# Multifractal Spectra of Branching Measure on a Galton-Watson Tree

Narn-Rueih SHIEH
*Department of Mathematics, National Taiwan University*
*Taipei, Taiwan*
E–mail: shiehnr@math.ntu.edu.tw
and
S. James TAYLOR
*School of Mathematics, University of Sussex*
*Falmer, Brighton, England*
E–mail: SJT4K@aol.com

**Abstract**

If $Z$ is the branching mechanism for a supercritical Glaton–Watson tree with a single progenitor and $E[Z \log Z] < \infty$, there is a branching measure $\mu$ defined on $\partial\Gamma$ the set of all path $\xi$ which has a unique node $\xi|n$ at each generation $n$. We use the natural metric $\rho(\xi, \eta) = e^{-n}$ where $n = \max\{k : \xi|k = \eta|k\}$ and observe that the local dimension index

$$d(\mu.\xi) = \lim_{n\to\infty} \frac{\log \mu(B(\xi|n))}{-n} = \alpha = \log m, \quad \mu - a.e. \ \ \xi.$$

Our objective is to consider the exceptional points where the above display may fail. There is a non–trivial "thin" spectrum for $\overline{d}(\mu, \xi)$ when $p_1 = P\{Z = 1\} > 0$ and $Z$ has finite moments of all positive orders. Because $\underline{d}(\mu, \xi) = \alpha$ for all $\xi$, we obtain a "thick" spectrum by introducing the "right" power of a logarithm. In both cases we find the Hausdorff dimension of the exceptional sets.

*Running Title:* Multifractal spectra on GWT
*AMS Subject Classifications(1991):* Primary 60G17, 60J80
*Key words:* branching measure, Galton–Watson tree,
Hausdorff dimension, packing dimension, multifractal spectrum.

# 1 Introduction

We are interested in supercritical Galton-Watson trees with a single progenitor. Let $Z = \{p_0, p_1, \cdots\}$ be the offspring distribution of a Galton–Watson branching process, defined on a probability space $(\Omega, P)$, with mean $m = \sum jp_j > 1$ and $\Gamma = \Gamma(\omega)$ denote the associated family tree. Let $\mu$ be the branching measure on the boundary $\partial\Gamma$; see Section 2 for more detailed descriptions. We remark that $\mu$ is a random measure on the (random) tree $\Gamma$ and our object in this paper is to find properties of the multifractal spectra of $\mu$ which are true with probability one. Put

$$\alpha = \log m > 0.$$

It is already known that, with probability one,

$$d(\mu, \xi) = \lim_{n\to\infty} \frac{\log \mu B(\xi|n)}{n} = -\alpha \quad \mu - a.e. \ \ \xi \in \partial\Gamma,$$

which can be translated, using the natural metric in $\partial\Gamma$, to

$$d(\mu, \xi) = \lim_{r\to 0} \frac{\log \mu B(\xi, r)}{\log r} = \alpha, \quad \mu - a.e. \ \ \xi.$$

The above display is the usual starting point for multifractal analysis of a locally finite Borel measure.

In a recent paper Liu[L3] showed that, if $p_0 = p_1 = 0$ and $Z$ has finite moments of all orders then, with probability one,

$$d(\mu, \xi) = \alpha, \quad \forall\xi.$$

Thus the ordinary multifractal spectrum is trivial in this case. However, even in this case, as was shown for the occupation measure of stable subordinator in Shieh–Taylor[ST] and of Brownian motion in Dembo–Peres–Rosen–Zeitouni[DPRZ], one can observe a non-trivial spectrum for "thick" points, by introducing an appropriate power of a logarithm. Results of this type are obtained in Section 4.

In [L3], Liu also observed that, if $Z$ has finite moments of all positive orders, then

$$\underline{d}(\mu, \xi) = \liminf_{n\to\infty} \frac{-\log \mu B(\xi|n)}{n} = \alpha \quad \forall\xi \in \partial\Gamma,$$

which implies that for $\beta \neq \alpha$ the set

$$E_\beta = \{\xi \in \partial\Gamma : \quad d(\mu, \xi) = \beta\} \quad \text{is empty},$$

so that the standard multifractal formalism cannot yield a spectrum. However, as in Perkins–Taylor[PT] for super–Brownian motion and Hu–Taylor[HT] for stable occupation measure, we prove that, when $0 < p_1 < 1$, there is a non–trivial spectrum for

$$\overline{d}(\mu, \xi) = \limsup_{n\to\infty} \frac{-\log \mu B(\xi|n)}{n}.$$

In Section 5, we obtain the Hausdorff dimension of

$$C_\beta = \{\xi : \quad \overline{d}(\mu, \xi) \geq \beta\},$$
$$D_\beta = \{\xi : \quad \overline{d}(\mu, \xi) = \beta\},$$

for an interval of values of $\beta$ in which these sets are non–empty. In both Sections 4 and 5, our method also yields the packing dimension of the relevant sets.

In this paper we again make use of the strong spherical porosity conditions first defined in [PT]. Section 2 defines this condition and its meaning on $\partial\Gamma$; in addition we recall the necessary preliminary definitions and results for Galton–Watson trees. In Dembo–Peres–Rosen–Zeitouni[DPRZ], and Khoshnevisan–Peres–Xiao[KPX], it is pointed out that many exceptional sets examined in random phenomena are of limsup type; they provide a useful theorem giving a lower bound for the Hausdorff dimension of such sets The exceptional sets which we examine now on $\partial\Gamma$ are again of limsup type. We therefore develop a version of the main theorem in [DPRZ], proved there for Euclidean cubes, which is valid in the context of a Galton–Watson tree. This is done in Section 3, and is used in both Sections 4 and 5.

# 2 Preliminaries

We begin with notation and results for Galton–Watson trees which we need in this paper; these are adapted from Pemantle–Peres[PP]. Let $Z = \{p_0, p_1, \cdots\}$ be the offspring distribution of a Galton–Watson branching process. We assume that $m := \sum_j jp_j < \infty$ and that $p_0 = 0, p_1 < 1$; thus $1 < m < \infty$, that is we are in the supercritical case(As pointed out in [PP], if $p_0 > 0$ and $m > 1$ there is positive probability that $\Gamma$ is finite. All our results remain true, condtioned on the event that $\Gamma$ is infinite. However we impose $p_0 = 0$ to eliminate the complication of conditioning). We also assume that $Z$ is not a constant, that is $p_j < 1, \quad \forall j$. Associated with each realization of the process, we have a (random) family tree, called a *Galton–Watson tree*(GWT), which we denote by $\Gamma = \Gamma(\omega)$. Let $\Gamma_n, n = 0, 1, 2, \cdots$ be the $n$–th level(generation) so that $\Gamma = \cup_n \Gamma_n$. Let $Z_n$ denote the cardinal number of $\Gamma_n$ and we assume that $Z_0 = \{\emptyset\}$(single progenitor). Assume moreover that $E[Z \log Z] < \infty$, then the limit

$$W := \lim_{n\to\infty} \frac{Z_n}{m^n}$$

exists and is finite and positive a.s., see Artheya–Ney[AN]. For a GWT $\Gamma$ there is associated a natural *boundary* $\partial\Gamma$, which is defined as the set of infinite self–avoiding paths from $\emptyset$ through the tree; we denote by $\xi$ a generic point in $\partial\Gamma$. For $\sigma \in \Gamma_n$, $|\sigma| = n$ denotes its length and $B(\sigma) = B(\sigma, r), \quad r = e^{-n}$, denotes the "ball" $\{\xi \in \partial\Gamma : \sigma$ is the ancestor of $\xi$ in $\Gamma_n\}$. We also use $\xi|n$ to denote the ancestor in $\Gamma_n$ of an $\xi \in \partial\Gamma$. We remark that $\partial\Gamma$ is a compact metric space under the metric $d(\xi_1, \xi_2) = e^{-n}$, where $n = \max\{k : \xi_1|k = \xi_2|k\}$. In this metric, the subtree consisting of the vertex $\sigma \in \Gamma_n$ and all its descendants is a ball in $\partial\Gamma$ of diameter of $e^{-n}$, as denoted above. Let $W(\sigma)$ be the shifted $W$ at the vertex $\sigma$, that is

$$W(\sigma) := \lim_{n\to\infty} \frac{\text{card}\{\eta \in \Gamma_n : \sigma \text{ is the ancester of } \eta\}}{m^{n-|\sigma|}}.$$

Since $\Gamma$ is a countable set, $W(\sigma)$ exists for all $\sigma \in \Gamma$ with probability one. By assuming the existence of $W(\sigma)$, we define *branching measure* on $\partial\Gamma$ as the unique (random) Borel measure $\mu$ on $\partial\Gamma$ such that

$$\mu B(\sigma) = m^{-n} W_\sigma, \quad \sigma \in \Gamma_n.$$

Note that $W(\sigma), \sigma \in \Gamma_n$, are iid with the same distribution as $W$, conditional on $Z_j, j \leq n$. Thus, the above display reflects the statistical self-similarity of the measure $\mu$.

In Perkins–Taylor[PT], the notion of a $\gamma - thin$ set for $\gamma > 1$ was defined. The definition was for $\mathbb{R}^d$ but it translates to any metric space, so we define it for $\partial\Gamma$.

**Definition 2.1** *Fix $\gamma > 1$. We say $E \subset \partial\Gamma$ is $\gamma - thin$ at $\xi$ if there is a sequence $r_i \downarrow 0$ such that*

$$E \cap [B(\xi, r_i) \backslash B(\xi, r_i^\gamma)] = \emptyset \quad \forall i.$$

*We say $E$ is a $\gamma - thin$ set if $E$ is $\gamma - thin$ at each $\xi \in E$.*

The nature of our metric on $\partial\Gamma$ then asserts that $\partial\Gamma$ is $\gamma - thin$ at $\xi$ if and only if

$$B(\xi|n_i) = B(\xi|[\gamma n_i])$$

for an increasing sequence of positive integers $n_i$($[\cdot]$: the greatest integer part). This is equivelent to saying that $\xi|n \in \Gamma_n$ has exactly one descendant in $\Gamma_{n+1}$ for $n_i \leq n \leq [\gamma n_i]$, $\quad i = 1, 2, \ldots$, This shows that $\partial\Gamma$ can have $\gamma-$thin points only if $p_1 > 0$, and in this case we denote the set of all $\gamma-$thin points for $\partial\Gamma$ by $S_\gamma$.

We now recall some facts first proved in [PT] for Euclidean space. It is easy to check that the results remain true for $\partial\Gamma$.

**Lemma 2.1** *For any $\gamma - thin$ set $E \subset \partial\Gamma$, we have*

$$\mathrm{Dim}E \geq \gamma \mathrm{dim}E.$$

Here. Dim stands for packing dimension and dim stands for Hausdorff dimension. One can refer to [PP] for the detailed definitions and properties of Dim and dim on GWT's.

**Lemma 2.2** *Let $\nu$ be any Borel measure on $\partial\Gamma$ and its support be $S = S(\nu)$. If $A \subset S$ is $\gamma - thin$, and*

$$\underline{d}(\nu, x) \geq a \quad \forall x \in A,$$

*then,*

$$\overline{d}(\nu, x) \geq \gamma a \quad \forall x \in A.$$

We note that, in the case of branching measure $\mu$, with probability one, every ball $B(\sigma)$ has positive $\mu$ measure. Hence $S(\mu) = \partial\Gamma$. We will see that there is a range of values of $\gamma$ for which $S_\gamma \neq \emptyset$, provided some simple conditions are satisfied.

We mention that the metric space $\partial\Gamma$ has fractal dimension $\alpha$: with probability one,

$$\mathrm{dim}\partial\Gamma = \mathrm{Dim}\partial\Gamma = \alpha.$$

# 3 Limsup fractals on Galton–Watson Trees

The following two propositions are a version of Theorem 2.1 of [DPRZ] for GWT. Firstly we note that $\partial\Gamma$ can be regarded as a random subset of the infinite sequence space $\mathbb{N}^\infty$.

We define a random mapping $\Psi$ on $\mathbb{N}^{\infty} \times \Omega$ so that $\Psi(B,\omega) = 0$, whenver $B \not\subset \partial\Gamma(\omega)$, and that $Z(B,\omega)$ is $\{0,1\}-$ valued, when $B \subset \partial\Gamma(\omega)$. Set

$$A = \limsup_n A(n),$$

where

$$A(n) = \cup_{\Psi(B(\sigma))=1, \ \ \sigma\in\Gamma_n} B(\sigma).$$

**Proposition 3.1** *Assume that*
*(i) the random variables* $\Psi(B(\sigma))$, $\sigma \in \Gamma_n$, *are independent;*
*(ii)the expectation*

$$q_n := E[\Psi(B(\sigma))] = P\{\Psi(B(\sigma)) = 1\}$$

*is the same for all* $\sigma \in \Gamma_n$, *and*
*(iii) there is a sequence of poitive integers* $n_k$ *which increase to* $\infty$ *rapidly enough so that* $m^{2n_k} \leq n_{k+1}$, $\forall k$, *such that* $q_n$ *satisfies the following lower bound estimate,*

$$cm^{-\delta n_k} \leq q_{n_k} \quad \forall k,$$

*where c is some absolute constant and* $0 < \delta < 1$. *Then, with probability one, the limsup set A defined above has infinite Hausdorff* $\phi-$*measure,* $\phi - m(A) = +\infty$, *where the gauge function* $\phi$ *is defined by*

$$\phi(x) = x^{(1-\delta)\alpha}(\log(1/x))^b, \quad 0 < x < 1, \quad b > 2.$$

*In particular,* $\dim A \geq (1-\delta)\alpha$.

**Proposition 3.2** *Under the conditions of Proposition 3.1, with probability one,*

$$\text{Dim}A = \alpha.$$

Propositions 3.1 and 3.2 can be proved using the methods of [DPRZ]. We remark that we do not need a condition which bounds the correlation since the sub-trees $B(\sigma), \sigma \in \Gamma_n$, are completely independent. We need modifications because we are now working on a random tree, rather than the binary tree; these can be made by suitable use of conditional independence. Moreover, the third condition in Proposition 3.1 is stated as required for all large $n$ in [DPRZ], yet in fact it is only needed for a sufficiently rapidly increasing sequence. We will leave the details to readers. We also mention that Theorem 2.1 of [DPRZ] has been further refined in [KPX].

# 4 Dimension Spectrum for thick behavior

From Section 1, we know that the "typical behavior" of the branching measure $\mu$ on $B(\sigma), \sigma \in \Gamma_n$, is $m^{-n}$, for all n large enough. To describe the behavior of $\mu$ which is "thicker", we introduce the following two (random) sets

$$A_\theta := \Big\{\xi \in \partial\Gamma : \ \limsup_n \frac{\mu(B(\xi|n))}{m^{-n}n^\lambda} \geq \theta\Big\},$$

and

$$B_\theta := \Big\{\xi \in \partial\Gamma : \ \limsup_n \frac{\mu(B(\xi|n))}{m^{-n}n^\lambda} = \theta\Big\}.$$

In the above, $\lambda$ is defined by

$$\lambda := 1 - \frac{\alpha}{\log ||Z||_\infty},$$

where $||\cdot||_\infty$ is the sup norm of the underlying probability space. Note that $0 < \lambda \leq 1$, and $\lambda < 1$ if and only if $Z$ is a finite distribution. To describe the dimension spectrum of $A_\theta$ and $B_\theta$, we need the following two parameters

$$\begin{aligned} r_0 &:= \liminf_{x\to\infty} \frac{-\log P[W > x]}{x^{1/\lambda}}, \\ r &:= \text{the one such that } \ P[W > x] \approx e^{-rx^{1/\lambda}}, \quad \text{as } x \uparrow \infty, \end{aligned}$$

here $a \approx b$ means that there are $c_1, c_2$ such that $c_1 a \leq b \leq c_2 a$. To discuss the dimension spectrum of $A_\theta$, we assume that $r_0$ is finite and positive, which is a quite mild assumption as we can see from Lemma 4.1 below. To discuss the dimension spectrum of $B_\theta$, we need to impose the stronger assumption that $r$ exists, and is finite and positive. This is a strong condition, yet it holds for the interesting case that $Z$ has a geometric distribution, which makes $W$ have an exponential distribution and then $\lambda = r = 1$. It holds also for the case in which $W$ has a gamma distribution, see Harris[H, p17]. Now we state our dimension spectrum separately for $A_\theta$ and $B_\theta$.

**Theorem 4.1** *Assume that $r_0$ is finite and positive, then, with probability one,*

$$\dim A_\theta = \alpha - r_0\theta^{1/\lambda}, \quad 0 \leq \theta \leq \Big(\frac{\alpha}{r_0}\Big)^\lambda.$$

**Theorem 4.2** *Assume that $r$ exists, and is finite and positive, then, with probability one,*

$$\dim B_\theta = \alpha - r\theta^{1/\lambda}, \quad 0 \le \theta \le \left(\frac{\alpha}{r}\right)^\lambda.$$

*Moreover, under the assumption of Theorem 4.1, resp. Theorem 4.2,*

$$\begin{aligned} \mathrm{Dim} A_\theta &= \alpha, \quad 0 < \theta < \left(\frac{\alpha}{r_0}\right)^\lambda, \\ \textit{resp. } \mathrm{Dim} B_\theta &= \alpha, \quad 0 < \theta < \left(\frac{\alpha}{r}\right)^\lambda. \end{aligned}$$

**Remark 1.** Since $B_\theta \subset A_\theta$ and $r$ is necessarily equal to $r_0$ under the existence assumption, Theorem 4.2 has a stronger assertion under stronger assumption, compared with Theorem 4.1.

**Remark 2.** By the following uniform law for $\mu$ proved in Liu–Shieh[LS],

$$\limsup_n \sup_{\xi \in \partial\Gamma} \frac{\mu(B(\xi|n))}{m^{-n} n^\lambda} = \left(\frac{\alpha}{r_0}\right)^\lambda,$$

the set $B_\theta = \emptyset$ for $\theta > \left(\frac{\alpha}{r_0}\right)^\lambda$. Thus, the range for $\theta$ in Theorems 4.1 and 4.2 is exact.

Firstly we state a lemma giving conditions which imply that $r_0$ is finite and positive. The lemma is a direct consequence of Liu[L1 Theorem 3.1 and L2], however it can be deduced from earlier results.

**Lemma 4.1** *The parameter $r_0$ is finite and positive, when $\lambda < 1$, or when $\lambda = 1$ and $E[e^{tZ}] < \infty$ for some, but not for all, $t > 0$.*

*Proofs of Theorems.* We concentrate first on the Hausdorff dimension, dim. We begin with the discussion of the extreme cases $\theta = 0$ and $\theta = \left(\frac{\alpha}{r_0}\right)^\lambda$. For $\theta = 0$, the asserton $\dim A_\theta = \dim B_\theta = \alpha$ is merely a consequence of the well–known result that $\dim \mu = \log m$ a.s.; see Hawkes[H] and Lyons–Pemantle–Peres[LPP]. For $\theta = \left(\frac{\alpha}{r_0}\right)^\lambda$, it is a consequence of letting a sequence $\theta_k$ strictly increase to $\theta$ and proving the spectrum for $\theta_k$. Therefore, henceafter we assume that $\theta$ is not at the endpoints of the range in the theorems.

To prove the upper bound of dim it suffices to show that, for any $b > \alpha - r_0\theta^{1/\lambda}$, the Hausdorff $b$–dimensional measure, $b - m$, of $A_\theta$ is zero. This proof is standard and was given in [LS, Section 3]; we include the proof here for completeness. We observe that, for $\epsilon : 0 < \epsilon < \theta$ and positive integer $k$,

$$A_\theta \subset \cup_{n \ge k} \left\{\xi \in \partial\Gamma : \frac{\mu(B(\xi|n))}{m^{-n} n^\lambda} > (\theta - \epsilon) C\right\},$$

where $C = \left(\frac{\alpha}{r_0}\right)^\lambda$. We consider the pre–Hausdorff $b$–dimensional measure at level $k$,

$$(b-m)_k(A_\theta) = \inf\Big\{ \sum_\sigma |B(\sigma)|^b : \quad A \subset \cup B(\sigma), \quad |\sigma| \geq k, \quad \forall\sigma \Big\}.$$

Recall that $|B(\sigma)| = e^{-k}$ when $\sigma \in \Gamma_k$. Let $I_k$ denote the random variable defined by

$$I_k = \sum_{|\sigma|=n} |B_\sigma| 1\Big\{ \frac{\mu(B(\xi|n))}{m^{-n}n^\lambda} > (\theta-\epsilon)C \Big\},$$

then, using the definition of $r_0$ we see that

$$EI_k \leq \sum_{n\geq k} e^{-(b-\alpha)n} e^{-(r_0-\epsilon)(\theta-\epsilon)^{1/\lambda}C^{1/\lambda}n},$$

when $k = k(\epsilon)$ is large enough. The series in the above display is convergent, so that $I_k$ tends to 0 a.s. as $k \uparrow \infty$. Since $\epsilon$ is arbitrary, we conclude that $b - m(A_\theta) = 0$.

To obtain the lower bound of dim, let $r_1$ be such that $r_0 < r_1$ and $\theta < \left(\frac{\alpha}{r_1}\right)^\lambda$. We prove that $\dim A_\theta \geq \alpha - r_1\theta^{1/\lambda}$ by proving that $A_\theta$ has infinite Hausdorff $\phi$–measure, where

$$\phi(x) = x^{\alpha - r_1\theta^{1/\lambda}}(\log(1/x))^b, \quad 0 < x < 1, \quad b > 2.$$

We apply Proposition 3.1 in the following way. Define the random mapping $\Psi(B,\omega)$, $B \subset \mathbb{N}^\infty$ and $\omega \in \Omega$, to be 1, only when $B = B(\sigma), \sigma \in \partial\Gamma_n(\omega)$ and when $W(\sigma,\omega) > n^\lambda\theta$; otherwise $\Psi$ takes value 0. Thus $q_n = E[\Psi(B(\sigma))] = P[W > n^\lambda\theta]$. By our definition of $r_0$ as a liminf and our choice of $r_1$ there exists a sequence $n_k \uparrow \infty$ such that $q_n \geq e^{-r_1 n\theta^{1/\lambda}}$, $\forall n = n_k$. We may well assume that $n_k$ satisfies the rapid growth condition in Proposition 3.1. Therefore Theorem 4.1 is an application of Proposition 3.1 with $\delta = \alpha^{-1}r_1\theta^{1/\lambda}$ there, and that $r_1$ can be arbitrarily close to $r_0$. To prove the lower bound for $B_\theta$, we need to use a strategy first used in [PT]. Under the stronger assumption on the existence of $r$, let now

$$\phi(x) = x^{\alpha - r\theta^{1/\lambda}}(\log(1/x))^b, \quad 0 < x < 1, \quad b > 2.$$

Then the above arguments assert that Hausdorff $\phi$–measure of $A_\theta$ is infinite while, by the upper bound proof given in the above, that of $A_{\theta+1/k}$ is 0 for all $k = 1, 2, \cdots$. Thus, $B_\theta = A_\theta \setminus \cup_k A_{\theta+1/k}$ is also of infinite Hausdorff $\phi$–measure; in particular we get the desired lower bound for $B_\theta$.

Application of Proposition 3.2 gives the assertion for Dim. □

**Remark.** We believe that the stronger assumption in Theorem 4.2 for $B_\theta$ is not needed; a proof would require a version of the limsup theorem based on Corollary 3.3 of [KPX], in which the target set satidfies a regularity condition so that we can use the value of $q_n$ in Proposition 3.1 on a subsequence. We have not formulated a precise theorem, so in this paper we use the methods of [DPRZ].

# 5 Multifractal spectrum for thin behavior

The reader is reminded that smaller than usual branching behavior is reflected in large values of $\overline{d}(\mu,\xi)$ defined in Section 1. As we have seen in Section 1 that $d(\mu,\xi)$, and hence so is $\overline{d}(\mu.\xi)$, is equal to $\alpha$ for all $\xi$ whenever $p_0 = p_1 = 0$; thus there is only a trivial mulftifractal spectrum in this case. In this section, we prove that, whenever $0 < p_1$ there is an interesting spectrum for $\overline{d}(\mu,\xi)$(we always assume that $p_0 = 0$ and that $p_j < 1, \ \forall j$). We need the following lemma which is in Liu[L3].

**Lemma 5.1** ([L3, Theorem 4.1(ii)) *If $Z$ has finite moments of all positive orders, then, with probability one,*

$$\underline{d}(\mu,\xi) = \alpha, \quad \forall \xi \in \partial\Gamma.$$

We introduce a new paremeter needed in this section, *assuming that* $p_1 > 0$.

$$\tau = -\frac{\log p_1}{\alpha}.$$

Note that

$$p_1 = e^{-\tau\alpha}.$$

The following small tail distribution of $W$ is known from Bingham[B, p. 217].

$$P[W \le x] \approx x^{\tau}, \quad x \downarrow 0. \tag{5.1}$$

We first observe that the probability of a long string of vertices giving rise to a single branch leads to the same estimate for $P[W \le x]$. For, if $k > n$ and $\sigma \in \Gamma_n$, then the event $E$ that there is only one $\eta \in \Gamma_k$ descended from $\sigma$ is

$$P(E) = p_1^{k-n}. \tag{5.2}$$

Now conditional on $E$, $W_\eta = m^{k-n} W_\sigma$, so that

$$P[W_\sigma \leq m^{-(k-n)} | E] = P[W_\eta \leq 1].$$

Thus, we have

**Lemma 5.2** *Under the above conditions, if $x \approx m^{-(k-n)}$, and $E$ is the event that each vertex starting with $\sigma \in \Gamma_n$ has only one descendant up to $\eta \in \Gamma_k$, then*

$$1 \geq \frac{P[\{W_\sigma \leq x\} \cap E]}{P(E)} \geq c_3.$$

We now see how to obtain an efficient cover for points in the thin spectrum.

**Lemma 5.3** *Suppose $\gamma > 1, 0 < \epsilon < (\gamma - 1)/3$. Then, with probability one, there is an $n_0 = n_0(\omega)$ such that every vertex $\sigma \in \Gamma_n$ with $n \geq n_0$ such that $W_\sigma \leq e^{-(\gamma-1)\alpha n}$ has fewer than $e^{\epsilon\alpha n}$ descendants at the level $k = [(\gamma - \epsilon)n]$.*

*Proof.* For each $\eta \in \Gamma_k$ descended from $\sigma \in \Gamma_n$ such that $W_\sigma \leq e^{-(\gamma-1)\alpha n}$, it is seen that

$$W_\eta \leq e^{-\epsilon\alpha n}.$$

By (5.1), the above has probability $\leq c_2 e^{-\epsilon\alpha\tau n}$. Putting $N_\sigma$ as the number of descendants of $\sigma$ at level $k$, we have then

$$P[N_\sigma > e^{\epsilon\alpha n} | W_\sigma \leq e^{-(\gamma-1)\alpha n}] \leq [c_2 e^{-\epsilon\alpha\tau n}]^{e^{\epsilon\alpha n}},$$

in which we have used the fact that $W_\eta$ for distinct $\eta \in \Gamma_k$ are iid. Recall that $Z_n$ counts the vertices $\sigma \in \Gamma_n$, thus the expected number of $\sigma$ such that $W_\sigma \leq e^{-(\gamma-1)\alpha n}$ and $N_\sigma > e^{\epsilon\alpha n}$ is

$$\leq E[Z_n] \cdot c_2 e^{-\epsilon\alpha\tau n \cdot e^{\epsilon\alpha n}}.$$

Since $E[Z_n] = e^{\alpha n}$ we deduce that the probability that there is at least one such vertex is bounded by $c_2 e^{\alpha n(1-\epsilon\tau \cdot e^{\epsilon\alpha n})}$, which is the general term of a convergent series. By the Borel–Cantelli Lemma we have proved the lemma. □

**Lemma 5.4** *If $0 < p_1 < 1$, then, with probability one,*

$$\text{(5.3)} \qquad \dim C_\beta \leq \alpha[\frac{\alpha}{\beta}(1+\tau) - \tau], \quad \alpha \leq \beta,$$

*where $C_\beta = \{\xi \in \partial\Gamma : \ \overline{d}(\mu, \xi) \geq \beta\}$.*
*When $\beta > \alpha(1 + 1/\tau)$, the right hand side of (5.3) is negative, and we interpret this as stating that $C_\beta = \emptyset$.*

*Proof.* (i) When $\beta = \alpha$, (5.3) is immediate.
(ii) When $\beta > \alpha(1 + 1/\tau)$, we will prove that $C_\beta = \emptyset$ a.s. Take $\zeta$ such that $\beta > \zeta > \alpha(1+1/\tau)$, then $\overline{d}(\mu,\xi) \geq \beta$ implies that $\mu B(\xi|n) < e^{-\zeta n}$ for infinitely many integers $n$. The expected number of those $\sigma \in \Gamma_n$ for which $W_\sigma < e^{(\alpha-\zeta)}$ is

$$E[Z_n] \cdot P[W < e^{(\alpha-\zeta)}] = e^{(\alpha+(\alpha-\zeta)\tau))n},$$

whcih is a negative power of $e^n$. By Borel–Cantelli Lemma, we deduce that, for $n \geq n_1 = n_1(\omega)$

$$\mu B(\xi|n) \geq e^{-\zeta n}, \quad \forall \xi.$$

By letting $\beta \downarrow \alpha(1+1/\tau)$ through a countable set, we deduce that, with probability one,

$$\overline{d}(\mu,\xi) \leq \alpha(1+1/\tau), \quad \forall \xi \in \partial\Gamma.$$

(iii) Now suppose that $\alpha < \beta < \alpha(1 + 1/\tau)$. Put $\gamma = \frac{\beta}{\alpha} > 1$. Instead of covering the vertices $\sigma \in \Gamma_n$ where $\mu B(\xi|n) < e^{-\beta n}$ by balls of diameter $e^{-n}$ we use the descendant vertices at level $k = [(\gamma - \epsilon)n]$ which can be covered by balls of diameters $e^{-k}$. By Lemma 5.3, when $n$ is large enough the number of such vertices is less than $e^{\epsilon\alpha n}$ so that the total number needed will be at most $e^{\epsilon\alpha n} \cdot T_n$ where $T_n$ is the number of those $\sigma \in \Gamma_n$ for which $W_\sigma < e^{-(\beta-\alpha)n}$. Now $E[T_n] = m^n \cdot e^{-(\beta-\alpha)\tau n}$, so that we obtain

$$E[s^\delta - m(C_\beta)] \leq \sum_{n=n_1}^{\infty} e^{-[(\gamma-\epsilon)n]\delta+\epsilon\alpha n-(\beta-\alpha)\tau n+\alpha n},$$

where $n_1$ is arbitrary. If the power of of $e^n$ in this series is negative, we deduce that $s^\delta - m(C_\beta) = 0$ a.s. This will be true if

$$\delta > \delta_\epsilon := \frac{1 + \epsilon - (\gamma - 1)\tau}{\gamma - \epsilon} \cdot \alpha.$$

Letting $\epsilon \downarrow 0$ through a countable set, we see that $s^\delta - m(C_\beta) = 0$ a.s. for $\delta \geq \frac{(1-(\gamma-1)\tau)\alpha}{\gamma}$. Substituing $\gamma = \beta/\alpha$ we prove the assertion. □.

We are now ready to prove that (5.3) gives the right answer for $\dim C_\beta$. However, if we are to obtain the same answer for $\dim D_\beta$, as in Section 4, we need to find a gauge function $\phi(s) = s^\Delta L(s)$ with $L(s)$ slowly varying, such that $\phi - m(C_\beta) = \infty$. We will prove this by applying Proposition 3.1, and the strategy is the same as that used in [PT]: we find a random Cantor–like subset $T_\gamma$ which is $\gamma$–thin and use Proposition 3.1 to find its Hausdorff $\phi-$ measure. This set $T_\gamma \subset C_\beta$ by Lemma 2.2 and Lemma 5.1. In order to apply Proposition 3.1, fix a $\gamma > 1$, we define the random mapping $\Psi$ there by defining $\Psi(B,\omega) = 1$ if and only if $B = B(\sigma)$, $\sigma \in \Gamma_n(\omega)$ is such that its ancestor in $\Gamma_{[n/\gamma]-1}(\omega)$ has a string of single branches stretching to $\sigma$. Denote the limsup set $A$ there now by $T_\gamma$. By (5.2), the probability $q_n$ in Proposition 3.1 is now

$$q_n \geq c \cdot p_1^{(1-1/\gamma)n} \cdot e^{\alpha(1-1/\gamma)n} = c \cdot e^{\alpha n(1-1/\gamma)(\tau+1)}.$$

We remark that the third factor in the middle term of the above display comes from the expected number of all the possible ancestors in $\Gamma_{[n/\gamma]-1}$, given an element in $\Gamma_n$. We can now calculate the $\delta$ in Proposition 3.1. Note that $T_\gamma$ is clearly a $\gamma-$thin subset of $\partial\Gamma$ by the construction. Thus we have

**Lemma 5.5** *Assume that $0 < p_1 < 1$ and $Z$ has finite moments of all positive orders; let $\beta$ be fixed, $\alpha < \beta < (1 + 1/\tau)\alpha$, and define $\gamma = \beta/\alpha$. Then the Hausdorff measure of the $\gamma-$thin set $T_\gamma$ defined above satisfies $\phi - m(T_\gamma) = +\infty$,*
*where the gauge function $\phi$ is $\phi(x) = x^\Delta(\log(1/x))^3$, with*

$$\Delta = \alpha[\frac{1}{\gamma}(1+\tau) - \tau]$$

We can now state our main decomposition.

**Theorem 5.1** *If $0 < p_1 < 1$ and $Z$ has finite moments of all positive orders, then the branching meassure $\mu$ has the following propoerties, with probability one. Set*

$$C_\beta = \{\xi : \ \overline{d}(\mu,\xi) \geq \beta\}, D_\beta = \{\xi : \ \overline{d}(\mu,\xi) = \beta\},$$

*then*

*(a) $C_\beta$ and therefore $D_\beta$ is empty for $\beta > \alpha(1 + \frac{1}{\tau})$.*

*(b) $D_\beta$ is non-empty for $\alpha \leq \beta \leq \alpha(1 + \frac{1}{\tau})$, and in this range*

$$\begin{aligned} \text{dim}C_\beta = \text{dim}D_\beta &= \alpha[\frac{\alpha}{\beta}(1+\tau) - \tau] \\ \text{Dim}C_\beta = \text{Dim}D_\beta &= \alpha. \end{aligned}$$

*Proof.* By Lemmas 2.2 and 5.1, $T_\gamma \subset C_\beta$, where $\gamma = \beta/\alpha$. By Lemma 5.5, the Hausdorff $\phi_\Delta$- measure of $C_\beta$ is $+\infty$, where $\phi_\Delta$ is the gauge function there in Lemma 5.5. Regard $\Delta$ as a function of $\beta$, it is strictly monotone. Lemma 5.4 then tells that $\phi_\Delta$ measure of $C_{\beta+1/k}$ is 0. Thus, arguing as in the proofs of Theorems 4.1 and 4.2, we see that $\dim D_\beta \geq \Delta$. Since $D_\beta \subset C_\beta$, we have completed the proof for $\alpha \leq \beta < \alpha(1+1/\tau)$. In the case where $\beta = \alpha(1+1/\tau)$ we only need to show that $D_\beta = C_\beta$ is non-empty. This will follow if we can construct $T_\gamma$ for $\gamma = 1+1/\tau$ by requiring the string of single branches to stretch from the level $[n/\gamma - \log n] - 1$ to $n$. This condition forces $\overline{d}(\mu,\xi) \geq \alpha(1+1/\tau)$, on using Lemmas 2.2 and 5.1. □

**Remark.** In Theorem 5.1 we assume that $Z$ has finite moments of all positive orders is mainly to apply Lemma 5.1. It seems possible that we may weaken the condition in Theorem 5.1 to, say, that $Z$ has finite moments up to a certain order $k_0$ greater than one and get a spectrum involving $p_+ = \sup\{a \geq 1 : EZ^a < \infty\}$(one critical value in [L3]).

**Acknowledgement.** This research was initiated while both authors attended the Newton Institute, Cambridge University, for the workshop on Fractals and Applications, and was completed while the second author visited Mathematical Institute of Academia Sinica at Taipei. Support from these institutes is appreciated. At the Newton Institute we benefited from extensive discussions with Yuval Peres about the limsup construction. The first author is partially supported by the grants NSC *1998/99 2115-M-002-017* and *1999/2000 2115-M-002-006*

# References

[AN] K.B. Athreya and P. Ney(1972). *Branching Processes.* Springer.

[B] N.H. Bingham(1988). *On the limit of a supercritical branching processes.* J. Appl. Probab. **25A** 215-228.

[DPRZ] A. Dembo, Y. Peres, J. Rosen and O. Zeitouni(1998). *Thick points for spatial Brownian motions: multifractal analysis of occupation measure.* Ann. Probab., to appear.

[HT] X. Hu and S.J. Taylor(1997). *The multifractal structure of stable occupation.* Stoch. Processes Appl. **66** 283-299.

[H] T.E. Harris(1963). *The Theory of Branching Processes.* Springer.

[KPX] D. Khoshnevisan, Y. Peres and Y. Xiao(1998). *Limsup random fractals.* Electr. J. Probab., to appear.

[L1] Q.S. Liu(1996). *The exact Hausdorff dimension of a branching set.* Prob. Theory and Rel. Fields. **104** 515-538.

[L2] Q.S. Liu(1996). *The growth of an entire characteristic function and the tail probabilities of the limit of a tree martingale.* In: *Trees*(ed. B. Chauvin *etal*) Birkhouser.

[L3] Q.S. Liu(1999). *Local dimension of the branching measure on a Galton-Watson tree.* Preprint submitted.

[LS] Q.S. Liu and N.R. Shieh(1998). *A uniform limit law for the branching measure on a Galton-Watson tree.* Asian J. Math., to appear.

[LPP] R. Lyons, R. Pemantle and Y. Peres(1995). *Ergodic theory on Galton–Watson trees.* Ergod. Th. Dynam. Sys. **15** 593-619.

[PP] R. Pemantle and Y. Peres(1995). *Galton-Watson trees with the same mean have the same polar sets.* Ann. Probab. **23** 1102-1124.

[PT] E.A. Perkins and S.J. Taylor(1998). *The multifractal structure of super Brownian motions.* Ann. Inst. H. Poincaré **34** 97-138.

[ST] N.R. Shieh and S.J. Taylor(1998) *Logarithmic multifractal spectrum of stable occupation measure.* Stoch. Processes Appl. **75** 249-261.