Duality Relations for nonlinear incompressible two dimensional elasticity

G.A. Francfort * and P.M. Suquet **

* L.P.M.T.M., Université Paris 13
Avenue J.B. Clément, 93430 Villetaneuse, France.

** Laboratoire de Mécanique et d'Acoustique, UPR CNRS 7051,
31 Chemin Joseph Aiguier, 13402 Marseille Cedex 20, France.

Abstract

Keller's phase interchange relation which computes the overall conductivity of a two dimensional checkerboard with alternating conductivity is revisited in the context of nonlinear incompressible elasticity. A general phase interchange relation is obtained in a monotone setting through H–convergence and in a convex setting through variational methods. Several Keller-like applications are presented.

1 Introduction

Duality or interchange relations are viewed as a precious tool in the study of the overall properties of heterogeneous media. They will indeed deliver explicit formulae macroscopically linking together apparently unrelated mixtures.

The archetype of such a relation is the so–called Keller phase interchange relation [10]: A conducting checkerboard with alternating conductivity α and β has $\sqrt{\alpha\beta}$ as overall conductivity.

This result was the starting point for a flurry of activities concentrating on the 2–d conductivity case or on a related topic, that of 2–d incompressible elasticity.

Let us mention for example that the effective shear modulus of an isotropic 2–d, two–phase mixture with isotropic incompressible phases of respective shear moduli μ and μ' is $\sqrt{\mu\mu'}$ (Lurié–Cherkaev [14]). The interested reader is refered to e.g. Benveniste [1] in the conductivity case and to Helsing, Milton and Movchan [9] in the elastic case.

The structural resemblance between those two settings should not come as a surprize; see Francfort [5] for a more abstract analysis of that correspondence.

Non linear contitutive behavior is a road far less traveled as far as duality relations are concerned. In the conductivity case, a recent systematic exploration of duality was undertaken in Levy and Kohn [12]. Of notable interest is their recovery of a curiosity evoked in Kozlov [11]: the overall behavior of a non–ohmic checkerboard with dual power–law behaviors is ohmic with overall conductivity 1!

Our goal in the present paper is to pursue the investigation in the context of 2–d nonlinear incompressible elasticity. Our analysis is limited to the case of monotone constitutive behaviors. In other words, it is assumed that the stress–strain relation is of the form:

$$\boldsymbol{\sigma} = p\boldsymbol{i} + \boldsymbol{s}, \quad \boldsymbol{s} = \boldsymbol{A}\left(\boldsymbol{x}, \boldsymbol{e}(\boldsymbol{u})\right), \tag{1.1}$$

where $\boldsymbol{\sigma}$ is the actual stress field, p is the pressure field, $\boldsymbol{i}$ is the identity matrix and $\boldsymbol{s}$ is the deviatoric stress field which satisfies

$$-\text{div } \boldsymbol{s} = \boldsymbol{f} + \text{grad } p.$$

In addition $\boldsymbol{e}(\boldsymbol{u}) = 1/2\left(\text{grad } \boldsymbol{u} + \text{grad } \boldsymbol{u}^{\mathsf{T}}\right)$ is the linearized strain associated to the incompressible displacement field $\boldsymbol{u}$ (div $\boldsymbol{u} = 0$), and $\boldsymbol{A}\left(\boldsymbol{x}, .\right))$ is, for each $\boldsymbol{x}$, a monotone mapping. Alternatively, $\boldsymbol{\sigma}$ can be viewed as

$$\boldsymbol{\sigma} = \frac{\partial w}{\partial \boldsymbol{e}}\left(\boldsymbol{x}, \boldsymbol{e}(\boldsymbol{u})\right),$$

where $w(\boldsymbol{x}, \boldsymbol{e})$, the elastic energy density, is assumed to be convex with respect to $\boldsymbol{e}$ and equal to $+\infty$ if tr$\boldsymbol{e} \neq 0$.

This bipolar viewpoint is reflected in Sections 2, 3 and 5 which respectively handle monotone constitutive laws and convex energy potentials. Specifically, Sections 2 and 3 are devoted to the monotone case. The setting is very general — no assumption of periodicity, or restriction on the number of phases — and the duality relation is established rigorously using the tools of H–convergence (Murat and Tartar [16]). The main result is Theorem 3.1; see also the following remark (Remark 3.2). Section 4 is devoted to two applications of theorem 3.1. In the first application two–phase isotropic mixtures of power–law materials, *i.e.* materials with

$$s = \alpha^p \, |e|^{p-2} \, e, \quad \text{resp.} \quad \beta^p \, |e|^{p-2} \, e, \quad 0 < \alpha, \beta < +\infty, \quad 1 < p < +\infty, \tag{1.2}$$

as constitutive relation, are investigated. It is then known that, in dimension 2, the resulting macroscopic behavior is also of the form

$$s = a^p(\alpha, \beta, p) \, |e|^{p-2} \, e,$$

with $a(\alpha, \beta, p)$ depending on α, β, p and the material distribution of each phase. Then, if $a(\beta, \alpha, p')$ denotes the coefficient corresponding to the macroscopic behavior of a two–phase isotropic mixture with constitutive relation

$$s = \beta^{p'} \, |e|^{p'-2} \, e, \quad \text{resp.} \quad \alpha^{p'} \, |e|^{p'-2} \, e, \quad p' = \frac{p}{p-1},$$

we obtain that (cf. (4.4))

$$a(\alpha, \beta, p) \; a(\beta, \alpha, p') = \alpha \; \beta,$$

which is a generalization of the two–phase incompressible polycrystal studied in Lurié and Cherkaev [14] and Helsing *et al* [9].

In the second application, it is shown, in the spirit of Kozlov [11], that nonlinearly incompressible elastic mixtures which are self-dual behave macroscopically like a linear material with shear modulus 1/2, a generalization of the conductivity result in Levy and Kohn [12] to the incompressible elastic case.

Finally, Section 5 investigates the variational standpoint, *i.e.* that where convex elastic energy densities are considered. In that section we favor simplicity over rigor and propose a fast derivation of the analogue of Theorem 3.1, namely Proposition 5.3. We finally conclude this study by revisiting the self–dual setting of Section 4 when both phases have Hölder–conjugate growth properties and rederive a similar result in that context. The proposed derivation could be rendered rigorous at the expense of an extensive use of Γ–convergence results of the type found in Braides and DeFranceschi [3].

Before closing this introduction we specify the notation. In all that follows, tensors are denoted by bold face letters. M_N denotes the subspace of all symmetric matrices on $\mathbb{R}^N$ $(\mathbb{R}_s^{N^2})$ that are trace free. If $\boldsymbol{a}, \boldsymbol{b}$ are two vectors in $\mathbb{R}^N$, $\boldsymbol{a} \otimes_s \boldsymbol{b} = 1/2\,(\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a})$. If $\boldsymbol{u}$ is a $\mathbb{R}^N$–valued vector field on $\mathbb{R}^N$,

$$e(u) = 1/2 \left(\text{grad } u + \text{grad } u^{\top}\right),$$

while if $\boldsymbol{e}$ is a $\mathbb{R}^{N^2}_s$– valued field on $\mathbb{R}^N$, comp$(\boldsymbol{e})$ is the $\mathbb{R}^{N^4}$–valued field defined as

$$\text{comp}(\boldsymbol{e})_{ijkh} = \frac{\partial^2 e_{ij}}{\partial x_k \partial x_h} + \frac{\partial^2 e_{kh}}{\partial x_i \partial x_j} - \frac{\partial^2 e_{ik}}{\partial x_j \partial x_h} - \frac{\partial^2 e_{jh}}{\partial x_i \partial x_k}.$$

Remark that, when $N = 2$, all components of comp$(\boldsymbol{e})$ are identically 0 except $\text{comp}(\boldsymbol{e})_{1122} = \text{comp}(\boldsymbol{e})_{2211} = \frac{\partial^2 e_{11}}{\partial x_2^2} + \frac{\partial^2 e_{22}}{\partial x_1^2} - 2\frac{\partial^2 e_{12}}{\partial x_1 \partial x_2}$; we will thus loosely identify comp$(\boldsymbol{e})$ with that component. The curl of a $\mathbb{R}^2$–valued field $\boldsymbol{v}$ is the scalar–valued field defined as

$$\text{curl}\ \boldsymbol{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.$$

Finally $\to$ will always denote convergence in the strong topology, whereas $\rightharpoonup$ will denote that in the weak topology. We assume familiarity with the usual Sobolev spaces.

2 Homogenization results in incompressible elasticity with power–law type constitutive behavior

Consider an ε–indexed sequence of elasticity functionals of power–law type. Specifically a sequence $\boldsymbol{A}^\varepsilon$ of Caratheodory functions from $\mathbb{R}^N \times M_N$ into M_N is introduced; $\boldsymbol{A}^\varepsilon(\boldsymbol{x}, .)$ is further assumed to be a monotone invertible mapping from M_N into itself that satisfies, for some $1 < p < +\infty$,

$$\left.\begin{aligned} (\boldsymbol{A}^\varepsilon(\boldsymbol{x}, \boldsymbol{\lambda}) - \boldsymbol{A}^\varepsilon(\boldsymbol{x}, \boldsymbol{\mu})) \,.\, (\boldsymbol{\lambda} - \boldsymbol{\mu}) &\ \geq\ \alpha\, |\boldsymbol{\lambda} - \boldsymbol{\mu}|^p, \\ ((\boldsymbol{A}^\varepsilon)^{-1}(\boldsymbol{x}, \boldsymbol{\lambda}) - (\boldsymbol{A}^\varepsilon)^{-1}(\boldsymbol{x}, \boldsymbol{\mu})) \,.\, (\boldsymbol{\lambda} - \boldsymbol{\mu}) &\ \geq\ \beta^{-1}\, |\boldsymbol{\lambda} - \boldsymbol{\mu}|^{p'}, \end{aligned}\right\} \tag{2.1}$$

with $0 < \alpha < \beta < +\infty$ and $1/p + 1/p' = 1$.

Let Ω be a fixed bounded and connected domain of $\mathbb{R}^N$ with smooth enough boundary $\partial\Omega$, so that Korn's theorem holds true on $L^q(\Omega; \mathbb{R}^N)$, $1 < q < +\infty$. We recall that Korn's theorem asserts that, for $1 < p < +\infty$, if $\boldsymbol{u} \in W^{-1,q}(\Omega; \mathbb{R}^N)$ and grad$\boldsymbol{u} \in W^{-1,q}(\Omega; \mathbb{R}^{N^2})$, then $\boldsymbol{u} \in L^q(\Omega; \mathbb{R}^N)$, provided that $\partial\Omega$ is smooth enough, while Korn's inequality asserts that $\|\boldsymbol{u}\|_{L^q(\Omega;\mathbb{R}^N)} + \|\boldsymbol{e}(\boldsymbol{u})\|_{L^q(\Omega;\mathbb{R}^{N^2})}$ is, on $W^{1,q}(\Omega; \mathbb{R}^N)$, a norm which is equivalent to $\|\boldsymbol{u}\|_{W^{1,q}(\Omega;\mathbb{R}^N)}$ (see Geymonat and Suquet [8] for a proof of Korn's theorem and Korn's inequality on $L^q(\Omega; \mathbb{R}^N)$).

Then, if $\boldsymbol{f} \in W^{-1,p'}(\Omega; \mathbb{R}^N)$, the problem

$$\left.\begin{aligned} -\text{div}\ \boldsymbol{A}^\varepsilon(\boldsymbol{x}, \boldsymbol{e}(\boldsymbol{u}^\varepsilon)) &= \boldsymbol{f} + \text{grad}\ p^\varepsilon && \text{in } \Omega, \\ \text{div}\ \boldsymbol{u}^\varepsilon &= 0 && \text{in } \Omega \\ \boldsymbol{u}^\varepsilon &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{2.2}$$

admits a unique solution $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ in $W^{1,p}_{0,\text{div}_0}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$ where

$$W^{1,p}_{0,\text{div}_0}(\Omega) = \{\boldsymbol{v} \in W^{1,p}_0(\Omega;\mathbb{R}^N);\ \text{div}\ \boldsymbol{v} = 0\}.$$

The existence proof for $\boldsymbol{u}^\varepsilon$ results from a classical theorem for coercive monotone operators on closed convex sets in reflexive Banach spaces (Theorem 8.2 in Lions [13]). Note that the coercivity is checked through the use of Korn's inequality on $W^{1,p}_0(\Omega;\mathbb{R}^N)$. Then the first equation of (2.2) is used to define the pressure field p^ε in $W^{-1,p'}(\Omega)$ which is found, via Korn's theorem, to be in $L^{p'}(\Omega)$.

As ε varies, a homogenization result in the spirit of Murat and Tartar [16] can be derived. Let us define H_p–convergence as follows in the present context.

Definition 2.1 *: A sequence $\boldsymbol{A}^\varepsilon$ satisfying (2.1) H_p–converges to $\boldsymbol{A}$ satisfying (2.1) $\left(\boldsymbol{A}^\varepsilon \stackrel{H_p}{\rightharpoonup} \boldsymbol{A}\right)$ if and only if, for any bounded and connected domain Ω of $\mathbb{R}^N$ with, say, C^∞ boundary $\partial\Omega$, and any $\boldsymbol{f}$ in $W^{-1,p'}(\Omega;\mathbb{R}^N)$ the solution $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ of (2.2) is such that*

$$\left.\begin{aligned} \boldsymbol{u}^\varepsilon &\rightharpoonup \boldsymbol{u} \quad \text{in } W^{1,p}(\Omega;\mathbb{R}^N),\\ p^\varepsilon &\rightharpoonup p \quad \text{in } L^{p'}(\Omega)/\mathbb{R},\\ \boldsymbol{A}^\varepsilon(\boldsymbol{x}, \boldsymbol{e}(\boldsymbol{u}^\varepsilon)) &\rightharpoonup \boldsymbol{A}(\boldsymbol{x}, \boldsymbol{e}(\boldsymbol{u})) \quad \text{in } L^{p'}(\Omega;\mathbb{R}^N) \end{aligned}\right\}$$

where $(\boldsymbol{u}, p)$ is the unique solution in $W^{1,p}_{0,\text{div}_0}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$ of

$$\left.\begin{aligned} -\text{div}\ \boldsymbol{A}(\boldsymbol{x}, \boldsymbol{e}(\boldsymbol{u})) &= \boldsymbol{f} + \text{grad}\ p && \text{in } \Omega,\\ \text{div}\ \boldsymbol{u} &= 0 && \text{in } \Omega,\\ \boldsymbol{u} &= 0 && \text{on } \partial\Omega. \end{aligned}\right\}$$

Then, the following theorem can be proved:

Theorem 2.2 *: If $\boldsymbol{A}^\varepsilon$ is a sequence that satisfies (2.1), then there exists a subsequence $\boldsymbol{A}^{\varepsilon'}$ ($\{\varepsilon'\} \subset \{\varepsilon\}$) and $\boldsymbol{A}$ satisfying (2.1) such that*

$$\boldsymbol{A}^\varepsilon \stackrel{H_p}{\rightharpoonup} \boldsymbol{A}$$

Remark 2.3 *:* It is not our purpose here to produce a proof of Theorem 2.2. The interested reader is invited to consult the literature. In a scalar setting (namely that of nonlinear conductivity), the same theorem is proved in Murat and Tartar [16] in the linear case, in Tartar [19], Murat [15], Chiado-Piat, Dal Maso and DeFranceschi [4] in the monotone $p = 2$ case, and in Fusco and Moscariello [7] in the monotone $p \neq 2$ case. An extension to the nonlinear but monotone compressible elasticity setting can be found in Suquet [18] (in a periodic framework and for $p = 2$). An actual proof of Theorem 2.2 cannot be found, to our knowledge, in the literature. It would however easily follow from the proofs given in the above mentioned references. The key difference stems from the incompressibility constraint which requires the following $L^{p'}(\Omega)$– estimate on p^ε,

$$\|p^\varepsilon\|_{L^{p'}(\Omega)/\mathbb{R}} \leq \|\text{grad } p^\varepsilon\|_{W^{-1,p'}(\Omega;\mathbb{R}^N)}, \tag{2.3}$$

itself a direct consequence of Korn's theorem on $L^{p'}(\Omega)$.

Next we introduce the notion of deformation–correctors.

Proposition 2.4 *: For any $\boldsymbol{\lambda} \in M_N$, any bounded and connected domain Ω of $\mathbb{R}^N$ with C^∞ boundary $\partial\Omega$, and any H_p–converging sequence $\boldsymbol{A}^\varepsilon$, there exists a pair–sequence $(\boldsymbol{P}^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda}), p^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda}))$ in $L^p(\Omega; M_N) \times L^{p'}(\Omega)/\mathbb{R}$ such that*

$$\left.\begin{aligned} \boldsymbol{P}^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda}) &\rightharpoonup \boldsymbol{\lambda} \quad \text{in } L^p(\Omega; M_N), \\ p^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda}) &\rightharpoonup 0 \quad \text{in } L^{p'}(\Omega)/\mathbb{R}, \end{aligned}\right\} \tag{2.4}$$

with further

$$\boldsymbol{A}^\varepsilon(\boldsymbol{x}, \boldsymbol{P}^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda})) \rightharpoonup \boldsymbol{A}(\boldsymbol{x},\boldsymbol{\lambda}) \quad \text{in } L^{p'}(\Omega; M_N), \tag{2.5}$$

while

$$\left.\begin{aligned} \text{comp}[\boldsymbol{P}^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda})] &\in \text{ compact of } W^{-2,p}(\Omega;\mathbb{R}^{N^4}), \\ \text{div } \boldsymbol{A}^\varepsilon(\boldsymbol{x},\boldsymbol{P}^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda})) + \text{grad } p^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda}) &\in \text{ compact of } W^{-1,p'}(\Omega;\mathbb{R}^N). \end{aligned}\right\} \tag{2.6}$$

Further, local uniqueness holds, i.e., if a pair–sequence $\left(\tilde{\boldsymbol{P}}^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda}), \tilde{p}^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda})\right)$ satisfies

$$\left.\begin{aligned} \tilde{\boldsymbol{P}}^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda}) \rightharpoonup \boldsymbol{\lambda} \quad \text{in } L^p(\Omega; M_N), \quad \tilde{p}^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda}) \rightharpoonup \tilde{p}(\boldsymbol{x},\boldsymbol{\lambda}) \quad \text{in } L^{p'}(\Omega)/\mathbb{R}, \\ \boldsymbol{A}^\varepsilon(\boldsymbol{x}, \tilde{\boldsymbol{P}}^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda})) \rightharpoonup \boldsymbol{B}(\boldsymbol{x},\boldsymbol{\lambda}) \quad \text{in } L^{p'}(\Omega;\mathbb{R}^N), \end{aligned}\right\} \tag{2.7}$$

together with (2.6), then

$$\tilde{p}(\boldsymbol{x},\boldsymbol{\lambda}) = 0, \quad \boldsymbol{A}(\boldsymbol{x},\boldsymbol{\lambda}) = \boldsymbol{B}(\boldsymbol{x},\boldsymbol{\lambda}),$$

$$\left(\boldsymbol{P}^\varepsilon - \tilde{\boldsymbol{P}}^\varepsilon, p^\varepsilon - \tilde{p}^\varepsilon\right) \to 0 \text{ in } L^p_{loc}(\Omega; M_N) \times L^{p'}_{loc}(\Omega)/\mathbb{R}$$

Proof: Imbed Ω into a ball $B(0,R)$; solve on $D(0,R) = B(0,2R)/\overline{B(0,R)}$ the following Stokes problem for $(\boldsymbol{d}, q)$:

$$\left.\begin{array}{rcll} -\Delta \boldsymbol{d} &=& \text{grad } q & \text{in } D(0,R), \\ \text{div } \boldsymbol{d} &=& 0 & \text{in } D(0,R) \\ \boldsymbol{d}|_{\partial B(0,R)} &=& \boldsymbol{\lambda}.\boldsymbol{x}, & \boldsymbol{d}|_{\partial B(0,2R)} = 0. \end{array}\right\}$$

Then, upon setting

$$\hat{\boldsymbol{d}}(\boldsymbol{x}) = \left\{\begin{array}{ll} \boldsymbol{\lambda}.\boldsymbol{x} & \text{on } B(0,R) \\ \boldsymbol{d}(\boldsymbol{x}) & \text{on } D(0,R), \end{array}\right.$$

we define $(\boldsymbol{P}^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda}), p^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda}))$ as

$$\boldsymbol{P}^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda}) \stackrel{def}{=} \boldsymbol{e}(\boldsymbol{w}^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda})), \tag{2.8}$$

where $(\boldsymbol{w}^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda}), p^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda}))$ is the solution in $W^{1,p}_{0,\text{div}_0}(B(0,2R)) \times L^{p'}(B(0,2R))/\mathbb{R}$ of

$$\left.\begin{array}{rcll} -\text{div } \boldsymbol{A}^\varepsilon(\boldsymbol{x}, \boldsymbol{e}(\boldsymbol{w}^\varepsilon)) &=& -\text{div } \boldsymbol{A}\left(\boldsymbol{x}, \boldsymbol{e}(\hat{\boldsymbol{d}})\right) + \text{grad } p^\varepsilon & \text{in } B(0,2R), \\ \text{div } \boldsymbol{w}^\varepsilon &=& 0 & \text{in } B(0,2R) \\ \boldsymbol{w}^\varepsilon &=& 0 & \text{on } \partial B(0,2R). \end{array}\right\} \tag{2.9}$$

Then, since $\boldsymbol{A}^\varepsilon$ H_p–converges to $\boldsymbol{A}$,

$$\left.\begin{array}{rcll} \boldsymbol{w}^\varepsilon & \rightharpoonup & \boldsymbol{w} & \text{in } W^{1,p}(B(0,2R);\mathbb{R}^N), \\ p^\varepsilon & \rightharpoonup & p & \text{in } L^{p'}(B(0,2R))/\mathbb{R}, \end{array}\right\}$$

with $(\boldsymbol{w}, p)$ unique solution in $W^{1,p}_{0,\text{div}_0}(B(0,2R)) \times L^{p'}(B(0,2R))/\mathbb{R}$ of

$$\left.\begin{array}{rcll} -\text{div } \boldsymbol{A}(\boldsymbol{x}, \boldsymbol{e}(\boldsymbol{w})) &=& -\text{div } \boldsymbol{A}\left(\boldsymbol{x}, \boldsymbol{e}(\hat{\boldsymbol{d}})\right) + \text{grad } p & \text{in } B(0,2R), \\ \text{div } \boldsymbol{w} &=& 0 & \text{in } B(0,2R), \\ \boldsymbol{w} &=& 0 & \text{on } \partial B(0,2R). \end{array}\right\}$$

Thus $\boldsymbol{w} = \hat{\boldsymbol{d}}$ and $p = 0$ modulo a constant, hence $\boldsymbol{w}|_\Omega = \boldsymbol{\lambda}.\boldsymbol{x}$, $p|_\Omega = 0$, *i.e.*,

$$\left.\begin{array}{rcll} \boldsymbol{P}^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda}) & \rightharpoonup & \boldsymbol{\lambda} & \text{in } L^p(\Omega; M_N), \\ p^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda}) & \rightharpoonup & 0 & \text{in } L^{p'}(\Omega)/\mathbb{R}, \\ \boldsymbol{A}^\varepsilon(\boldsymbol{x}, \boldsymbol{P}^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda})) & \rightharpoonup & \boldsymbol{A}(\boldsymbol{x},\boldsymbol{\lambda}) & \text{in } L^{p'}(\Omega; M_N), \end{array}\right\}$$

which is precisely (2.4), (2.5) while (2.6) is satisfied in view of (2.8) (2.9).

Finally, let $\left(\tilde{\boldsymbol{P}}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}),\tilde{p}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})\right)$ satisfy (2.4) (2.6) and (2.7). Then, for any $\varphi \in C_0^{\infty}(\Omega)$,

$$\alpha\ \overline{\lim_{\varepsilon}}\left\|\varphi^{2/p}\left(\boldsymbol{P}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})-\tilde{\boldsymbol{P}}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})\right)\right\|^{p}_{L^p(\Omega;M_N)}$$

$$\leq \overline{\lim_{\varepsilon}}\int_{\Omega}\varphi^2\left(\boldsymbol{A}^{\varepsilon}(\boldsymbol{x},\boldsymbol{P}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}))-\boldsymbol{A}^{\varepsilon}(\boldsymbol{x},\tilde{\boldsymbol{P}}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}))\right).\left(\boldsymbol{P}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})-\tilde{\boldsymbol{P}}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})\right)\,d\boldsymbol{x}$$

$$=\overline{\lim_{\varepsilon}}\int_{\Omega}\varphi^2\left(\{\boldsymbol{A}^{\varepsilon}(\boldsymbol{x},\boldsymbol{P}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}))+p^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})\boldsymbol{i}\}-\{\boldsymbol{A}^{\varepsilon}(\boldsymbol{x},\tilde{\boldsymbol{P}}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}))+\tilde{p}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})\boldsymbol{i}\}\right).$$

$$\left(\boldsymbol{P}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})-\tilde{\boldsymbol{P}}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})\right)\,d\boldsymbol{x}. \tag{2.10}$$

An elementary application of compensated compactness (Tartar [20] or Francfort and Murat [6] in the specific context of linearized elasticity) permits to pass to the limit in the last term in (2.10) and to obtain that

$$\boldsymbol{P}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})-\tilde{\boldsymbol{P}}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})\to 0 \quad \text{in } L^p_{loc}(\Omega;M_N),$$

from which it is immediately deduced that $\boldsymbol{A}(\boldsymbol{x},\boldsymbol{\lambda})=\boldsymbol{B}(\boldsymbol{x},\boldsymbol{\lambda})$ and by application of estimate (2.3) in Remark 2.3 that

$$p^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})-\tilde{p}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})\to 0 \quad \text{in } L^{p'}_{loc}(\Omega)/\mathbb{R}.$$

The proof of Proposition 2.4 is complete.

Stress–correctors are in turn defined through the following

Proposition 2.5 : *For any $\boldsymbol{\lambda}\in M_N$, there exists a pair–sequence $(\boldsymbol{Q}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}),q^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}))$ in $L^{p'}(\Omega;M_N)\times L^{p'}(\Omega)/\mathbb{R}$ such that*

$$\left.\begin{array}{rcl}\boldsymbol{Q}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}) & \rightharpoonup & \boldsymbol{\lambda} \quad \text{in } L^{p'}(\Omega;M_N),\\ q^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}) & \rightharpoonup & 0 \quad \text{in } L^{p'}(\Omega)/\mathbb{R},\end{array}\right\} \tag{2.11}$$

with further

$$\left(\boldsymbol{A}^{\varepsilon}\right)^{-1}(\boldsymbol{x},\boldsymbol{Q}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}))\rightharpoonup(\boldsymbol{A})^{-1}(\boldsymbol{x},\boldsymbol{\lambda}) \quad \text{in } L^p(\Omega;M_N), \tag{2.12}$$

while

$$\left.\begin{array}{rcl} \operatorname{div} \boldsymbol{Q}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}) + \operatorname{grad} q^{\varepsilon} & \in & \text{compact of } W^{-1,p'}(\Omega;\mathbb{R}^N), \\ \operatorname{comp}[(\boldsymbol{A}^{\varepsilon})^{-1}(\boldsymbol{x},\boldsymbol{Q}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})] & \in & \text{compact of } W^{-2,p}(\Omega;\mathbb{R}^{N^4}). \end{array}\right\} \tag{2.13}$$

Further local uniqueness holds as in Proposition 2.4.

Proof: We consider the solution $\boldsymbol{w}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})$ of the following system

$$\left.\begin{array}{rcl} -\operatorname{div} \boldsymbol{A}^{\varepsilon}[\boldsymbol{x}, \boldsymbol{e}(\boldsymbol{w}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})) + (\boldsymbol{A})^{-1}(\boldsymbol{x},\boldsymbol{\lambda})] & = & \operatorname{grad} q^{\varepsilon} \quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{w}^{\varepsilon} & = & 0, \quad \text{in } \Omega \\ \boldsymbol{w}^{\varepsilon}|_{\partial\Omega} & = & 0, \end{array}\right\}$$

and set

$$\boldsymbol{Q}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}) \overset{def}{=} \boldsymbol{A}^{\varepsilon}\left[\boldsymbol{x}, \boldsymbol{e}(\boldsymbol{w}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})) + (\boldsymbol{A})^{-1}(\boldsymbol{x},\boldsymbol{\lambda})\right],$$

so that

$$\left.\begin{array}{rcl} \operatorname{div} \boldsymbol{Q}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}) & + & \operatorname{grad} q^{\varepsilon} = 0 \quad \text{in } \Omega, \\ (\boldsymbol{A}^{\varepsilon})^{-1}(\boldsymbol{x},\boldsymbol{Q}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})) & = & \boldsymbol{e}(\boldsymbol{w}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})) + (\boldsymbol{A})^{-1}(\boldsymbol{x},\boldsymbol{\lambda}), \end{array}\right\} \tag{2.14}$$

hence (2.13) is satisfied. Further $\boldsymbol{w}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})$ is bounded in $W_0^{1,p}(\Omega;\mathbb{R}^N)$, thus, for a subsequence $(\boldsymbol{w}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}), \boldsymbol{Q}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}), q^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}))$ still indexed by ε,

$$\left.\begin{array}{rcl} \boldsymbol{w}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}) & \rightharpoonup & \boldsymbol{w}(\boldsymbol{x},\boldsymbol{\lambda}) \quad \text{in } W^{1,p}(\Omega;\mathbb{R}^N), \\ \boldsymbol{Q}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}) & \rightharpoonup & \boldsymbol{Q}(\boldsymbol{x},\boldsymbol{\lambda}) \quad \text{in } L^{p'}(\Omega;M_N), \\ q^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}) & \rightharpoonup & q(\boldsymbol{x},\boldsymbol{\lambda}) \quad \text{in } L^{p'}(\Omega)/\mathbb{R}, \end{array}\right\} \tag{2.15}$$

and

$$(\boldsymbol{A}^{\varepsilon})^{-1}(\boldsymbol{x},\boldsymbol{Q}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})) \rightharpoonup \boldsymbol{e}(\boldsymbol{w})(\boldsymbol{x},\boldsymbol{\lambda}) + (\boldsymbol{A})^{-1}(\boldsymbol{x},\boldsymbol{\lambda}) \quad \text{in } L^p(\Omega;M_N). \tag{2.16}$$

We now compute $\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{\lambda})$. Take an arbitrary $\boldsymbol{v} \in W^{1,p}_{0,\mathrm{div}_0}(\Omega)$ and solve for $(\boldsymbol{v}^{\varepsilon}, r^{\varepsilon})$

$$\left.\begin{array}{rcl} -\operatorname{div} \boldsymbol{A}^{\varepsilon}(\boldsymbol{x},\boldsymbol{e}(\boldsymbol{v}^{\varepsilon})) & = & -\operatorname{div} \boldsymbol{A}(\boldsymbol{x},\boldsymbol{e}(\boldsymbol{v})) + \operatorname{grad} r^{\varepsilon} \quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v}^{\varepsilon} & = & 0 \quad \text{in } \Omega, \\ \boldsymbol{v}^{\varepsilon}|_{\partial\Omega} & = & 0. \end{array}\right\}$$

so that, since $\boldsymbol{A}^\varepsilon$ H_p–converges to $\boldsymbol{A}$,

$$\left.\begin{aligned} \boldsymbol{v}^\varepsilon &\rightharpoonup \boldsymbol{v} \quad \text{in } W_0^{1,p}(\Omega;\mathbb{R}^N), \\ \boldsymbol{A}^\varepsilon(\boldsymbol{x},\boldsymbol{e}(\boldsymbol{v}^\varepsilon)) &\rightharpoonup \boldsymbol{A}(\boldsymbol{x},\boldsymbol{e}(\boldsymbol{v})) \quad \text{in } L^{p'}(\Omega;M_N), \\ r^\varepsilon &\rightharpoonup 0 \quad \text{in } L^{p'}(\Omega)/\mathbb{R}. \end{aligned}\right\}$$

Take $\varphi \geq 0$ in $C_0^\infty(\Omega)$. Then, by virtue of the monotone character of $\boldsymbol{A}^\varepsilon$,

$$\begin{aligned} 0 \ &\leq \ \underline{\lim}_\varepsilon \int_\Omega \varphi\,(\boldsymbol{Q}^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda}) - \boldsymbol{A}^\varepsilon(\boldsymbol{x},\boldsymbol{e}(\boldsymbol{v}^\varepsilon)))\,.\,(\boldsymbol{e}(\boldsymbol{w}^\varepsilon) - \boldsymbol{e}(\boldsymbol{v}^\varepsilon) + (\boldsymbol{A})^{-1}(\boldsymbol{x},\boldsymbol{\lambda}))\ d\boldsymbol{x} \\ &= \int_\Omega \varphi\,(\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{\lambda}) - \boldsymbol{A}(\boldsymbol{x},\boldsymbol{e}(\boldsymbol{v})))\,.(\boldsymbol{A})^{-1}(\boldsymbol{x},\boldsymbol{\lambda})\ d\boldsymbol{x} \\ &+ \underline{\lim}_\varepsilon \Big\{\int_\Omega \varphi\ (\boldsymbol{Q}^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda}) - \boldsymbol{A}^\varepsilon(\boldsymbol{x},\boldsymbol{e}(\boldsymbol{v}^\varepsilon)))\,.\,(\boldsymbol{e}(\boldsymbol{w}^\varepsilon) - \boldsymbol{e}(\boldsymbol{v}^\varepsilon))\ d\boldsymbol{x} \\ &+ \int_\Omega \varphi(q^\varepsilon - r^\varepsilon)(\mathrm{div}\boldsymbol{w}^\varepsilon - \mathrm{div}\boldsymbol{v}^\varepsilon)\ d\boldsymbol{x}\Big\} \\ &= \int_\Omega \varphi\,(\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{\lambda}) - \boldsymbol{A}(\boldsymbol{x},\boldsymbol{e}(\boldsymbol{v})))\,.(\boldsymbol{A})^{-1}(\boldsymbol{x},\boldsymbol{\lambda})\ d\boldsymbol{x} \\ &+ \int_\Omega \varphi\,(\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{\lambda}) - \boldsymbol{A}(\boldsymbol{x},\boldsymbol{e}(\boldsymbol{v})) + q\boldsymbol{i})\,.(\boldsymbol{e}(\boldsymbol{w}) - \boldsymbol{e}(\boldsymbol{v}))\ d\boldsymbol{x}, \end{aligned} \tag{2.17}$$

where the first equality holds since $\boldsymbol{w}^\varepsilon$ and $\boldsymbol{v}^\varepsilon$ are divergence–free fields while the second is easily obtained through an elementary application of compensated compactness. Since $\boldsymbol{w}$ and $\boldsymbol{v}$ are divergence–free, (2.17) finally becomes

$$0 \leq \int_\Omega \varphi\,(\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{\lambda}) - \boldsymbol{A}(\boldsymbol{x},\boldsymbol{e}(\boldsymbol{v})))\,.\,\big(\boldsymbol{e}(\boldsymbol{w}) + (\boldsymbol{A})^{-1}(\boldsymbol{x},\boldsymbol{\lambda}) - \boldsymbol{e}(\boldsymbol{v})\big)\ d\boldsymbol{x},$$

and, since φ is arbitrary,

$$(\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{\lambda}) - \boldsymbol{A}(\boldsymbol{x},\boldsymbol{e}(\boldsymbol{v})))\,.\,\big(\boldsymbol{e}(\boldsymbol{w}) + (\boldsymbol{A})^{-1}(\boldsymbol{x},\boldsymbol{\lambda}) - \boldsymbol{e}(\boldsymbol{v})\big) \geq 0, \quad \text{a.e. on } \Omega. \tag{2.18}$$

Choosing, for $\boldsymbol{x}_0 \in \Omega$ such that $\boldsymbol{A}^{-1}(\boldsymbol{x}_0,.)$ is continuous and $\mathrm{div}\boldsymbol{w}(\boldsymbol{x}_0) = 0$, $\boldsymbol{E} \in M_N$,

$$\boldsymbol{v}(\boldsymbol{x}) = \begin{cases} (\boldsymbol{e}(\boldsymbol{w})(\boldsymbol{x}_0) + (\boldsymbol{A})^{-1}(\boldsymbol{x}_0,\boldsymbol{\lambda}) + t\boldsymbol{E})\,.\boldsymbol{x} & \text{in a small ball around } \boldsymbol{x}_0, \\ 0 & \text{on } \partial\Omega, \end{cases}$$

(which is always possible by a construction similar to that which led to (2.7)), the continuous character of $\boldsymbol{A}^{-1}(\boldsymbol{x}_0,.)$ yields

$$\big[\boldsymbol{Q}(\boldsymbol{x}_0,\boldsymbol{\lambda}) - \boldsymbol{A}\left(\boldsymbol{x}_0, \boldsymbol{e}(\boldsymbol{w})(\boldsymbol{x}_0) + (\boldsymbol{A})^{-1}(\boldsymbol{x}_0,\boldsymbol{\lambda})\right)\big]\,.\boldsymbol{E} = 0.$$

But $\boldsymbol{x}_0$ and $\boldsymbol{E}$ are arbitrary and $\boldsymbol{Q}(\boldsymbol{x}_0,\boldsymbol{\lambda}) \in M_N$. Thus

$$\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{\lambda}) = \boldsymbol{A}\left(\boldsymbol{x}, \boldsymbol{e}(\boldsymbol{w})(\boldsymbol{x}) + (\boldsymbol{A})^{-1}(\boldsymbol{x},\boldsymbol{\lambda})\right) \quad \text{a.e. on } \Omega. \tag{2.19}$$

Passing to the limit in (2.14) yields, in view of (2.15),(2.19),

$$\left.\begin{aligned} &\operatorname{div} \boldsymbol{A}\left(\boldsymbol{x}, \boldsymbol{e}(\boldsymbol{w}) + (\boldsymbol{A})^{-1}(\boldsymbol{x},\boldsymbol{\lambda})\right) + \operatorname{grad} q = 0 \quad \text{in } \Omega, \\ &\operatorname{div} \boldsymbol{w} = 0 \quad \text{in } \Omega, \\ &\boldsymbol{w}|_{\partial\Omega} = 0. \end{aligned}\right\}$$

which admits as unique solution

$$\boldsymbol{w} = 0, \quad q = 0 \quad (\text{modulo } \mathbb{R}). \tag{2.20}$$

Thus (2.16) reads as

$$(\boldsymbol{A}^\varepsilon)^{-1}\left(\boldsymbol{x}, \boldsymbol{Q}^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda})\right) \rightharpoonup (\boldsymbol{A})^{-1}(\boldsymbol{x},\boldsymbol{\lambda}) \quad \text{in } L^p(\Omega; M_N),$$

and (2.12) is satisfied, while (2.11) is a direct consequence of (2.15) and (2.20). Note that since $(\boldsymbol{w}, q, \boldsymbol{Q})$ are uniquely defined there is no need for subsequence extraction in (2.15). The existence of stress–correctors is established. Local uniqueness would follow by an argument similar to that used at the end of the proof of proposition 2.4.

3 Duality relations for incompressible nonlinear elastic materials in dimension 2

We now specialize the dimension to be $N = 2$. Consider an *initial* material with $\boldsymbol{s} = \boldsymbol{A}(\boldsymbol{x}, \boldsymbol{e}(\boldsymbol{u}))$ as stress–strain relation. The *dual* material is defined as that material with $\boldsymbol{s} = (\boldsymbol{A})^{-1}(\boldsymbol{x}, \boldsymbol{e}(\boldsymbol{u}))$ as stress–strain relation, where $\boldsymbol{A}^{-1}(\boldsymbol{x},\boldsymbol{\lambda})$ is for every $\boldsymbol{x}$ the inverse of $\boldsymbol{A}(\boldsymbol{x},\boldsymbol{\lambda})$ as an operator on M_2. For instance, if $\boldsymbol{A}(\boldsymbol{x},\boldsymbol{\lambda}) = \alpha^p |\boldsymbol{\lambda}|^{p-2} \boldsymbol{\lambda}$, then $\boldsymbol{A}^{-1}(\boldsymbol{x},\boldsymbol{\lambda}) = \alpha^{-p'} |\boldsymbol{\lambda}|^{p'-2} \boldsymbol{\lambda}$, with $1/p + 1/p' = 1$.

The aim of this section is to derive a general relation between the H_p – limit of the initial sequence of operators $\boldsymbol{A}^\varepsilon(\boldsymbol{x},\boldsymbol{\lambda})$ and the $H_{p'}$ – limit of the associated dual sequence of operators $(\boldsymbol{A}^\varepsilon)^{-1}(\boldsymbol{x},\boldsymbol{\lambda})$.

Consider the linear mapping $\mathcal{R}$ from M_2 into itself defined as

$$\mathcal{R}\begin{pmatrix} \sigma & \tau \\ \tau & -\sigma \end{pmatrix} \stackrel{def}{=} \begin{pmatrix} -\tau & \sigma \\ \sigma & \tau \end{pmatrix}$$

Note that $\mathcal{R}^{-1} = \mathcal{R}^{\top} = -\mathcal{R}$. Also note that, if $\boldsymbol{S}(\boldsymbol{x})$ is a M_2–valued distribution,

$$\left.\begin{aligned} \text{comp }(\mathcal{R}\boldsymbol{S}) &= -\text{curl }(\text{div }\boldsymbol{S}), \\ \text{curl }(\text{div }(\mathcal{R}\boldsymbol{S})) &= \text{comp }(\boldsymbol{S}). \end{aligned}\right\} \tag{3.1}$$

Further, if $\boldsymbol{A}(\boldsymbol{x}, \boldsymbol{\lambda})$ is a Caratheodory function from $\mathbb{R}^2 \times M_2$ into M_2 that satisfies (2.1) for some $1 < p < +\infty$, then the mapping $\hat{\boldsymbol{A}}$ defined as

$$\hat{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{\mu}) \stackrel{def}{=} \mathcal{R}\,(\boldsymbol{A})^{-1}\,(\boldsymbol{x}, -\mathcal{R}\boldsymbol{\mu}), \quad \boldsymbol{x} \in \mathbb{R}^2, \quad \mu \in M_2, \tag{3.2}$$

satisfies, for all $\boldsymbol{\mu}, \boldsymbol{\nu}$ in M_2,

$$\left.\begin{aligned} \left(\hat{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{\mu}) - \hat{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{\nu})\right).(\boldsymbol{\mu} - \boldsymbol{\nu}) &\geq \beta^{-1}\,|\boldsymbol{\mu} - \boldsymbol{\nu}|^{p'}, \\ \left(\hat{\boldsymbol{A}}^{-1}(\boldsymbol{x}, \boldsymbol{\mu}) - \hat{\boldsymbol{A}}^{-1}(\boldsymbol{x}, \boldsymbol{\nu})\right).(\boldsymbol{\mu} - \boldsymbol{\nu}) &\geq \alpha\,|\boldsymbol{\mu} - \boldsymbol{\nu}|^{p}. \end{aligned}\right\} \tag{3.3}$$

Remark that $\hat{\boldsymbol{A}}^{-1}(\boldsymbol{x}, \boldsymbol{\mu}) = \mathcal{R}\boldsymbol{A}(\boldsymbol{x}, -\mathcal{R}\boldsymbol{\mu})$.

Consider now a sequence $\boldsymbol{A}^\varepsilon$ satisfying (2.1) that H_p–converges to $\boldsymbol{A}$ satisfying (2.1), and, for any $\boldsymbol{\lambda} \in M_2$, the associated stress–correctors $(\boldsymbol{Q}^\varepsilon(\boldsymbol{x}, \boldsymbol{\lambda}), q^\varepsilon(\boldsymbol{x}, \boldsymbol{\lambda}))$ satisfying (2.11)–(2.13). In view of (3.1), $\mathcal{R}\boldsymbol{Q}^\varepsilon(\boldsymbol{x}, \boldsymbol{\lambda})$ satisfies

$$\mathcal{R}\boldsymbol{Q}^\varepsilon(\boldsymbol{x}, \boldsymbol{\lambda}) \rightharpoonup \mathcal{R}\boldsymbol{\lambda} \quad \text{in } L^{p'}(\Omega; M_2), \tag{3.4}$$

$$\left.\begin{aligned} &\text{comp}\,[\mathcal{R}\boldsymbol{Q}^\varepsilon(\boldsymbol{x}, \boldsymbol{\lambda})] = -\text{curl }(\text{div }\boldsymbol{Q}^\varepsilon(\boldsymbol{x}, \boldsymbol{\lambda})) \\ &= -\text{curl }(\text{div }\boldsymbol{Q}^\varepsilon(\boldsymbol{x}, \boldsymbol{\lambda}) + \text{grad } q^\varepsilon(\boldsymbol{x}, \boldsymbol{\lambda})) \in \text{ compact of } W^{-2,p'}(\Omega; \mathbb{R}), \end{aligned}\right\} \tag{3.5}$$

$$\hat{\boldsymbol{A}}^\varepsilon\,(\boldsymbol{x}, \mathcal{R}\boldsymbol{Q}^\varepsilon(\boldsymbol{x}, \boldsymbol{\lambda})) \rightharpoonup \hat{\boldsymbol{A}}\,(\boldsymbol{x}, \mathcal{R}\boldsymbol{\lambda}) \quad \text{in } L^p(\Omega; M_2), \tag{3.6}$$

$$\left.\begin{aligned} \text{curl }\left(\text{div }\hat{\boldsymbol{A}}^\varepsilon\,(\boldsymbol{x}, \mathcal{R}\boldsymbol{Q}^\varepsilon(\boldsymbol{x}, \boldsymbol{\lambda}))\right) &= \text{comp}\,[(\boldsymbol{A}^\varepsilon)^{-1}\,(\boldsymbol{x}, \boldsymbol{Q}^\varepsilon(\boldsymbol{x}, \boldsymbol{\lambda}))] \\ &\in \text{ compact of } W^{-2,p}(\Omega; \mathbb{R}), \end{aligned}\right\}$$

where $\hat{\boldsymbol{A}}^\varepsilon$ and $\hat{\boldsymbol{A}}$ are defined from $\boldsymbol{A}^\varepsilon$ and $\boldsymbol{A}$ in (3.2). But if $\boldsymbol{D}^\varepsilon$ is in $W^{-1,p}(\Omega; R^2)$ (where Ω is a smooth connected open set) and if curl $\boldsymbol{D}^\varepsilon$ belongs to a compact set of $W^{-2,p}(\Omega; \mathbb{R})$, then there exists a field d^ε in $L^p(\Omega)$ such that $\boldsymbol{D}^\varepsilon - \text{grad } d^\varepsilon$ belongs to a compact set of $W^{-1,p}(\Omega; \mathbb{R}^2)$. Note that the existence of d^ε uses once again Korn's theorem of $L^{p'}(\Omega)$ and (2.3) with p' replaced by p.

Thus, there exists $\hat{p}^\varepsilon$ in $L^p(\Omega)/\mathbb{R}$ such that

$$\mathrm{div}\ \hat{\boldsymbol{A}}^\varepsilon(\boldsymbol{x}, \mathcal{R}\boldsymbol{Q}^\varepsilon(\boldsymbol{x}, \boldsymbol{\lambda})) + \mathrm{grad}\ \hat{p}^\varepsilon(\boldsymbol{x}, \boldsymbol{\lambda}) \in \text{compact of } W^{-1,p}(\Omega; M_2), \tag{3.7}$$

But in view of (3.3) applied to $\hat{\boldsymbol{A}}^\varepsilon$, a subsequence of $\hat{\boldsymbol{A}}^\varepsilon$, denoted by $\hat{\boldsymbol{A}}^{\varepsilon_j}$, $H_{p'}$–converges to $\boldsymbol{C}$ satisfying (3.3). By virtue of (3.4), (3.5,) (3.6), (3.7) and the local uniqueness in Proposition (2.4), $(\mathcal{R}\boldsymbol{Q}^{\varepsilon_j}(\boldsymbol{x}, \boldsymbol{\lambda}), \hat{p}^{\varepsilon_j}(\boldsymbol{x}, \boldsymbol{\lambda}))$ is, for that subsequence, a deformation–corrector — with respect to $\mathcal{R}\boldsymbol{\lambda}$ — and thus

$$\hat{\boldsymbol{A}}(\boldsymbol{x}, \mathcal{R}\boldsymbol{\lambda}) = \boldsymbol{C}(\boldsymbol{x}, \mathcal{R}\boldsymbol{\lambda}). \tag{3.8}$$

Since the $H_{p'}$–limit $\boldsymbol{C}$ is, in view of (3.8), independent of the extracted subsequence, the whole sequence $\hat{\boldsymbol{A}}^\varepsilon(\boldsymbol{x}, \mathcal{R}\boldsymbol{\lambda})$ $H_{p'}$–converges to $\hat{\boldsymbol{A}}(\boldsymbol{x}, \mathcal{R}\boldsymbol{\lambda})$. We have proved the following

Theorem 3.1 : *If $\boldsymbol{A}^\varepsilon$ H_p –converges to $\boldsymbol{A}$ and $N = 2$, then*

$$\hat{\boldsymbol{A}}^\varepsilon(\boldsymbol{x}, \boldsymbol{\mu}) \overset{def}{=} \mathcal{R}(\boldsymbol{A}^\varepsilon)^{-1}(\boldsymbol{x}, -\mathcal{R}\boldsymbol{\mu})$$

$H_{p'}$–converges to

$$\hat{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{\mu}) \overset{def}{=} \mathcal{R}(\boldsymbol{A})^{-1}(\boldsymbol{x}, -\mathcal{R}\boldsymbol{\mu}).$$

Remark 3.2 : Note that the action of the linear mapping $\mathcal{R}$ on an element $\boldsymbol{\Sigma}$ of M_2 may be rewritten as

$$\mathcal{R}\boldsymbol{\Sigma} = \mathbf{R}^\mathsf{T}\boldsymbol{\Sigma}\mathbf{R}, \quad -\mathcal{R}\boldsymbol{\Sigma} = \mathbf{R}\boldsymbol{\Sigma}\mathbf{R}^\mathsf{T},$$

where $\mathbf{R} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ is the rotation matrix associated to a rotation of $\pi/4$ in the plane.

In such a setting, Theorem 3.1 reads as

$$\mathbf{R}^\mathsf{T}(\boldsymbol{A}^\varepsilon)^{-1}(\boldsymbol{x}, \mathbf{R}\boldsymbol{\mu}\mathbf{R}^\mathsf{T})\mathbf{R} \overset{H_{p'}}{\rightharpoonup} \mathbf{R}^\mathsf{T}(\boldsymbol{A})^{-1}(\boldsymbol{x}, \mathbf{R}\boldsymbol{\mu}\mathbf{R}^\mathsf{T})\mathbf{R}.$$

In mechanical terms, the homogenized behavior of the dual composite is the dual of the homogenized behavior of the initial composite, provided that the microscopic and macroscopic deformations are rotated by $\pi/4$.

4 Two applications of the duality relation

This short section is devoted to two applications of Theorem 3.1.

4.1 Two–phase isotropic mixtures of incompressible power–law materials

Assume that, for some $1 < p < +\infty$,

$$\boldsymbol{A}^{\varepsilon}_{\alpha,\beta,p}(\boldsymbol{x},\boldsymbol{\lambda}) = (\alpha^p \chi^\varepsilon(\boldsymbol{x}) + \beta^p(1-\chi^\varepsilon(\boldsymbol{x})))\,|\boldsymbol{\lambda}|^{p-2}\,\boldsymbol{\lambda}, \tag{4.1}$$

where $0 < \alpha < \beta < +\infty$ and χ^ε is a sequence of characteristic functions on $\mathbb{R}^2$.

Assume further that $\boldsymbol{A}^{\varepsilon}_{\alpha,\beta,p}$ H_p–converges — which is always the case, up to the possible extraction of a subsequence, according to Theorem 2.2 — and that the H_p-limit $\boldsymbol{A}_{\alpha,\beta,p}$ is isotropic. Thus, in dimension 2, $\boldsymbol{A}_{\alpha,\beta,p}$ reads as (see Boehler [2] for instance)

$$\boldsymbol{A}_{\alpha,\beta,p}(\boldsymbol{x},\boldsymbol{\lambda}) = \varphi_1 \boldsymbol{i} + \varphi_2 \boldsymbol{\lambda}, \quad \varphi_i = \varphi_i(\boldsymbol{x}, \mathrm{tr}\boldsymbol{\lambda}, \mathrm{tr}\boldsymbol{\lambda}^2), \quad \mathrm{tr}\boldsymbol{\lambda}^2 = |\boldsymbol{\lambda}|^2.$$

It follows from the incompressibility constraint that φ_i does not depend on $\mathrm{tr}\boldsymbol{\lambda}$. Then, a simple dilation argument would show that $\boldsymbol{A}_{\alpha,\beta,p}$ is positively homogeneous of degree $p-1$ in $\boldsymbol{\lambda}$. $\boldsymbol{A}_{\alpha,\beta,p}$ finally reads as

$$\boldsymbol{A}_{\alpha,\beta,p}(\boldsymbol{x},\boldsymbol{\lambda}) = a^p\,(\boldsymbol{x},\alpha,\beta,p)\,|\boldsymbol{\lambda}|^{p-2}\,\boldsymbol{\lambda}, \tag{4.2}$$

for some measurable function a, with $\alpha \le a\,(\boldsymbol{x},\alpha,\beta,p) \le \beta$. In all rigor the above derivation presupposes some degree of smoothness on $\boldsymbol{A}_{\alpha,\beta,p}(\boldsymbol{x},.)$; a scrupulous reader is thus at liberty to postulate the form (4.2) for $\boldsymbol{A}_{\alpha,\beta,p}$.

In such a setting $\hat{\boldsymbol{A}}_{\alpha,\beta,p}(\boldsymbol{x},\boldsymbol{\mu})$ defined in (3.2) reduces to

$$\hat{\boldsymbol{A}}_{\alpha,\beta,p}(\boldsymbol{x},\boldsymbol{\mu}) = \left(\alpha^{-p'}\chi^\varepsilon(\boldsymbol{x}) + \beta^{-p'}(1-\chi^\varepsilon(\boldsymbol{x}))\right)|\boldsymbol{\mu}|^{p'-2}\,\boldsymbol{\mu},$$

with $1/p + 1/p' = 1$. Theorem 3.1 asserts that

$$\hat{\boldsymbol{A}}^{\varepsilon}_{\alpha,\beta,p}(\boldsymbol{x},\boldsymbol{\mu})) \stackrel{H_p}{\rightharpoonup} a^{-p'}\,(\boldsymbol{x},\alpha,\beta,p)\,|\boldsymbol{\mu}|^{p'-2}\,\boldsymbol{\mu}.$$

But

$$\hat{\boldsymbol{A}}^{\varepsilon}_{\alpha,\beta,p}(\boldsymbol{x},\boldsymbol{\mu})) = \frac{1}{(\alpha\beta)^{p'}}\left(\beta^{p'}\chi^\varepsilon(\boldsymbol{x}) + \alpha^{p'}(1-\chi^\varepsilon(\boldsymbol{x}))\right)|\boldsymbol{\mu}|^{p'-2}\,\boldsymbol{\mu} = \boldsymbol{A}^{\varepsilon}_{\beta,\alpha,p'}(\boldsymbol{x},\boldsymbol{\mu})),$$

which thus $H_{p'}$–converges to $\dfrac{1}{(\alpha\beta)^{p'}} a^{p'}\,(\boldsymbol{x},\beta,\alpha,p')\,|\boldsymbol{\mu}|^{p'-2}\,\boldsymbol{\mu}$.

Remark 4.1 : It is here implicitely assumed that the microgeometry — the sequence $\chi^\varepsilon(\boldsymbol{x})$ — is such that any H_p-converging sequence of the form (4.1) has an isotropic H_p-limit, for any admissible triplet α, β, p.

We thus obtain the following duality relation

$$a(\boldsymbol{x},\alpha,\beta,p)\ a(\boldsymbol{x},\beta,\alpha,p') = \alpha\ \beta. \tag{4.3}$$

Finally, if the microgeometry is invariant under permutation of the phases, (4.3) becomes

$$a(\alpha,\beta,p)\ a(\alpha,\beta,p') = \alpha\ \beta. \tag{4.4}$$

As such, (4.4) is a generalization of formula (2.46) in Helsing *et al* [9]. In particular if $p = p' = 2$ we obtain as overall shear modulus

$$a(\alpha,\beta,2) = \sqrt{\alpha\beta},$$

as already observed in Lurié and Cherkaev [14] (see also formula (2.47) in [9]).

4.2 Two-phase macroscopic interchangeability with $\pi/4$ macroscopic invariance

We call *self–dual* a material which is its own dual in the sense of the beginning of section 3, that is a material such that its stress–strain relation satisfies

$$\boldsymbol{A}(\boldsymbol{x},\boldsymbol{\lambda}) = \boldsymbol{A}^{-1}(\boldsymbol{x},\boldsymbol{\lambda}). \tag{4.5}$$

But (4.5) immediately implies that $\boldsymbol{A}(\boldsymbol{x},\boldsymbol{\lambda}) = \boldsymbol{\lambda}$. Thus the only self–dual material is the linear incompressible and isotropic material with shear modulus 1/2 (recall that the shear modulus μ of a linear isotropic material is defined as $\boldsymbol{s} = 2\mu\ \boldsymbol{e}$).

We now investigate the existence of self–dual mixtures. Consider a two-phase mixture, the stiffness of which is given by $\boldsymbol{B}(\boldsymbol{\lambda})$ in phase 1 and $\boldsymbol{B}^{-1}(\boldsymbol{\lambda})$ in phase 2, *i.e.*,

$$\boldsymbol{A}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}) = \chi^{\varepsilon}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{\lambda}) + (1-\chi^{\varepsilon}(\boldsymbol{x})\,(\boldsymbol{B})^{-1}(\boldsymbol{\lambda}). \tag{4.6}$$

In (4.6), $\boldsymbol{B}(\boldsymbol{\lambda})$ is assumed to satisfy (2.1) with $p = 2$, so that $\boldsymbol{B}^{-1}(\boldsymbol{\lambda})$ also satisfies (2.1) with the same p. We also assume microscopic $\pi/4$ material invariance that is

$$\boldsymbol{B}(\mathbf{R}\boldsymbol{\lambda}\mathbf{R}^{\mathsf{T}}) = \mathbf{R}\boldsymbol{B}(\boldsymbol{\lambda})\mathbf{R}^{\mathsf{T}}, \quad \boldsymbol{\lambda} \in M_2. \tag{4.7}$$

Note that (4.7) also holds true if $\boldsymbol{B}$ is replaced by $\boldsymbol{B}^{-1}$. Define the *interchanged* mixture as that where the phases are interchanged, that is,

$$\tilde{\boldsymbol{A}}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}) = \chi^{\varepsilon}(\boldsymbol{x})\,(\boldsymbol{B})^{-1}(\boldsymbol{\lambda}) + (1-\chi^{\varepsilon}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{\lambda}) = (\boldsymbol{A}^{\varepsilon})^{-1}(\boldsymbol{x},\boldsymbol{\lambda});$$

the interchanged mixture then coincides with the dual mixture. Assume — which is, as already seen several times, no restriction — that

$$\left.\begin{array}{l} \boldsymbol{A}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}) \xrightarrow{H_2} \boldsymbol{A}(\boldsymbol{x},\boldsymbol{\lambda}), \\[2ex] \tilde{\boldsymbol{A}}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}) \xrightarrow{H_2} \tilde{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{\lambda}). \end{array}\right\}$$

Now, in view of (4.7), $\hat{\boldsymbol{A}}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda})$ defined in the previous section satisfies

$$\hat{\boldsymbol{A}}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}) = \tilde{\boldsymbol{A}}^{\varepsilon}(\boldsymbol{x},\boldsymbol{\lambda}),$$

so that upon application of Remark 3.2

$$\mathbf{R}^{\mathsf{T}}\boldsymbol{A}^{-1}(\boldsymbol{x},\mathbf{R}\boldsymbol{\lambda}\mathbf{R}^{\mathsf{T}})\mathbf{R} = \tilde{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{\lambda}). \tag{4.8}$$

If the mixture is *interchangeable*, i.e., if

$$\boldsymbol{A}(\boldsymbol{x},\boldsymbol{\lambda}) = \tilde{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{\lambda}), \tag{4.9}$$

we thus obtain that

$$\mathbf{R}^{\mathsf{T}}\boldsymbol{A}^{-1}(\boldsymbol{x},\mathbf{R}\boldsymbol{\lambda}\mathbf{R}^{\mathsf{T}})\mathbf{R} = \boldsymbol{A}(\boldsymbol{x},\boldsymbol{\lambda}). \tag{4.10}$$

If in addition the mixture is macroscopically invariant by rotation of $\pi/4$, (4.10) implies that it is self–dual hence that $\boldsymbol{A}(\boldsymbol{x},\boldsymbol{\lambda}) = \boldsymbol{\lambda}$. This result is along the lines of similar results in Kozlov [11] or in Levy and Kohn [12] for the conductivity case.

Remark 4.2 *:* The reader is referred to (5.15) in Section 5 for a similar example in the case of behaviors with non-quadratic growth. Note however that Theorem 3.1 cannot apply in the latter setting for want of an estimate of the form (2.1) on the constitutive law.

5 The duality relation viewed from an energetic standpoint

This section may be seen as a revisiting of Section 3 in a variational framework. We do not attempt here to provide full mathematical justification for the argument, but rather strive for an expeditious derivation of the analogue of Theorem 3.1. We further assume that all mixtures are periodic, so that the relevant macroscopic behaviors — the corresponding Γ–limits — are homogeneous, *i.e.* do not explicitly depend upon the spatial variable $\boldsymbol{x}$.

The microstructure is then defined on the periodic cell $Y = [0,1]^2$ through its elastic energy $w(\boldsymbol{x},\boldsymbol{e})$, a $\overline{\mathbb{R}}$-valued, convex Caratheodory function on $Y \times \mathbb{R}^4_s$ such that $w(\boldsymbol{x},\boldsymbol{e})$ is finite if and only if $\boldsymbol{e} \in M_2$.

The effective primal and dual energies are given, for any $\boldsymbol{E}, \boldsymbol{\Sigma}$ in M_2 by

$$W(\boldsymbol{E}) = \operatorname{Inf}\left\{\int_Y w(\boldsymbol{x}, \boldsymbol{e}(\boldsymbol{v}))d\boldsymbol{x},\ \boldsymbol{v} = \boldsymbol{E}.\boldsymbol{x} + \boldsymbol{v}^*,\ \boldsymbol{v}^* \text{ periodic, div } \boldsymbol{v}^* = 0\right\}, \quad (5.1)$$

$$\begin{aligned} W^*(\boldsymbol{\Sigma}) = \operatorname{Inf}\Big\{\int_Y w^*(\boldsymbol{x}, \boldsymbol{s})\ d\boldsymbol{x}, \quad & \boldsymbol{s} = \boldsymbol{\Sigma} + \boldsymbol{s}^*, \\ & \textstyle\int_Y \boldsymbol{s}^*\ d\boldsymbol{x} = 0, \quad \boldsymbol{s}^*.\boldsymbol{n} \text{ antiperiodic, div } \boldsymbol{s}^* = 0\Big\}. \end{aligned} \quad (5.2)$$

In (5.2), $w^*(\boldsymbol{x}, \boldsymbol{s})$ is the Fenchel transform of $w(\boldsymbol{x}, \boldsymbol{e})$ with respect to $\boldsymbol{e}$ and $\boldsymbol{n}$ is the outwardly directed normal to ∂Y.

Remark 5.1 *:* Under appropriate growth and coercivity conditions the heuristic homogenization process above can be fully justified in the context of Γ-convergence (see e.g. Braides and DeFrancheschi [3], ch. 14 and references herein). For example it suffices to assume that

$$\alpha\,|\boldsymbol{e}|^p \leq w(\boldsymbol{x}, \boldsymbol{e}) \leq \beta(1 + |\boldsymbol{e}|^p), \quad \boldsymbol{e} \in M_2, \quad (5.3)$$

with $1 < p < +\infty$, in which case W is the integrand associated to the $\Gamma(L^p)$-limit of $\displaystyle\int_Y w(\boldsymbol{x}/\varepsilon, \boldsymbol{e}(\boldsymbol{u}))\ d\boldsymbol{x}$.

Remark 5.2 *:* Since w takes the value $+\infty$ outside M_2, it is easily seen that in (5.1) (5.2), w can be restricted to elements of M_2, while w^* does not depend on tr$\boldsymbol{s}$, or, in other words, w is a function of e_{11}, e_{12} while w^* is a function of $1/2(s_{11} - s_{22}), s_{12}$.

Let us further investigate (5.1) and (5.2). A divergence-free vector $\boldsymbol{v}^*$ on Y can be expressed as

$$v_1^* = \frac{\partial \varphi}{\partial x_2}, \quad v_2^* = -\frac{\partial \varphi}{\partial x_1},$$

with φ defined on Y; since $\boldsymbol{v}^*$ is periodic, φ may be further chosen to be such that gradφ is periodic. Thus, in view of Remark 5.2, (5.1) can be replaced by

$$\left.\begin{aligned} W(\boldsymbol{E}) = \operatorname{Inf}\int_Y w\left(\boldsymbol{x}, E_{11} + \frac{\partial^2\varphi}{\partial x_1 \partial x_2}; E_{12} + \frac{1}{2}\left(\frac{\partial^2\varphi}{\partial x_2^2} - \frac{\partial^2\varphi}{\partial x_1^2}\right)\right) dx_1 dx_2, \\ \varphi \text{ such that grad } \varphi \text{ is } Y-\text{periodic.} \end{aligned}\right\} \quad (5.4)$$

In a similar manner a divergence-free tensor field $\boldsymbol{s}^*$ on Y can be expressed as

$$s_{11}^* = \frac{\partial^2\psi}{\partial x_2^2}, \quad s_{12}^* = -\frac{\partial^2\psi}{\partial x_1 \partial x_2}, \quad s_{22}^* = \frac{\partial^2\psi}{\partial x_1^2}.$$

From the condition that $\int_Y \boldsymbol{s}^* \, d\boldsymbol{x} = 0$ and that $\boldsymbol{s}^*.\boldsymbol{n}$ is Y-antiperiodic on ∂Y, it is easily deduced through straight integration that gradψ must be Y–periodic. Thus, in view of Remark 5.2, (5.2) can be replaced by

$$\left.\begin{array}{c} W^*(\boldsymbol{\Sigma}) = \text{Inf} \displaystyle\int_Y w^* \left(\boldsymbol{x}, \frac{1}{2}(\Sigma_{11} - \Sigma_{22} + \frac{\partial^2 \psi}{\partial x_2^2} - \frac{\partial^2 \psi}{\partial x_1^2}); \Sigma_{12} - \frac{\partial^2 \psi}{\partial x_1 \partial x_2} \right) \, d\boldsymbol{x}, \\ \psi \text{ such that grad } \psi \text{ is } Y - \text{periodic.} \end{array}\right\} \tag{5.5}$$

Define, for any $\boldsymbol{e}$ in $\mathbb{R}^4_s$,

$$\hat{w}(\boldsymbol{x}, \boldsymbol{e}) = w^*(\boldsymbol{x}, \mathbf{R}\boldsymbol{e}\mathbf{R}^{\mathsf{T}}), \tag{5.6}$$

where $\mathbf{R}$ is the rotation matrix of angle $\pi/4$ introduced in Remark 3.2. Then the associated effective primal energy is, for any $\boldsymbol{E}$ in M_2 defined as (see (5.4))

$$\left.\begin{array}{c} \hat{W}(\boldsymbol{E}) = \text{Inf} \displaystyle\int_Y \hat{w} \left(\boldsymbol{x}, E_{11} + \frac{\partial^2 \varphi}{\partial x_1 \partial x_2}; E_{12} + \frac{1}{2}(\frac{\partial^2 \varphi}{\partial x_2^2} - \frac{\partial^2 \varphi}{\partial x_1^2}) \right) \, d\boldsymbol{x}, \\ \varphi \text{ such that grad } \varphi \text{ is } Y - \text{periodic.} \end{array}\right\} \tag{5.7}$$

Thus, in view of (5.6), (5.7) reads as

$$\left.\begin{array}{c} \hat{W}(\boldsymbol{E}) = \text{Inf} \displaystyle\int_Y w^* \left(\boldsymbol{x}, E_{12} + \frac{1}{2}(\frac{\partial^2 \varphi}{\partial x_2^2} - \frac{\partial^2 \varphi}{\partial x_1^2}); -E_{11} - \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} \right) \, d\boldsymbol{x}, \\ \varphi \text{ such that grad } \varphi \text{ is } Y - \text{periodic.} \end{array}\right\} \tag{5.8}$$

But, according to (5.5), (5.8) is exactly $W^*(\mathbf{R}\boldsymbol{E}\mathbf{R}^{\mathsf{T}})$. Thus

$$\hat{W}(\boldsymbol{E}) = W^*(\mathbf{R}\boldsymbol{E}\mathbf{R}^{\mathsf{T}}). \tag{5.9}$$

In other words we have established the following result, in the context of *periodic* homogenization,

Proposition 5.3 *: Let $w(\boldsymbol{x}, \boldsymbol{e})$ be a $\overline{\mathbb{R}}$-valued convex Caratheodory function on $Y \times \mathbb{R}^4_s$ with $w(\boldsymbol{x}, \boldsymbol{e}) = +\infty$ if* tr$\boldsymbol{e} \neq 0$. *If $W(\boldsymbol{E})$ is, for any $\boldsymbol{E}$ in M_2, the effective primal energy associated to $w(\boldsymbol{x}, \boldsymbol{e})$, then the primal energy associated to $\hat{w}(\boldsymbol{e}) = w^*(\mathbf{R}\boldsymbol{e}\mathbf{R}^T)$, seen as a primal energy (and not a dual energy) is $W^*(\mathbf{R}\boldsymbol{E}\mathbf{R}^T)$.*

Remark 5.4 *:* The above result would still hold true for any kind of boundary conditions on ∂Y, as long as Hill's lemma also holds true ([17]).

Remark 5.5 : Note that this result is consistent with that of Theorem 3.1. Indeed, if

$$\boldsymbol{A}(\boldsymbol{x},\boldsymbol{\lambda})=\frac{\partial W}{\partial \boldsymbol{E}}(\boldsymbol{x},\boldsymbol{\lambda}),$$

then

$$\hat{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{\lambda})=\frac{\partial \hat{W}}{\partial \boldsymbol{E}}(\boldsymbol{x},\boldsymbol{\lambda})=\mathbf{R}^{\mathsf{T}}\frac{\partial W^*}{\partial \boldsymbol{S}}(\boldsymbol{x},\mathbf{R}\boldsymbol{\lambda}\mathbf{R}^{\mathsf{T}})\mathbf{R}=\mathbf{R}^{\mathsf{T}}\boldsymbol{A}^{-1}(\boldsymbol{x},\mathbf{R}\boldsymbol{\lambda}\mathbf{R}^{\mathsf{T}})\mathbf{R}.$$

Remark 5.6 : In the context of Remark 5.1, Proposition 5.3 can easily be turned into a theorem on the $\Gamma(L^p)$–limit of $\int_Y w(\boldsymbol{x}/\varepsilon,\boldsymbol{e}(\boldsymbol{u})\ d\boldsymbol{x}$ under the same hypothesis (5.3).

Remark 5.7 : In mechanical terms the dual composite (i.e., that with elastic energy density $w^*(\boldsymbol{x},\boldsymbol{e})$) has for effective behavior the dual of the effective energy density provided that the macroscopic and microscopic deformations are rotated by $\pi/4$.

We close this section with two applications of Proposition 5.3. The first application deals with a periodic two-phase composite; both phases are power-law materials with the same exponent p and material constants α in phase 1 and β in phase 2. An in-plane shear deformation

$$\boldsymbol{E}=E_{12}\,(\boldsymbol{e}_1\otimes\boldsymbol{e}_2+\boldsymbol{e}_2\otimes\boldsymbol{e}_1)$$

is applied. The homogenized strain and stress energies of the composite read as

$$W(\boldsymbol{E})=\frac{1}{p}\ a^p\,|E_{12}|^p\,,\quad W^*(\Sigma)=\frac{1}{p'}\ \frac{1}{a^{p'}}\,|\Sigma_{12}|^{p'}\,,\quad \frac{1}{p}+\frac{1}{p'}=1.$$

The dual composite is made of power-law phases with exponent p' and material constants $1/\alpha$ and $1/\beta$ respectively. The rotated strain reads

$$\mathbf{R}\boldsymbol{E}\mathbf{R}^{\mathsf{T}}=E_{12}\,(\boldsymbol{e}_1\otimes\boldsymbol{e}_1-\boldsymbol{e}_2\otimes\boldsymbol{e}_2)\,,$$

and the strain energy of the rotated dual composite reads as

$$\hat{W}(\boldsymbol{E})=\frac{1}{p'}\ \hat{a}^{p'}\,|E_{12}|^{p'}\,.$$

It follows from Proposition 5.3 that $\hat{a}=1/a$.

The second application considers a setting similar to that of Subsection 4.2 in a non-quadratic framework. Specifically it is assumed that

$$w(\boldsymbol{x},\boldsymbol{e})=\chi(\boldsymbol{x})z(\boldsymbol{e})+(1-\chi(\boldsymbol{x}))z^*(\boldsymbol{e}),\tag{5.10}$$

where $z(\boldsymbol{e})$ is a convex elastic density on $\mathbb{R}^4_s$, finite on and only on M_2 and $\chi(\boldsymbol{x})$ a characteristic function on Y. Further, we assume that

$$z(\mathbf{R}\boldsymbol{e}\mathbf{R}^{\mathsf{T}}) = z(\boldsymbol{e}), \quad \boldsymbol{e} \in M_2, \tag{5.11}$$

where $\mathbf{R}$ is, as before, the rotation matrix of angle $\pi/4$. Finally we assume interchangeability, *i.e.*, that upon setting

$$\tilde{w}(\boldsymbol{x}, \boldsymbol{e}) = \chi(\boldsymbol{x})z^*(\boldsymbol{e}) + (1 - \chi(\boldsymbol{x}))z(\boldsymbol{e}),$$

we have

$$W(\boldsymbol{E}) = \tilde{W}(\boldsymbol{E}), \tag{5.12}$$

where $\tilde{W}$ is the effective primal energy associated to $\tilde{w}$ through (5.1) (or (5.4)). Thus, in view of (5.11) (which also holds true for z^*)

$$\hat{w}(\boldsymbol{x}, \boldsymbol{e}) = \tilde{w}(\boldsymbol{x}, \boldsymbol{e}),$$

so that,

$$\hat{W}(\boldsymbol{E}) = \tilde{W}(\boldsymbol{E}). \tag{5.13}$$

But, according to Proposition 5.3 and equation (5.9),

$$\hat{W}(\boldsymbol{E}) = W^*(\mathbf{R}\boldsymbol{E}\mathbf{R}^{\mathsf{T}}),$$

so that in view of (5.12),(5.13)

$$W(\boldsymbol{E}) = W^*(\mathbf{R}\boldsymbol{E}\mathbf{R}^{\mathsf{T}}). \tag{5.14}$$

If, finally, W is invariant under rotation of $\pi/4$, *i.e.*, if

$$W(\boldsymbol{E}) = W(\mathbf{R}\boldsymbol{E}\mathbf{R}^{\mathsf{T}}), \quad \boldsymbol{E} \in M_2,$$

(5.14) yields

$$W(\boldsymbol{E}) = W^*(\boldsymbol{E}), \quad \boldsymbol{E} \in M_2,$$

from which it is immediately concluded that

$$W(\boldsymbol{E}) = \frac{1}{2}\, \boldsymbol{E}.\boldsymbol{E}. \tag{5.15}$$

Thus in the terminology of Section 4.2, an interchangeable material of the form (5.10) with microscopic and macroscopic $\pi/4$ invariance behaves macroscopically like a linear incompressible isotropic material with shear modulus $1/2$.

Remark 5.8 : The result above generalizes that of subsection 4.2 in the case of non-quadratic growth.

Remark 5.9 : The result above can be turned into a theorem provided that the formulae yielding the effective energies are justified in the present context. Note that the energy density $w(\boldsymbol{x}, \boldsymbol{e})$ satisfies the non-standard growth and coercivity condition,

$$\alpha\,|\boldsymbol{e}|^{p} \leq w(\boldsymbol{x}, \boldsymbol{e}) \leq \beta(1+|\boldsymbol{e}|^{p'}), \quad \frac{1}{p}+\frac{1}{p'}=1,$$

for some $0<\alpha<\beta<+\infty$, provided that, for some $0<\alpha'<\beta'<+\infty$,

$$\alpha'\,|\boldsymbol{e}|^{p} \leq z(\boldsymbol{e}) \leq \beta'(1+|\boldsymbol{e}|^{p}),$$

and also that $p \leq 2$.

In such a case the elastic energy $W(\boldsymbol{E})$ may be proved to be the energy density associated to the $\Gamma(L^p)$-limit of $\int_Y w(\boldsymbol{x}/\varepsilon, \boldsymbol{e}(\boldsymbol{u}))\,d\boldsymbol{x}$ whenever $\frac{4}{3}<p\leq 2$ (cf. Braides and DeFranceschi [3] ch. 21).

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