Non-local constitutive response of a random laminate subjected to configuration-dependent body force

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Abstract

The nonlocal constitutive behaviour of an infinite composite laminate is analyzed. The laminate is treated as random and is subjected to a combination of deterministic and configuration-dependent body force. In this case, in addition to the effective non local elastic operator, other non local constitutive operators must be considered in order to define the mean response of the body. For a laminate subjected to forces that vary only in the direction of lamination, these operators are obtained explicitly. The Hashin–Shtrikman principles developed by Luciano and Willis (2000a), which provide bounds for the operators for general composites, are shown to generate exactly the two operators that define the stress, while giving only bounds for the remaining operator that appears in the expression for the total energy. The case of a two-phase laminate with the layers arranged periodically is presented as an example.

Keywords: A, microstructures ; B, constitutive behaviour, inhomogeneous material; C, energy methods, integral transforms, probability and statistics.

1 Introduction

Luciano and Willis (2000a) considered the non-local constitutive response of a composite medium, with random microgeometry, loaded by a body-force that depended on the realisation of the medium. An example is provided by gravity loading: the body force then has the form ρg, where g is sure but the mass density ρ at position x depends on the material at x. More generally, however, $\rho(x)$ is simply a given property of the material. Introduction of such a field is a necessity if the medium is porous, since force cannot be applied within any cavity. Boutin (1996) considered such body force in the context of periodic homogenization and noted some difficulty in deriving a consistent effective stress-strain relation. The difficulty was resolved by Luciano and Willis (2000a) who observed that an *additional* term arose through the presence of the additional random field ρ. They derived bounds of Hashin–Shtrikman type (Hashin and Shtrikman, 1962a, b) for the effective operators relating the ensemble mean of the stress to the mean strain and the body force. They noted, in addition, that the energy stored in the composite contained a contribution from the interaction of the field produced by the fluctuating part of the body force with itself, and gave Hashin–Shtrikman bounds for this also.

The problem addressed in the present work is a one-dimensional realisation of the general problem discussed by Luciano and Willis (2000a). A laminate is considered, whose tensor of elastic moduli L varies with only one coordinate, x say. The medium is loaded by body force $f(x) + \rho(x)g(x)$, directed either parallel or transverse to the direction of the x-axis. The tensor L is taken to be orthotropic, with one plane of material symmetry normal to the x-axis. Then, the displacement induced by the body force only has a single non-zero component, $u(x)$, parallel to the force, and only a single stress component $\sigma(x)$ appears in the equation of equilibrium. This problem is solved explicitly, first directly and then employing the Hashin–Shtrikman formalism developed by Luciano and Willis (2000a). It emerges that the effective modulus operator is local, and identical with that obtained from conventional laminate theory. The effective operator linking stress to body force is non-local, however. In this simple problem it depends on the properties of the medium taken two points at a time. In consequence, the Hashin–Shtrikman variational approximations (which, in general, provide bounds which incorporate two-point statistics) deliver this new effective operator exactly, as well as the effective modulus. The other new operator, that gives the additional term in the energy, involves points taken three at a time and hence the Hashin–Shtrikman variational approximations deliver only bounds for this. The case of a periodic two-component laminate is treated explicitly, as an example.

2 Formulation

For the one-dimensional problem under discussion, the single non-trivial equilibrium equation is

$$\sigma' + f + \rho g = 0. \tag{2.1}$$

Here, the prime signifies differentiation with respect to x. The 'sure' component f of the body force is carried at no extra cost for the analysis. The stress component σ satisfies

$$\sigma(x) = L(x)e(x); \quad e(x) = u'(x), \tag{2.2}$$

in which $L(x)$ denotes the single relevant component of the elastic constant tensor. The function $\rho(x)$ is taken to be a property of the medium: as already remarked, it could be (but does not have to be) the mass density. The functions L and ρ are assumed to be stationary random variables, taking only positive values. More specifically, for the n-component laminate considered below,

$$L(x) = \sum_{r=1}^{n} L_r \chi_r(x), \quad \rho(x) = \sum_{r=1}^{n} \rho_r \chi_r(x), \tag{2.3}$$

where L_r and ρ_r are properties of material of type r and χ_r is the characteristic function of the region occupied by that material. It is assumed that f and g have compact support, contained within the interval $(-l, l)$, say, and that

$$\int (f + \langle\rho\rangle g)\, dx = 0, \tag{2.4}$$

where the angled brackets denote ensemble average.

The problem of concern is provided by equations (2.1) and (2.2) for $-d < x < d$, with boundary conditions

$$u(-d) = u(d) = 0. \tag{2.5}$$

The length d will be assumed much greater than the dimension l characterising the support of f and g, which is in turn greater than the "correlation length" of the medium. The quantities of interest are the mean values $\langle\sigma\rangle$, $\langle e\rangle$ and $\langle u\rangle$. Averaging equation (2.1) gives

$$\langle\sigma\rangle'(x) + f(x) + \langle\rho\rangle(x)g(x) = 0. \tag{2.6}$$

Also, $\langle e\rangle(x) = \langle u\rangle'(x)$. What is lacking is an "effective" constitutive relation linking $\langle\sigma\rangle$ and $\langle e\rangle$. Luciano and Willis (2000a) argued that this would have the form

$$\langle\sigma\rangle = L^{\text{eff}}\langle e\rangle + R^{\text{eff}} g, \tag{2.7}$$

where L^{eff} and R^{eff} are non-local operators, expressible in the form of integrals. A local operator L fits into this framework, in the sense that $[Le](x) = \int L(x)\delta(x - x')e(x')\, dx'$.

Now introduce a comparison medium, with elastic constant L_0 that is independent of x, and set

$$\sigma = Le = L_0 e + \tau, \quad \tau = (L - L_0)e. \tag{2.8}$$

It follows from (2.1) and (2.8) that

$$L_0 u'' + f + \rho g + \tau' = 0, \tag{2.9}$$

and hence that

$$u = u_0 - E_0^\dagger \tau, \tag{2.10}$$

where

$$u_0 = G_0(f + \rho g). \tag{2.11}$$

G_0 and $E_0^\dagger$ are operators. The kernel of G_0 is the Green's function $G_0(x, x')$ associated with the problem (2.9), (2.5), and the kernel of $E_0^\dagger$ is

$$E_0^\dagger(x, x') = \partial G_0(x, x')/\partial x'. \tag{2.12}$$

Explicitly,

$$G_0(x, x') = -\tfrac{1}{2} L_0^{-1} |x - x'| + G_0^{\mathrm{im}}(x, x'), \tag{2.13}$$

where

$$G_0^{\mathrm{im}}(x, x') = -\frac{x x'}{2 L_0 d} + \frac{d}{2 L_0}. \tag{2.14}$$

It follows now, by differentiating (2.11) and employing the second form of (2.8), that

$$(L - L_0)^{-1} \tau + \Gamma_0 \tau = e_0, \tag{2.15}$$

where $e_0 = u_0'$, so that

$$e_0 = E_0(f + \rho g). \tag{2.16}$$

In these equations, the operator E_0 has kernel

$$E_0(x, x') = \partial G_0(x, x')/\partial x, \tag{2.17}$$

and Γ_0 has kernel

$$\Gamma_0(x, x') = \frac{\partial^2 G_0}{\partial x \partial x'} = L_0^{-1}\{\delta(x - x') - 1/(2d)\}. \tag{2.18}$$

Thus, by substituting (2.18) into (2.15),

$$L_0^{-1}(L - L_0)^{-1} L \tau - L_0^{-1} \overline{\tau} = e_0, \tag{2.19}$$

where

$$\overline{\tau} = \frac{1}{2d}\int_{-d}^{d} \tau(x)\, dx. \tag{2.20}$$

Solving equation (2.19) gives

$$\tau = (1 - L_0 L^{-1})\Big\{L_0 e_0 + (\overline{L^{-1}})^{-1}(\overline{e}_0 + L_0 \overline{L^{-1} e_0})\Big\}. \tag{2.21}$$

In principle, it would be possible now to employ the relations developed above to calculate $\langle\tau\rangle$, $\langle\sigma\rangle$ and $\langle e\rangle$, and hence deduce the precise form of the effective constitutive relation (2.7). The relation becomes much simpler, however, if boundary effects are ignored. This can be done by letting $d/l \to \infty$, so that quantities such as $\overline{e}_0$, $\overline{\tau}$ tend to zero. In this limit, equation (2.21) reduces to

$$\tau = (1 - L_0 L^{-1})(L_0 E_0)(f + \rho g), \tag{2.22}$$

where

$$L_0 E_0(x, x') = -\tfrac{1}{2}\mathrm{sgn}(x - x'). \tag{2.23}$$

It follows from the second equation of (2.8) that

$$e = (L - L_0)^{-1}\tau = L^{-1}(L_0 E_0)(f + \rho g), \tag{2.24}$$

and therefore

$$\sigma = Le = (L_0 E_0)(f + \rho g). \tag{2.25}$$

Thus, by ensemble averaging (2.24) and (2.25), it follows that

$$\langle\sigma\rangle = \langle L^{-1}\rangle^{-1}\langle e\rangle - \langle L^{-1}\rangle^{-1}\langle L^{-1}(L_0 E_0)(\rho - \langle\rho\rangle)g\rangle. \tag{2.26}$$

Thus,

$$L^{\mathrm{eff}} = \langle L^{-1}\rangle^{-1}, \quad R^{\mathrm{eff}} = -\langle L^{-1}\rangle^{-1}\langle L^{-1}(L_0 E_0)(\rho - \langle\rho\rangle)\rangle. \tag{2.27}$$

The operator L^{eff} is *local*. The operator R^{eff} is non-local; the choice of L_0 is immaterial, on account of (2.23).

3 Alternative solution

With the objective of expressing the mean stress directly in terms of the mean strain, it is helpful to express e_0 in the form

$$e_0 = \langle e\rangle + E_0(\rho - \langle\rho\rangle)g + \Gamma_0\langle\tau\rangle. \tag{3.1}$$

Then, equation (2.15) can be written

$$(L-L_0)^{-1}\tau+\Gamma_0(\tau-\langle\tau\rangle)=\langle e\rangle+E_0(\rho-\langle\rho\rangle)g=:\varepsilon, \tag{3.2}$$

the latter equality defining ε. This equation will now be solved, in the limit $d/l\to\infty$.[1] First, substituting for Γ_0 as before,

$$L_0^{-1}(L-L_0)^{-1}L\tau=\varepsilon+L_0^{-1}\langle\tau\rangle. \tag{3.3}$$

Therefore,

$$\tau=(1-L_0L^{-1})(L_0\varepsilon+\langle\tau\rangle) \tag{3.4}$$

and so

$$\langle\tau\rangle=\langle L^{-1}\rangle^{-1}(\langle\varepsilon\rangle-L_0\langle L^{-1}\varepsilon\rangle). \tag{3.5}$$

Hence, reverting to (3.4),

$$\tau=(1-L_0L^{-1})\Big\{L_0\varepsilon+\langle L^{-1}\rangle^{-1}(\langle\varepsilon\rangle-L_0\langle L^{-1}\varepsilon\rangle)\Big\}. \tag{3.6}$$

Luciano and Willis (2000a) designated the solution of (3.2), not just for a laminate,

$$\tau=T\varepsilon. \tag{3.7}$$

Equation (3.6) gives T explicitly, for the laminate. It is a local operator in this case. Substituting ε from its definition (*c.f.* (3.2)) into (3.7), in conjunction with $\langle\sigma\rangle=L_0\langle e\rangle+\langle\tau\rangle$ gives

$$L^{\mathrm{eff}}=L_0+\langle T\rangle,\quad R^{\mathrm{eff}}=\langle TE_0(\rho-\langle\rho\rangle)\rangle. \tag{3.8}$$

Evaluation of the mean values in (3.8) reproduces (2.27).

For an n-component laminate, as specified by equation (2.3),

$$T(x)=\sum_{r=1}^{n}\sum_{s=1}^{n}(\chi_r(x)/p_r)T_{rs}p_s, \tag{3.9}$$

where

$$T_{rs}:=(1-L_0L_r^{-1})\Big[L_0\delta_{rs}/p_r+\langle L^{-1}\rangle^{-1}(1-L_0L_s^{-1})\Big]. \tag{3.10}$$

Here, p_r is the probability of finding material of type r at x:

$$p_r=\langle\chi_r(x)\rangle. \tag{3.11}$$

It is independent of x, and so coincides with the volume fraction of material of type r, on account of the assumed statistical uniformity.

[1]In fact, *all* subsequent reasoning will be for this limiting case, which will not be mentioned explicitly each time.

4 Hashin–Shtrikman approximation

Luciano and Willis (2000a) developed Hashin–Shtrikman approximations for a general n-component composite which includes, as a special case, the laminate of present concern. Substitution of the trial field

$$\tau(x) = \sum_{r=1}^{n} \tau_r(x)\chi_r(x) \tag{4.1}$$

into the stochastic version of the Hashin–Shtrikman variational principle and optimizing over the functions τ_r requires that τ_r satisfy the equations

$$p_r(L_r - L_0)^{-1}\tau_r + \sum_{s=1}^{n} \{\Gamma_0\psi_{rs}\}\tau_s = p_r\varepsilon_r, \tag{4.2}$$

where

$$p_r\varepsilon_r := p_r\langle e\rangle + \sum_{s=1}^{n} \{E_0\psi_{rs}\}\rho_s g. \tag{4.3}$$

$\{\Gamma_0\psi_{rs}\}$ denotes an operator whose kernel is $\Gamma_0(x-x')\psi_{rs}(x-x')$, and $\{E_0\psi_{rs}\}$ is defined similarly.

$$\psi_{rs}(x-x') = p_{rs}(x-x') - p_r p_s, \tag{4.4}$$

where $p_{rs}(x-x')$ is the probability of finding simultaneously material r at x and material s at x':

$$p_{rs}(x-x') = \langle\chi_r(x)\chi_s(x')\rangle. \tag{4.5}$$

It is a function of $(x-x')$ only, on account of the assumed statistical homogeneity.

Since, for the present problem, $\Gamma_0(x,x') = L_0^{-1}\delta(x-x')$, the operator $\{\Gamma_0\psi_{rs}\}$ has kernel $L_0^{-1}\delta(x-x')\psi_{rs}(0) = L_0^{-1}\delta(x-x')p_r(\delta_{rs}-p_s)$ (no sum on r). Therefore, equation (4.2) can be solved, for a laminate, in parallel to the solution (3.6) of (3.2):

$$\tau_r = \sum_{s=1}^{n} T_{rs}p_s\varepsilon_s, \tag{4.6}$$

where T_{rs} is as specified in equation (3.10).

In the Hashin–Shtrikman approximation, therefore,

$$\langle\tau\rangle = \sum_{r=1}^{n} p_r\tau_r = (\langle L^{-1}\rangle^{-1} - L_0)\langle e\rangle - \langle L^{-1}\rangle^{-1}\sum_{r=1}^{n}\sum_{s=1}^{n} L_r^{-1}\{L_0E_0\psi_{rs}\}\rho_s g. \tag{4.7}$$

This yields, as Hashin–Shtrikman approximations to L^{eff} and R^{eff} for the n-component laminate, the expressions (2.27) *exactly*.

5 Energy

The energy of the system comprising the composite and its loading mechanism is

$$\mathcal{E} = -\tfrac{1}{2}\int u(f+\rho g)\,dx, \tag{5.1}$$

and this has ensemble mean

$$\langle\mathcal{E}\rangle = -\tfrac{1}{2}\int \Big[\langle u\rangle(f+\langle\rho\rangle g) + \langle u(\rho-\langle\rho\rangle)\rangle g\Big]\,dx. \tag{5.2}$$

As shown by Luciano and Willis (2000a), this can be expressed in the form

$$\begin{aligned}\langle\mathcal{E}\rangle \;=\; & -\tfrac{1}{2}\int \Big[(f+\langle\rho\rangle g)G^{\text{eff}}(f+\langle\rho\rangle g) - 2(f+\langle\rho\rangle g)E^{\text{eff}\dagger}R^{\text{eff}}g \\ & +g(S^{\text{eff}} + R^{\text{eff}\dagger}\Gamma^{\text{eff}}R^{\text{eff}})g\Big]\,dx,\end{aligned} \tag{5.3}$$

where G^{eff}, $E^{\text{eff}\dagger}$ and Γ^{eff} are defined like G_0, $E_0^\dagger$ and Γ_0, except that L^{eff} replaces L_0. The operator S^{eff} is

$$S^{\text{eff}} = \langle(\rho-\langle\rho\rangle)(G_0 - E_0^\dagger T E_0)(\rho-\langle\rho\rangle)\rangle. \tag{5.4}$$

Luciano and Willis (2000a) made the definition

$$V^{\text{eff}} = S^{\text{eff}} + (R^{\text{eff}})^\dagger\Gamma^{\text{eff}}R^{\text{eff}}. \tag{5.5}$$

The term V^{eff} must be independent of L_0, because it was derived from expression (5.2). In the context of a laminated medium, T is local but E_0 is non-local. Therefore, $S^{\text{eff}}$and so also V^{eff}, samples points in the medium up to three at a time. The corresponding Hashin–Shtrikman approximation to S^{eff} (equation (57) of Luciano and Willis (2000a)) is

$$S_{HS} = \sum_{r=1}^{n}\sum_{s=1}^{n}\Big(\rho_r\{G_0\psi_{rs}\}\rho_s - \sum_{t=1}^{n}\sum_{k=1}^{n}(T_{rs}\{E_0\psi_{st}\}\rho_t)\{E_0\psi_{rk}\}\rho_k\Big). \tag{5.6}$$

This employs only two-point information and so cannot be exact, even though the Hashin–Shtrikman approximation delivers the exact L^{eff} and R^{eff}. By appropriate choices of the modulus L_0 of the comparison medium, the Hashin–Shtrikman approximation provides *bounds* for V^{eff}, as discussed by Luciano and Willis (2000a).

6 Two phase laminate

In this section two phase laminates are considered and it is assumed that the elastic moduli of the two phases are such that $L_1 > L_2$. First, the exact expressions of L^{eff} and R^{eff} are obtained. By using $(2.27)_1$, L^{eff} becomes

$$L^{\text{eff}} = (p_1 L_1^{-1} + p_2 L_2^{-1}) = \frac{(L_1 L_2)}{p_1 L_2 + p_2 L_1}. \tag{6.1}$$

On the other hand, R^{eff} is a nonlocal operator that can be expressed via $(2.27)_2$ as

$$R^{\text{eff}}(x - x') = \frac{(L_1 - L_2)(\rho_1 - \rho_2)}{(p_1 L_2 + p_2 L_1)} S_0(x - x'), \tag{6.2}$$

where

$$S_0(x - x') = -\tfrac{1}{2}\text{sgn}(x - x')\psi_{11}(x - x') \tag{6.3}$$

and $\psi_{11}(x - x')$ is defined in (4.4). Further, in (6.2), the identities $\psi_{11}(x - x') = \psi_{22}(x - x') = -\psi_{12}(x - x') = -\psi_{21}(x - x')$, valid for two phase composites, have been used.

As explained in the previous section only *bounds* on the nonlocal operator V^{eff} can be obtained by using the Hashin–Shtrikman approximation. For this reason, two reference materials will be adopted in the following.

First, $L_0 = L_2$ is considered. In this case only T_{11} (see (3.10)) is different from zero; it is equal to

$$T_{11} = \frac{1}{p_1}(\langle L^{-1}\rangle^{-1} - L_2). \tag{6.4}$$

Then, by using (5.6), the following expression for S^{eff}_{HS} is obtained

$$S^{\text{eff}}_{HS}(x - x') = \frac{(\rho_2 - \rho_1)^2}{L_2}\Big[S_g(x - x') + (\frac{(\langle L^{-1}\rangle^{-1} - L_2)}{p_1^2 L_2})(S_0 * S_0)(x - x')\Big], \tag{6.5}$$

where $*$ means convolution and

$$S_g(x - x') = -\tfrac{1}{2}|x - x'|\psi_{11}(x - x'). \tag{6.6}$$

Finally, the Hashin–Shtrikman approximation to the nonlocal operator V^{eff} delivers the following *upper bound*

$$V^{-}_{HS}(x - x') = S^{\text{eff}}_{HS}(x - x') - \frac{((L_1 - L_2)(\rho_1 - \rho_2))^2}{L_1 L_2(p_1 L_2 + p_2 L_1)}(S_0 * S_0)(x - x'), \tag{6.7}$$

and, hence, by substituting the expression (6.5) for S^{eff}_{HS},

$$\begin{aligned} V^{-}_{HS}(x-x') &= (\rho_2-\rho_1)^2\Big[\frac{S_g(x-x')}{L_2}+ \\ &+\frac{(p_2L_1^2-p_1L_2^2+(p_1-p_2)L_1L_2)}{p_1L_1L_2(p_1L_2+p_2L_1)}(S_0*S_0)(x-x')\Big]. \end{aligned} \tag{6.8}$$

Here the notation of Luciano and Willis (2000a) is used: the superscript $^-$ indicates that L_0 is chosen as L_2, which is the smaller of L_1 and L_2 and yields in general a lower bound for L^{eff}, although in this case it gives L^{eff} exactly.

Next, $L_0 = L_1$ is considered. In this case only T_{22} (see (3.10)) is different from zero and it is equal to

$$T_{22} = \frac{1}{p_2}(\langle L^{-1}\rangle^{-1} - L_1). \tag{6.9}$$

Then, by using (5.6), S^{eff}_{HS} becomes

$$S^{\text{eff}}_{HS}(x-x') = \frac{(\rho_2-\rho_1)^2}{L_1}\Big[S_g(x-x') + (\frac{(\langle L^{-1}\rangle^{-1}-L_1)}{p_2^2L_1})(S_0*S_0)(x-x')\Big]. \tag{6.10}$$

This yields a *lower bound* for the nonlocal operator V^{eff}

$$V^{+}_{HS}(x-x') = S^{\text{eff}}_{HS}(x-x') - \frac{((L_1-L_2)(\rho_1-\rho_2))^2}{L_1L_2(p_1L_2+p_2L_1)}(S_0*S_0)(x-x'). \tag{6.11}$$

By substituting the expression (6.10) for S^{eff}_{HS}, this becomes

$$\begin{aligned} V^{+}_{HS}(x-x') &= (\rho_2-\rho_1)^2\Big[\frac{S_g(x-x')}{L_1}+ \\ &-\frac{(p_2L_1^2-p_1L_2^2+(p_1-p_2)L_1L_2)}{p_2L_1L_2(p_1L_2+p_2L_1)}(S_0*S_0)(x-x')\Big]. \end{aligned} \tag{6.12}$$

The operators R^{eff} and the upper and lower bounds on V^{eff} can be expressed easily in the Fourier domain if the Fourier transforms of S_0 and S_g are known. It turns out that, since R^{eff} is odd, its Fourier transform is imaginary. On the other hand, the Fourier transforms of V^{-}_{HS} and V^{+}_{HS} are real.

7 Example

In this section a periodic laminate, characterized by two phases with volume fraction p_1 equal to 0.4 and period $p = 2\pi$ is considered.

Appendix A records the expressions of the Fourier transforms of the characteristic functions and of the two point correlation functions for periodic laminates, together with other useful relations corresponding to the particular laminate considered in this section. In the computations the elastic moduli and the mass densities of the two phases have been taken to be equal to $E_1 = 16$, $E_2 = 8$, $\rho_1 = 200$ and $\rho_2 = 100$.

The Fourier transforms $\tilde{G}^{\text{eff}}(k)$, $\tilde{R}^{\text{eff}}(k)$, $\tilde{V}^{\pm}_{HS}(k)$, and $\tilde{E}^{\text{eff}\dagger}(k)\tilde{R}^{\text{eff}}(k)$ have been calculated using the formulae obtained in the previous section and Appendix A.

The result for $\tilde{R}^{\text{eff}}(k)$ is displayed in Fig.1 even though the operator R^{eff} is given exactly by (6.2) and (6.3), because no previous work has identified this operator. Figure 2 gives the upper and lower bounds $\tilde{V}^{-}_{HS}(k)$ and $\tilde{V}^{+}_{HS}(k)$ for $\tilde{V}^{\text{eff}}_{HS}(k)$. The singularities shown in both of these figures are a consequence of the periodicity of the medium.

Boutin (1996) formally developed solutions for periodic media subjected to configuration-dependent body force, following the methodology of periodic homogenization (Bakhavalov and Panasenko, 1984). However, presumably because of difficulty of interpretation, he presented fully explicit results for a periodic laminate, only in the case of configuration-independent body force ($f \neq 0$, $g = 0$ in present notation). Thus, he gave explicit formulae for components of the tensor L^{eff}, developed in a gradient approximation valid for gradually-varying mean fileds, as far as second gradients. The components L^{eff}_{1111}, L^{eff}_{1212}, L^{eff}_{1313}, corresponding to the cases of loading considered here, contain no terms involving derivatives with respect to $x_1 = x$, consistent with the exact result (6.3). If, however, body force is applied which varies in the x_2 and/or x_3 directions, even the components L^{eff}_{1111}, L^{eff}_{1212}, L^{eff}_{1313} become non-local. The Hashin-Shtrikman methodology of Luciano and Willis (2000a) provides bounds for their Fourier components. Figure 3 displays bounds for $\tilde{L}^{\text{eff}}_{1111}(0, k_2, 0)$, corresponding to a body force with components $(0, f_2(x_2), 0)$. The bounds coincide in the local approximation, corresponding to $k_2 = 0$. Their asymptotic development for $|k_2| << 1$, up to order k_2^2, is consistent with the result of Boutin (1996). A more complete discussion of bounds for periodic composites (not just laminates) will be presented separately (Luciano and Willis, 2000b).

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Appendix A: Relations related to periodic laminates

This appendix develops some relations and examples useful for laminates with periodic microstructure. Consider, therefore, a laminate composed of two phases in a periodic arrangment with period p. For convenience, the length scale is chosen so that $p = 2\pi$. It follows that the characteristic functions and the two point correlation functions which characterize the geometry of the laminate, are periodic with period 2π.

The laminate, even though exactly periodic, is taken to be random in the sense that the exact location of any one interface is unknown. The characteristic function χ_1 is taken to depend on a parameter α that locates an interface. Then, with a slight compression of notation,

$$\chi_1(x,\alpha) = \chi_1(x+\alpha,0) = \chi_1(x+\alpha) = \sum_{\zeta\in Z} \tilde{\chi}_1(\zeta)e^{i\zeta(x+\alpha)} \tag{A.1}$$

where Z denotes the set of integers, $\tilde{\chi}_1(\zeta)$ is the $\zeta - th$ Fourier coefficient of χ_1 and α indicates the single realization of our laminate. The random variable α is taken to be uniformly distributed on $[0, 2\pi]$. Then, via (4.3), the two point correlation function p_{11} is

$$p_{11}(x - x') = \frac{1}{2\pi} \int_{2\pi} \chi(x + \alpha)\chi(x' + \alpha)\, d\alpha, \tag{A.2}$$

and so, from (A.1),

$$p_{11}(x - x') = \sum_{\zeta \in Z} \tilde{\chi}_1(\zeta)\tilde{\chi}_1(-\zeta) e^{i\zeta(x-x')}. \tag{A.3}$$

Correspondingly, the function $\psi_{11}(x)$ of (4.4) can be expressed in the Fourier series:

$$\psi_{11}(x) = \sum_{\zeta \in Z - \{0\}} \tilde{\psi}_{11}(\zeta) e^{ix\zeta} = \sum_{\zeta \in Z - \{0\}} \tilde{\chi}_1(\zeta)\tilde{\chi}_1(-\zeta) e^{ix\zeta}. \tag{A.4}$$

The Fourier coefficients of the series (A.4), for a laminate, are:

$$\tilde{\psi}_{11}(\zeta) = \tilde{\chi}_1(\zeta)\tilde{\chi}_1(-\zeta) = \left(\frac{\sin(p_1\zeta)}{\pi\zeta} \right)^2. \tag{A.5}$$

Further, if the Fourier transform of any function f is defined as

$$\tilde{f}(k) = \int f(x) e^{-ikx} dx, \tag{A.6}$$

it easy to prove that the Fourier transform of $\psi_{11}(x)$ is

$$\tilde{\psi}_{11}(k) = \sum_{\zeta \in Z - \{0\}} \tilde{\psi}_{11}(\zeta)\delta(k - \zeta), \tag{A.7}$$

where δ denotes the Dirac function. Then, the Fourier transform of $S_0(x) = -\frac{i}{2}\mathrm{sgn}(x)\psi_{11}(x)$ and $S_g(x) = -\frac{1}{2}|x|\psi_{11}(x)$ can be expressed by using (A.7). In fact, since the Fourier transform of $\mathrm{sgn}(x)$ is $-\frac{2i}{k}$ with $i = \sqrt{-1}$ we have

$$\tilde{S}_0(k) = \frac{i}{k} * \tilde{\psi}_{11}(k) = i \sum_{\zeta \in Z - \{0\}} \frac{1}{(k - \zeta)} \tilde{\psi}_{11}(\zeta), \tag{A.8}$$

where $*$ denotes convolution with respect to k. Analogously, since the Fourier transform of $|x|$ is $-\frac{2}{k^2}$, we have

$$\tilde{S}_g(k) = \frac{1}{k^2} * \tilde{\psi}_{11}(k) = \sum_{\zeta \in Z - \{0\}} \frac{1}{(k - \zeta)^2} \tilde{\psi}_{11}(\zeta). \tag{A.9}$$

Finally, the Fourier transform of $S_\gamma(x) = \delta(x)\psi_{11}(x)$ is:

$$\tilde{S}_\gamma(k) = \sum_{\zeta \in Z - \{0\}} \tilde{\psi}_{11}(\zeta) = \psi_{11}(0) = p_1 p_2. \tag{A.10}$$

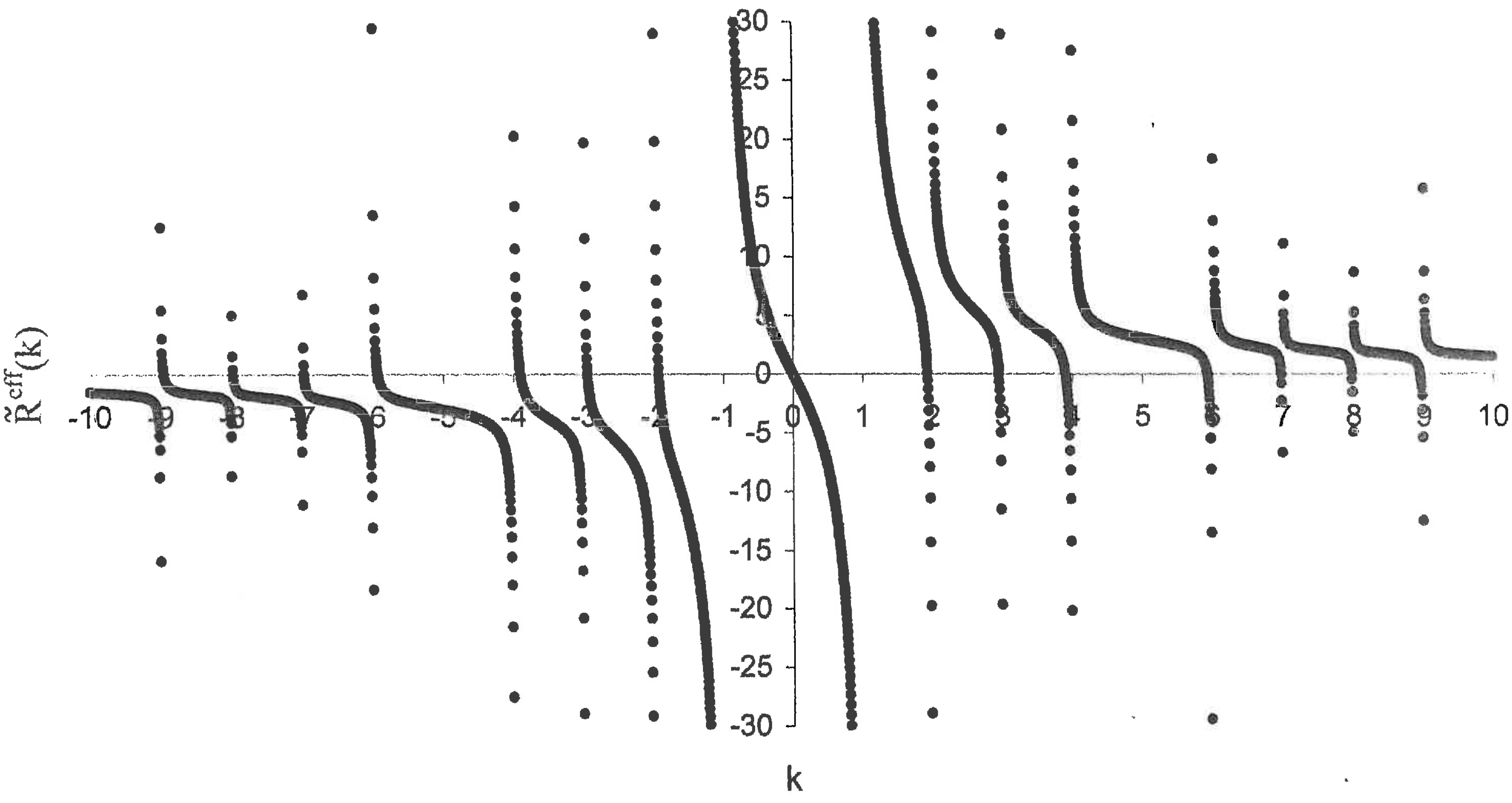

Figure 1. $\tilde{R}^{eff}(k)$.

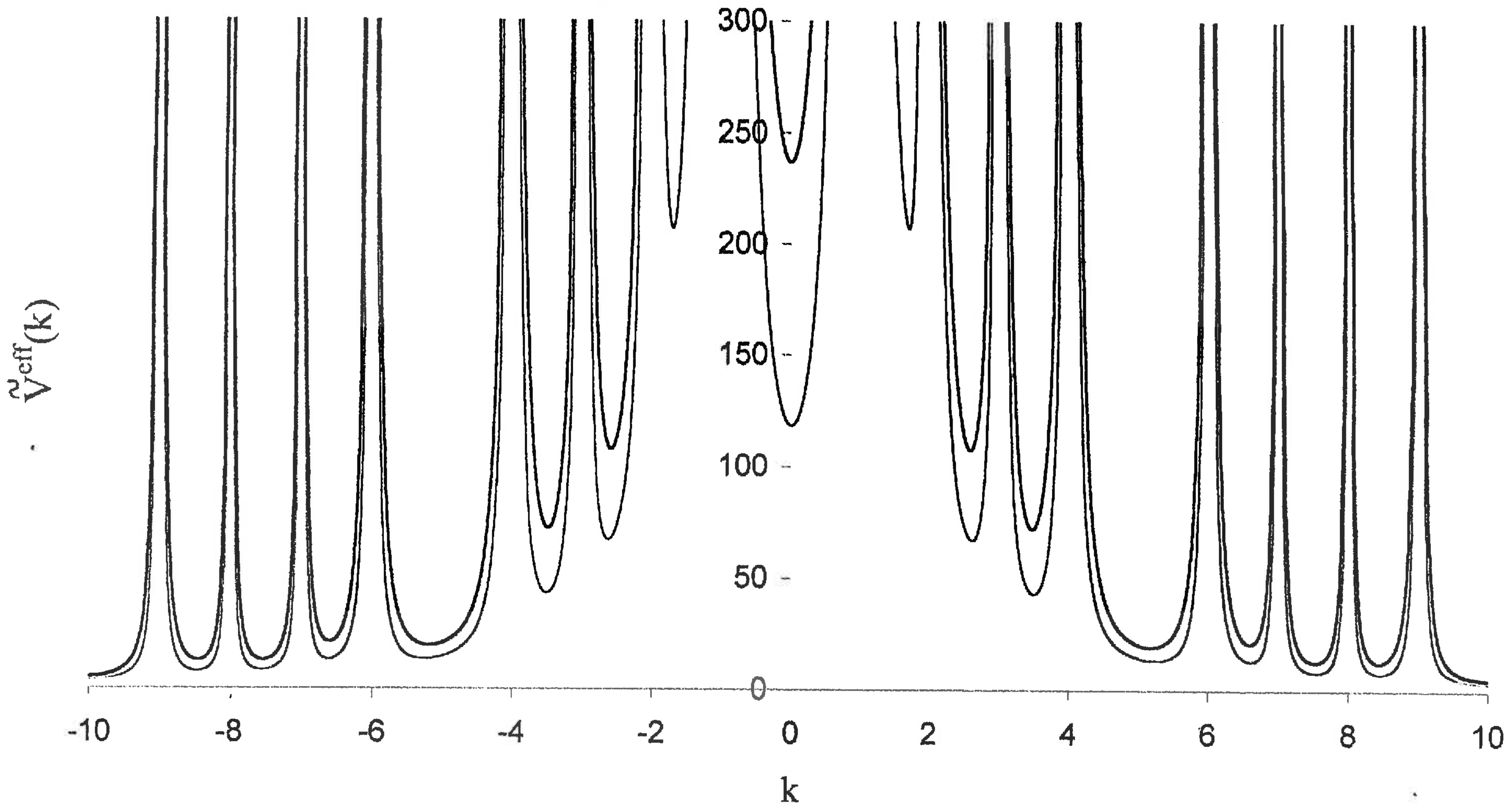

Figure 2. Bounds for $\tilde{V}^{eff}(k)$.

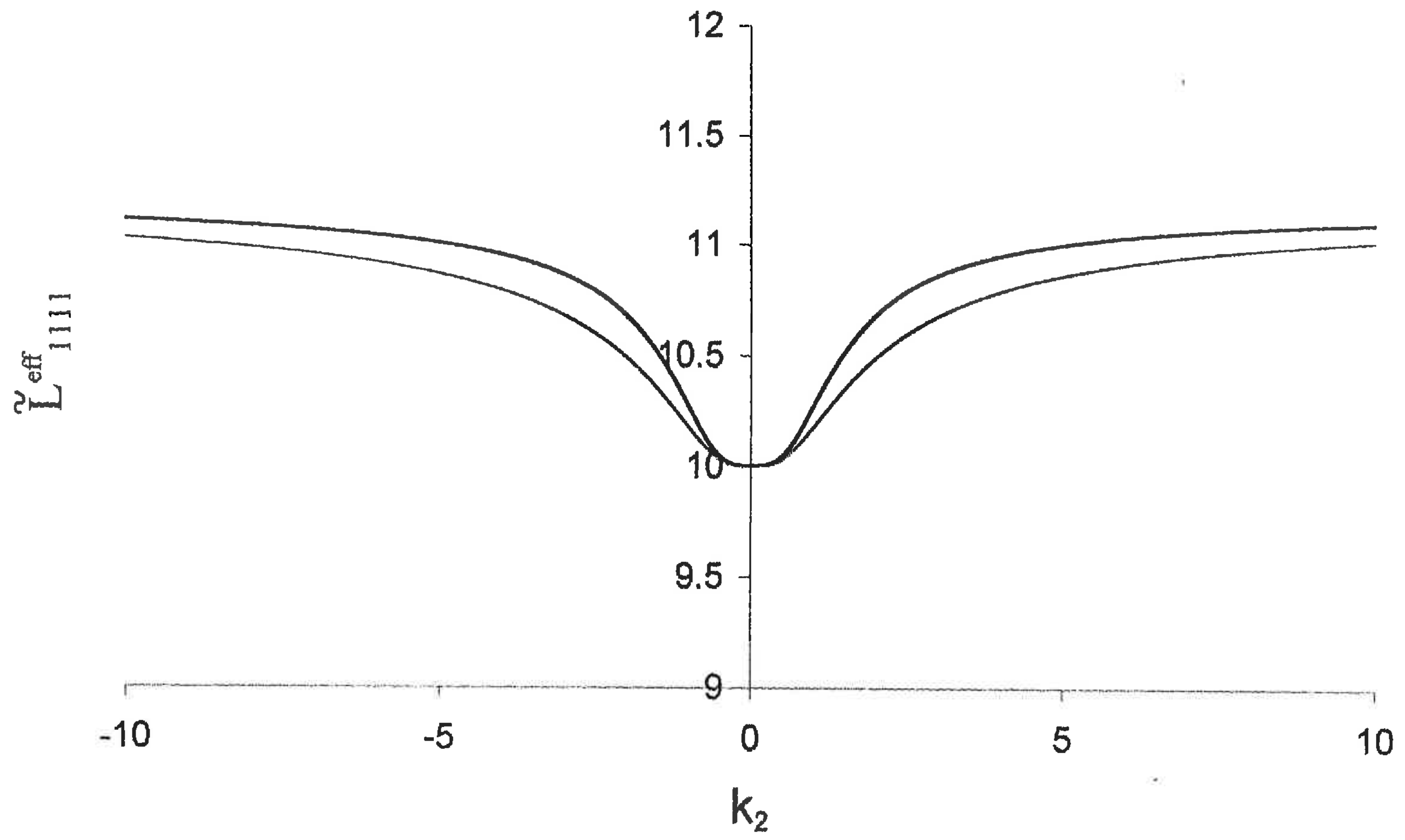

Figure 3. Bounds for $\tilde{L}^{eff}_{1111}(0, k_2, 0)$.

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