\sigma \cdot d}$.

Definition of $J_n(\psi)$ Let ψ be any element of \hat{M}_n ; this will be fixed from now on. If d is a divisor of n , we often identify ψ with the element $\psi|_{M_d}$ of \hat{M}_d . Define the class-function $J_n(\psi)$ on $G_n(k)$ as follows. If $c \in \text{ccl } G_n(k)$ is not primary, then $J_n(\psi)\{c\} = 0^3$. If f^σ is the primary class described above, with $d = d(f)$ and σ a partition of $e = n/d$, then

$$1.3 \quad J_n(\psi)\{f^\sigma\} = \psi(f) \cdot k(\sigma:q^d),$$

where the symbols $\psi(f)$ and $k(\sigma:T)$ have the following meanings. If $y \in k_d$ is a zero of $f(t)$, so that $f(t) = (t-y)(t-y^q)\dots(t-y^{q^{d-1}})$, then we define $\psi(f) := \psi(y) + \psi(y^q) + \dots + \psi(y^{q^{d-1}})$. If $\sigma = (\sigma_1, \dots, \sigma_b)$ is any partition, then the polynomial $k(\sigma:T) \in \mathbf{Z}[T]$ is defined to be $(1-T)(1-T^2)\dots(1-T^{b-1})$ if σ has $b > 1$ parts, and to be 1 if $b = 1$.

$J_n(\psi)$ is a generalized character of $G_n(k)$ for any $\psi \in \hat{M}_n$, and if ψ is *primitive* (or is *in general position*; this means that $\psi, \psi^q, \dots, \psi^{q^{n-1}}$ are distinct elements of \hat{M}_n) then $(-1)^{n-1} J_n(\psi)$ is irreducible [GL, pp.431, 433, 430]. The distinct irreducible characters which you get by taking all primitive $\psi \in \hat{M}_n$ comprise the discrete series for $G_n(k)$. However in the rest of this paper ψ will be an arbitrary element of \hat{M}_n .

2 The subgroups $\boxed{H_{d,n}(k)}$ of $\boxed{G_n(k)}$

A class-function F on $G_n(k)$ is said to *have primary support* if $F\{c\} \neq 0$ implies that the class c is primary. A *primary subgroup* H of $G_n(k)$ is one whose elements all lie in primary classes of $G_n(k)$. Clearly $J_n(\psi)$ has primary support, and any character of $G_n(k)$, which is induced from a character of a primary subgroup H , has primary support. In this section we define a family $\mathbf{F}(n)$ of primary subgroups of $G_n(k)$, and we show later that $J_n(\psi)$ can be expressed as a \mathbf{Z} -linear combination of characters induced from groups H of $\mathbf{F}(n)$.

³To avoid a confusing forest of parentheses, the value of a class-function F at a class c is given as $F\{c\}$; or sometimes as $F\{g\}$, where g is an element of c .

Let d be a positive integer. The field k_d may be regarded as a k -algebra. It becomes a (simple) left k_d -module by multiplication ($a \in k_d$ acts on $v \in k_d$ to give av). Then each k -basis $\{v_1, \dots, v_d\}$ of k_d provides a k -algebra monomorphism $j_d: k_d \rightarrow \text{Mat}_d(k)$, which takes $a \in k_d$ to the k -matrix (a_{ij}) given by the equations $av_j = \sum_i a_{ij}v_i$. If we use a different basis of k_d , then j_d is replaced by $\gamma \circ j_d: k_d \rightarrow \text{Mat}_d(k)$, where γ is conjugation by some element of $G_d(k)$.

Now let e be a positive integer. The map $j_d: k_d \rightarrow \text{Mat}_d(k)$ induces a group monomorphism $G_e(k_d) \hookrightarrow G_{de}(k)$ which takes $(b_{ij}) \rightarrow (j_d(b_{ij}))$; we denote this also by j_d .

For any field K , let $Z_e(K)$ and $P_e(K)$ denote, respectively, the centre of $G_e(K)$ and the upper unitriangular subgroup of $G_e(K)$. Let $H_e(K)$ be the group $Z_e(K)P_e(K)$ (this is, of course, the direct product of $Z_e(K)$ and $P_e(K)$).

Now suppose that $d \mid n$, and that $e = \frac{n}{d}$. Then we define $Z_{d,n}(k)$, $P_{d,n}(k)$, $H_{d,n}(k)$ and $G_{d,n}(k)$ to be the images under the map $j_d: G_e(k_d) \hookrightarrow G_n(k)$ of $Z_e(k_d)$, $P_e(k_d)$, $H_e(k_d)$ and $G_e(k_d)$ respectively.

Examples If $d = 1$, $e = n$ we take the monomorphism $j_1: G_n(k) \hookrightarrow G_n(k)$ to be the identity map, so that $Z_{1,n}(k) = Z_n(k)$, $P_{1,n}(k) = P_n(k)$ and $H_{1,n}(k) = H_n(k)$. If $d = n$, $e = 1$ then $P_{n,n}(k) = \{1\}$, and $Z_{n,n}(k) = H_{n,n}(k)$ is the image of $j_n: k_n^\times \hookrightarrow G_n(k)$, which is a “maximal torus” of $G_n(k)$ (see [M], p.273), and has order $|k_n^\times| = q^n - 1$.

Definition Let $F(n) = \{H_{d,n}(k) \mid d \text{ any positive divisor of } n\}$.

The set $F(n)$ has the following virtues, proved in the lemma below: (i) each member $H_{d,n}(k)$ of $F(n)$ is a primary subgroup of $G_n(k)$, and (ii) every primary class of $G_n(k)$ meets $H_{d,n}(k)$ for at least one divisor d of n .

2.1 Lemma (i) Let d be a divisor of n , $e = \frac{n}{d}$ and let $h \in H_{d,n}(k)$. Then the conjugacy class c of $G_n(k)$ which contains h has the form $c = f^\sigma$, where

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2.1a $m = d(f)$ divides d , and

2.1b There exists a partition $\pi \vdash e$ such that $\sigma = \pi \cdot \frac{d}{m}$ (for notation, see section 1).

(ii) Each primary class f^σ of $G_n(k)$ contains an element of $H_{d,n}(k)$, for $d = d(f)$.

Proof of (i) Each element of $H_e(k_d)$ has the form $x = \zeta \mathcal{I}_e \cdot u$, where $\zeta \in k_d^\times$ and $u \in P_e(k_d)$. Let $X = (\zeta) \in G_1(k_d)$. Then (using the Jordan normal form) x is conjugate in $G_e(k_d)$ to an element of the form $X_{(\pi)}$, where $\pi = (\pi_1, \dots, \pi_b)$ is some partition of e (we use here the notation 1.1, 1.2 of section 1). Therefore $h = j_d(x)$ is conjugate in $G_n(k)$ to $j_d(\zeta)_{(\pi)}$.

Now let m be the degree of ζ over k , and let $f \in \Phi(k)$ be the minimal polynomial of ζ over k . Then $m = d(f)$ divides d , and the k -subfield $k(\zeta)$ of k_d which is generated by ζ is isomorphic to k_m , hence is equal to k_m (because this is the only subfield of k_d of order q^m). As left k_m -module, k_d may be written as direct sum of $\frac{d}{m}$ submodules, each isomorphic to k_m . Therefore if we take a k -basis $\{v_1, \dots, v_d\}$ of k_d adapted to this direct sum decomposition, we can arrange that

$$2.2 \quad j_d(\zeta) = \begin{pmatrix} C(f) & 0 & \dots & 0 \\ 0 & C(f) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & C(f) \end{pmatrix} \in G_d(k),$$

where $C(f) \in G_m(k)$ has characteristic polynomial equal to f , and there are $\frac{d}{m}$ diagonal blocks $C(f)$. The reader will now be able to see that $j_d(\zeta)_{(\pi)}$ is conjugate in $G_n(k)$, by an element which permutes suitably the rows and columns of $j_d(\zeta)_{(\pi)}$, to a diagonal sum of $\frac{d}{m}$ copies of the matrix $C(f)_{(\pi)}$. This clearly lies in the conjugacy class f^σ , where $\sigma = \pi \cdot \frac{d}{m}$.

This proves part (i) of the lemma.

Proof of (ii) This comes very easily from what has just been proved. If f^σ is a primary class of $G_n(k)$, then $d = d(f)$ divides n , and $\sigma \vdash e = \frac{n}{d}$. Let ζ be a zero of f . Then $\zeta \in k_d$ and f is the minimum polynomial of ζ over k . Let $X = (\zeta) \in G_1(k_d)$ and let $x = X_{(\sigma)}$. This

is an element of $H_e(k_d)$. The proof of (i), where we now have $m=d$ and $\pi = \sigma$, shows that the class in $G_n(k)$ of $j_d(x)$ is $f^{\sigma^{-1}} = f^\sigma$. Hence this class meets $H_{d,n}(k)$.

3 The characters $X_{d,n}(\psi, \lambda)$ of $G_n(k)$ There is a bijection $\lambda \leftrightarrow J(\lambda)$ between the set of all compositions $\lambda = (n_1, \dots, n_b)$ of n , and the set of all subsets of the set $I = \{1, \dots, n-1\}$ (if $n = 1$, take $I = \emptyset$), as follows: if $\lambda = (n)$, then $J(\lambda) = I$, otherwise

$$J(\lambda) = I \setminus \{n_1, n_1 + n_2, \dots, n_1 + n_2 + \dots + n_{b-1}\}.$$

Notice that $J((1^n)) = \emptyset$.

For any field K and any $\lambda \models n$ let $\theta_\lambda : P_n(K) \rightarrow \mathbf{C}^\times$ denote the linear character of $P_n(K)$ which takes each (upper unitriangular) matrix $(a_{ij}) \in P_n(K)$ to $\omega_1(a_{12}) \dots \omega_{n-1}(a_{n-1,n})$, where $\omega_1, \dots, \omega_{n-1}$ are elements of the character group $\hat{K}^+ = \text{Hom}(K^+, \mathbf{C}^\times)$ which satisfy the condition

$$3.1 \quad \omega_j \neq 1 \text{ if and only if } j \in J(\lambda).$$

Examples $\theta_{(n)}$ is a *non-degenerate* character of $P_n(K)$, i.e. $\omega_j \neq 1$ for all $j \in I$. The induced character $\text{Ind}_{P_n(K)}^{G_n(K)}(\theta_{(n)})$ is called the *Gelfand–Graev* character of $G_n(K)$ (see section 6, also [DL], p.155 or [Ca], p.254). $\theta_{(1^n)}$ is the trivial (unit) character of $P_n(K)$.

Now let d be a divisor of n , and let $e = \frac{n}{d}$. Recall that $\psi : M_n \rightarrow \mathbf{C}^\times$ is a fixed character of $M_n = k_n^\times$. For each $\lambda \models e$ we define a linear character $\psi \cdot \lambda$ of $H_e(k_d) = Z_e(k_d) P_e(k_d)$ as follows: if $z = \zeta \cdot I_d \in Z_e(k_d)$ (so that $\zeta \in k_d^\times$), and if $a \in P_e(k_d)$, let $(\psi \cdot \lambda)(za) = \psi(\zeta) \theta_\lambda(a)$. Composing $\psi \cdot \lambda$ with the inverse of the map $j_d : H_e(k_d) \rightarrow H_{d,n}(k)$ we get a linear character of the subgroup $H_{d,n}(k)$ of $G_n(k)$, also denoted $\psi \cdot \lambda$.

Finally we make the

Definition $X_{d,n}(\psi, \lambda) := \text{Ind}_{H_{d,n}(k)}^{G_n(k)}(\psi \cdot \lambda)$.

Remarks (1) $X_{d,n}(\psi, \lambda)$ is independent of the choice of the ω_j , provided that these satisfy 3.1 (see section 6).

(2) If λ and λ' are compositions of the same integer, write $\lambda \approx \lambda'$ to mean that λ' can be obtained from λ by permuting its components. Each \approx -class contains exactly one partition. It will turn out that $X_{d,n}(\psi, \lambda) = X_{d,n}(\psi, \lambda')$ if $\lambda' \approx \lambda$ (see section 7). Therefore we may confine ourselves to $X_{d,n}(\psi, \lambda)$ for which $\lambda \vdash n$.

The main result of this paper is that $J_n(\psi)$ is a \mathbf{Z} -linear combination of these induced characters $X_{d,n}(\psi, \lambda)$ of $G_n(k)$. More precisely, we have the following theorem, whose proof will occupy the rest of this paper.

3.2 Theorem *There exist polynomials $r_\lambda(T) \in \mathbf{Z}[T]$, one for each partition λ , such that for each positive integer n , for each field k of finite order q , and for each $\psi \in \hat{M}_n = \text{Hom}(M_n, \mathbf{C}^\times)$ there holds*

$$3.3 \quad J_n(\psi) = \sum_{d|n} \sum_{\lambda \vdash nd} r_\lambda(q^d) X_{d,n}(\psi, \lambda).$$

The polynomials $r_\lambda(T)$ are determined uniquely by the equations (3.3)

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4 Some properties of the characters $X_{d,n}(\psi, \lambda)$

Since $X_{d,n}(\psi, \lambda)$ is an induced character from the subgroup $H_{d,n}(k)$ of $G_n(k)$, its value at a class c of $G_n(k)$ is zero unless c contains an element of $H_{d,n}(k)$, i.e. unless c has the form f^σ , where f and σ satisfy conditions 2.1a, 2.1b (see lemma 2.1). This proves statement (i) in the next proposition; the proof of statement (ii) and the definition of the polynomials $x_{\lambda, \pi, l}(T)$ will be deferred to section 7.

4.1 Proposition For each pair λ, π of partitions for which $|\lambda|=|\pi|$, and for each positive integer l , there exists a polynomial $x_{\lambda, \pi, l}(T) \in \mathbf{Z}[T]$, such that for all positive integers d, n with $d | n$, for all partitions $\lambda \vdash e = n/d$ and for all $c \in \text{ccl}G_n(k)$, there hold

- (i) $X_{d, n}(\psi, \lambda) \{c\} = 0$ unless $c = f^\sigma$, where f, σ satisfy conditions 2.1a and 2.1b, and
- (ii) If f, σ satisfy conditions 2.1a and 2.1b, so that $m = d(f)$ divides d , and there exists $\pi \vdash e$ such that $\sigma = \pi \cdot d/m$, then $X_{d, n}(\psi, \lambda) \{f^\sigma\} = \psi(f) \cdot x_{\lambda, \pi, d/m}(q^m)$.

It is sometimes convenient to augment the definition of $x_{\lambda, \pi, l}(T)$ by making the convention: for any partitions λ, σ and any $l \in \mathbf{Q}$, $x_{\lambda, \sigma/l, l}(T)$ is zero unless $l \in \mathbf{Z}$ and there exists π such that $\sigma = \pi \cdot l$, in which case $x_{\lambda, \sigma/l, l}(T) := x_{\lambda, \pi, l}(T)$. Then we have a “short” version of proposition 4.1, namely

4.1a If $d | n$, $\lambda \vdash n/d$ and $f^\sigma \in \text{ccl}G_n(k)$ then $X_{d, n}(\psi, \lambda) \{f^\sigma\} = \psi(f) \cdot x_{\lambda, \sigma/l, l}(q^{d(f)})$, where $l = d/f$.

In particular we see that $X_{1, n}(\psi, \lambda) \{f^\sigma\} = 0$ if $d(f) \neq 1$, while if $d(f) = 1$, i.e. if $f(t) = t - x$ for some $x \in k^\times$, then

$$4.2 \quad X_{1, n}(\psi, \lambda) \{(t-x)^\sigma\} = \psi(x) \cdot x_{\lambda, \sigma, 1}(q), \text{ for all } \lambda, \sigma \vdash n \text{ and all } x \in k^\times.$$

We shall see later (section 7) that the polynomials $x_{\lambda, \pi, l}(T)$ are determined in a simple way by the $x_{\lambda, \pi, 1}(T)$. If we take $\psi = 1$ in 4.2 we get

$$4.3 \quad X_{1, n}(1, \lambda) \{(t-x)^\sigma\} = x_{\lambda, \sigma, 1}(q), \text{ for all } \lambda, \sigma \vdash n \text{ and all } x \in k^\times.$$

For each $\mu \vdash n$ there is an irreducible character I^μ of $G_n(k)$, denoted $I_1^0[\mu]$ in [GL], p.437, and first discovered by R. Steinberg (see [S], p. 275. In Steinberg’s notation, $I^\mu = \Gamma(\nu)$, where $\nu = (\mu_n, \mu_{n-1}, \dots, \mu_1)$). We have the following important relation between the $X_{1, n}(1, \lambda)$ and the I^μ . Let \bullet, \circledast denote the usual scalar product on class-functions on $G_n(k)$ (see 8.3).

4.4 Proposition *Let $W(\lambda, \mu) = \bullet X_{1,n}(1, \lambda)$, $I^\mu \otimes$, for any $\lambda, \mu \vdash n$. Then $W(\lambda, \mu) \in \mathbf{Z}$ is independent of k (i.e. of q), and the matrix $(W(\lambda, \mu))_{\lambda, \mu \vdash n}$ is unimodular.*

It will be useful to record here some information about the character I^μ . From its definition as $I^\mu = I_1^0[\mu]$, using [GL], p.441, lemma 8.2 together with [GL], p.423, definition (18), we may verify the first equality in

$$4.5 \quad I^\mu \{(t-x)^\pi\} = \sum_{\rho \vdash n} \frac{1}{z_\rho} Q_\rho^\pi(q) \cdot \chi_\rho^\mu = q^{n(\pi)} K_{\mu\pi}(q^{-1}),$$

where the $Q_\rho^\pi[T] \in \mathbf{Z}[T]$ are certain polynomials introduced in [GL]—some of whose properties we shall recall in section 8—and χ_ρ^μ is standard notation (see e.g. [Le] or [M]) for the value at class ρ of the irreducible character χ^μ of the symmetric group $S(n)$. The second equality in 4.5 comes by applying [M], p.248, (7.11), and using the orthogonality relations for the characters of $S(n)$. The polynomials $K_{\mu\sigma}(T) \in \mathbf{Z}[T]$ are defined in [M], p.239; the expression $T^{n(\sigma)} K_{\mu\sigma}(T^{-1})$ is also a polynomial in $\mathbf{Z}[T]$, see [M], p.248.

4.6 Proposition *With n given, $\{X_{d,n}(1, \lambda) \mid d \mid n, \lambda \vdash n/d\}$ is a linearly independent set of class-functions on $G_n(k)$.*

Propositions 4.4 and 4.6 will be proved in section 9.

5 Proof of theorem 3.2

In this section we prove theorem 3.2, on the assumption that the propositions in section 4 are true. It is clear that equation 3.3 holds for $n = 1$ in any case, by taking $r_{(1)}(T) = 1$. For we have $G_1(k) = k^\times = M_1$ and $J_1(\psi) = \psi|_{M_1} = X_{1,1}(\psi, (1))$.

We proceed by induction on n . Suppose that $n > 1$, and that we have already defined polynomials $r_\lambda(T) \in \mathbf{Z}[T]$ for all $\lambda \vdash \bar{n}$ and all $\bar{n} < n$ in such a way that 3.3 holds, for any appropriate ψ and k , with n replaced by any $\bar{n} < n$. To prove theorem 3.2 for n , we must

show that there exist $r_\lambda(T) \in \mathbf{Z}[T]$ for all $\lambda \vdash n$ so that 3.3 holds using these new $r_\lambda(T)$ (together, of course, with the $r_\lambda(T)$ already defined).

Let s be a divisor of n . Then we define the class-function $R_s(k)$ on $G_n(k)$ by

$$5.1 \quad R_s(k) = J_n(\psi) - \sum_{s|d|n} \sum_{\lambda \vdash n/d} r_\lambda(q^d) X_{d,n}(\psi, \lambda),$$

where the first sum is over all divisors d of n which are divisible by s . Notice that theorem 3.2 is equivalent to the statement that polynomials $r_\lambda(T)$ exist, such that $R_1(k) = 0$ for all n , ψ and k .

5.2 Lemma *Let $s \neq 1$ be a divisor of n . Then $R_s(k) \{f^\sigma\} = 0$ for all class-functions f^σ of $G_n(k)$ such that $s \mid d(f)$.*

Proof Let f^σ be a class of $G_n(k)$ as described, and let $m = d(f)$. By 1.3 and 4.1a,

$$R_s(k) \{f^\sigma\} = \psi(f) U_s(k), \text{ where}$$

$$5.3 \quad U_s(k) = k(\sigma : q^m) - \sum_{s|d|n} \sum_{\lambda \vdash n/d} r_\lambda(q^d) x_{\lambda, \sigma/(d/m), d/m}(q^m).$$

Notice that s divides all the integers n , m , d appearing in 5.3. Write $\bar{n} = n/s$, $\bar{m} = m/s$, $\bar{d} = d/s$. Take any $\bar{f} \in \Phi(k_s)$ of degree $d(\bar{f}) = \bar{m}$ (for example, we could take \bar{f} to be the minimal polynomial over k_s , of an element $\eta \in k_d$ whose minimum polynomial over k is f), and consider the class \bar{f}^σ of $G_{\bar{n}}(k_s)$. The class-function $R_1(k_s)$ on $G_{\bar{n}}(k_s)$ is zero by our induction hypothesis. On the other hand, the analogue of 5.3 gives us, writing $\bar{q} = q^s = |k_s|$,

$$5.4 \quad U_1(k_s) = k(\sigma : \bar{q}^{\bar{m}}) - \sum_{\bar{d}|\bar{n}} \sum_{\lambda \vdash \bar{n}/\bar{d}} r_\lambda(\bar{q}^{\bar{d}}) x_{\lambda, \sigma/(\bar{d}/\bar{m}), \bar{d}/\bar{m}}(\bar{q}^{\bar{m}}).$$

But it is clear that $U_1(k_s) = U_s(k)$. Since $0 = R_1(k_s) = \bar{\psi}(f) \cdot U_1(k_s)$ holds for any $\bar{\psi} \in \hat{M}_n$ (including $\bar{\psi} = 1$) we have $0 = U_1(k_s) = U_s(k)$, and so $R_s(k)\{f^\sigma\} = \psi(f) U_s(k)$ is zero, which proves the lemma.

Next we define a class-function $B_n(\psi)$ on $G_n(k)$ by

$$5.5 \quad B_n(\psi) = J_n(\psi) - \sum_{d|n, d \neq 1} \sum_{\lambda \vdash n/d} r_\lambda(q^d) X_{d,n}(\psi, \lambda).$$

5.6 Lemma $B_n(\psi)$ is zero on all classes f^σ of $G_n(k)$ for which $d(f) \neq 1$.

Proof Suppose f^σ is a class on $G_n(k)$ for which $d(f) = s \neq 1$. Then by proposition 4.1(i), $X_{d,n}(\psi, \lambda)\{f^\sigma\} = 0$, for all $d|n$ and $\lambda \vdash n/d$ such that s does not divide d . Therefore $B_n(\psi)\{f^\sigma\} = R_s(k)\{f^\sigma\}$, which is zero by lemma 5.2. ◀

In order to complete the proof of theorem 3.2, we must construct polynomials $r_\lambda(T) \in \mathbf{Z}[T]$ such that

$$5.7 \quad B_n(\psi) = \sum_{\lambda \vdash n} r_\lambda(q) X_{1,n}(\psi, \lambda)$$

for all $\psi \in \hat{M}_n$. It is enough that 5.7 should hold for $\psi=1$. For by 1.3 and 4.1, we have $B_n(\psi)\{(t-x)^\sigma\} = \psi(x) \cdot B_n(1)\{(t-1)^\sigma\}$ and $X_{1,n}(\psi, \lambda)\{(t-x)^\sigma\} = \psi(x) \cdot X_{1,n}(1, \lambda)\{(t-1)^\sigma\}$, for all x and σ , and both sides of 5.7 are zero on all classes f^σ of $G_n(k)$ with $d(f) \neq 1$ (see proposition 4.1(i)). Define the class function $B = B_n(1)$. From 1.3 and 4.1a we get, for all $x \in k^\times$ and $\sigma \vdash n$,

$$5.8 \quad B\{(t-x)^\sigma\} = B\{(t-1)^\sigma\} = k(\sigma : q) - \sum_{d|n, d \neq 1} \sum_{\lambda \vdash n/d} r_\lambda(q^d) x_{\lambda, \sigma/d, d}(q),$$

Using the notation \bullet , \otimes for the scalar product on class-functions on $G_n(k)$ (see 8.3) we have by 8.4 the following lemma.

5.9 Lemma *If the class-function F on $G_n(k)$ is zero on all classes f^σ with $d(f) \neq 1$, and satisfies $F\{(t-x)^\sigma\} = F\{(t-1)^\sigma\}$ for all $x \in k^\times$ and $\sigma \vdash n$, then*

$$5.10 \quad \bullet F, I^\mu \otimes = (q-1) \sum_{\sigma \vdash n} \frac{1}{a_\sigma(q)} \cdot F\{(t-1)^\sigma\} \cdot I^\mu \{(t-1)^\sigma\},$$

for all $\mu \vdash n$.

5.11 Corollary *If F is as above, and if $\bullet F, I^\mu \otimes = 0$ for all $\mu \vdash n$, then $F = 0$.*

Proof By [M], p.239 the matrix $(q^{n(\sigma)} K_{\mu, \sigma}(q^{-1}))_{\mu, \sigma \vdash n}$ is non-singular, hence by 4.5 the matrix $(I^\mu \{(t-1)^\sigma\})_{\mu, \sigma \vdash n}$ is non-singular. So $\bullet F, I^\mu \otimes = 0$ for all $\mu \vdash n \iff F\{(t-1)^\sigma\} = 0$ for all $\sigma \vdash n$ (see 5.10) $\square F = 0$.

Now we define, for each $\lambda \vdash n$,

$$5.12 \quad r_\lambda(k) = \sum_{\mu \vdash n} \bullet B, I^\mu \otimes V(\mu, \lambda),$$

where $(V(\lambda, \mu))$ is the inverse of the matrix $(W(\lambda, \mu))$ of proposition 4.4.

5.13 Lemma $B = \sum_{\lambda \vdash n} r_\lambda(k) X_{1,n}(1, \lambda)$.

Proof Let S denote the right side of the equation above. We check immediately from 4.4 that $\bullet S, I^\tau \otimes = \bullet B, I^\tau \otimes$ for all $\tau \vdash n$. Hence $S = B$ by corollary 5.11.

We must now show that each coefficient $r_\lambda(k)$ defined by 5.12 “belongs to $\mathbf{Z}[q]$ ” in the sense that there exists a polynomial $r_\lambda(T) \in \mathbf{Z}[T]$ such $r_\lambda(k) = r_\lambda(q)$, for each field k of order q . By 5.12 and 4.4, it will be enough to prove that each $\bullet B, I^\mu \otimes$ “belongs to $\mathbf{Z}[q]$ ” in this sense. We may apply lemma 5.9 to $F = B$. Then 5.10 gives

$$5.14 \quad \bullet B, I^\mu \otimes = (q-1) \sum_{\sigma \vdash n} \frac{1}{a_\sigma(q)} \cdot B\{(t-1)^\sigma\} \cdot I^\mu \{(t-1)^\sigma\},$$

for all $\mu \vdash n$. But 5.8 shows that $B\{(t-1)^\sigma\}$ “belongs to $\mathbf{Z}[q]$ ”, because $k(\sigma : T)$, $r_\lambda(T^d)$ and $x_{\lambda, \sigma/d, d}(T)$ all belong to $\mathbf{Z}[T]$, for all divisors $d \neq 1$ of n and all $\lambda \vdash n/d$. Also

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$I^\mu \{(t-1)^\sigma\} = q^{n(\sigma)} K_{\mu\sigma}(q^{-1})$ “belongs to $\mathbf{Z}[q]$ ”, see the end of section 4. Of course we have in 5.14 denominators $a_\sigma(q)$. But the polynomials $a_\sigma(T)$ lie in $\mathbf{Z}[T]$ and are monic (see section 8), and we know that $\bullet B, I^\mu \otimes \in \mathbf{Z}$ for fields k of all p -power orders q , because B is a generalized character of $G_n(k)$ (see 5.5) and I^μ is a character of $G_n(k)$. Therefore we deduce that $\bullet B, I^\mu \otimes$ “belongs to $\mathbf{Z}[q]$ ” from 5.14 and the following elementary lemma (whose proof we leave to the reader).

5.15 Lemma *Let $\alpha(T)$ and $\beta(T)$ belong to $\mathbf{Z}[T]$, with $\beta(T)$ monic. Let $\kappa(T) = \frac{\alpha(T)}{\beta(T)}$, and suppose that $\kappa(q) \in \mathbf{Z}$ for infinitely many distinct integers q . Then $\kappa(T) \in \mathbf{Z}[T]$.*

We have now proved 5.7, hence that equations 3.3 hold. It remains to prove that the $r_\lambda(T)$ are determined uniquely by these equations. But this follows from the case $\psi = 1$ of 3.3, together with proposition 4.6.

Remark We can prove that $\bullet B, I^\mu \otimes \in \mathbf{Z}$ for all k , without appealing to fact that $J_n(\psi)$ (and in particular $J_n(1)$) is a generalized character. For it is easy to check that $\bullet J_n(\psi), I^\mu \otimes \in \mathbf{Z}$ by direct calculation, using the definition 1.3 of $J_n(\psi)$. Of course $\bullet X_{d,n}(1, \lambda), I^\mu \otimes \in \mathbf{Z}$ for all $d | n$ and $\lambda \vdash n/d$, because the $X_{d,n}(1, \lambda)$ are characters of $G_n(k)$ by definition. We then have $\bullet B, I^\mu \otimes \in \mathbf{Z}$ as before, from the definition 5.5 of $B_n(\psi)$.

Therefore theorem 3.2 provides a proof (even if rather indirect!) that the functions $J_n(\psi)$ are generalized characters. But, as Robert Steinberg has remarked, this could be proved by a direct application to $J_n(\psi)$ of Brauer’s characterization of characters.

6 Gelfand-Graev character for $G_n(k)$, I

The Gelfand–Graev character Γ_n of $G_n(k)$ is by definition the induced character $\text{Ind}_P^G(\theta_n)$, where $G = G_n(k)$, $P = P_n(k)$, and θ_n is any non-degenerate linear character of P , see 3.1. (Γ_n is independent of the characters $\omega_j \in \hat{k}^+$, provided these are all $\neq 1$; see

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[C], p.254.) Clearly $\Gamma_n \{c\}$ is zero, unless the conjugacy class c meets P , i.e. unless $c = (t-1)^\pi$ for some $\pi \mid n$. For brevity, we shall henceforth write $F\{\pi\}$ for $F\{(t-1)^\pi\}$, for any class-function F on G .

Deligne and Lusztig have discovered an important property of Γ_n , which they prove for a large class of finite reductive groups ([DL], p. 155, Prop. 10.3). In our case, Deligne–Lusztig’s result may be written

$$6.1 \quad \sum_{J \subset I} (-1)^{|J|} \Gamma_J = \Delta_n,$$

where the class-function Δ_n on G is given by

$$6.2 \quad \Delta_n \{c\} = 0 \text{ unless } c = (t-1)^{(n)}, \text{ and } \Delta_n((t-1)^{(n)}) = \Delta_n((n)) = q^{n-1}(q-1).$$

To define Γ_J , we recall (section 3) that to each subset J of $I = \{1, \dots, n-1\}$ is associated a composition λ of n , which we shall here denote $\lambda = (n_1, \dots, n_b)$. To this is associated the parabolic subgroup $P(J)$ of G consisting of all matrices

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1b} \\ 0 & A_{22} & \dots & A_{2b} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_{bb} \end{pmatrix}$$

of $G = G_n(k)$ such that $A_{jj} \in G_{n_j}(k)$ ($j = 1, \dots, b$). If χ_j is a character on $G_{n_j}(k)$ ($j = 1, \dots, b$), define the character $\chi_1 \circ \dots \circ \chi_b$ on G to be $\text{Ind}_{P(J)}^G(\chi)$, where χ is the character $\chi(A) = \chi_1(A_{11}) \dots \chi_b(A_{bb})$ on $P(J)$. We recall from [GL], p. 411 that this “circle-product” \circ is multilinear, associative and commutative. It is easy to see that the character denoted $\text{Ind}_{P(J)^F}^{G^F}(\Gamma_{L(J)})$ in [DL], (10.3.1), is in our notation,

$$6.3 \quad \Gamma_J := \Gamma_{n_1} \circ \dots \circ \Gamma_{n_b}.$$

To calculate \circ products like this, we have the formula (see [GL], p.410):

if F_j is a class-function on $G_{n_j}(k)$ ($j=1, \dots, b$) then for all $\pi \vdash n = n_1 + \dots + n_b$

$$6.4 \quad (F_1 \circ \dots \circ F_b)\{\pi\} = \sum g_{\pi_1 \dots \pi_b}^\pi(q) F_1\{\pi_1\} \dots F_b\{\pi_b\},$$

where the sum is over all rows π_1, \dots, π_b of partitions of n_1, \dots, n_b respectively, and the integer $g_{\pi_1 \dots \pi_b}^\pi(q)$ is the value at $T = q$ of Hall's polynomial $g_{\pi_1 \dots \pi_b}^\pi(T) \in \mathbf{Z}[T]$ (see [M], p.188).

6.5 Lemma *For each $J \subset I = \{1, \dots, n-1\}$ and each $\pi \vdash n$ there exists a polynomial $c_{J,\pi}(T) \in \mathbf{Z}[T]$ such that $\Gamma_J\{\pi\} = (q-1) c_{J,\pi}(q)$ for all fields k of order q .*

Proof Deligne and Lusztig ([DL], p. 155, 10.4) give the following formula, a “dual” to 6.1,

$$6.6 \quad \Gamma_n = \sum_{J \subset I} (-1)^{|J|} \Delta_J,$$

where $\Delta_J = \Delta_{n_1} \circ \dots \circ \Delta_{n_b}$. By 6.2, there is, for each positive integer n , a polynomial $d_\pi(T) \in \mathbf{Z}[T]$ such that $\Delta_n\{\pi\} = (q-1) d_\pi(q)$ for all fields k of order q . Therefore by 6.4 there is, for each $J \subset I$ and each $\pi \vdash n$, a polynomial $d_{J,\pi}(T)$ such that $\Delta_J\{\pi\} = (\Delta_{n_1} \circ \dots \circ \Delta_{n_b})\{\pi\} = (q-1) d_{J,\pi}(q)$ for all fields k of order q . Then 6.6 shows that $c_\pi(T) = \sum_{J \subset I} (-1)^{|J|} d_{J,\pi}(T) \in$

$\mathbf{Z}[T]$ has the property that $\Gamma_n\{\pi\} = (q-1) c_\pi(q)$, and we may use this, together with 6.4 again, to construct the polynomials $c_{J,\pi}(T)$ required by the lemma.

Remark We often write $c_{J,\pi}(T) = c_{\lambda,\pi}(T)$, if J and λ are related as in section 3.

It is clear that the property $\Gamma_J\{\pi\} = (q-1) c_{J,\pi}(q)$, i.e.

$$6.7 \quad c_{\lambda,\pi}(q) = \frac{1}{q-1} (\Gamma_{n_1} \circ \dots \circ \Gamma_{n_b})\{\pi\},$$

defines the polynomial $c_{\lambda,\pi}(T)$ uniquely. Because the \circ product is commutative, $c_{\lambda,\pi}(T) = c_{\lambda',\pi}(T)$ whenever $\lambda' \approx \lambda$ (see section 3, remark (2)).

7 Proof of proposition 4.1(ii)

We want to calculate the values of the character $X_{d,n}(\psi, \lambda) = \text{Ind}_{H_{d,n}(k)}^{G_n(k)}(\psi \cdot \lambda)$ defined in section 3, and we start with the special case $d = 1$. According to the definitions in section 2, $H_{1,n}(k) = H_n(k)$, $Z_{1,n}(k) = Z_n(k)$ and $P_{1,n}(k) = P_n(k)$ (we identify $\text{Mat}_1(k)$ with k , so that j_1 is the identity map). Write these groups H , Z and P for short, and write $G_n(k) = G$. Observe that $H = ZP$ has order $(q-1)|P|$.

7.1 Lemma *For all partitions $\lambda, \pi \vdash n$ and for all $x \in k^\times$*

$$7.2 \quad X_{1,n}(\psi, \lambda)\{(t-x)^\pi\} = \psi(x) \cdot c_{\lambda, \pi}(q).$$

Proof Let u be an element of the class $(t-1)^\pi$. Then $g = xI_n \cdot u$ is an element of the class $(t-x)^\pi$, and we have $X_{1,n}(\psi, \lambda)\{(t-x)^\pi\} = \text{Ind}_{ZP}^G(\psi \cdot \lambda)\{g\} = \frac{1}{|ZP|} \sum_{s \in G, s^{-1}gs \in H} (\psi \cdot \lambda)(s^{-1}gs) = \frac{\psi(x)}{q-1} \text{Ind}_P^G(\theta_\lambda)\{(t-1)^\pi\}$, because $s^{-1}gs = xI_n \cdot s^{-1}us$, hence

$s^{-1}gs \in H$ if and only if $s^{-1}us \in P$, and in that case $(\psi \cdot \lambda)(s^{-1}gs) = \psi(x) \cdot \theta_\lambda(s^{-1}us)$.

To calculate $\text{Ind}_P^G(\theta_\lambda)\{(t-1)^\pi\}$, notice that from the definition of θ_λ (section 3), and in the notation of section 6, $\text{Ind}_P^{P(J)}(\theta_\lambda)$ takes $A \in P(J)$ to $\Gamma_{n_1}(A_{11}) \dots \Gamma_{n_b}(A_{bb})$. Therefore

$$7.3 \quad \text{Ind}_P^G(\theta_\lambda) = \Gamma_{n_1} \circ \dots \circ \Gamma_{n_b},$$

and now lemma 7.1 follows from 6.7.

Now consider the situation of proposition 4.1(ii). We have a divisor d of n , partitions λ, π of $e = n/d$, and a polynomial $f \in \Phi$ of degree m which divides d . Let $\sigma = \pi \cdot d/m$, so that $\sigma \vdash n/m$ and f^σ is a conjugacy class of $G_n(k)$. From the proof of 2.1(i) we know that f^σ contains an element $h = j_d(\zeta)_{(\pi)} \in H_{d,n}(k)$, where $\zeta \in k_m^\times$ is a zero of $f(t) = (t-\zeta)(t-\zeta^q) \dots (t-\zeta^{q^{m-1}})$. The semisimple part of $(\zeta)_{(\pi)}$ is $(\zeta)_{(1^{nd})}$, and its unipotent part is $(1)_{(\pi)}$. The following lemma is an elementary consequence of this, together with the discussion following 2.2.

7.4 Lemma *If $h = h_{p'} h_p$ is the Jordan decomposition of h , then (i) $h_{p'} = j_d((\zeta)_{1^{nd}}) = C(f)_{(1^{nd} m)}$ satisfies the equation $f(h_{p'}) = 0$, and the k -algebra generated by $h_{p'}$ (in $\text{Mat}_n(k)$) is a field; (ii) $h_p = j_d(1)_{(\pi)}$ is unipotent of type $\sigma \cdot m = \pi \cdot d$.*

It is clear that

$$7.5 \quad X_{d,n}(\psi, \lambda) \{f^\sigma\} = \text{Ind}_{G_{d,n}(k)}^{G_n(k)}(Y)\{h\},$$

where

$$7.6 \quad Y := \text{Ind}_{H_{d,n}(k)}^{G_{d,n}(k)}(\psi \cdot \lambda).$$

From the standard definition of induced character we have

$$7.7 \quad \text{Ind}_{G_{d,n}(k)}^{G_n(k)}(Y)\{h\} = \frac{1}{|G_{d,n}(k)|} \sum_{s \in \Theta} Y(s^{-1}hs),$$

where

$$7.8 \quad \Theta = \{s \in G_n(k) \mid s^{-1}hs \in G_{d,n}(k)\}.$$

7.9 Lemma *If $s \in \Theta$, then $(s^{-1}hs)_{p'} = (h_{p'})^{q^j}$ for some $j \in \{0, \dots, m-1\}$.*

Proof Since $s^{-1}hs \in G_{d,n}(k)$, then also $(s^{-1}hs)_{p'} \in G_{d,n}(k)$. Let $z \in G_e(k_d)$ be such that $(s^{-1}hs)_{p'} = s^{-1}h_p s = j_d(z)$. By 7.4(i), z satisfies $f(z) = 0$, and the k -subalgebra (of $\text{Mat}_e(k_d)$) generated by z is a field. But this means that the minimum polynomial of z over k_d is irreducible, and it divides $f(t) = (t - \zeta)(t - \zeta^q) \dots (t - \zeta^{q^{m-1}})$. It follows that $z = a \cdot \zeta^{q^j}$, for some $a \in k_d^\times$ and some $j \in \{0, \dots, m-1\}$. But since $j_d(z)$ has the same eigenvalues as $h_{p'} = j_d(\zeta)_{1^{nd}}$, we must have $a = 1$.

Since the elements ζ^{q^j} ($j = 0, \dots, m-1$) are distinct, we deduce from this lemma that $\Theta = \Theta_0 \cup \dots \cup \Theta_{m-1}$ (disjoint union), where

$$7.10 \quad \Theta_j = \{s \in G_n(k) \mid s^{-1}hs \in G_{d,n}(k), (s^{-1}hs)_{p'} = h_{p'}^{q^j}\}.$$

7.11 Lemma Let F denote the Frobenius endomorphism $(a_{ij}) \rightarrow (a_{ij}^q)$ on $G_e(k_d)$. Then there exists a matrix $M \in G_n(k)$ such that

$$7.12 \quad M \cdot j_d(a) \cdot M^{-1} = j_d(a^F) \text{ for all } a \in G_e(k_d)$$

Proof Let $\{v_1, \dots, v_d\}$ be the k -basis of k_d which was used to define the k -algebra map $j_d: k_d \rightarrow \text{Mat}_d(k)$ (section 2). Then it is easy to verify that the matrix $M^0 = (m_{ij}) \in G_d(k)$ defined by the equations $v_j^q = \sum_i m_{ij} v_i$ ($j = 1, \dots, d$) has the property: $M^0 \cdot j_d(\alpha) \cdot (M^0)^{-1} = j_d(\alpha^q)$ for all $\alpha \in k_d$. Therefore the matrix $M = (M^0)_{(1^d)} \in G_n(k)$ (i.e. M is the diagonal sum of n/d copies of M^0) has the property 7.12.

7.13 Corollary $\Theta_0 M^{-j} = \Theta_j$, for all $j = 0, \dots, m-1$.

Proof 7.12 implies that M normalizes $G_{d,n}(k)$, and also that $M^j h_{p'} M^{-j} = h_{p'}^{q^j}$, because $h_{p'} = j_d((\zeta)_{(1^d)})$ by 7.4. The corollary now follows from the definition 7.10.

Now suppose that $s \in \Theta_0$. Then $s^{-1} h_{p'} s = h_{p'}$. But by 7.4(i) we know that $h_{p'}$ is the diagonal sum of n/m copies of $C(f)$, and $C(f)$ is the image of $\zeta \in k_m$ under the k -algebra map $j_m: k_m \rightarrow \text{Mat}_m(k)$ determined by the part $\{v_1, \dots, v_m\}$ of the k -basis $\{v_1, \dots, v_d\}$ of k_d which was used to obtain 2.2. So the centralizer of $h_{p'}$ in $G_n(k)$ consists of all non-singular matrices $B = (B_{rs})_{r,s=1, \dots, n/m}$, in which each B_{rs} is an $m \times m$ matrix belonging to the centralizer of $C(f)$ in $\text{Mat}_m(k)$. But this latter centralizer is exactly $\text{Im } j_m$ (it consists of those matrices which correspond to elements of the k_m -endomorphism algebra of the left k_m -module k_m ; but since k_m is commutative, this is the same as the algebra of all left multiplications by elements of k_m). This proves that

$$7.14 \quad \text{The centralizer of } h_{p'} \text{ in } G_n(k) \text{ is } G_{m,n}(k).$$

The groups which we are dealing with are shown in the diagram below.

$$\begin{array}{ccc}
& & G_n(k) \\
& & \downarrow \\
& & G_{m,n}(k) \\
& & \uparrow \\
G_e(k_d) & \xrightarrow{j_d} & G_{d,n}(k) \\
\downarrow & & \downarrow \\
H_e(k_d) & \xrightarrow{j_d} & H_{d,n}(k)
\end{array}$$

Let R denote the conjugacy class of $G_{n,d}(k)$ which contains $h_p = j_d((1)_\pi)$; this consists of all unipotent elements of $G_{n,d}(k)$ of type $\pi \cdot d$. Clearly $R = j_d(S)$, where S is the conjugacy class of $G_e(k_d)$ which contains $(1)_{(\pi)}$.

Let $\rho \vdash N$. We shall recall in section 8 that there is a monic polynomial $a_\rho(T) \in \mathbf{Z}[T]$, such that the order of the centralizer in $G_N(K)$ of the unipotent element $(1)_{(\rho)}$ is $a_\rho(Q)$, for any field K of finite order Q (see [GL], p.409, or [M], p.181).

7.15 Lemma (i) The order of Θ_0 is $a_{\pi \cdot d/m}(q^m) / a_\pi(q^d)$ times $|G_{n,d}(k)|$.

(ii) For each $s \in \Theta_0$, $s^{-1}hs$ lies in the class $j_d((t - \zeta)^\pi)$ of $G_{n,d}(k)$. Hence

$$7.16 \quad \frac{1}{|G_{d,n}(k)|} \sum_{s \in \Theta_0} Y(s^{-1}hs) = \psi(\zeta) \cdot \frac{a_{\pi \cdot d/m}(q^m)}{a_\pi(q^d)} \cdot c_{\lambda, \pi}(q^d).$$

Proof (i) From the definition 7.10 of Θ_0 , together with 7.14, we see that Θ_0 consists of all $s \in G_{m,n}(k)$ such that $s^{-1}h_p s \in G_{d,n}(k)$, i.e. such that $s^{-1}h_p s \in R$. The order of R is the same as that of S , i.e. $|G_e(k_d)| / a_\pi(q^d) = |G_{d,n}(k)| / a_\pi(q^d)$. But the number of $s \in G_{m,n}(k)$ for which $s^{-1}h_p s$ has a given value, is the order of the centralizer in $G_{m,n}(k)$ of h_p . Since $h_p = j_d((1)_\pi)$ is conjugate to $j_m((1)_{\pi \cdot d/m})$, this order is $a_{\pi \cdot d/m}(q^m)$. This proves (i).

(ii) If $s \in \Theta_0$ then $s^{-1}hs$ has semisimple part $h_p = j_d(\zeta)$, and unipotent part in $j_d(S)$. This proves the first statement of (ii). But $Y = \text{Ind}_{H_{d,n}(k)}^{G_{d,n}(k)}(\psi \cdot \lambda)$ maps $\{j_d((t - \zeta)^\pi)\}$ to $\text{Ind}_{H_e(k_d)}^{G_e(k_d)}(\psi \cdot \lambda)\{(t - \zeta)^\pi\}$, which equals $\psi(\zeta) \cdot c_{\lambda, \pi}(q^d)$, by 7.2. From this, 7.16 follows.

It is now easy to prove, using 7.13, that 7.16 remains true, if we replace Θ_0 by Θ_j and ζ by ζ^j , for any $j \in \{0, \dots, m-1\}$. Add the m equations which result; we get the

7.17 Proposition *Suppose d divides n , and $\lambda, \pi \vdash e = n/d$, and $f \in \Phi$ has degree m which divides d . Let $\sigma = \pi \cdot d/m$, so that $\sigma \vdash n/m$, and f^σ is a conjugacy class of $G_n(k)$. Then for any $\psi \in \hat{M}_n$*

$$7.18 \quad X_{d,n}(\psi, \lambda)\{f^\sigma\} = \psi(f) \cdot \frac{a_\sigma(q^m)}{a_\pi(q^d)} \cdot c_{\lambda, \pi}(q^d)$$

7.19 Lemma $\frac{a_{\pi \cdot l}(T)}{a_\pi(T^l)} \in \mathbf{Z}[T]$, for all partitions π and all positive integers l .

Proof With the notation of lemma 7.17, $a_\sigma(q^m)$, $a_\pi(q^d)$ are the orders of the centralizers of h_p in $G_{m,n}(k)$, $G_{d,n}(k)$, respectively. Since $G_{d,n}(k)$ is a subgroup of $G_{m,n}(k)$, it follows that $a_\sigma(q^m)/a_\pi(q^d) \in \mathbf{Z}$ for all prime-powers q . Now take $m = 1$, and write l for d . Since $\sigma = \pi \cdot d/m = \pi \cdot l$, this shows that $a_{\pi \cdot l}(q)/a_\pi(q^l) \in \mathbf{Z}$ for all prime-powers q . Now lemma 7.19 follows from lemma 5.15.

Thus 7.18 proves proposition 4.4(ii); the polynomials $x_{\lambda, \pi, l}(T)$ are given by

$$7.20 \quad x_{\lambda, \pi, l}(T) = \frac{a_{\pi \cdot l}(T)}{a_\pi(T^l)} \cdot c_{\lambda, \pi}(T^l).$$

8 Gelfand–Graev character for $G_n(k)$, II.

To prove propositions 4.4 and 4.6 (and hence complete the proof of theorem 3.2) we need an explicit formula for the polynomials $c_{\lambda, \pi}(T)$, which were defined rather indirectly in

section 6. For this purpose we shall make use of the polynomials $Q_\rho^\lambda(T)$ which were introduced in [GL, p.420], and later defined in a different way by D.E. Littlewood ([Li]; see [M], p.246).

The polynomials $Q_\rho^\lambda(T) \in \mathbf{Z}[T]$ are defined for all partitions λ, ρ of n , and satisfy the following orthogonality relations:

$$8.1 \quad \sum_{\pi \vdash n} \frac{1}{a_\pi(T)} Q_\rho^\pi(T) Q_\sigma^\pi(T) = \delta_{\rho, \sigma} \frac{z_\rho}{c_\rho(T)}, \text{ for all } \rho, \sigma \vdash n,$$

and

$$8.2 \quad \sum_{\rho \vdash n} \frac{c_\rho(T)}{z_\rho} Q_\rho^\pi(T) Q_\rho^\tau(T) = \delta_{\pi, \tau} a_\pi(T), \text{ for all } \pi, \tau \vdash n.$$

The coefficients appearing in 8.1 and 8.2 are defined as follows.

$$a_\rho(T) = T^{|\rho|+2n(\rho)} \prod_i \left(1 - \frac{1}{T}\right) \left(1 - \frac{1}{T^2}\right) \dots \left(1 - \frac{1}{T^{r_i}}\right),$$

$$c_\rho(T) = \prod_i (T^i - 1)^{r_i},$$

and

$$z_\rho = \prod_i i^{r_i} r_i!,$$

for any partition $\rho = 1^{r_1} 2^{r_2} \dots$ of n . Notice that $a_\rho(T) \in \mathbf{Z}[T]$ and is monic; we have already used the fact that $a_\rho(q)$ is the order of the centralizer in $G_n(k)$ of an element of the class $(t-1)^\rho$ ([GL], p.409; [M], p. 181).

The scalar product \bullet, \circledast on the space of class-functions on $G_n(k)$ is defined by

$$8.3 \quad \bullet F, F' \circledast = \sum_c \frac{1}{a(c)} F\{c\} F'\{\bar{c}\},$$

where the sum is over all classes c of $G_n(k)$, $a(c)$ denotes the order of the centralizer in $G_n(k)$ of an element of c , and $\bar{c} = c^{-1}$. Define U_n to be the space of all class-functions F of *unipotent support*, i.e. such that $F\{c\} \neq 0 \iff c = (t-1)^\pi$ for some $\pi \vdash n$. As in section 6, we write $F\{(t-1)^\pi\} = F\{\pi\}$. Then 8.3 becomes, when at least one of F, F' belongs to U_n ,

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$$8.4 \quad \bullet F, F' \circledast = \sum_{\pi \vdash n} \frac{1}{a(\pi)} F\{\pi\} F'\{\pi\}.$$

8.5 Lemma For each $\rho \vdash n$ define $\mathbf{Q}_\rho \in U_n$ by setting $\mathbf{Q}_\rho\{\pi\} = Q_\rho^\pi(q)$. Then for any $F \in U_n$ there holds

$$8.6 \quad F = \sum_{\rho \vdash n} \frac{c_\rho(q)}{z_\rho} \bullet F, \mathbf{Q}_\rho \circledast \mathbf{Q}_\rho.$$

Proof Use 8.2 to evaluate the right-hand side of 8.4.

The following proposition provides convenient rules for calculating the coefficients $\bullet F, \mathbf{Q}_\rho \circledast$ of the ‘‘Fourier expansion’’ 8.6 of F in terms of the \mathbf{Q}_ρ . Let $\rho = 1^{r_1} 2^{r_2} \dots$, $\sigma = 1^{s_1} 2^{s_2} \dots$ and $\tau = 1^{t_1} 2^{t_2} \dots$ be partitions, and recall the definition $\rho + \sigma = 1^{r_1+s_1} 2^{r_2+s_2} \dots$

8.7 Proposition (i) If $\rho \vdash l$, $\sigma \vdash m$ then $\mathbf{Q}_\rho \circ \mathbf{Q}_\sigma = \mathbf{Q}_{\rho+\sigma}$.

(ii) If $\rho, \sigma \vdash n$ then $\bullet \mathbf{Q}_\rho, \mathbf{Q}_\sigma \circledast = \delta_{\rho, \sigma} \frac{z_\rho}{c_\rho(q)}$.

(iii) If $f \in U_l$ and $g \in U_m$ then for any $\tau \vdash l+m$

$$8.8 \quad \bullet f \circ g, \mathbf{Q}_\tau \circledast = \sum_{\rho+\sigma=\tau} \left[\begin{array}{c} \tau \\ \rho, \sigma \end{array} \right] \bullet f, \mathbf{Q}_\rho \circledast \bullet g, \mathbf{Q}_\sigma \circledast,$$

where the sum is over all pairs (ρ, σ) such that $\rho \vdash l$, $\sigma \vdash m$ and $\rho + \sigma = \tau$, and

$$8.9 \quad \left[\begin{array}{c} \tau \\ \rho, \sigma \end{array} \right] = \prod_i \frac{t_i!}{r_i! s_i!}.$$

Proof (i) Use 6.4, together with lemma 4.4 of [GL], p.420.

(ii) Follows at once from 8.4 and 8.1.

(iii) Since both sides of 8.8 are linear in both f and g , it is enough verify that 8.8 holds when $f = \mathbf{Q}_\lambda$ and $g = \mathbf{Q}_\mu$, where $\lambda \vdash l$ and $\mu \vdash m$. This is a routine calculation using (i) and (ii).

The next proposition shows the connection between the Gelfand-Graev character Γ_n , the $Q_\lambda^\rho(q)$, and the characters of $S(n)$.

8.10 Proposition (i) $\bullet\Delta_n, \mathbf{Q}_\rho \otimes = 1$, for all $\rho \vdash n$.

(ii) $\bullet\Gamma_n, \mathbf{Q}_\rho \otimes = \varepsilon_\rho$, where $\varepsilon_\rho = (-1)^{r_2 + r_4 + \dots}$ is the value of the alternating character ε of $S(n)$ at the conjugacy class ρ of $S(n)$.

Proof (i) By 8.4 $\bullet\Delta_n, \mathbf{Q}_\rho \otimes = \sum_{\pi \vdash n} \frac{1}{a_\pi(q)} \cdot \Delta_n \{ \pi \} \cdot Q_\rho^\pi(q)$. Using 6.2, this reduces to

$$\frac{1}{a_{(n)}(q)} \cdot q^{n-1}(q-1) \cdot Q_\rho^{(n)}(q), \text{ which equals 1, because } a_{(n)}(q) = q^{n-1}(q-1), \text{ and } Q_\rho^{(n)}(q)$$

= 1 for all $\rho \vdash n$ ([GL], p.445, or [M], p.248, Ex. 1).

(ii) Let $\lambda = (n_1, \dots, n_b)$ be the composition of n associated to a subset J of $I = \{1, \dots, n-1\}$.

We consider the function $\Delta_J = \Delta_{n_1} \circ \dots \circ \Delta_{n_b}$, as in section 6. Using the evident extension of 8.8 to several factors, together with (i), we find that $\bullet\Delta_J, \mathbf{Q}_\rho \otimes = H(J, \rho)$, where

$$8.11 \quad H(J, \rho) = \sum_{\mathbf{p}} \prod_i \frac{r_i!}{r_i(1)! \dots r_i(b)!},$$

the sum being over all vectors $\mathbf{p} = (\rho(1), \dots, \rho(b))$ such that $\rho(j) = 1^{r_1(j)} 2^{r_2(j)} \dots \vdash n_j$ for all

$j = 1, \dots, b$, and $\rho(1) + \dots + \rho(b) = \rho$. Frobenius showed, in his classic paper [F] on the

characters of $S(n)$, that $H(J, \rho)$ is the value at class ρ (of $S(n)$) of the character $\chi_J =$

$\text{Ind}_{S(J)}^{S(n)}(1_{S(J)})$, where $S(J)$ is the subgroup of $S(n)$ consisting of all permutations of $\{1, \dots, n\}$

which leave fixed each of the subsets $\{1, \dots, n_1\}$, $\{n_1 + 1, \dots, n_2\}$, \dots , $\{n_1 + \dots + n_{b-1} + 1, \dots, n\}$

(see [F], p.149, or [Le], p.103). L. Solomon has proved a formula ([S], theorem 2), which

gives as special case the following equation on characters of $S(n)$

$$8.12 \quad \sum_{J \subset I} (-1)^{|J|} \chi_J = \varepsilon.$$

If we combine 8.12 with Deligne–Lusztig’s formula 6.6 for Γ_n , we get

$$\bullet\Gamma_n, \mathbf{Q}_\rho \otimes = \sum_{J \subset I} (-1)^{|J|} \bullet\Delta_J, \mathbf{Q}_\rho \otimes = \sum_{J \subset I} (-1)^{|J|} H(J, \rho) = \sum_{J \subset I} (-1)^{|J|} \chi_J \{ \rho \} = \varepsilon_\rho.$$

8.13 Corollary For any $J \subset I$ and $\rho \vdash n$ there holds

$$\bullet \Gamma_J, \mathbf{Q}_\rho^{\otimes} = \bullet \Gamma_{n_1} \circ \dots \circ \Gamma_{n_b}, \mathbf{Q}_\rho^{\otimes} = H(J, \rho) \cdot \varepsilon_\rho.$$

Proof This follows from 8.10(ii) and 8.8.

From now on we shall always write F_λ , $H(\lambda, \pi)$ and $c_{\lambda, \pi}$ instead of F_J , $H(J, \pi)$ and $c_{J, \pi}$, when $\lambda \vdash n$ is the composition associated to J . Notice that each of these is unchanged if λ is replaced by any $\lambda' \approx \lambda$, so we lose nothing if we assumed that $\lambda \vdash n$. From 8.13 and 8.6 we have

$$8.14 \quad \Gamma_\lambda \{\pi\} = \sum_{\rho \vdash n} \frac{c_\rho(q)}{z_\rho} \cdot H(\lambda, \rho) \cdot \varepsilon_\rho \cdot Q_\rho^\pi(q),$$

for all $\lambda, \pi \vdash n$. Notice that all the polynomials $c_\rho(T)$ are divisible by $T - 1$. Therefore, if we write $\hat{c}_\rho(T) = \frac{1}{T-1} \cdot c_\rho(T)$, we have from 8.14 and 6.7

$$8.15 \quad c_{\lambda, \pi}(T) = \sum_{\rho \vdash n} \frac{\hat{c}_\rho(T)}{z_\rho} \cdot H(\lambda, \rho) \cdot \varepsilon_\rho \cdot Q_\rho^\pi(T).$$

9 Proof of propositions 4.4 and 4.6

We need to connect the numbers $H(\lambda, \pi)$ with the characters χ^λ of $S(n)$. Using Macdonald's notation for symmetric functions (see [M]), we have equations

$$p_\rho(x) = \sum_{\lambda \vdash n} H(\lambda, \rho) m_\lambda(x) \text{ (see [F], p.149, or [Le], p.103). If we combine these with}$$

$$p_\rho(x) = \sum_{\lambda \vdash n} \chi_\rho^\lambda s_\lambda(x) \text{ and } s_\lambda(x) = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu(x) \text{ (see [M], p.101; the } K_{\tau\lambda} \text{ are the "Kostka}$$

numbers") we get

$$9.1 \quad H(\lambda, \pi) = \sum_{\tau \vdash n} \chi_\rho^\tau K_{\tau\lambda}.$$

9.2 Proof of proposition 4.4 We want to calculate $W(\lambda, \mu) = \bullet X_{1,n}(1, \lambda), I^\mu \otimes$. From 7.2, $X_{1,n}(1, \lambda)$ is zero on all classes except the classes $(t-x)^\pi$, $x \in M_1 = k^\times$, $\pi \vdash n$. Moreover, for fixed π , $X_{1,n}(1, \lambda) \{(t-x)^\pi\} = c_{\lambda, \pi}(q)$, for all $x \in M_1$. From 4.5, $I^\mu \{(t-x)^\pi\} = \sum_{\rho \vdash n} \frac{1}{z_\rho} \cdot Q_\rho^\pi(q) \cdot \chi_\rho^\mu$ for all $x \in M_1$. Then it follows easily from 8.15 that

$$9.3 \quad \bullet X_{1,n}(1, \lambda), I^\mu \otimes = (q-1) \bullet \sum_{\rho \vdash n} \frac{\hat{c}_\rho(q)}{z_\rho} \cdot \varepsilon_\rho \cdot H(\lambda, \rho) \cdot \mathbf{Q}_\rho, \sum_{\sigma \vdash n} \frac{1}{z_\sigma} \cdot \chi_\sigma^\mu \cdot \mathbf{Q}_\sigma \otimes,$$

and by 8.7(ii) this reduces to $\sum_{\rho \vdash n} \frac{1}{z_\rho} \cdot \varepsilon_\rho \cdot H(\lambda, \rho) \cdot \chi_\rho^\mu$. Since $\varepsilon_\rho \chi_\rho^\mu = \chi_{\tilde{\mu}}^\mu$, where $\tilde{\mu}$ is the

conjugate of μ ([Le], p.135), we get from 9.3

$$9.4 \quad W(\lambda, \mu) = \sum_{\rho, \tau \vdash n} \frac{1}{z_\rho} \cdot K_{\tau \lambda} \cdot \chi_\rho^\tau \chi_\rho^{\tilde{\mu}} = K_{\tilde{\mu} \lambda}.$$

But the matrix $(K_{\lambda \mu})$ has integer coefficients and is unimodular ([M], p.101), and therefore the same is true of the matrix $(W(\lambda, \mu)) = (K_{\tilde{\mu} \lambda})$.

9.5 Proof of proposition 4.6 If this proposition is false, there exist complex numbers $h_{d,\lambda}$, not all zero, such that

$$9.6 \quad \sum_{d|n} \sum_{\lambda \vdash n/d} h_{d,\lambda} X_{d,n}(1, \lambda) = 0.$$

Let d_0 be the largest divisor of n such that $h_{d_0,\lambda} \neq 0$ for some $\lambda \vdash \frac{n}{d_0}$. Now take any $f \in \Phi(k)$ such that $d(f) = d_0$, and any $\sigma \vdash \frac{n}{d_0}$.

Let d be any divisor of n . If $d > d_0$, then $h_{d,\lambda} = 0$, by the definition of d_0 . If $d < d_0$, then $d(f) = d_0$ does not divide d , hence $X_{d,n}(1, \lambda) \{f^\sigma\} = 0$ by 4.1(i). Therefore if we evaluate 9.6 at the class f^σ , we get

$$9.7 \quad \sum_{\lambda \vdash n/d_0} h_{d_0,\lambda} X_{d,n}(1, \lambda) \{f^\sigma\} = 0, \text{ for all } \sigma \vdash \frac{n}{d_0}.$$

But 7.18 tells us that $X_{d_0,n}(1, \lambda) \{f^\sigma\} = c_{\lambda, \sigma}(q^{d_0})$ (notice that $d(f) = m = d_0$, hence $\sigma = \pi$), and by 8.16 the matrix $(c_{\lambda, \sigma}(q^{d_0}))$ is non-singular. Thus 9.7 implies that $h_{d_0,\lambda} = 0$ for all λ .

This contradiction proves proposition 4.6.

Appendix: some $r_\lambda(T)$ These are found by the inductive construction in section 5. Values of the characters $X_{d,n}(\psi, \lambda)$ are calculated from formulae 7.18 and 8.15 .

λ	(1)	(2)	(1 ²)	(3)	(21)	(1 ³)	(4)	(31)	(2 ²)	(21 ²)	(1 ⁴)
$r_\lambda(T)$	1	-1	0	$2-T$	-1	0	$T-2$	1	0	0	0

λ	(5)	(41)
$r_\lambda(T)$	$4-3T-T^3+2T^5-T^6$	$-3+2T-T^2+T^3+T^4-T^5$

λ	(32)	(31 ²)	(2 ² 1)	(21 ³)	(1 ⁵)
$r_\lambda(T)$	$-3+T+T^2+T^3-T^4$	$2-T+T^2-T^3$	$2-T^2$	-1	0

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