SHAPE SENSITIVITY OF CURVILINEAR CRACKS

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Summary

We consider the linear elasticity model of a 2D-solid with a crack under the stress-free boundary condition at the crack. To analyze the sensitivity of a crack shape, one needs to define variations by distributed parameters. We apply the technique using the global variations of a solution and obtain the expansion of the potential energy functional of an arbitrary order with respect to the function describing the crack shape. By the Griffith criterion, from the representation of the total potential energy up to the second order terms we deduce the parametric problem for local optimization of the crack shape and condition of its solvability. This allows us to study also a quasi-static model depending on the loading parameter.

The shape sensitivity was studied in Simon [1], Sokolowski and Zolesio [2], Khludnev and Sokolowski [3]; variations of a crack were considered in Rice [4], Khludnev and Kovtunenko [5], Bach, Khludnev and Kovtunenko [6].

1. Variation of a crack shape

Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with a boundary Γ of the class $C^{0,1}$, and $\overline{\Omega} = \Omega \cup \Gamma$. We define a crack Γ_0 as the segment $\{0 < x_1 < l,\, x_2 = 0\}$ of the x_1-axis in $\mathbf{R}^2$ which lies inside some subdomain B of Ω. We consider a body with a crack occupying the domain $\Omega_0 = \Omega \setminus \overline{\Gamma}_0$.

Let $f \in [C^\infty(\overline{\Omega})]^2$ be a given force, and let the solid possess the usual linear Hooke law

$$\sigma_{ij}(u) = c_{ijkl}\varepsilon_{kl}(u), \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i,j = 1,2,$$

$$c_{ijkl} = c_{jikl} = c_{klij}, \quad c_1\xi_{ij}\xi_{ij} \le c_{ijkl}\xi_{kl}\xi_{ij} \le c_2\xi_{ij}\xi_{ij}, \quad c_1, c_2 > 0,$$

with the constant coefficients c_{ijkl} for simplicity. We look for the displacement vector $u = (u_1, u_2)$ in the functional space

$$\widetilde{H}^1(\Omega_0) = \{u \in [H^1(\Omega_0)]^2, \quad u = 0 \quad \text{on } \Gamma\},$$

which includes the zero displacement condition at the external boundary Γ. At the crack faces $\Gamma_0^{\pm}$ split by the normal vector $(0,1)$ to Γ_0 we assume the usual stress-free boundary condition of the Neumann type $\sigma_{12}(u)=\sigma_{22}(u)=0$.

Introduce the potential energy functional of the body with the crack

$$\Pi(u;\Omega_0)=\frac{1}{2}\int\limits_{\Omega_0}\sigma_{ij}(u)\varepsilon_{ij}(u)-\int\limits_{\Omega_0}f_i u_i. \tag{1}$$

The equilibrium problem is equivalent to the minimization of this functional and gives the variational equation

$$\int\limits_{\Omega_0}\sigma_{ij}(u^0)\varepsilon_{ij}(v)=\int\limits_{\Omega_0}f_i v_i \quad \forall v\in\widetilde{H}^1(\Omega_0). \tag{2}$$

By the well-known properties of Π and by the Korn inequality provided that the functions are zero on Γ, there exists a unique solution $u^0\in\widetilde{H}^1(\Omega_0)$ of (2). The corresponding boundary value problem for u^0 is of the form:

$$-\sigma_{ij,j}(u^0)=f_i,\quad i=1,2,\quad \text{in }\Omega_0;$$

$$\sigma_{i2}(u^0)=0,\quad i=1,2,\quad \text{on }\Gamma_0^{\pm};\quad u^0=0\quad \text{on }\Gamma.$$

Let us now introduce a curvilinear crack Γ_ψ close to Γ_0 given by a smooth function $\psi\in C^1(\mathbf{R})$, $\psi(x_1)\equiv 0$ outside $(0,l)$, namely

$$\Gamma_\psi=\{0<x_1<l,\quad x_2=\psi(x_1)\}.$$

We suppose that $\overline{\Gamma}_\psi\subset B$ and $\overline{B}\subset\Omega$. The normal vector ν to Γ_ψ defines its positive Γ_ψ^+ and negative Γ_ψ^- faces. In the domain $\Omega_\psi=\Omega\setminus\overline{\Gamma}_\psi$ with the perturbed crack Γ_ψ we have the potential energy functional

$$\Pi(u;\Omega_\psi)=\frac{1}{2}\int\limits_{\Omega_\psi}\sigma_{ij}(u)\varepsilon_{ij}(u)-\int\limits_{\Omega_\psi}f_i u_i \tag{3}$$

on the functional space

$$\widetilde{H}^1(\Omega_\psi)=\{u\in[H^1(\Omega_\psi)]^2,\quad u=0\quad \text{on }\Gamma\}.$$

By the same reasons, there exists a unique solution $u^\psi\in\widetilde{H}^1(\Omega_\psi)$ of the equilibrium problem

$$\int\limits_{\Omega_\psi}\sigma_{ij}(u^\psi)\varepsilon_{ij}(v)=\int\limits_{\Omega_\psi}f_i v_i \quad \forall v\in\widetilde{H}^1(\Omega_\psi), \tag{4}$$

which fulfils the following relations

$$-\sigma_{ij,j}(u^\psi)=f_i,\quad i=1,2,\quad \text{in }\Omega_\psi;$$

$$\sigma_{ij}(u^\psi)\nu_j = 0, \quad i = 1,2, \quad \text{on } \Gamma_\psi^\pm; \quad u^\psi = 0 \quad \text{on } \Gamma.$$

We are now going to construct a local coordinate transformation Λ mapping Ω_ψ onto the fixed domain Ω_0. Choose a smooth cut-off function η such that $supp\, \eta \subset \Omega$ and $\eta(x) \equiv 1$ on B. We denote $\Psi(x) \equiv \psi(x_1)\eta(x)$ for simplicity and define the transformation

$$\Lambda^{-1}: \; y_1 = x_1, \quad y_2 = x_2 + \Psi(x), \quad x \in \Omega_0, \quad y \in \Omega_\psi. \tag{5}$$

Its Jacobian $J = 1 + \Psi_{,2}$ is strictly positive for any ψ with $\|\psi\|_{C([0,l])} \le \varepsilon_0$ for some ε_0 small enough. Therefore, the correspondence Λ^{-1} is one-to-one and $\Lambda: \Omega_\psi \to \Omega_0$. In the future we will denote by 'hat' the transformed function $\Lambda \circ u$, namely

$$u(y) = u(x_1, x_2 + \Psi(x)) \equiv \widehat{u}(x), \quad x \in \Omega_0, \quad y \in \Omega_\psi. \tag{6}$$

One can easily calculate the inverse matrix of the transformation (5) as

$$\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{J}\Psi_{,1} & 1 - \frac{1}{J}\Psi_{,2} \end{pmatrix},$$

which leads to the following relation between derivatives

$$\frac{\partial u}{\partial y_i} = \frac{\partial \widehat{u}}{\partial x_i} - \frac{1}{J}\Psi_{,i}\frac{\partial \widehat{u}}{\partial x_2}, \quad i = 1,2.$$

Consequently, we can write the transformation of the strain and stress tensors

$$\varepsilon_{ij}(u) = \varepsilon_{ij}(\widehat{u}) - \frac{1}{J}E_{ij}(\Psi;\widehat{u}_{,2}), \quad \sigma_{ij}(u) = \sigma_{ij}(\widehat{u}) - \frac{1}{J}\Sigma_{ij}(\Psi;\widehat{u}_{,2}) \tag{7}$$

with the new forms

$$E_{ij}(\Psi;w) = \frac{1}{2}\Big(\Psi_{,i}w_j + \Psi_{,j}w_i\Big), \quad \Sigma_{ij}(\Psi;w) = c_{ijkl}E_{kl}(\Psi;w), \quad i,j = 1,2.$$

Applying the transformation (5) to the integrals in (4), in view of (6) and (7) we rewrite the equation (4) on the fixed domain Ω_0:

$$\int_{\Omega_0} \sigma_{ij}(\widehat{u}^\psi)\varepsilon_{ij}(v) + A(\Psi;\widehat{u}^\psi,v) + B\Big[\frac{1}{J}\Big](\Psi^2;\widehat{u}^\psi,v) = \int_{\Omega_0} J\widehat{f}_i v_i \quad \forall v \in \widetilde{H}^1(\Omega_0), \tag{8}$$

where the bilinear forms A and $B[\cdot]$ are given as

$$\begin{aligned} A(\Psi;u,v) &= \int_{\Omega_0} \Big(\Psi_{,2}\sigma_{ij}(u)\varepsilon_{ij}(v) - \Sigma_{ij}(\Psi;u_{,2})\varepsilon_{ij}(v) - \sigma_{ij}(u)E_{ij}(\Psi;v_{,2})\Big), \\ B[w](\Psi^2;u,v) &= \int_{\Omega_0} w \cdot \Sigma_{ij}(\Psi;u_{,2})E_{ij}(\Psi;v_{,2}). \end{aligned} \tag{9}$$

Thus, the transformed solution $\widehat{u}^\psi \in \widetilde{H}^1(\Omega_0)$ is the unique solution of the problem (8).

2. Global variations of the solution

We seek an expansion of the solution $\widehat{u}^\psi$ of (8) in the form of a series

$$\widehat{u}^\psi = u^0 + \sum_{n=1}^{\infty} \frac{1}{n!} \overset{(n)}{u} (\psi^n), \quad \overset{(0)}{u} (\psi^0) \equiv u^0. \tag{10}$$

The functions $\dot{u}(\psi)$, $\ddot{u}(\psi^2)$, ..., $\overset{(n)}{u} (\psi^n)$ are called the global variations of a solution with respect to ψ of the corresponding order n. To obtain them, we need to decompose the equation (8) with respect to ψ.

First of all, from (6) one can easily write the expansion of the transformed force $\widehat{f}$ by (5) as

$$\widehat{f} = f + \sum_{n=1}^{\infty} \frac{\Psi^n}{n!} \frac{\partial^n f}{\partial x_2^n} \tag{11}$$

with the obvious estimate

$$\Big\| \widehat{f} - \sum_{k=0}^{n} \frac{\Psi^k}{k!} \frac{\partial^k f}{\partial x_2^k} \Big\|_{[L^2(\Omega)]^2} \leq \|\psi\|^{n+1}_{C^1([0,l])}, \quad n = 0, 1, \ldots. \tag{12}$$

Therefore, it follows from (11) that

$$J\widehat{f}_i = (1 + \Psi_{,2})\widehat{f}_i = f_i + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\Psi^n \frac{\partial^{n-1} f_i}{\partial x_2^{n-1}} \right)_{,2}, \quad i = 1, 2. \tag{13}$$

Also we get

$$\frac{1}{J} = 1 - \frac{\Psi_{,2}}{J} = 1 + \sum_{n=1}^{\infty} (-\Psi_{,2})^n. \tag{14}$$

Let us substitute formally (10), (13), (14) in (8) to obtain

$$\int_{\Omega_0} \sigma_{ij}(u^0)\varepsilon_{ij}(v) + \sum_{n=1}^{\infty} \frac{1}{n!} \Big(\int_{\Omega_0} \sigma_{ij}(\overset{(n)}{u} (\psi^n))\varepsilon_{ij}(v) + nA(\Psi; \overset{(n-1)}{u} (\psi^{n-1}), v)$$

$$+ n(n-1) \sum_{k=0}^{n-2} \frac{(n-2)!}{k!} B\Big[(-\Psi_{,2})^{n-2-k} \Big] (\Psi^2; \overset{(k)}{u} (\psi^k), v) \Big)$$

$$= \int_{\Omega_0} f_i v_i + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Omega_0} \left(\Psi^n \frac{\partial^{n-1} f_i}{\partial x_2^{n-1}} \right)_{,2} v_i.$$

We define the functions $\overset{(n)}{u}(\psi^n) \in \widehat{H}^1(\Omega_0)$, $n = 1, 2, ...,$ as the unique solutions of the following elasticity problems

$$\int_{\Omega_0} \sigma_{ij}(\overset{(n)}{u}(\psi^n))\varepsilon_{ij}(v) = \int_{\Omega_0} \left(\Psi^n \frac{\partial^{n-1} f_i}{\partial x_2^{n-1}}\right)_{,2} v_i - nA(\Psi; \overset{(n-1)}{u}(\psi^{n-1}), v) \tag{15}$$

$$-n(n-1)\sum_{k=0}^{n-2} \frac{(n-2)!}{k!} B\Big[\big(-\Psi_{,2}\big)^{n-2-k}\Big](\Psi^2; \overset{(k)}{u}(\psi^k), v) \quad \forall v \in \widetilde{H}^1(\Omega_0).$$

That is an iterative procedure for finding of the functions $\dot{u}(\psi)$, $\ddot{u}(\psi^2)$, ..., $\overset{(n)}{u}(\psi^n)$ with the initial value $\overset{(0)}{u}(\psi^0) = u^0$ being the solution of the same elasticity problem (2). In particular, for $n = 1$ we have

$$\int_{\Omega_0} \sigma_{ij}(\dot{u}(\psi))\varepsilon_{ij}(v) = \int_{\Omega_0} (\Psi f_i)_{,2} v_i - A(\Psi; u^0, v) \quad \forall v \in \widetilde{H}^1(\Omega_0). \tag{16}$$

We intend now to prove the correctness of the expansion (10). If we substitute $v = \widehat{u}^\psi$ in (8) and apply the Korn and Hölder inequalities, then for any ψ such that $||\psi||_{C^1([0,l])} \leq \varepsilon_1$ with $\varepsilon_1 > 0$ small enough, the uniform estimate follows

$$||\widehat{u}^\psi||_{\widetilde{H}^1(\Omega_0)} \leq \text{c}. \tag{17}$$

Subtracting (2) from (8), one gets

$$\int_{\Omega_0} \sigma_{ij}(\widehat{u}^\psi - u^0)\varepsilon_{ij}(v) = \int_{\Omega_0} (\widehat{f}_i - f_i + \Psi_{,2}\widehat{f}_i)v_i - A(\Psi; \widehat{u}^\psi, v) - B[\frac{1}{J}](\Psi^2; \widehat{u}^\psi, v).$$

Substitute $v = \widehat{u}^\psi - u^0$ here, apply again the Korn and Hölder inequalities and use (12), (14), (17) to deduce that

$$||\widehat{u}^\psi - u^0||_{\widetilde{H}^1(\Omega_0)} \leq c||\psi||_{C^1([0,l])}. \tag{18}$$

Analogously, the subtraction of (2) and (16) from (8) leads to the relation

$$\int_{\Omega_0} \sigma_{ij}(\widehat{u}^\psi - u^0 - \dot{u}(\psi))\varepsilon_{ij}(v) = \int_{\Omega_0} \big((\widehat{f}_i - f_i - \Psi f_{i,2}) + \Psi_{,2}(\widehat{f}_i - f_i)\big)v_i$$

$$-A(\Psi; \widehat{u}^\psi - u^0, v) - B[\frac{1}{J}](\Psi^2; \widehat{u}^\psi, v)$$

and, in view of (18), to the next estimate

$$||\widehat{u}^\psi - u^0 - \dot{u}(\psi)||_{\widetilde{H}^1(\Omega_0)} \leq c||\psi||^2_{C^1([0,l])}.$$

In the general case, the above procedure takes the form:

$$\int_{\Omega_0} \sigma_{ij}\Big(\widehat{u}^\psi - u^0 - \sum_{k=1}^{n}\frac{1}{k!}\overset{(k)}{u}(\psi^k)\Big)\varepsilon_{ij}(v) = \int_{\Omega_0}\Big(\Big(\widehat{f}_i - f_i - \sum_{k=1}^{n}\frac{\Psi^k}{k!}\frac{\partial^k f_i}{\partial x_2^k}\Big)$$

$$+\Psi_{,2}\Big(\widehat{f}_i - f_i - \sum_{k=1}^{n-1}\frac{\Psi^k}{k!}\frac{\partial^k f_i}{\partial x_2^k}\Big)\Big)v_i - A\Big(\Psi;\widehat{u}^\psi - u^0 - \sum_{k=1}^{n-1}\frac{1}{k!}\overset{(k)}{u}(\psi^k),v\Big)$$

$$-\sum_{m=0}^{n-2} B\Big[\big(-\Psi_{,2}\big)^{n-2-m}\Big]\Big(\Psi^2;\widehat{u}^\psi - \sum_{k=0}^{m}\frac{1}{k!}\overset{(k)}{u}(\psi^k),v\Big)$$

$$-B\Big[\frac{1}{J}\big(-\Psi_{,2}\big)^{n-1}\Big](\Psi^2;\widehat{u}^\psi,v).$$

This provides the fulfilment of the inequalities

$$||\widehat{u}^\psi - \sum_{k=0}^{n}\frac{1}{k!}\overset{(k)}{u}(\psi^k)||_{\widetilde{H}^1(\Omega_0)} \le c||\psi||^{n+1}_{C^1([0,l])}, \quad n = 0,1,..., \tag{19}$$

and proves the following theorem.

Theorem 1. *For* $||\psi||_{C^1([0,l])}$ *small enough, there exist the global variations* $\dot{u}(\psi)$, $\ddot{u}(\psi^2)$, ..., $\overset{(n)}{u}(\psi^n) \in \widetilde{H}^1(\Omega_0)$ *given in* (15), (16), *which yield the expansion* (10) *with the estimate* (19).

3. Variations of the energy functional

We can define the potential energy as a functional $\mathcal{P} : C_0^1([0,l]) \to \mathbf{R}$ by substituting the solution u^ψ of (4) in (3):

$$\mathcal{P}(\psi) \equiv \Pi(u^\psi;\Omega_\psi) = -\frac{1}{2}\int_{\Omega_\psi} f_i u_i^\psi. \tag{20}$$

For $\psi = 0$ one needs to substitute the solution u^0 of (2) in (1) to deduce from (20) that

$$\mathcal{P}(0) \equiv \Pi(u^0;\Omega_0) = -\frac{1}{2}\int_{\Omega_0} f_i u_i^0. \tag{21}$$

Applying the transformation (5) to the integral in (20), we have

$$\mathcal{P}(\psi) = -\frac{1}{2}\int_{\Omega_0} J\widehat{f}_i\widehat{u}_i^\psi,$$

and by Theorem 1 one can substitute the representations (10), (13) to deduce the expansion of $\mathcal{P}$,

$$\mathcal{P}(\psi) = \mathcal{P}(0) + \sum_{n=1}^{\infty} \frac{1}{n!} \mathcal{P}_0^{(n)}(\psi^n), \tag{22}$$

where the variations of the order n, $n = 1, 2, ...$, are as follows:

$$\mathcal{P}_0^{(n)}(\psi^n) = -\frac{1}{2} \int_{\Omega_0} \Big(\sum_{k=0}^{n-1} \frac{n!}{k!(n-k)!} \Big(\Psi^{n-k} \frac{\partial^{n-k-1} f_i}{\partial x_2^{n-k-1}} \Big)_{,2} \overset{(k)}{u}_i (\psi^k) \tag{23}$$

$$+ f_i \overset{(n)}{u}_i (\psi^n) \Big).$$

It follows from (15) that

$$\| \overset{(n)}{u} (\psi^n) \|_{\widetilde{H}^1(\Omega_0)} \le c \|\psi\|^n_{C^1([0,l])}, \quad n = 1, 2, ...,$$

and therefore, from (22), (23) the evident estimates can be also deduced

$$\Big| \mathcal{P}(\psi) - \mathcal{P}(0) - \sum_{k=1}^{n} \frac{1}{k!} \mathcal{P}_0^{(k)}(\psi^k) \Big| \le c \|\psi\|^{n+1}_{C^1([0,l])}, \quad n = 1, 2, ..., \tag{24}$$

which prove the correctness of the expansion (22) of $\mathcal{P}$ with respect to ψ. Let us note that the integrals in (20) and (21) do not depend on the cut-off function η, consequently, all the functionals $\mathcal{P}_0'(\psi)$, $\mathcal{P}_0''(\psi^2)$, ... do not depend on η in $\Psi = \psi\eta$, but only on ψ.

One can reduce the order n of the variations included in formula (23) to the order $n-1$. Indeed, if we take $v = \overset{(n)}{u} (\psi^n)$ in (2) and of $v = u^0$ in (15), then

$$\int_{\Omega_0} f_i \overset{(n)}{u}_i (\psi^n) = \int_{\Omega_0} \sigma_{ij}(u^0) \varepsilon_{ij}(\overset{(n)}{u} (\psi^n)) = \int_{\Omega_0} \sigma_{ij}(\overset{(n)}{u} (\psi^n)) \varepsilon_{ij}(u^0)$$

$$= \int_{\Omega_0} \Big(\Psi^n \frac{\partial^{n-1} f_i}{\partial x_2^{n-1}} \Big)_{,2} u_i^0 - nA(\Psi; \overset{(n-1)}{u} (\psi^{n-1}), u^0)$$

$$-n(n-1) \sum_{k=0}^{n-2} \frac{(n-2)!}{k!} B\Big[(-\Psi_{,2})^{n-2-k} \Big] (\Psi^2; \overset{(k)}{u} (\psi^k), u^0),$$

and its substitution in (23) yields for $n = 2, 3, ...$

$$\mathcal{P}_0^{(n)}(\psi^n) = -\frac{1}{2} \sum_{k=1}^{n-1} \frac{n!}{k!(n-k)!} \int_{\Omega_0} \Big(\Psi^{n-k} \frac{\partial^{n-k-1} f_i}{\partial x_2^{n-k-1}} \Big)_{,2} \overset{(k)}{u}_i (\psi^k)$$

$$-\int\limits_{\Omega_0}\left(\Psi^n\frac{\partial^{n-1}f_i}{\partial x_2^{n-1}}\right)_{,2}u_i^0+\frac{n}{2}A(\Psi;\overset{(n-1)}{u}(\psi^{n-1}),u^0) \tag{25}$$

$$+\frac{n(n-1)}{2}\sum_{k=0}^{n-2}\frac{(n-2)!}{k!}B\Big[(-\Psi_{,2})^{n-2-k}\Big](\Psi^2;\overset{(k)}{u}(\psi^k),u^0),$$

and for $n=1$

$$\mathcal{P}_0'(\psi)=-\int\limits_{\Omega_0}(\Psi f_i)_{,2}u_i^0+\frac{1}{2}A(\Psi;u^0,u^0). \tag{26}$$

In particular, for $n=2$ we have

$$\mathcal{P}_0''(\psi^2)=-\int\limits_{\Omega_0}\Big(\big(\Psi^2f_{i,2}\big)_{,2}u_i^0+(\Psi f_i)_{,2}\dot{u}_i(\psi)\Big)+A(\Psi;\dot{u}(\psi),u^0))+B[1](\Psi^2;u^0,u^0).$$

Moreover, let us take $v=\dot{u}(\psi)$ in (16) to deduce from

$$\int\limits_{\Omega_0}\sigma_{ij}(\dot{u}(\psi))\varepsilon_{ij}(\dot{u}(\psi))=\int\limits_{\Omega_0}(\Psi f_i)_{,2}\dot{u}_i(\psi)-A(\Psi;\dot{u}(\psi),u^0)$$

the next expression

$$\mathcal{P}_0''(\psi^2)=-\int\limits_{\Omega_0}\Big(\big(\Psi^2f_{i,2}\big)_{,2}u_i^0+\sigma_{ij}(\dot{u}(\psi))\varepsilon_{ij}(\dot{u}(\psi))\Big)+B[1](\Psi^2;u^0,u^0). \tag{27}$$

Summarizing the discussion above, we have proven the following result.

Theorem 2. *There exist the variations $\mathcal{P}_0'(\psi)$, $\mathcal{P}_0''(\psi^2)$, ..., $\mathcal{P}_0^{(n)}(\psi^n)$ of the energy functional with respect to the crack shape ψ given by* (23) *or* (25), (26) *such that the expansion* (22) *with the estimate* (24) *holds.*

Let us define the functional of the total potential energy $\mathcal{U}:C_0^1([0,l])\to\mathbf{R}$ by adding the surface energy of the crack,

$$\mathcal{U}(\psi)=\mathcal{P}(\psi)+\gamma\,\mathrm{meas}\,(\Gamma_\psi),\quad \gamma>0,\quad \mathrm{meas}\,(\Gamma_\psi)=\int\limits_0^l\sqrt{1+\psi'(x_1)^2}\,dx_1,$$

and for $\psi=0$

$$\mathcal{U}(0)=\mathcal{P}(0)+\gamma\,\mathrm{meas}\,(\Gamma_0),\quad \mathrm{meas}\,(\Gamma_0)=l.$$

When $\|\psi\|_{C^1([0,l])}$ is small enough, we have the expansion

$$\int\limits_0^l\sqrt{1+(\psi')^2}=l+\frac{1}{2}\int\limits_0^l(\psi')^2+o\big(\|\psi\|^3_{C^1([0,l])}\big).$$

This relation together with (22) provides by Theorem 2 the following representation

$$\mathcal{U}(\psi) = \mathcal{U}(0) + \mathcal{P}_0'(\psi) + \frac{1}{2}\Big(\mathcal{P}_0''(\psi^2) + \gamma \int_0^l (\psi')^2\Big) + o\big(\|\psi\|^2_{C^1([0,l])}\big). \tag{28}$$

To find the locally optimal crack shape ψ, one needs to minimize the functional $\mathcal{U}(\psi)$ by the Griffith criterion on the set $\psi \in C_0^1([0,l])$, $\overline{\Gamma}_\psi \subset B$. The functional $\mathcal{P}_0'(\psi)$ is associated with the linear continuous form $\mathcal{L}_1 : C_0^1([0,l]) \to \mathbf{R}$,

$$\mathcal{L}_1(\psi) = \int_{\Omega_0} \Big(\Big(-(\eta f_i)_{,2} u_i^0 + \frac{1}{2}\eta_{,2}\sigma_{ij}(u^0)\varepsilon_{ij}(u^0) - \Sigma_{ij}(\eta; u^0_{,2})\varepsilon_{ij}(u^0)\Big) \cdot \psi \tag{29}$$

$$-\eta \Sigma_{ij}(x_1; u^0_{,2})\varepsilon_{ij}(u^0) \cdot \psi'\Big),$$

because of the formula

$$E_{ij}(\Psi; w) = \psi E_{ij}(\eta; w) + \eta\psi' E_{ij}(x_1; w), \quad i,j = 1,2.$$

By the same reason, the functional $\mathcal{P}_0''(\psi^2)$ is associated with the symmetric bilinear continuous form $\mathcal{L}_2 : C_0^1([0,l]) \times C_0^1([0,l]) \to \mathbf{R}$,

$$\mathcal{L}_2(\psi_1, \psi_2) = \int_{\Omega_0} \Big(\Big(-(\eta^2 f_{i,2})_{,2} u_i^0 + \Sigma_{ij}(\eta; u^0_{,2}) E_{ij}(\eta; u^0_{,2})\Big) \cdot \psi_1\psi_2$$

$$+\eta \Sigma_{ij}(\eta; u^0_{,2}) E_{ij}(x_1; u^0_{,2}) \cdot (\psi_1\psi_2)' + \eta^2 \Sigma_{ij}(x_1; u^0_{,2}) E_{ij}(x_1; u^0_{,2}) \cdot \psi_1'\psi_2' \tag{30}$$

$$-\sigma_{ij}(\dot{u}(\psi_1))\varepsilon_{ij}(\dot{u}(\psi_2))\Big),$$

where the functions $\dot{u}(\psi_1)$ and $\dot{u}(\psi_2)$, which are linear and continuous in their arguments, are obtained as unique solutions of the following problem for $\psi = \psi_1$ and $\psi = \psi_2$, respectively,

$$\int_{\Omega_0} \sigma_{ij}(\dot{u}(\psi))\varepsilon_{ij}(v) = \int_{\Omega_0} \Big(\Big((\eta f_i)_{,2} v_i - \eta_{,2}\sigma_{ij}(u^0)\varepsilon_{ij}(v)$$

$$+\Sigma_{ij}(\eta; u^0_{,2})\varepsilon_{ij}(v) + \sigma_{ij}(u^0) E_{ij}(\eta; v_{,2})\Big) \cdot \psi \tag{31}$$

$$+\eta\Big(\Sigma_{ij}(x_1; u^0_{,2})\varepsilon_{ij}(v) + \sigma_{ij}(u^0) E_{ij}(x_1; v_{,2})\Big) \cdot \psi'\Big) \quad \forall v \in \widetilde{H}^1(\Omega_0).$$

But the positiveness and the convexity properties of the form $\mathcal{L}_2$ remain unknown. To describe the local minimization problem approximately, we propose the procedure of parametric optimization.

4. Parametric optimization of a crack shape

Let $\{\rho_k\}_{k=1}^N$ be some finite basis of the functions such that $\rho_k \in C^1(\mathbf{R})$, $\rho_k(x_1) \equiv 0$ outside $(0,l)$, $k = 1, ..., N$, and let the graphs of ρ_k lie inside B. For example, we can take the system of local functions on $[0,l]$:

$$\begin{cases} \rho_k(x_1) = A\cos^2 \frac{\pi(x_1 - s_k)}{2\delta} & \text{as } x_1 \in [s_k - \delta, s_k + \delta], \\ \rho_k(x_1) \equiv 0 & \text{otherwise,} \end{cases} \tag{32}$$

where $s_k = k\delta$, $\delta = \frac{l}{N+1}$ and $A > 0$ is some normalizing factor. Let us consider the test function as the linear combination

$$\psi(x_1) = \psi_k \rho_k(x_1), \quad \psi_1, ..., \psi_N \in \mathbf{R}, \tag{33}$$

with N unknown parameters ψ_k. Substituting ρ_k instead ψ in (31), we can find N functions $\dot{u}(\rho_k) \in \widetilde{H}^1(\Omega_0)$, $k = 1, ..., N$, as its unique solutions. By the linearity of (31) it follows obviously that

$$\dot{u}(\psi_k \rho_k) = \psi_k \dot{u}(\rho_k). \tag{34}$$

Therefore, taking the form $\mathcal{L}_2$ given by (30) on the test element (33), we have in view of (34) that $\mathcal{L}_2(\psi_k \rho_k, \psi_n \rho_n) = b_{kn} \psi_k \psi_n$ with the symmetric coefficients

$$b_{nk} = b_{kn} = \mathcal{L}_2(\rho_k, \rho_n), \quad k, n = 1, ..., N.$$

Analogously, we introduce the coefficients

$$c_k = \mathcal{L}_1(\rho_k), \quad k = 1, ..., N,$$

where the form $\mathcal{L}_1$ is given in (29), and thanks to its linearity have $\mathcal{L}_1(\psi_k \rho_k) = c_k \psi_k$. Finally, one can also easily deduce $\gamma \int_0^l (\psi_k \rho_k)'(\psi_n \rho_n)' = a_{kn} \psi_k \psi_n$ with the symmetric coefficients

$$a_{nk} = a_{kn} = \gamma \int_0^l \rho_k' \rho_n', \quad k, n = 1, ..., N.$$

Thus, from (28) we have the approximate relation

$$\mathcal{U}(\psi_k \rho_k) \approx \mathcal{U}(0) + c_k \psi_k + \frac{1}{2}(a_{kn} + b_{kn}) \psi_k \psi_n \equiv U(\psi_1, ..., \psi_N)$$

and can consider the minimization problem

$$U(\psi_1, ..., \psi_N) \ \rightarrow \ \inf. \tag{35}$$

The extremum of the quadratic function U over ψ_k yields $dU/d\psi_k = 0$, $k = 1, ..., N$, and leads to the linear algebraic system of N equations with N unknowns

$$c_n + (a_{kn} + b_{kn})\psi_k = 0, \quad n = 1, ..., N, \tag{36}$$

with the symmetric matrix $\{a_{kn} + b_{kn}\}_{k,n=1}^N$ and the right-hand side $\{-c_n\}_{n=1}^N$. The solvability condition of (36) is

$$\det \{a_{kn} + b_{kn}\}_{k,n=1}^N \neq 0.$$

A minimum in (35) is guaranteed by the conditions

$$\det \{a_{kn} + b_{kn}\}_{k,n=1}^m > 0, \quad m = 1, ..., N. \tag{37}$$

Let us now investigate the system of local functions (32). In this case the matrix $\{a_{kn}\}_{k,n=1}^N$ is of the form

$$\frac{\gamma A^2 \pi^2}{8\delta} \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ & . & . & . & \\ 0 & 0 & 0 & \dots & 2 \end{pmatrix}$$

and has the determinant $\left(\frac{\gamma A^2 \pi^2}{8\delta}\right)^N (N+1) > 0$. If we take $f \equiv 0$ then $u^0 \equiv 0$, and it follows from (29)–(31) that $\{b_{kn}\} = 0$, $\{c_n\} = 0$; consequently, all the conditions (37) are fulfilled and $\psi_k = 0$, $k = 1, ..., N$, is a solution of the system (36) providing the minimum of U. In a standard way, from (2) the estimate can be obtained

$$||u^0||_{\widetilde{H}^1(\Omega_0)} \leq c||f||_{[L^2(\Omega)]^2}.$$

Therefore, for $||f||_{[L^2(\Omega)]^2}$ small enough, we have

$$\det \{a_{kn} + b_{kn}\}_{k,n=1}^m = \det \{a_{kn}\}_{k,n=1}^m + O\big(||f||_{[L^2(\Omega)]^2}\big)$$

due to the representation of the coefficients b_{kn}, that leads to the fulfilment of (37), and the solution of (36) solves (35) also.

The above consideration can be summarized in the following theorem.

Theorem 3. *If the condition*

$$\det \{a_{kn}\}_{k,n=1}^m > 0, \quad m = 1, ..., N,$$

holds, then at least for $||f||_{[L^2(\Omega)]^2}$ *small enough the system* (36) *is solvable and its solution* $\psi_1, ..., \psi_N$ *minimizes the function* U.

On the basis of Theorem 3 we propose a quasi-static model of the local optimization of the crack shape. Introduce a loading parameter $t > 0$ and consider the t-dependent force $f(t) = tf$. By the linearity of the problem (2) with respect to f we have then $u^0(t) = tu^0$. Substituting these values in the expressions for the coefficients, we have $\{b_{kn}(t)\} = t\{b_{kn}\}$, $\{c_n(t)\} = t\{c_n\}$, and therefore, the system (36) reduces to

$$t^2 c_n + (a_{kn} + t^2 b_{kn})\psi_k(t) = 0, \quad n = 1, ..., N. \tag{38}$$

Theorem 3 guarantees the solvability of (38) at least for small t, and its solution $\psi_1(t), ..., \psi_N(t)$ minimizes the function $U(t)$ for every such t.

In conclusion, we can see that to fulfil the above procedure of a local parametric optimization of a crack shape, a unique requirement is to be able to solve the elasticity problem

$$\int_{\Omega_0} \sigma_{ij}(u)\varepsilon_{ij}(v) = F(v) \quad \forall v \in \widetilde{H}^1(\Omega_0)$$

in the fixed domain Ω_0 with the crack Γ_0 for the various right-hand sides F.

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