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Photon Bubbles in Accretion Discs

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ABSTRACT

We show that radiation dominated accretion discs are likely to suffer from a "photon bubble" instability similar to that described by Arons in the context of accretion onto neutron star polar caps. The instability requires a magnetic field for its existence. In an asymptotic regime appropriate to accretion discs, we find that the overstable modes obey the remarkably simple dispersion relation

$$\omega^2 = -igkF(\mathbf{B},\mathbf{k}).$$

Here g is the vertical gravitational acceleration, $\mathbf{B}$ the magnetic field, and F is a geometric factor of order unity that depends on the relative orientation of the magnetic field and the wavevector. In the nonlinear outcome it seems likely that the instability will enhance vertical energy transport and thereby change the structure of the innermost parts of relativistic accretion discs.

Key words:

1 INTRODUCTION

Compact objects such as black holes and neutron stars are among the most interesting objects in astrophysics because of their exotic strong-field gravitational physics and because they are likely at the center of some of the most luminous and energetic objects in the universe. Since it is mainly their accretion flows, and not the compact objects themselves, that are readily observed, the dynamics and radiative properties of the accretion flows are the focus of much theoretical attention.

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One of the most important models for accretion flows is the thin disc (Shakura & Sunyaev 1973, Lynden-Bell & Pringle 1974). This model predicts that for rapidly accreting objects the inner parts of the disc are radiation pressure dominated. It was realized soon after the model was first proposed, however, that radiation pressure dominated discs are unstable, both viscously (Lightman & Eardley 1974) and thermally (Shakura & Sunyaev 1975). This suggests that the standard thin disc model is not self-consistent, and so attention has turned to other models such as advection-dominated flows (see Narayan 1997 for a review).

Despite theoretical arguments that the thin disc is unstable, thin disc spectra are widely used to fit observations of black hole candidates. Indeed, there is indirect evidence for (theoretically unstable) thin discs in that some observations are well fit by thin disc spectra. For example, some galactic black hole candidates such as Nova Muscae have X-ray spectra that are well fit by multi-temperature thin disc models (R. Narayan, private communication).

One can stabilize radiation pressure dominated discs by modifying the usual prescription for the shear stress $t_{r\phi} \sim \alpha p$ so that $t_{r\phi} \sim \alpha p_g$ ($p_g \equiv$ gas pressure) or some combination of gas and radiation pressure (e.g. Lightman & Eardley 1974, Piran 1978). Arguments have been advanced in favour of such a modification by Eardley & Lightman (1975), Coroniti (1981), Sakimoto & Coroniti (1981,1989), and Stella & Rosner (1984). These arguments rely on the thermodynamic peculiarities of magnetic buoyancy in a radiation dominated plasma. More recent work has vastly increased our understanding of magnetically driven angular momentum diffusion in discs (see the review of Balbus & Hawley 1997). Numerical experiments suggest that $t_{r\phi}$ is limited by Lorentz forces rather than by buoyant escape of magnetic fields (Stone et al. 1996), although these experiments do not include radiation pressure and radiative diffusion.

An alternative route to viscous stability (e.g. Liang 1977) is to modify the disc cooling law using convection, although this is not successful in eliminating the thermal instability (Piran 1978). Modification of the disc cooling rate has not seemed a promising approach.

Recently, however, while investigating neutron star polar cap accretion, Klein & Arons (1989, 1991) noticed the development of evacuated regions, or "photon bubbles"* in their radiation hydrodynamics simulations. A subsequent linear analysis (Arons 1992) revealed that the source of the photon bubbles was an overstable mode present in radiation dominated, *magnetized* atmospheres. Earlier incarnations of the photon bubble instability in an

* Bubble is something of a misnomer, since surface tension plays no role. The phenomenon is really more like convection.

unmagnetized plasma have turned out to be flawed; Marzec (1978) gives a full discussion of this point. Later numerical studies of the nonlinear evolution of the photon bubble instability (Hsu et al. 1997) showed that vertical transport of energy is enhanced in the nonlinear outcome.

One is naturally led to inquire whether this same photon bubble instability is present in the radiation dominated parts of accretion discs, and if so, what the consequences might be for disc structure. Unfortunately Arons's (1992) analysis is not immediately applicable to discs. It considers physical parameters relevant to neutron star polar cap accretion: a thermal timescale long compared to the dynamical timescale, negligible gas pressure, and superthermal magnetic field. Our purpose in this paper is to generalize Arons's work to the regime appropriate to discs.

First we evaluate conditions inside accretion discs using a standard α model (§2). Then we write down a set of governing equations appropriate to these conditions (§3). In §4 we construct a model equilibrium to perturb about, and in §§5 and 6 we work out the linear theory in the WKB limit. The astrophysical implications of the result depend on the nonlinear outcome, about which we speculate in §7. §8 contains a summary.

2 CONDITIONS IN ACCRETION DISCS

What conditions are relevant to a study of the photon bubble instability in accretion discs? Consider accretion onto a black hole of mass $m\,M_\odot$ at a rate $\dot{m}\,L_{edd}c^2$, where $L_{edd} \equiv 4\pi GMc/\kappa_{es}$, and κ_{es} is the electron scattering opacity. Using the standard thin-disc, one-zone model (Shakura & Sunyaev 1973), assuming radiation pressure is dominant and $\kappa \simeq \kappa_{es} \simeq 0.4\,\mathrm{g\,cm^{-2}}$, we find that at radius rGM/c^2

$$\Sigma \simeq 0.37\ \alpha^{-1}\dot{m}^{-1}r^{3/2}\ \mathrm{g\,cm^{-2}}, \tag{1}$$

$$T \simeq 2.4\times 10^{7}\ \alpha^{-1/4}m^{-1/4}r^{-3/8}\ \mathrm{K}, \tag{2}$$

$$\rho \simeq 4.2\times 10^{-7}\ \alpha^{-1}m^{-1}\dot{m}^{-2}r^{3/2}\ \mathrm{g\,cm^{-3}}. \tag{3}$$

Here and throughout we ignore corrections due to general relativity and the inner boundary of the disc. Radiation-dominated accretion discs have entropy profiles that suggest convective instability (e.g. Bisnovatyi-Kogan & Blinnikov 1977). Calculations of disc vertical structure using mixing-length theory (Shakura, Sunyaev, & Zilitinkevich 1978) show, however, that

the vertical radiative flux of energy always dominates the vertical advective flux and so the mean structure of the disc is not likely to be very different from the values calculated above.

The electron scattering optical depth is

$$\tau_s \simeq \Sigma\kappa_{es}/2 = 0.074\ \alpha^{-1}\dot{m}^{-1}r^{3/2}, \tag{4}$$

while the "true" optical depth, which measures the extent to which radiation is thermalized inside the disc, depends on the absorption opacity κ_a:

$$\tau^* = \Sigma\sqrt{\kappa_{es}\kappa_a}/2. \tag{5}$$

The absorption opacity is a complicated function of density, temperature, metallicity and the radiation spectrum. Taking $\kappa_a \simeq \kappa_P$, the Planck mean opacity, and evaluating κ_P from tables for a solar composition gas (Magee et al. 1995) we find that τ^* ranges from $\lesssim 1$ to $\gg 1$. For example, at $r = 50$ from a $10\,M_\odot$ black hole accreting at $\dot{m} = 1$, and taking $\alpha = 0.1$, we find $\tau^* \simeq 14.8$. Discs around supermassive black holes have somewhat higher effective optical depths, since they are cooler and the Planck mean opacity increases sharply below 3×10^6 K due to bound-free absorption by metals. All this implies that the true optical depth of perturbations that are smaller than the scale height can be small, and so the radiation field must be treated using an approximation that is valid in this regime.

The ratio of gas pressure p_g to total pressure P is

$$\beta_r \equiv p_g/P = 1.6\times10^{-6}\ \alpha^{-1/4}m^{-1/4}\dot{m}^{-2}r^{21/8}, \tag{6}$$

so the boundary of the radiation-dominated region lies at

$$r_c \simeq 160\ \alpha^{2/21}m^{2/21}\dot{m}^{16/21}. \tag{7}$$

Also the sound speed c_s is

$$c_s/c = 3.0\ \dot{m}r^{-3/2}. \tag{8}$$

The radiative diffusivity is $D_0 \equiv c/(\kappa\rho)$. In dimensionless form,

$$M_0 \equiv \frac{D_0}{c_s H} \simeq 4.5\ \alpha, \tag{9}$$

where $H \equiv$ disc scale height. Thus $M_0 \sim 1$, while in neutron star polar cap accretion $M_0 \ll 1$. This difference changes the character of the instability significantly, and is the most important difference between our work and Arons's work.

We also need an estimate for the magnetic field strength. We use the prescription $\beta_M \equiv c_i^2/v_A^2 = (4\alpha)^{-1}$ ($c_i^2 \equiv P/\rho$ is the isothermal sound speed; $v_A \equiv$ Alfvén speed), consistent with the simulations of Hawley, Gammie, & Balbus (1995). So if $\alpha = 0.1$, $\beta_M = 2.5$. For

reasonable values of α, then, $\beta_M \sim 1$. Arons (1992) focused on the case $\beta_M \ll 1$, but developed a more general analysis in an appendix.

Finally, heat conduction, radiative viscosity, ordinary viscosity, and ordinary resistivity are all completely negligible.

To summarize the results of this section, accretion discs have approximately thermal magnetic fields, so $\beta_M \equiv c_i^2/v_A^2 \sim 1$. In their inner region radiation pressure dominates, so $\beta_r \ll 1$. The thermal timescale is comparable to the dynamical timescale (to within a factor of α) so the dimensionless radiative diffusion rate $M_0 \equiv (c/(\kappa\rho))/(c_i H) \sim 1$. Finally, the thermalization length $l_* \equiv (\rho\sqrt{\kappa\kappa_a})^{-1}$ varies widely but can be $\sim H$. Since we will consider perturbations on scales small compared to the scale height, the perturbations can also have a scale small compared to the thermalization length. Thus it is not clear *a priori* that it is appropriate to use the Rosseland diffusion approximation for the radiation field.

3 BASIC EQUATIONS

Since we cannot use the Rosseland diffusion approximation for the radiation field, we turn to the more general flux-limited nonequilibrium diffusion approximation (see Mihalas & Mihalas 1984 and references therein). This approximation is very similar to the Rosseland diffusion approximation (it still sets $K_{ij} = J\delta_{ij}/3$, where K and J are the second and zeroth angular moments of the intensity), but, loosely speaking, it allows for the possibility that the radiation field has a different "temperature" than the gas. More precisely, it does not require that $J = \sigma T_g^4/\pi$, where $T_g \equiv$ gas temperature. Because it is a diffusion approximation, it is strictly valid only on scales large compared to the photon mean free path $(\rho\kappa_{es})^{-1}$, but it gives qualitatively sensible results on smaller scales.

The governing equations for the gas, then, are the continuity equation,

$$D_t\rho = -\rho(\nabla\cdot\mathbf{v}), \tag{10}$$

where $D_t \equiv \partial_t + (\mathbf{v}\cdot\nabla)$, the momentum equation,

$$\rho D_t\mathbf{v} = -\nabla p_g - \rho\nabla\phi + \frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{4\pi} - \frac{\nabla B^2}{8\pi} + \frac{4\pi\kappa\rho}{c}\mathbf{H}, \tag{11}$$

where $\mathbf{H} \equiv \int_{d\Omega} I\mathbf{n}/(4\pi)$ † is the frequency-integrated flux and ϕ is the gravitational potential, and the gas energy equation ($u \equiv$ internal energy per unit volume),

† H denotes the scale height and H_i a component of the flux. Likewise B denotes the thermal mean intensity and B_i a component of the magnetic field. The difference should be clear in context.

$$D_t u = -\gamma u(\nabla \cdot \mathbf{v}) + 4\pi\kappa_a\rho(J - B), \tag{12}$$

where J is the frequency-integrated mean intensity, $B \simeq \sigma T_g^4/\pi$, and we approximate the absorption opacity by the Planck mean opacity. The magnetic field evolution is governed by the induction equation

$$D_t \mathbf{B} = -\mathbf{B}(\nabla \cdot \mathbf{v}) + (\mathbf{B} \cdot \nabla)\mathbf{v} \tag{13}$$

and the constraint $\nabla \cdot \mathbf{B} = 0$. The mean intensity evolution is given by

$$\frac{1}{c} D_t J - \left(\frac{4J}{3\rho c}\right) D_t \rho = -\nabla \cdot \mathbf{H} + \kappa_a \rho (B - J), \tag{14}$$

and the flux evolution is given by

$$\frac{1}{c} D_t \mathbf{H} = -\frac{1}{3}\nabla J - \kappa\rho\mathbf{H}, \tag{15}$$

where we approximate κ by the Rosseland mean opacity $\simeq \kappa_{es}$.

4 MODEL PROBLEM: STRATIFIED ATMOSPHERE

Magnetized, radiation-dominated accretion discs are dynamically evolving flows, since they are subject to the magnetorotational instability (Balbus & Hawley 1991) and, possibly, ordinary convective instability. Linear theory therefore cannot even in principle provide a rigorous guide to their dynamics. The best we can hope for is to find a model problem that captures the essence of physical conditions in accretion discs and is simple enough to solve.

The model problem we have chosen is a nonrotating stratified atmosphere with $\nabla\phi = g\hat{\mathbf{z}} = const.$, a *uniform* magnetic field, and a constant flux H_z from below. This is the least complicated model that is potentially subject to the photon bubble instability. It allows us to focus on photon bubbles alone, disentangled from the magnetorotational instability, magnetic Rayleigh-Taylor instability, and convective instability.

The model equilibrium is determined by the vertical momentum equation

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{4\pi\kappa\rho}{c} H_z - g = 0, \tag{16}$$

the gas energy equation

$$J = B, \tag{17}$$

the intensity equation

$$\nabla \cdot \mathbf{H} = 0, \tag{18}$$

and the vertical component of the flux equation

$$-\frac{1}{3}\frac{dJ}{dz} = -\kappa\rho H_z. \tag{19}$$

This set of equations admits a two-parameter family of solutions. A natural set of parameters is β_r evaluated at $z = z_o$, and $\mathcal{L} = H_z 4\pi\kappa/(cg)$, which is the ratio of the flux to the critical flux where the effective gravity vanishes (i.e., the local Eddington limit). Defining the pressure scale height $H \equiv c_i^2/g$, at $z = z_o$ we have

$$h_T \equiv -H\frac{\partial \ln T_g}{\partial z} = \frac{1}{4}\frac{\mathcal{L}}{1-\beta_r}, \tag{20}$$

$$h_\rho \equiv -H\frac{\partial \ln \rho}{\partial z} = \frac{1}{\beta_r}\left(1 - \mathcal{L}\frac{(1-3\beta_r/4)}{(1-\beta_r)}\right), \tag{21}$$

which imply

$$h_r \equiv -H\frac{\partial \ln P_r}{\partial z} = \frac{\mathcal{L}}{1-\beta_r}, \tag{22}$$

$$h_g \equiv -H\frac{\partial \ln P_g}{\partial z} = \frac{1-\mathcal{L}}{\beta_r}. \tag{23}$$

The entropy profile suggests convective stability (entropy increases upwards) if

$$\mathcal{L} < \mathcal{L}_{crit} = \frac{4(\gamma-1)(4-7\beta_r+3{\beta_r}^2)}{(\gamma-1)(16-12\beta_r-3{\beta_r}^2)+{\beta_r}^2} \tag{24}$$

(Kutter 1970, Wentzel 1970, Tayler 1954). In the limit of small β_r this amounts to

$$\mathcal{L} < 1 - \beta_r + O({\beta_r}^2). \tag{25}$$

For model atmospheres in which $M_0 \ll 1$ this implies convective stability. In discs the question of convective stability is more subtle, and outside the scope of this paper, because of the presence of radiative damping and rotation.

5 LINEAR THEORY

We will consider only short-wavelength (WKB) perturbations. Longer-wavelength modes are global in nature and depend on the details of the equilibrium; but the careful construction of disc equilibria is a futile endeavour since all weakly magnetized discs are unstable. The perturbations have the form

$$\delta f \sim \exp\left[i(k_x x + \int^z k_z(z')dz' - \omega t)\right], \tag{26}$$

where f is any one of the perturbed variables. Like Arons, we take $k_x = (1-\mu^2)^{1/2}k$ and $k_z = \mu k$. The basic small parameter of the WKB approximation is $(kH)^{-1} \sim \epsilon \ll 1$ (we assume that $\mu \sim 1$).

Retaining all terms that are potentially of leading order, the linearized continuity equation is

$$-i\omega\delta\rho - \delta v_z \frac{\rho h_\rho}{H} = -i\rho\mathbf{k}\cdot\delta\mathbf{v}, \tag{27}$$

the momentum equation is

$$\begin{aligned} -i\omega\rho\delta\mathbf{v} = &-i\mathbf{k}\delta p_g - i\mathbf{k}\tfrac{\mathbf{B}\cdot\delta\mathbf{B}}{4\pi} + i\delta\mathbf{B}\tfrac{\mathbf{k}\cdot\mathbf{B}}{4\pi} \\ &+\tfrac{4\pi}{c}\left(\delta\kappa\rho\mathbf{H} + \kappa\delta\rho\mathbf{H} + \kappa\rho\delta\mathbf{H}\right) - g\delta\rho\,\hat{\mathbf{z}}, \end{aligned} \tag{28}$$

the induction equation is

$$-i\omega\delta\mathbf{B} = -i\mathbf{B}(\mathbf{k}\cdot\delta\mathbf{v}) + i(\mathbf{k}\cdot\mathbf{B})\delta\mathbf{v}, \tag{29}$$

the gas energy equation is

$$-i\omega\delta u - \delta v_z \frac{u h_g}{H} = -i\gamma u(\mathbf{k}\cdot\delta\mathbf{v}) + 4\pi\kappa_a\rho(\delta J - \delta B), \tag{30}$$

the intensity equation is

$$\begin{aligned} \tfrac{1}{c}(-i\omega\delta J - \delta v_z \tfrac{h_r B}{H}) - \tfrac{4J}{3\rho c}(-i\rho\mathbf{k}\cdot\delta\mathbf{v}) = \\ -i\mathbf{k}\cdot\delta\mathbf{H} + \kappa_a(\delta B - \delta J), \end{aligned} \tag{31}$$

and the flux equation is

$$\frac{1}{c}(i\omega\delta\mathbf{H}) = -\frac{1}{3}i\mathbf{k}\delta J - \kappa\rho\,\delta\mathbf{H} - \boldsymbol{\kappa}\,\delta\rho\,\mathbf{H} - \delta\kappa\,\rho\mathbf{H}. \tag{32}$$

Together with the constraint $k\cdot\delta\mathbf{B} = 0$, these equations imply a complicated eleventh-order dispersion relation $D(\omega,\mathbf{k}) = 0$ which we shall not record here (the most compact form of the full relation is the above equations). We have confirmed that it contains the dispersion relations for magnetohydrodynamic (MHD) waves, magnetoatmospheric waves (e.g. Thomas 1982), internal waves, sound waves in a radiating fluid (Mihalas & Mihalas 1984), and the overstable photon bubble mode of Arons (1992) as special cases.

6 OVERSTABILITY

6.1 Numerical Solution

The full dispersion relation is analytically intractable. In the end we will need to expand it in a small parameter to retrieve the relevant pieces that describe the photon bubble mode. To motivate an asymptotic approach, we shall first solve the dispersion relation numerically. The parameters are those appropriate to a disc around a $10\,M_\odot$ black hole accreting at

the Eddington rate, at $r = 50GM/c^2$. Assume $\alpha = 0.1$, so $\beta_M = 2.5$, and that the field is purely vertical. Then $\beta_r = 0.05$, $c/c_i = 120$, $l_* = 0.07H$, $\gamma = 5/3$, $M_0 = 0.4$. Take $\mathcal{L} = 1 - 3\beta_r/2 = 0.93$ so that convection is absent. We consider a set of modes with varying k and $\mu = 1/3$. The real part of the full dispersion relation is shown in Figure 1. The real part of the overstable mode is marked with a heavy solid line; the imaginary part has comparable magnitude.

At large wavenumber the dispersion relation is easily interpreted because it becomes analytic. For general μ,

$$\begin{aligned}&(\omega^2 - \tfrac{1}{3}c^2k^2)(\omega^2 - \tfrac{1}{\beta_M}c_i^2k^2\mu^2) \times \\ &(\omega^4 - (\beta_r\gamma c_i^2k^2 + \tfrac{1}{\beta_M}c_i^2k^2)\omega^2 + \tfrac{1}{\beta_M}\beta_r\gamma c_i^2k^4\mu^2) \times \\ &(\omega + \tfrac{ic^2}{D_0})^2(\omega + \tfrac{48i(1-\beta_r)(\gamma-1)D_0}{\beta_r\gamma l_*^2}) = 0.\end{aligned} \tag{33}$$

The first term in parentheses is the flux-limited electromagnetic wave with phase velocity $c/\sqrt{3}$ (see Mihalas & Mihalas 1984). The second term is the Alfvén wave. The third term contains the fast and slow MHD modes. These are labeled in Figure 1. Notice that the overstable mode becomes the slow MHD mode at large wavenumber. The final three modes are strongly damped entropy modes.

Notice that the slow MHD mode has rather low frequency. This is because at short wavelengths radiation diffuses rapidly out of the perturbation and so the radiation pressure perturbation is nil. Thus only gas pressure provides a restoring force for this mode. For $\beta_r \ll 1$, the slow mode has $\omega^2 \approx \beta_r c_i^2 \gamma k^2 \mu^2$, so the slow mode velocity is directly related to the sound speed associated with the gas pressure alone. A radiation pressure dominated fluid is thus rather delicate on small scales in that it is easily compressed.

6.2 Vertical Magnetic Field

We have shown that an overstable mode exists; we will now demonstrate the existence of this mode analytically. Again we consider only the simple case of a purely vertical magnetic field. The basic small parameter is $\epsilon = (kH)^{-1}$. The survey of conditions in accretion discs (§2) then suggests the following scalings for the other parameters in the problem. We take $H \sim 1, c_i \sim 1, c_i/c \sim \epsilon, \beta_M \sim 1, l_* \sim 1, \beta_r \sim \epsilon^{3/2}, M_0 \sim 1$, $\mathcal{L} = 1 - O(\beta_r)$, and $\mu \sim 1$. Using a little asymptotic foresight, we also take $\omega \sim \epsilon^{-1/2}$.

Expanding $D(\omega, \mathbf{k})$ through leading order in ϵ, we find the remarkably simple relation

$$\omega^2 = -igk\mu(1-\mu^2). \tag{34}$$

One root of this equation, the one with negative phase velocity, describes the overstable photon bubble mode.

What happens if we increase the importance of gas pressure? Suppose that $\beta_r \sim \epsilon$. Then we find

$$\omega^2 = -igk\mu(1-\mu^2) + \beta_r c_i^2 \mu^2 k^2. \tag{35}$$

The first term is the photon bubble term, while the second is that of a slow MHD mode in which gas pressure provides the only restoring force. If we increase $\beta_r/(kH)$ still further, then the first term becomes subdominant. This branch of the dispersion relation then becomes the slow mode, which is stable to leading order in WKB.

Since the overstability is no longer present to leading order in ϵ when $k \gtrsim (\beta_r H)^{-1}$, and $k \gtrsim 1/H$, the overstability fails in a WKB sense when $\beta_r \sim 1$. This provides an approximate limit on the overstable region in parameter space.

Now suppose that the diffusion time is long compared to a dynamical time, i.e. $M_0 \sim \epsilon^{1/2}$. Again expanding $D(\omega, \mathbf{k})$ to leading order in ϵ we find

$$\omega^2 + i\omega \frac{4c_i^2\mu^2}{D_0} + igk\mu(1-\mu^2) = 0. \tag{36}$$

The new term causes damping. It shows that instability is present in a WKB sense only on scales such that $k \gtrsim 1/(HM_0^2)$.

6.3 General Discussion

We can obtain a better physical understanding of the origin of the photon bubble instability, and a more general dispersion relation, by expanding the linearized equations in ϵ. This is not trivial because we must assign an explicit relative ordering in ϵ to all the perturbed variables. We do this by solving for the eigenvectors $\delta f(\delta\rho)$ (f represents any of the perturbed variables), and then determining the relative ordering. We use the same ordering for the model parameters as that described at the beginning of §6.2. This procedure reveals that the photon bubble mode consists predominantly of motions along the magnetic field. This is easy to understand because the frequency of the mode, $\sim \sqrt{gk}$, is smaller by order $\epsilon^{1/2}$ than the Alfvén frequency at the same scale, $\sim kv_A$. Thus the field is stiff enough to resist any motion that bends the field lines.

Writing out the dominant terms in full and allowing for a general orientation of the initially uniform magnetic field, the continuity equation becomes

$$-i\omega\delta\rho = -i\rho\mathbf{k}\cdot\delta\mathbf{v}. \tag{37}$$

The perturbed velocities perpendicular to the magnetic field vanish. Denoting the unit vector parallel to the field by $\hat{\mathbf{b}}$, the parallel component of the velocity is governed by

$$-i\omega\delta\mathbf{v}\cdot\hat{\mathbf{b}} = \frac{4\pi\kappa\rho}{c}\delta\mathbf{H}\cdot\hat{\mathbf{b}}, \tag{38}$$

so the material feels only the perturbed radiation force. The induction equation is irrelevant, since the field is stiff. The perturbed flux obeys

$$\mathbf{k}\cdot\delta\mathbf{H} = 0. \tag{39}$$

The perturbed flux is conserved, so there is no exchange of energy between the radiation field and the fluid on a timescale ω^{-1}. Combining this result with the perturbed intensity equation, we find

$$\delta\mathbf{H} = -\frac{\delta\rho}{\rho}\left(\mathbf{H} - \mathbf{k}\frac{(\mathbf{k}\cdot\mathbf{H})}{k^2}\right). \tag{40}$$

In words, the fluid feels a radiation force that is directed parallel to the wave crests and is inversely proportional to the density perturbation. Radiation escapes more rapidly along density minima, while energy flow is impeded along density maxima. Combining eqns.(37), (38), and (40), and denoting the unit vector along $\mathbf{k}$ by $\hat{\mathbf{k}}$, we find the general dispersion relation

$$\omega^2 = -ikg(\hat{\mathbf{k}}\cdot\hat{\mathbf{b}})\left[\hat{\mathbf{b}}\cdot\hat{\mathbf{z}} - (\hat{\mathbf{k}}\cdot\hat{\mathbf{z}})(\hat{\mathbf{k}}\cdot\hat{\mathbf{b}})\right]. \tag{41}$$

This implies that the growth rate of the instability is largest for $k_z = 0$ (where the WKB approximation is not strictly valid) and for $\mathbf{B}$ at an angle of $\pm\pi/4$ to the wavevector.

It is readily shown that if the fluid is allowed to move freely, unconstrained by the magnetic field, then the mode frequency vanishes if we start from the reduced set of equations above. More rigorously, if we begin with the full set of linearized equations and turn off the magnetic field, then the stability criterion reduces to the condition that the Brunt-Vaisala frequency be real.

To understand the overstability on a more qualitative level, consider the history of a single fluid element in the case $k_z = 0$ where $\mathbf{B}$ makes an angle $\pi/4$ with the vertical. Suppose the fluid element initially lies in a density minimum, so it feels an increased radiation force from below. It is accelerated upward along the field line. This soon puts it in a region of

convergent flow and then in a density maximum: the phase velocity of the overstable mode is such that the density maxima progress downward along field lines. It is now shadowed by the density maximum, the radiation flux drops somewhat, and it falls downward along the field line. This puts it in a region of divergent flow and then a density minimum, so it is accelerated upward again. Each round of acceleration is larger than the one before, and so an overstability results.

The photon bubble instability described by Arons is physically similar to ours, but there are two differences. The most important is that Arons considers the asymptotic regime $M_o \ll 1$ appropriate to neutron star polar caps. This leads to a rather more complicated dispersion relation for the overstable modes, with the growth rate proportional to M_0^3. In this limit, there is a small phase offset between the radiation flux and the density, rather than the large phase offset that initiates the instability in the limits $M_o \sim 1$. Another difference is that Arons uses the Rosseland diffusion approximation, which requires that the gas and radiation have the same temperature, while we use a nonequilibrium diffusion approximation, which allows the gas and radiation "temperature" to differ. In the end this turns out to have no effect whatsoever on the overstable modes, because gas pressure (and hence gas temperature) is completely negligible for modes with frequency and wavelength comparable to the photon bubble mode. Finally, we note that in the appropriate limit we are able to recover Arons's dispersion relation, thus confirming his analysis.

7 NONLINEAR OUTCOME

The astrophysical implications of the photon bubble instability depend on the nonlinear outcome, to which the linear theory is an unreliable guide. Numerical experiments by Hsu et al. (1997) show that in the regime $M_0 = c/(\tau_{es} c_i) \ll 1$, $\beta_M \ll 1$ the photon bubble instability leads to greatly enhanced vertical transport of energy. It thus seems likely that the photon bubble instability will enhance vertical energy transport in discs.

It is tempting to make an analogy between photon bubbles and convection. Convection generates such efficient transport of energy that it erases the inverted entropy gradient that initiated it and, if forced, maintains the convective fluid in a marginally stable state. It is natural to think that the photon bubble instability drives the disc toward a marginally stable state as well. Our analysis does not give any rigorous stability criteria, but it does show that when $\beta_r \sim 1$ the instability is no longer present at leading order. Thus the instability is likely

to be absent or greatly reduced in strength when $\beta_r \sim 1$. If the disc is initially radiation dominated, photon bubbles might then transport energy efficiently out of the disc, lowering the temperature and raising β_r toward 1.

The nonlinear outcome of the photon bubble instability can be characterized by a cooling rate $Q^-(\Sigma, T_c)$, which is the energy lost per unit time per unit disc area. The thermal and viscous stability properties of the disc depend on how Q^- varies with temperature and surface density (Piran 1978). Ultimately this can only be evaluated from a fully nonlinear theory, but if Q^- is a steep enough function of T_c, then the disc can be viscously and thermally stable.

Our analysis also shows that when the dimensionless radiative diffusion rate M_0 decreases the instability weakens, in that instability is only present for $kH \gtrsim M_0^{-2}$. This effect may also shut off photon bubbles. At low accretion rates, for example, a disc with $\beta_r \sim 1$ will have large M_0, while at larger accretion rates M_0 is smaller.

Photon bubbles might also change the emergent spectrum. The bubbles make the disc porous to radiation, so a photon traverses a shorter path to the surface than it would if the disc were subject only to ordinary radiative difffusion. The photon distribution therefore has less opportunity to thermalize. In addition, the vertical transport might be more episodic, enhancing variability.

This discussion is speculative. Other physical processes may contribute to vertical energy transport in discs: ordinary convection (but see the discussion of Rees 1987) or magnetic Rayleigh-Taylor instability (see the flux-tube calculation of Sakimoto & Coroniti 1989) may dominate photon bubbles. In addition, MHD turbulence initiated by the Balbus-Hawley instability must coexist with photon bubbles. We have developed a linear theory only when these effects are absent. The full nonlinear development of the radiation dominated disc can probably only be studied realistically via three dimensional numerical experiments. Numerical methods exist for treating the radiative transfer in a flux-limited Rosseland diffusion approximation. Since the character of the unstable mode is identical in the nonequilibrium and Rosseland diffusion approximations, numerical studies of the nonlinear evolution of a radiation dominated disc may be immediately practical.

8 SUMMARY

We have considered the linear theory of a radiation dominated atmosphere with spatially constant magnetic field as a model for the radiation dominated inner parts of thin accretion discs around compact objects. The model is subject to an overstable photon bubble mode that tends to separate radiation and matter. The photon bubble dispersion relation for a general orientation of the magnetic field is given by eqn.(41). Vertical energy transport is likely to be enhanced in the nonlinear outcome. The disc may then deflate until it is no longer radiation pressure dominated.

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FIGURE CAPTIONS

Figure 1. The real part of the WKB dispersion relation for a radiation-dominated atmosphere with uniform vertical field. See text for details. The abscissa is the wavenumber in units of the scale height; the ordinate is the phase velocity in units of the isothermal sound speed $(P/\rho)^{1/2}$. The real part of the phase velocity for the photon bubble mode is shown as a heavy line. The growth rate (imaginary part) is comparable in magnitude.

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