Global Problems in Mathematical Relativity

8 August to 23 December 2005

Report from the Organisers:
PT Chruściel (Tours), H Friedrich (Golm) and P Tod (Oxford)

S Wilkinson

P Tod, H Friedrich and PT Chruściel

Scientific Background

Einstein's theory of Special Relativity celebrated its centenary in 2005. So completely successful has it been, that it is impossible to imagine physics without it. It is so much a part of our framework of thought that we almost don't see it as a theory. Special Relativity is completely understood and straightforward enough to be taught to second-year undergraduates, and it is hard to imagine that there could be unsolved mathematical problems in the theory awaiting solution.

Its younger sibling, Einstein's theory of General Relativity, turned 90 in 2005 and enjoys a rather different reputation. When Eddington, in the early days, was asked if it was true that only three people in the world understood Relativity, it was General Relativity that he was being asked about. For a long time it was seen as the last word in mathematical complexity, but along with that it was always clearly a theory of the physical world, and in fact a very successful one. Predictions of the theory are verified to many significant figures, classically in the solar system, particularly in the observation of planetary orbits, and more recently in observations of binary pulsars. These latter observations provide indirect evidence of the existence of gravitational radiation, behaving just as the theory predicts.

Now, in contrast to Special Relativity, there are very definitely mathematical problems in General Relativity awaiting solution, and that was the topic of this programme.

There are two approaches here: one may ask how this geometrically-based subject relates to other mathematical disciplines involving geometry, and how one sets about using the theory. For the first, it can be said that almost any idea useful in differential geometry will find application in General Relativity, but often with a distinctive slant because the geometry of GR is Lorentzian instead of Riemannian. For the second, one can think of GR as determining a space-time as a geometry evolving from suitable initial data. Now there are many problems: How does one construct the data? How does the evolution proceed and how may one reliably compute it, to extract quantitative predictions? What singularities may form, and are they 'censored' inside black holes? Indeed, what kinds of black holes are there?

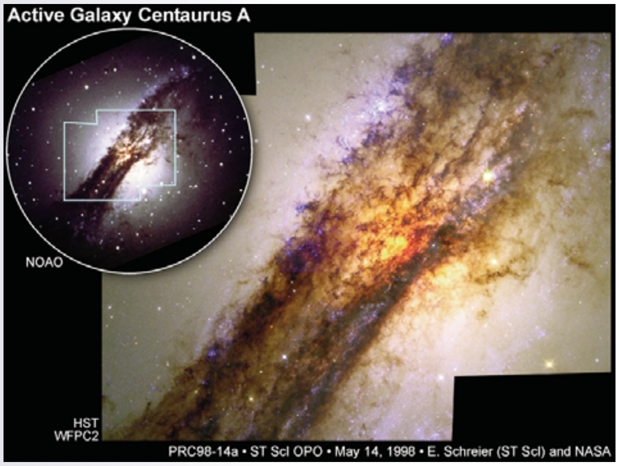
The aim of our programme was to address all of the above questions.

Structure of the Programme

The programme was attended by 61 long-stay participants and 91 short-stay ones, with a core of 7 researchers, 2 affiliates and 2 graduate students who attended the whole programme. The two single events that were most enthusiastically received by the participants were the conference on *Global General Relativity* and the Spitalfields Day, *Einstein and Beyond*. A satellite meeting, *New Directions in Numerical Relativity*, took place in Southampton. The programme was closed by a topical conference, *Einstein Constraint Equations*.

The programme itself was structured into emphasis weeks, which can be grouped roughly into five categories: evolution problems (including numerical

Space Telescope Science Institute and NASA



A massive black hole hidden at the centre of the Centaurus A galaxy (NGC 5128), feeding on a smaller galaxy

ones), constraint problems, global problems, quantum problems, and the remaining approaches (Riemannian and Lorentzian geometry, inverse scattering methods). Ninety-one talks were given during these emphasis weeks, outside of the workshops.

Workshops

New Directions in Numerical Relativity

Satellite Meeting at the University of Southampton, 18–19 August 2005

Organisers: C Gundlach and H Friedrich

The aim of this satellite meeting was to discuss mathematical aspects of the continuum and discrete models and to present simulations at the edge of what is currently possible.

It was attended by 46 researchers, including 18 members of the parent programme at the Newton Institute and 7 participants from Southampton. The programme consisted of 12 invited plenary talks of 45 minutes each, with 15 minutes' discussion. The simulations of binary black hole space-times presented by F Pretorius attracted a lot of interest, and his talk was considered a highlight of the meeting.

In spite of the tight schedule the participants had lively discussions after the lectures and during the breaks. The convenient accommodation and dinner arrangements allowed them to extend their discussions during and after the meals. The meeting was generally considered to be very stimulating and useful. It provided an opportunity for several younger participants (PhD students from Germany, Poland and Greece) to meet leading researchers in the field and share their ideas.

A special issue of the journal *Classical and Quantum Gravity* on numerical General Relativity, centred around the Southampton meeting together with a meeting that took place in Banff in February 2005, is in preparation.

Global General Relativity

Marie Curie Conference, 22–26 August 2005

Organisers: PT Chruściel, H Friedrich and P Tod

The aim of this conference was to give a wide-ranging review of the current status of General Relativity, with emphasis on the mathematical aspects but including observational and numerical results. There were 97 participants in total, including 17 graduate students and 11 post-doctoral researchers. The conference was filled to capacity, with several well qualified applicants turned down because of lack of space.

The 18 talks were of a very high standard. Taken as a whole, they were pedagogically accessible and covered the field, with excursions into nearby subjects from which the motivation for mathematical relativity may be drawn (specifically astrophysics and cosmology, numerical relativity, and higher-dimensional space-times inspired by string theory). We shall briefly describe the organisers' favourites.

Sir Martin Rees reviewed knowledge about black holes derived from observations, distinguishing stellar ($3–100M_\odot$), intermediate ($10^2–10^4M_\odot$) and massive ($10^4–10^{10}M_\odot$), discussing evidence for each class. This was a wide-ranging talk of great scope and skill. Of particular interest to this audience was the possibility of establishing that some black holes are rotating, using astrophysical signals derived from properties of the Kerr solution.

Greg Galloway gave a review of 'dynamical horizons' and his recent work with Ashtekar, which introduces real mathematical substance into this physically-motivated area of investigation. In one of several talks on the theory of partial differential equations as it illuminates General Relativity (the other talks being by Klainerman, Bizoń, Dafermos and Tao), Igor Rodnianski reviewed his 'new, economical' proof of the stability of Minkowski space. That is to say, data close to data for Minkowski space evolve to a give a space-time globally close to Minkowski space. The proof, with Lindblad, is remarkable in exploiting harmonic coordinates, which, as the conference heard from several speakers, are re-emerging as a valuable tool in both mathematical and numerical relativity. Other highlights included talks by Bob Wald on significant recent progress towards a rigorously defined, interacting quantum field theory in curved space-times using the ideas of micro-local analysis and local and covariant quantum fields; and Rick Schoen on analytic problems in the solution of the vacuum constraint equations, a subject which has made great progress in the last five years.

It is invidious to choose highlights, as very few talks were less than excellent. This view from the organisers was confirmed in conversations with participants and by comments in the end-of-conference questionnaires.

Einstein and Beyond

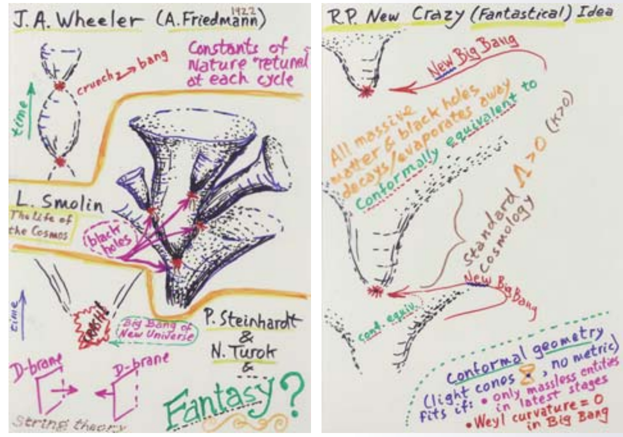
Spitalfields Day, 7 November 2005

Organiser: P Tod

This Spitalfields Day, sponsored by the LMS, was intended to mark the 90th birthday of General Relativity, and consisted of three talks on it and its extensions.

Under the title *Quantum Riemannian Geometry and its Ramifications*, Abhay Ashtekar described the programme for quantum gravity pursued by him and collaborators. Their aim is to construct a rigorous, background-independent quantisation of General Relativity. This leads to a striking new view of the physical universe in which, for example, area is quantised. For a system with finitely many degrees of freedom, it leads to a modified quantum mechanics and, in an application to cosmology, it becomes possible to evolve through the Big Bang, an idea which recurs below.

Karsten Danzman gave a very informative talk on *Gravitational wave astronomy: The large detectors are going into operation!*, with excellent graphics, including Einstein riding a laser beam on a bicycle. The subject is full of exciting prospects, as the observing run of the Geo 600 detectors starts at the



R Penrose

Two slides from Sir Roger Penrose's Spitalfields Day lecture. The entire lecture can be heard on the web *(see page 15)*

end of this year and the project will be fully operational and collaborating with the LIGO project from next spring. Already, the teams are looking forward to the next generation of detectors, which includes the ambitious LISA project for a vast detector based on an array of satellites.

Finally, the hall was packed for Sir Roger Penrose on *Before the big bang? A new perspective on the Weyl curvature hypothesis*. Accepting the observations of a positive cosmological constant, and assuming that all massive particles decay eventually, Sir Roger proposes a new view of the Universe at very late times. The matter content is solely massless particles and radiation, and space-time is very simple conformally. Now, according to his Weyl Curvature Hypothesis, the initial singularity of the Universe has finite or possibly zero Weyl curvature and so, at the level of conformal structure, the very early and very late Universe are very similar. They are distinguished by the behaviour of the conformal factor but, Sir Roger suggests, the conformal structure at the end of a phase of expansion may be continued through infinity as a new Big Bang.

These were three excellent talks, whose different styles complemented each other well.

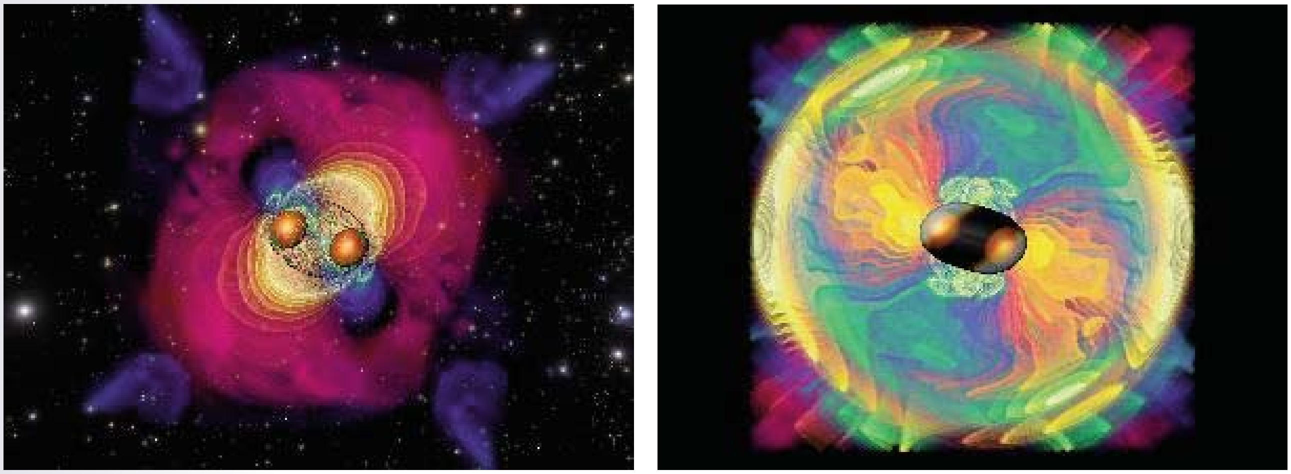
Einstein Constraint Equations

Marie Curie Conference, 12–16 December 2005

Organisers: PT Chruściel and J Isenberg

This conference was attended by 65 researchers, and attracted substantially more interest than expected.

Albert Einstein Institute



Numerical simulation of a grazing collision of two black holes

Unlike the earlier wide-ranging *Global General Relativity* conference which was held at the beginning of the programme, this conference was focussed on just one area of mathematical relativity: the mathematical study of the Einstein constraint equations. All the lectures were related to this topic in some way, and the meeting had a working character. There were 18 talks on a variety of issues, including: (i) use of the conformal method to obtain rough solutions and solutions with scalar fields; (ii) mass and quasi-local mass; (iii) gluing; (iv) preservation of the constraints during numerical evolution; (v) the Yamabe problem; and (vi) the mathematical description of the space of solutions of the constraints in phase space.

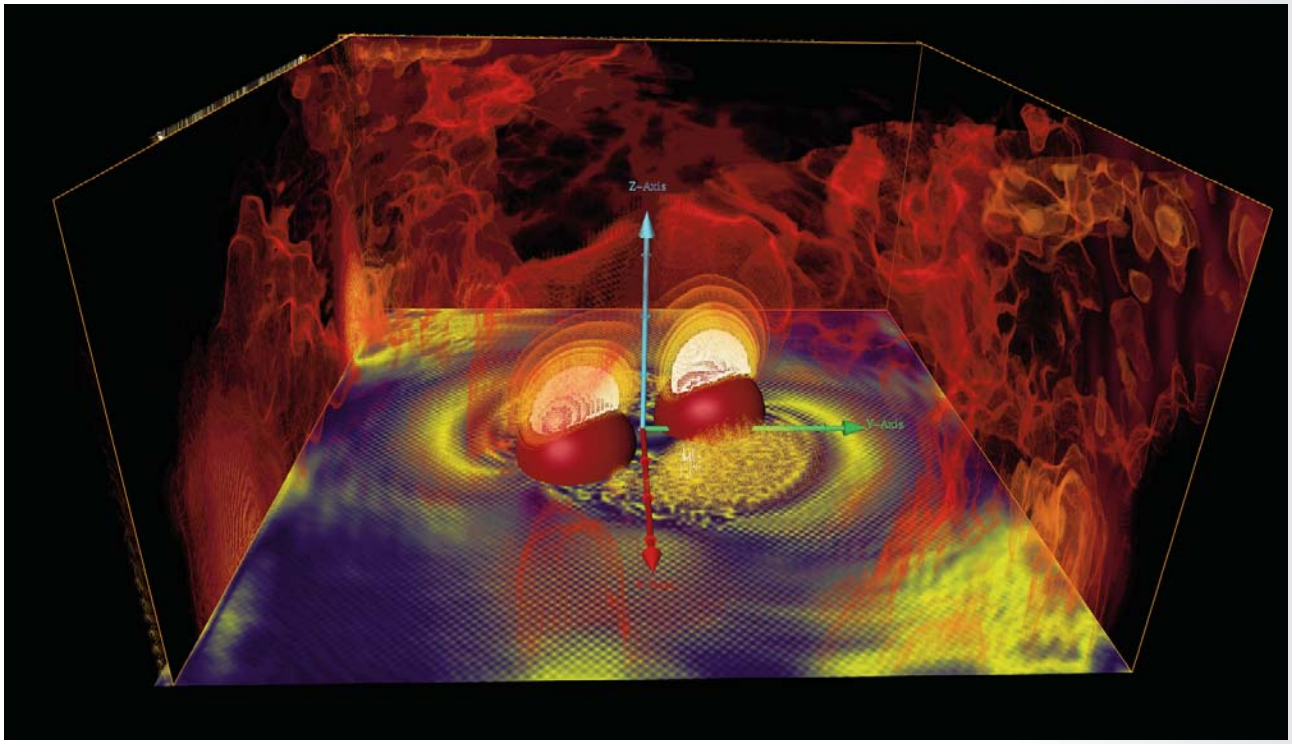
The talks of Dan Pollack, David Maxwell and Justin Corvino dealt directly with methods for finding solutions of the constraints. They showed that the conformal method for solving the constraints can now produce solutions of remarkably low differentiability, and can handle most matter source fields (including some that have caused difficulty in the past). They also showed how recent developments in the technology of scalar curvature deformation can be used to glue together solutions of the constraints in interesting ways.

Throughout the history of studies of solutions of the Einstein constraints, the concept of 'mass' has played a central role. At least five of the talks at the conference reflected the importance of mass. Noteworthy were Greg Galloway's discussion of a nonspinorial way to prove a positive mass theorem for hyperboloidal initial data, the analyses by Pengzi Miao and Niall O'Murchadha of difficulties with various notions of quasi-local mass, and Sergio Dain's study of the relation between spin and mass in axisymmetric black holes. Gerhard Huisken's excellent discussion of heat flow methods for generating special radial foliations of asymptotically flat initial data sets relates to the issue of mass as well.

Since the study of the constraint equations via the conformal method depends upon understanding conformal deformations of scalar curvature, there were three talks at the conference which discussed recent progress on the Yamabe problem. Taken together, the talks of Abbas Bahri, Simon Brendle, Marcus Khuri and Frank Pacard gave a rather complete picture of what is currently known about both singular and regular solutions of the Yamabe equation.

The constraint equations play a major role in current efforts to numerically model astrophysical events such as black hole collision. Two talks at the conference concerned the numerical treatment of the constraints. Lee Lindblom's talk provided some hope that we might some day be able to handle one of the big problems of numerical relativity: the exponential growth of the constraint functions during numerical evolution. The talk of Robert Bartnik, although not focused on numerical issues, provided a very nice mathematical structure for studying space-times in which the Einstein constraint equations are not satisfied.

The talks were all well attended by the conference participants, and the consensus was that they were generally of high quality.



Laboratoire de l'Univers et ses Théories, Meudon

Gravitational waves emitted by a binary system of neutron stars

Outcome and Achievements

The programme attracted a large number of participants, experts in all aspects of the field. Special care was taken to include promising young scientists, either as participants, junior members, affiliates or workshop participants. (All the graduate students who took part were enthusiastic about the insights they gained and the new perspectives opened for them.) This will have a long-term influence on the development of the field.

During the preparation of the programme it was recognised that the field is under-represented in UK universities, except perhaps in Oxford and in Cambridge itself. Efforts were made to alleviate this by inviting a wide spectrum of UK participants. In additions to the seminars at the Newton Institute, members of the programme gave 34 seminars throughout the UK.

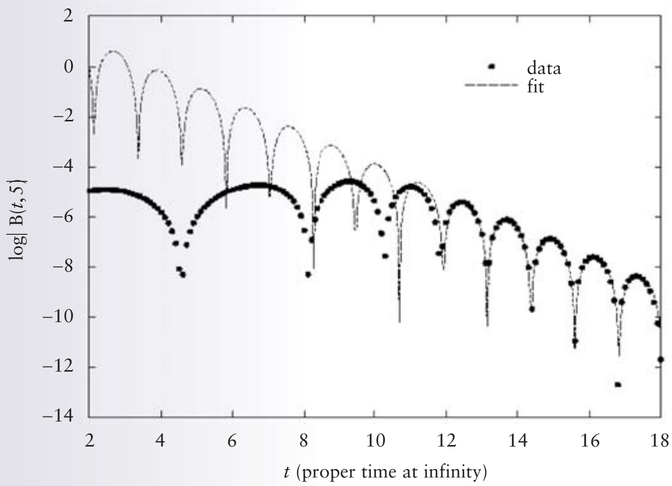
There was consensus about the outstanding level of many lectures. New collaborations were initiated, many more were continued, and extremely positive feedback was received concerning the impact of the programme on the research of many participants.

The participants of the programme submitted 31 papers to the Newton Institute preprint series.

Several very significant publications have appeared as a direct result of the research during the programme:

Mihalis Dafermos and Igor Rodnianski proved uniform decay bounds for solutions of the scalar wave equation on four-dimensional Schwarzschild space-time. This is a spectacular achievement, as this problem has been unsuccessfully studied by many researchers for years. The result is the starting point for any nonlinear stability analysis of black hole space-times. In related work, Mihalis Dafermos and Gustav Holzegel proved restricted nonlinear stability of a class of five-dimensional black holes found in June 2005 by Piotr Bizoń, Tadeusz Chmaj and Bernd Schmidt.

These solutions provide very interesting toy models, for which heuristic results from four dimensions can be rigorously established (the higher dimension making things easier): e.g., the existence of quasi-normal modes as illustrated in the figure overleaf. The stability of those black holes has been hinted at in numerical work by Piotr Bizoń during the programme. This is the first nonlinear stability result for the usual space-like Cauchy problem for black hole space-times. A follow-up to this work is another Newton Institute preprint by Piotr Bizoń



Quasi-normal ringing of Bizoń–Chmaj–Schmidt black holes (courtesy of Piotr Bizoń)

et al., analysing a nine-dimensional analogue of the Bizoń–Chmaj–Schmidt black holes. These papers will certainly start an avalanche of follow-up studies, including ongoing work by Bizoń and Gibbons.

Greg Galloway and Richard Schoen generalised Hawking's black hole topology theorem to all higher dimensions, showing that black hole horizons are necessarily of positive Yamabe type. This is a beautiful result, with an elegant proof published in the Newton Institute preprint series, of fundamental importance for the study of black holes.

Helmut Friedrich found necessary and sufficient conditions for the convergence of multipole expansions for static vacuum space-times. This closes a gap in our understanding of such metrics which has been open since the pioneering work by Geroch in 1970, together with its partial solution by Beig and Simon in 1980.

Piotr Chruściel and Paul Tod have closed the last gap of the 'static electro-vacuum no-hair theory', showing that the only regular static electro-vacuum black holes with degenerate components of the event horizon are the Majumdar–Papapetrou ones.

Vincent Moncrief finished writing his long-awaited paper on integral representation formulae for the curvature tensor with the associated *a priori* estimates. The result, published in a Newton Institute preprint, was discussed in detail during his lecture as Rothschild Visiting Professor at the Institute. The representation formula is widely expected to play a major role in the analysis of the dynamical properties of the Einstein equations. Variations and applications of Moncrief's result have already been discussed by Sergiu Klainerman in his lecture during the *Global General Relativity* conference. Another application, to self-force calculations, has been presented by Moncrief in a preprint.

Yvonne Choquet-Bruhat, Jim Isenberg and Dan Pollack have devised a very elegant (and natural) approach for describing the solvability of the constraint equations in the presence of a scalar field. This is done by introducing a generalised Yamabe-type invariant of the initial data, which provides necessary and sufficient conditions for the construction of the initial data using the conformal method.

The programme has been extremely stimulating. Several participants have indicated that the programme has already considerably affected their research, and there are clear indications that the programme will have a lasting impact.