Integrable Systems

23 July - 19 December 2001

Report from the organisers: Professor JC Eilbeck *(Heriot-Watt)*, Professor AV Mikhailov *(Leeds)*,
Professor PM Santini *(Rome)*, Professor VE Zakharov *(Moscow)*

Scientific Background

Many natural systems can be modelled by partial differential equations (PDEs), especially systems exhibiting wave-like phenomena. Such systems often have quantities that are conserved in time, common examples being energy or momentum. Often such systems are nonlinear; small changes in input can produce large changes in output, or vice versa. Mathematically, such nonlinearities make such systems difficult to study except using computer simulations.

Rather surprisingly, relatively sizable classes of nonlinear systems are found to have an extra property, integrability, which changes the picture completely. Integrable systems have a rich mathematical structure, which means that many interesting exact solutions to the PDEs can be found. Although important in their own right, these systems form an archipelago of solvable models in a sea of unknown, and can be used as stepping stones to investigate properties of \nearby" non-integrable systems.

A typical feature of integrable nonlinear wave equation is the existence of multi-soliton solutions, where a soliton is a stable solitary wave. The velocity of the soliton depends on its amplitude, so a bigger soliton can overtake a smaller one: the surprise is that after the collision the two waves separate with unchanged form, except for a phase shift.

Now that properties of most of the “standard" physical models (KdV, sine-Gordon, Non-linear Schr Ä dinger, etc.) exhibiting integrable behaviour are known, interest has shifted to more abstract mathematical questions, and to the remarkable connections between integrable system theory and many other core areas of mathematics: for example algebraic geometry, differential geometry, group theory, invariant theory, spectral theory, etc.

For a given PDE or discrete system, there are a variety of partially understood methods to deter mine the integrability or otherwise of the system. We would like to understand these methods better, and to understand the deep links that must lie between them, and to fully classify all integrable systems in some sense. Although it would be false to claim that the field was now completely unified and clarified after the meeting, much progress was made during the programme, and a large number of new collaborations were started which will bear fruit over the next few years.

Structure of the Programme

During the meeting there were a regular series of seminars, at least two per week on Mondays and Wednesdays. In addition there were a number of workshops and Conferences with up to a hundred participants for the major meetings.

Several visitors to the programme have travelled to other UK Universities to give seminars (Bogdanov, Boiti, Buchstaber, Calogero, Degasperis, Enolskii, Krichever Kruskal, MatinezAlonso, Tamizhmani, Tsarev, Zakharov).

A total of 53 long stay visitors attended the programme, 14 of whom were from the UK. There werein addition approximately 57 short stay visitors, 19of which were from the UK.

Workshops and Conferences

EuroConference: Nonlinear Evolution Equations and Dynamical Systems (NEEDS) XVth Meeting.
25 July - 31 July 2001
Organisers: M Bruschi, F Calogero, AV Mikhailov and PM Santini.

In the tradition of the NEEDS meetings, there were an eclectic mix of 78 half-hour talks plus a poster session in a very crowded but enjoyable schedule. The meeting covered a range of theory for integrable and near-integrable systems, with examples drawn from fluid mechanics, plasma, physics, nonlinear optics, general relativity, and BÄose-Einstein condensation. Techniques considered included spectral methods; direct methods (dbardressing, finite gap method, bilinear methods); integrability criteria and tests; bi-Hamiltonian systems and separation of variables; integrable quantizations, and integrable functional equations.

Particular highlights included the developmentof a Painlevé test for difference equations (Costin), the discovery of a new class of nonlinear evolution equations possessing many periodic trajectories (Calogero), a new theory of separation of variables for bi-Hamiltonian systems (Pedroni and Falqui), a dbar approach for the dispersionless KP hierarchy (Matinez Alonso), and a spectral approach to boundary value problems for linear and nonlinear PDEs (Pelloni).

The proceedings of this meeting will be published in two special issues of the Journal ofTheoretical and Mathematical Physics, edited by Mikhailov and Santini.

Euro Summer School: What is integrability?
13 - 24 August 2001
Organisers: A Degasperis, A Hone, AV Mikhailov and A Shabat

One of the main themes of this School was how to test for integrability in ordinary and partial differential equations. One type of test involves perturbative or asymptotic methods; these were represented in the lectures of Degasperis on multi-scale expansions, Zakharov on multidimensional perturbbation theory, and Kodama on normal forms. A second approach is algebraic, involving the classification of symmetries, described by Sokolov and Mikhailov, with connections to number theory, as discussed by Saunders. Yet another method is based on the analytic behaviour of solutions in the complex domain, the technique of Painlevé analysis were covered in the lectures by Kruskal.

Other major aspects of integrability were also treated. Olver lectured on multi-Hamiltonian structures, Shabat described BÄacklund transformations within the framework of dressing chains for linear operators, and Hietarinta described the Hi-rota bilinear method and the

connection with taufunctions. The relationship between integrability and geometry was explored in Mason's lectures and twistors and self-dual Yang-Mills equations, while Novikov discussed discrete symmetries and discrete systems on planar graphs. Miwa also gave an introductory course on quantum integrable systems. Other lecturers (Ablowitz, Clarkson, Flashka, Hone, and Winternitz) gave shorter courses and tutorials describing more recent developments in this field.

Selected lecture courses will be published in the book “Integrability" by Princeton University Press, edited by Mikhailov.

EuroWorkshop: Discrete systems and integrability

3 - 14 September 2001

Organisers: FW Nijhoff, J Hietarinta and PM Santini

There were 36 talks, almost all of them one-hour talks, organised in thematic sessions (most of which lasted a full working day), linked by an organized discussion. The themes of the sessions could be roughly divided by the following headings:

1. Discrete Painlevé equations and affne Weyl groups
2. Integrable lattices (partial difference equations)
3. Discrete and difference geometry
4. Algebraic integrability and computational aspects
5. Integrable mappings
6. Discrete Painlevé property
7. Quantum many-body systems and special functions
8. Inverse problems and solutions
9. Symmetries of difference equations
10. Cellular automata and applications

The talks were all of a very high level, and a large number of new results and ideas were put in front of the audience. In particular we mention the following contributions:

- M. Noumi (Kobe): “q-Painlevé” equations arising from a q-version of the modified KP hierarchy" (on the similarity reduction of the KP hierarchy and the emergence of discrete Painlevé equations form it).
- Veselov (Loughborough): “Discrete hydrodynamics and Monge-Ampere equations"
- C. Viallet (Paris VI): \Complexity, singularity and integrability of maps" (on the algebra-geometric analysis of singularities in birational mappings and their resolution through the blowing-up procedure)
- M.D. Kruskal: “Equivalent of the Painlevé property for difference equations and study of their solvability" (on a new class of “analysable" functions and their role in the definition of integrability in the discrete domain).

Workshop: Geometrical aspects of integrability

17 - 18 September 2001

Organisers: N Manton, L Mason and R Ward

Approximately 53 participants registered forthis meeting and there were several more attending many of the lectures. There were 9 one hour lectures over the two days including a selection from the leaders in the field and from younger up and coming researchers. One of the main foci was the interaction with equations coming from particle physics. Nigel Hitchin gave a new overview of various integrable geometric structures on the moduli space of Calabi Yau manifolds that have been discovered in the context of string theory. Nick Manton reviewed the theory of gauged vortices (not, strictly speaking integrable, but there are nevertheless many exact analytic results). Roger Bielawski reviewed the theoryof hyperkahler structures as an integrable system providing new methods for their study. Both

Mason and Calderbank reviewed different aspects of reductions of anti-self-dual conformal structures, the former from the point of view of the intrinsic equations and the latter from the point of view of the integrable equations that the background geometry can support. Ward's lecture concerned the analysis of a case of this, being the Bogomolny equations on a space of constant curvature and focussed on some of the new features that the background curvature leads to.

On more traditional topics, there were two lectures on Painlevé equations, Mazzocco's investigating when solutions can be obtained in terms of `classical functions' and Woodhouse's studying the isomonodromy problem using methods from twistor theory. Professor Zakharov's lecture concerned the integrability of a classical problem in geometry: finding metrics with `diagonal' curvature.

The meeting brought together people on the programme who's interest was from more of a traditional applied maths background with those whose primary interest was in geometry and physics, and this led to significant cross-fertilization.

Workshop: Computer algebra in applications to the problems of integrability
16 - 17 November 2001
Organiser: T Wolf

The workshop brought together users of computer algebra programs related to integrable systems (mainly among the audience), people involved more with the design of algorithms and others with more emphasis on the implementation of algorithms. The scope of talks reflected this wider scope and a number of people were happy to get in personal contact for the first time with others they knew only through their published papers. A novel aspect was that many computer packages in this are were made available during the meeting for participants to experiment with, also the organisers encouraged participants to suggest suitable computational challenges which could be addressed during and after the meeting.

Workshop: Algebraic aspects of integrability
26-30 November
Organisers: Yu Berest and AP Veselov

One of the highlights of the Programme was the special week devoted to Algebraic Aspects of Integrable Systems. The principal speakers were Pavel Etingof (MIT), Atsushi Matsuo (Tokyo) and Sergei Barannikov (ENS, Paris). They gave series of review lectures: Etingof on rational Cherednik algebras in relation to quantum Calogero-Moser systems, Matsuo on vertex operator algebras and the moonshine module, Barannikov on quantum periods and integrable hierarchies. Other speakers include Ch. Athorne (Glasgow), T. Brzezinski (Swansea), V.M. Buchstaber (Moscow), O.A. Chalykh and M.V. Feigin (Loughborough), V.Z. Enolski (Kiev).

The week culminated in the LMS Spitalfields Day at Loughborough University (November 30) where the most important recent achievements in the area were presented. P. Etingof gave a talk on his joint results with V. Ginzburg on symplectic reflection algebras, A. Matsuo on parafermion algebras and the Monster group, G. Wilson (IC) on his joint work with Yu. Berest in noncommutative projective geometry and O. Chalykh on his recent proofs of the Macdonald conjectures. These results demonstrated once again the importance of the fruitful interaction between the theory of integrable systems and other branches of Mathematics.

Workshop: Classical and Quantum Integrable Systems
2 - 8 December 2001
Organisers: R Weston, E Corrigan, JC Eilbeck and T Miwa

This was a Satellite Meeting hosted by the International Centre for Mathematical Sciences, Edinburgh, and took place on the Heriot-Watt University campus, close to the famous Union canal of soliton fame.

Approximately 70 participant took part in this lively meeting. There were 34 talks. 24 of these were by the main invited speakers. The remaining10 shorter talks were by specifically young participants. This mix was very effective in bringing new faces and new ideas into the meeting.

In the short talks category, the presentations by Caux, Castro-Alvaredo and Doikou notably inspired many scientific conversations. Many of the main talks presented important new results. The talks by Cardy, McCoy, Smirnov and Shiraishi might perhaps be particularly singled out. Cardy's talk showed that they are still many interesting physical systems for which novel exact results can be obtained via conformal field theory. McCoy presented his recent work in which, remarkably, he has discovered a new fundamental symmetry of our most thoroughly studied solvable lattice model.

Smirnov discussed recent important results in his ongoing programme to construct a more algebraic-geometrical description of quantum integrable systems. Shiraishi described a recent breakthrough that has enabled him to construct, after 10 years of labour, a free-field realisation of the algebra associated with the 8-vertex model. This construction enables him to compute exact correlation functions in this model.

There was much animated discussion of these and many other results at the meeting. Several collaborations certainly grew out of these discussions; two we know of involve Delius and Nepomechie, and Konno and McCoy.

In recent years, through the work of Sklyanin, Smirnov and others there have started to be signs of some convergence of the fields of classical and quantum integrability. The coming together of the two communities at this meeting, and the participation of many participants from the main Newton Institute programme, have certainly helped this trend. The organisers of the workshop believe that this will have a beneficial and lasting influence on both fields.

Workshop: Boundary valued problems
17 -18 December
Organisers: T Fokas, B Pelloni and AV Mikhailov

The workshop had been planned during the first meeting of the semester dedicated to Integrable Systems, when various short talks had focussed on boundary value problems, notably for the nonlinear Schroedinger equation (NLS), and it was felt that time had not allowed a proper discussion. Approximately 30 participant took part in this meeting, with 10 talks and plenty of discussions. Various approaches were discussed and compared, for specific examples as well as in talks surveying old and new results in this area. In particular, the results for NLS on the half line obtained by the approach of Fokas and his collaborators were presented alongside the results of Degasperis, Manakov, and Santini; this motivated a lively discussion among all the participants.

The possibility of exposition and discussion among few researchers interested in similar problems was very welcome by all, coming as it did at the end of a semester of concentration on integrable systems in general. It also stimulated the possibility of collaboration among different approaches; at least one such collaboration, between Pelloni and Jerome on the

sine-Gordon equation, was a direct result of the meeting. More generally, the workshop took advantage and fully exploited the opportunity offered by the presence at the Newton Institute of most of the people involved in this area of research.

Outcome and Achievements

The organisers and participants felt the programme was very successful, in bringing together a large number of key figures in the area, and in attracting a promising number of younger researchers. Apart from the hectic workshop schedule, much progress was made by the long- and short-stay participants in the invigorating surroundings e Newton Institute. Some long-standing problems were solved, and other new areas opened up for further investigation.

Martin Kruskal and KM Tamizhmani and others made progress in towards developing a “simple"proof of the Painlevé property for the six Painlevé equations.

Inspired by the close proximity to the haunts of the famous Cambridge Mathematician H. F. Baker, Chris Athorne, Chris Eilbeck, and Victor Enolskii developed a fully SL(2) invariant theory of genus 2 hyperelliptic } functions, opening up new areas inthis field as well as clarifying some rather obscure steps in Baker's original treatment. Athorne also worked with Sanders and with Hietarinta on generalizations of Hirota's bilinear derivative.

Tmara Grava, with Victor Enolskii, found new explicit solutions for the Riemann-Hilbert problem involving algebro-geometrci integration of Fuchsian systems and the Schlesinger equations. Mikhailov, Sanders and Wang discovered a O(N) invariant integrable generalisation of the famous sine-Gordon equation. This equation has important applications in the Riemanian geometry. Wang has also found an explicit generalisation of the Hasimoto transformation to the N-dimensional case.

Harry Braden made progress on a number of problems, including twistor theoies and the Calogero-Moser model (with Lionel Mason), StÄ ackel systems (with Alan Fordy), and Toda theory and Nahm monopoles (with Hermann Flaschka and Nick Ercolani). Flaschka also worked on combinatorial rules for the decomposition of tensor products of compact Lie groups, and related problems on the distribution of eigenvalues of random matrices.

Francesco Calogero worked on a number of ODE's and also on “cool" irrational numbers and their approximations. Darryl Holm performed a breakthrough numerical simulation showing that the NS-alpha model stimulates decay of a turbulent shear layer at least as well as highly tuned Large Eddy Simulation models.

Using asymptotic methods and the Inverse Scattering Transform, Mikhailov and Novokshenov found a new approach to the problem of DM solitons in non-linear optics. The analytically predicted shape of the soliton fits well with the correspondingnumerical simulations.

Andrew Hone, together with Degasperis and Holm, developed the integrability theory for a new equation proposed by Degasperis at the School. Mikhailov and Novikov formulated a perturbative version of the Symmetry Approach which is suitable for non-evolutionary and non-local equations, including multi-dimensional equations, and proved that the Camassa-Holm equation and a new equation dicovered by Degasperis are the only integrable equations in a certain class. They made a complete classification of integrable Benjabin-Ono type equations.

Andrei Kapeav, with Alexander Its, worked on assymptotics of the second Painlevé transcendent. Yuri Kodama started a new collaboration on the dispersionless KP equations, and worked with Mikhailov on normal forms and the symmetry approach for near integrable systems. Boris Konopelchenko worked with Martinez Alonso on integrable dispersionless

hierarchies, quaso-conformal maps, and -dressing method. Decio Levi, with Pavel Winternitz and Rafael Heredo, worked on Lie symmetries of difference equations. Vladimir Matveev worked on topological results for geodesically equivalent Riemannian metrics. Fank Nijhof and Hietarinta developed new ideas involving the characterisation of integrable mappings in one and two dimensions through the commutativity of discrete flows. Beatrice Pelloni worked on boundary value problems for the sine-Gordon and Schrodinger equations. Pogrebkov and Fokas studied initial value problems for the KPI equations with initial conditions given by single soliton solution plus a rapidly decaying term. Pogrebkov also constructed a hierarchy of quantum explicitly solvable models which can be considered as a quantum version of the dispersionless KdV hierarchy. A. B. Shabat studied a discrete version of the SchrÄ odinger spectral problem, and the connection between certain isospectral flows and infinite-dimensional hydrodynamic type systems.

Vladimir Sokolov, with Thomas Wolf, completed an investigation of vector (1 + 1) dimensional integrable models. With Andrey Tsiganov developed new integrable poynomila deformations of known integrable models from rigid body dynamics, includinga new integrable case fro the classical Kirchoff problem of motion of a rigid body.

Sergei Tsarev completed a comprehensive survey of work on integrable exponential systems and a review on algorithmic methods of integration of nonlinear ODEs. Vladimir Zakharov worked with a numer of people: with Tom Bridges on a hamiltonian description of a “renormalized" fluid; with Konopoelchenko and others on the dispersionless KP hierarchy, with Frederick Diaz on one-dimensional turbulence, with Fokas on boundary problems for integrable systems, with Jerry Griffiths on diagonal metrics in general relativity, and in addition found time to serach for new integrable solutions of the Einstein equations.

Back to Top