Ergodic Theory and Geometric Rigidity and Number Theory
5 January to 7 July 2000
Report from the Organisers

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Scientific background and objectives

The central scientific theme of this programme was the recent development of applications of ergodic theory to other areas of mathematics, in particular, the connections with geometry, group actions and rigidity, and number theory. The potential of ergodic theory as a tool in number theory was emphasized by Furstenberg's proof of Szemerdi's theorem on arithmetic progressions.

Ergodic theory is an area of mathematics with all of its roots and development contained within the 20th century. Strands of the modern theory can be traced back to the work of Poincaré, but the subject began to take a more recognizable form through the seminal work of von Neumann, Birkhoff and Kolmogorov. The impetus to these developments was the important concept of 'ergodicity' in dynamical systems - by which the temporal evolution of the system, though averaging over typical orbits (almost every orbit in the measure theoretic sense), corresponds to spatial averages over the system. An important concept in physical systems, it also set the foundation for applications to other branches of mathematics, most notably geometry and number theory.

Foremost amongst the recent contributions of ergodic theory to number theory is the solution of the Oppenheim Conjecture, a problem on quadratic forms which had been open since 1929. This conjecture was solved by Margulis, and a particular special case is the following:

The Oppenheim Conjecture. *Consider an indefinite ternary quadratic form, for example, the quadratic form*

$$Q(x,y,z) = ax^2 + by^2 - cz^2$$

in three variables, with a,b,c > 0 for the purposes of illustration. The original conjecture of Oppenheim was that the values Q(l,m,n) can be made arbitrarily close to 0 by taking choices of non-zero triples of integers $(l,m,n) \in \mathbf{Z}^3 - (0,0,0)$. *This problem was extensively studied by Davenport, using purely number theoretic methods. The final solution of the conjecture was achieved by Margulis by using a reformulation of the problem into the ergodic theory of homogeneous flows on lattices.*

Of equal importance is the role of ergodic theory in geometry and the rigidity of actions. In recent years there have been diverse results, including rigidity results for higher rank abelian groups, and results on the classification of geodesic flows on manifolds of non-positive curvature. This is a quickly evolving area of research. In more recent years, the richness of the applications to geometry have become more apparent. This is illustrated by the famous Mostow rigidity theorem, by which the geometry of certain manifolds is completely determined by their topology (i.e. two manifolds with the same fundamental groups are isometric).

Rigidity of Anosov actions. *In the context of actions, there is a very well developed programme of Katok, Spatzier, and others, to show local* C^∞ *rigidity of algebraic Anosov actions of* $\mathbf{Z}^k$ *and* $\mathbf{R}^k$ *on compact manifolds as well as orbit foliations of such actions. More precisely, two actions of a group G agree up to an automorphism if the second action can be obtained from the first one by composition with an automorphism of the underlying group. Call a* C^∞*-action of a Lie group G locally* C^∞*-rigid if any perturbation of the action which is* C^1*-close on a compact generating set is* C^∞*-conjugate up to an automorphism. Katok and Spatzier proved* C^∞*-local rigidity of most known irreducible Anosov actions of* $\mathbf{Z}^k$ *and* $\mathbf{R}^k$

(as well as the orbit foliations).
A related theme is that of paucity of invariant measures.

The Furstenberg Conjecture. *Consider the two transformations on the unit circle S, T : K* $\longrightarrow$ *K defined by* $Sz = z^2$ *and* $Tz = z^3$. *The only ergodic invariant probability measures which are simultaneously preserved by both S and T are either Haar measure, or measures supported on finitely many points.*

Whereas this famous conjecture remains open, it is known (by work of Rudolph) that the conclusion is true if we restrict to measures of positive entropy. This result was generalized to $\mathbf{Z}^n$-actions by Anosov toral automorphisms, and other more general settings, by Katok and Spatzier. A well-known cross-discipline application lies in the connection with Quantum Chaos. In particular, a quantitative version of the Oppenheim Conjecture gives a proof of the Berry Conjecture on the eigenvalues of the Laplacian on flat tori. The programme explored these and other emerging applications of ergodic theory. It brought together both national and international experts in ergodic theory and related disciplines, as well as other members from the wider UK and international mathematical communities. In particular, a major aim of the programme was to bring together people with different interests and backgrounds, and to promote the use of ergodic theory techniques.

Organisation

The overall planning for the programme was undertaken by all three of the organisers. The day-to-day organization of the programme was undertaken by M Pollicott from January to early April, and again from late May to the end. During this absence, his duties were undertaken by
G Margulis. There was also very able assistance from A Eskin and M Burger for specific workshops within the programme. In addition to the workshops and meetings within the programme, there was a regular research seminar on Tuesday afternoon, and a more informal seminar on Thursday morning. There were also other seminars scheduled as required by the participants or the organisers.

In May there was a Spitalfields Day, with talks by A Eskin, A Katok and G Margulis.
During the month of June there was a one-day meeting in Dynamical Systems (sponsored by the LMS and organised by Sharp and Walkden, from Manchester), with talks by R Itturiaga (INI and Heriot-Watt), M Urbanski (North Texas) and T Ward (UEA).

In June the frequency of talks increased and the last week of the month was designated a Special Emphasis Week, prior to the final Euro-workshop in July. During this special week there were 3 talks each day by participants, focusing particularly on results obtained by longer term visitors.

A Katok and H Furstenberg also spoke in the Institute’s Monday Seminar series, addressing a wider audience.

Participation

The programme hosted in excess of 50 long term visitors, at various times, and 75 short term visitors. In addition, there was very strong participation in the workshops and other activities during the programme. The three organisers each spent a substantial period of the programme in residence. In addition, H Furstenberg was a Rothschild Professor and made an invaluable contribution to the programme during his month in residence.

As the participant list shows, the majority of experts in the field visited the Institute during this programme, and a large number of leading experts in related fields attended. There was a strong presence from Europe and North America, as well as a substantial presence from the former Soviet Union helped by the generous support from the Leverhulme Trust.

Many of the meetings and individual talks attracted mathematicians from both Cambridge and other British universities. The Junior Membership scheme, support from the NSF and support from the EU for two of the workshops was particularly useful in encouraging participation from PhD students and younger mathematicians. A number of participants went to give talks in other UK departments (e.g. Manchester, Surrey, QMW and Warwick).

Meetings and workshops

Lectures on Ergodic Theory, Geometry and Lie Groups (10-14 January 2000):

A Katok (Penn State) and M Pollicott (Manchester)

The first meeting was designed to provide a firm foundation for the programme, and to help set the agenda for subsequent activities. It was also intended to provide an introduction to the subject for a broad audience, particularly younger mathematicians and non-experts from related areas.

The meeting consisted of five short lecture courses by acknowledged experts in the area. These were: M Burger (ETH Zurich), Cohomological aspects of lattices and applications to products of trees; R Feres (St. Louis), Topological superrigidity and differential geometry; H Furstenberg (Jerusalem), Ergodic theory and the geometry of fractals; A Katok (Penn State), Dynamics and ergodic theory of smooth actions of higher tank abelian groups and lattices in semi-simple Lie groups; and D Kleinbock (Rutgers), Interactions between homogeneous dynamics and number theory.The meeting successfully achieved all of its aims. Participants were mathematicians, predominantly graduate students, postdoctoral fellows and younger researchers.

Euroworkshop: Rigidity in Dynamics and Geometry (27-31 March 2000):

G Margulis (Yale), assisted by A Eskin (Chicago) and M Burger (ETH Zurich)

The second workshop was devoted to applications of ergodic theory to locally symmetric spaces, geometric rigidity, and number theory. This represented one of the most intense periods of activity during the programme.The subjects covered in the meeting included such important recent developments as the classification of the actions of higher rank groups, unipotent flows on homogeneous spaces and the Oppenheim conjecture.The meeting consisted of 33 lectures from an international audience. Those participating were a broad mix of senior mathematicians and younger researchers and students.

The main speakers included: D Gaboriau (ENS, Lyon);B Kleiner (Michigan); F Labourie (Orsay); A Zorich (Rennes I); L Mosher (Rutgers), B Weiss (SUNY, StoneyBrook); H Pajot (Cergy-Pointoise); N Monod (ETH Zurich); A Iozzi (Maryland); B Leeb (Tubingen);A Karlsson (Yale); G Knieper (Bochum); A Furman (Illinois); B Goldman (Maryland); A Adams (Minnesota); H Abels (Bielefeld); M Skriganov (Steklov); A Stepin (Moscow State); M Dodson (York); V Kaimanovich (Rennes/Manchester); Y Guivarc'h (Rennes); G Tomanov (Claude Bernard, Lyon), Benoist (ENS, Paris), D Witte (Oklahoma State), A Parreau (Orsay); B Remy (Henri Poincaré, Nancy); F. Paulin (Orsay);
Y Shalom (Yale); A Török (Houston); A Nevo (Technion); D Fisher (Yale); H Oh (Princeton); A Zuk (ENS, Lyon).

Ergodic Theory of $\mathbf{Z}^d$-actions (3-7 April 2000):

M Pollicott (Manchester), K Schmidt (Vienna) and P Walters (Warwick)

The third meeting in the programme was held in collaboration with the Mathematics Institute at the University of Warwick, and the venue was Warwick University. This meeting specialized more in the specific topic of $\mathbf{Z}^d$-actions, an area in which there was a very successful symposium at Warwick in 1993-94. This meeting focused on developments over the intervening six years, and showed that the area was still very active. There were 27 talks, all of 45 minutes. The total number of participants exceeded 75.

Speakers included: Aaronson (Tel Aviv), Auslander (Maryland), Bergeleson (Ohio), Bufetov (Moscow), Burton (Ohio), Dani (Tata), Einsiedler (UEA), Feres (Washington), Friedland (Chicago), Hurder (Illinois), Johnson (UNC), A Katok (Penn State), S Katok (Penn State), Kaminski (Krakow), Kitchens (IBM), Lind (Seattle), Margulis (Yale), Mozes (Jerusalem), Petersen (North Carolina), Putnam (Victoria), Schmidt (Vienna), Shah (Tata), Thouvenot (Paris), Tuncel (Seattle), Vershik (St. Petersburg), Ward (UEA).

Euro-conference on Ergodic Theory, Riemannian Geometry and Number Theory (3-7 July 2000):

A Katok (Penn State), G Margulis (Yale) and M Pollicott (Manchester)

This was the final meeting of the programme and served, in part, to review the achievements made during the previous six months. A number of speakers took the opportunity to present work which they had carried out while in residence at the Institute. The meeting encompassed much research activity, and marked the culmination of the programme.

There were a total of 29 talks, all of 45 minutes duration. The meeting attracted more than 100 participants. The speakers included: H Furstenberg (Jerusalem); G Margulis (Yale); H Masur (UIC), A Windsor (Penn State); A Gamburd (Dartmouth); D Burago (Penn State); E Ghys (ENS Lyon); B Klingler (INI); M Burger(ETH Zurich); C Drutu (Lille, MPI Bonn); G Soifer (Bar-Ilan); A Lubotzky (Jerusalem); L Clozel (Paris Sud); D Kleinbock (Rutgers); J Marklof (Bristol); M Kanai (Nagoya); U Hamenstadt (Bonn); M Babillot (Orleans); P Pansu (Paris Sud); B Kalinin (Penn State); C Walkden (Manchester); F Dal'bo (Rennes); Y Cheung (UIC CMI); J Schmeling (Berlin); C Connell (UIC); R Sharp (Manchester); S Katok (Penn State); U Bader (Technion); R Zimmer (Chicago).

Outcome and achievements

The main achievement of this programme was that it brought together both established experts in the field, as well as younger researchers, from home and abroad, in an effort to promote scientific research and training of the highest quality. To this end it was remarkably successful, with progress being made on a large number of problems, in a diverse number of different directions.

One of the topics where there was most progress was in the area of intersection between Lie groups and Ergodic theory. Abels, Margulis and Soifer worked intensively on the classical Auslander and Milnor conjectures and considerable progress was made on these problems. In addition, Abels also made progress, with Margulis, on another classical problem (due to Siegel) regarding metrics on reductive groups.

At a more algebraic level, Shalom made substantial progress in understanding to what extent lattices in higher rank Lie groups differ from lattices in the rank one Lie groups $Sp(n,1)$, in terms of their representation theory. On the face of it all of these lattices share Kazhdan's property (T), which dominates their behaviour. However, looking at uniformly bounded, rather than unitary, representations on Hilbert spaces reveals some fundamental differences.

Adams and Witte studied the classification of the homogeneous spaces of $SO(2,n)$ and $SO(1,n)$ that have Lorentz forms. This is an important ingredient in showing that $SL(2,\mathbf{R})$ is the only simple Lie group that can act non-tamely on a Lorentz manifold. Witte also collaborated with Lifschitz, one of the younger participants, on establishing rigidity of lattices in nilpotent algebraic actions over local fields of positive characteristic. He also completed work with Iozzi describing the Cartan-decomposition subgroups of $SU(2,n)$. They also completed a related project on tesselations of homogeneous spaces of $SU(2,n)$.

Another area which was emphasized during the programme was the action by groups on manifolds. Dani worked on ergodic $\mathbf{Z}^d$-actions on Lie groups by their automorphisms. Skriganov worked with Margulis on proving ergodic theorems for submanifolds generated by nilpotent subgroups in SL(n). These are results which arise naturally in applications of ergodic theory to lattice point problems.

The period spent by Witte and Zimmer at the INI allowed them to complete a long-term project describing actions of semi-simple Lie groups on compact principal bundles. Zimmer also took the opportunity to develop work with Nevo on properties of smooth projective factors for actions with stationary measure, with Fisher on actions on compact principal bundles, and with Labourie and Mozes on the study of locally homogeneous spaces with symmetry.

An important recurrent theme in the meeting was the application of ideas from the ergodic theory of homogeneous flows to number theory. Of particular timeliness was the refinement of the ergodic theoretic proof of the following famous conjecture of Baker and Sprindzhuk.
The Baker-Sprindzhuk Conjecture. *Given* $\mathbf{x} \in \mathbf{R}^n$, *let us write* $\Pi(x) = \prod_{i=1}^{n} |x_i|$. *We say that a vector* $\mathbf{x} \in \mathbf{R}^n$ *is very well multiplicatively approximated if for some* $\varepsilon > 0$ *there are infinitely many* $q \in \mathbf{Z}$ *and* $\mathbf{p} \in \mathbf{Z}^n$ *such that* $\Pi(q\mathbf{x} + \mathbf{p})|q| \leq |q|^{-\varepsilon}$.

For the curve $M_0 = \{(t,t^2,v,t^n), t \in \mathbf{R}\} \subset \mathbf{R}^n$, *Baker conjectured that almost all points on* M_0 *are not very well approximated. For* $n = 2$ *this was proved by Schmidt in 1964, and for* $n = 3$ *by Beresnevich-Bernik in 1996. More generally, let* $f_1,...,f_n$ *be real-analytic functions on a domain* $\mathbf{U} \subset \mathbf{R}^d$ *which, together with* 1, *are linearly independent over* $\mathbf{R}$. *A stronger conjecture (formulated by Sprindzhuk) states that almost all points of* M_0 *are not very well multiplicatively approximated. This was proved recently by Kleinbock-Margulis.*
The proof of the full conjecture by Kleinbock and Margulis uses the ergodic theory of homogeneous actions on the space $SL(n+1, \mathbf{R})/SL(n+1, \mathbf{Z})$. More precisely, given $\mathbf{y} \in \mathbf{R}^n$ we can define a lattice $\Lambda_y = ({}_0^1\ y^T\ I_n)\ \mathbf{Z}^{n+1}$. Then, for any vector $\mathbf{t}=(t_1, \ldots, t_n)$, $t_i \geq 0$, we can denote
$t = \sum_{i=1}^{n} t_i$ and $g_t = \mathrm{diag}(e^t, e^{-t_1}, \ldots, e^{-t_n}) \in SL(n+1,\mathbf{R})$ and can consider the family of lattices $g_t\Lambda_y$.
The dynamical characterization used by Kleinbock and Margulis for $\mathbf{y} \in \mathbf{R}^n$ to be very well multiplicatively approximated is that there exist
$\gamma > 0$ and infinitely many $\mathbf{t} \in \mathbf{Z}_+^n$ such that $\delta(g_t \Lambda_y) \leq e^{-\gamma t}$.
This approach has been used to give ergodic reformulations (and potentially accessible versions) of many important conjectures in number theory (e.g. the Littlewood Conjecture in simultaneous Diophantine approximation). In particular, Kleinbock, Bernik and Margulis developed further these techniques and were able to establish Khintchine-type theorems on manifolds (more precisely the convergence cases for the standard and multiplicative versions). Using more classical techniques, Velani and Pollington obtained estimates on the size of the badly simultaneously approximable set.
During his extended stay at the Institute, Dani also worked on applications of flows on homogeneous spaces to Diophantine approximation, related to earlier work of Margulis on the Oppenheim conjecture. In particular, results were obtained concerning diophantine solutions of quadratic inequalities, again related to the Oppenheim Conjecture, in narrow strips in the associated quadratic space.
There was also good progress in applications to number theory over different fields. Kleinbock and Tomanov successfully worked on proving the natural p-adic and S-arithmetic versions of problems in Diophantine approximation.
A surprising application of these number theoretical results is to spectral theory on certain special manifolds. Eskin, Margulis and Mozes made substantial progress in studying quadratic forms of signature (2,2) and the difficult eigenvalue spacing on flat 2 tori. This is closely related to the well known Berry Conjecture in quantum chaos.

In another direction, there was also substantial progress on geometric problems using ergodic theoretic approaches. For example, Klingler worked on finding a criterion for arithmeticity of complex hyperbolic lattices, which is still in progress. He also collaborated with Maubon to work on the well-known Katok conjecture that the topological entropy of a compact manifold is always strictly larger than the Liouville entropy, unless the metric is locally symmetric.

During his visit, Furstenberg worked on the use of ergodic theory in the geometry of fractals and geometric Ramsey theory. This constitutes the most substantial progress on these problems since his own seminal paper in 1969.

One of the geometers who participated in the programme, Burago, made progress with two problems: establishing ‘kick-stability’ (in the sense of Polterovich) of a parabolic subgroup of $SL(2, \mathbf{R})$, and examples of metrics on non-negatively curved manifolds with positive metric entropy. This involves products of hyperbolic matrices whose stable-unstable decompositions are not coherent. This is a delicate problem, the main difficulty being to destroy the degenerate situation by certain perturbations.

Sharp, one of the long-term British participants, worked on applications of ergodic theory to geometry. This included the study of minimizing measures for geodesic flows on negatively curved manifolds and new results on sector estimates for orbit counting for Kleinian groups.

Three of the other younger participants, Feres, Fischer and Benveniste worked on stratified rigid structures and, in particular, on the construction of examples of rigid geometric structures in the sense of Gromov, with specified types of degeneracies.

There was also considerable progress in terms of understanding which groups act on a circle.R Feres and D Witte extended recent work of Ghys on groups actions on the circle, to actions by automorphisms of foliations of co-dimension one. Shalom and Witte initiated work on the problem of showing that Kazhdan groups cannot act smoothly on the circle. This work is a long term project, but which is already beginning to bear fruit.

Another topic which attracted considerable interest during this meeting was that of polygonal billiards. A difficult problem is that of counting asymptotically the number of periodic trajectories of a billiard on a polygonal rectangular table, where the polygon is rational (i.e. all angles are rational multiples of π). The trajectories correspond to the motion of a particle inside the polygon with elastic collision at the boundaries.

Eskin and Mazur obtained results using a geometric approach. Gluing together several copies one can associate a surface S with a flat structure, and counting families of periodic trajectories of the billiard is equivalent to counting cylinders of closed geodesics on the associated surface. For large T, the number of cylinders of closed geodesics of length at most T is shown to be asymptotic to $\pi C T^2$, for some $C > 0$. The value of C can be computed for some specific surfaces.

The programme also brought together a number of leading experts in abstract ergodic theory, and it was natural that progress was also made on important problems in this underlying subject. For example, Thouvenot worked on the structure of measure-preserving transformations in the positive entropy case, which is related to splitting measure-preserving transformation. This may even prove relevant to the other themes of the programme, since ‘rigidity’ in abstract ergodic theory can be thought of as having a ‘trivial centralizer’ (whereas an irrational rotation can induce a rigid transformation, this is impossible starting from a Bernoulli shift). He made progress in understanding the spectral theory of $\mathbf{R}$ actions of the rationals, in particular, the connection with the property of Lebesgue spectrum.

Two other long term visitors, Goldsheid and Guivarc’h, collaborated to show estimates on the dimension of the Gaussian law with products of random matrices. This is closely linked to a classical problem studied by Furstenberg on such random products. In a somewhat different direction, Goldsheid also worked with Khoruzhenko on the distribution of eigenvalues in the Non-Hermitian Anderson model, an important classical model.

Another of the British long term visitors, Nair, showed that for various natural probability measures on the space of increasing sequences of integers almost all sequences are multiplicatively intersective. In connection with this problem, Nair and Weber (Strasbourg) are continuing to investigate the stability of multiple intersectivity under perturbation by integer-valued independent identically distributed random variables.

Finally, Kaimanovich and Schmidt took the opportunity to continue their work on the ergodicity of horocycle foliations of certain non-negatively curved manifolds, by extending the types of covers of compact manifolds for which they can show the horocycle action is ergodic. In particular, they have developed an approach which generalizes earlier results of Babillot-Ledrappier and Pollicott (all of whom participated in the programme). During the programme, Ledrappier and Pollicott extended some of these results to the case of stable manifold foliations of frame flows.

In summary, this programme made significant contributions to the study of Ergodic Theory and its applications to a range of important areas.

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